The Electromagnetic Equations of the Supraconductor

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Electric currents are commonly believed to persist in a supra-conductor without being maintained by an electromagnetic field. Thus the relation between the field strength **E** and the current density **J** in a supraconductor has sometimes been described† by means of an " acceleration equation," of the form

$$\Lambda \dot{\mathbf{J}} = \mathbf{E} \; ; \quad \Lambda = m/ne^2. \tag{1}$$

This equation, which might replace Ohm's law for supraconductors, simply expresses the influence of the electric part of the Lorentz force on freely movable electrons of the mass m and charge e, the number per cm^3 being n (we use rational units). By definition the constant Λ must be positive. As a direct consequence of this equation (1) stationary currents in supraconductors are possible when $\mathbf{E} = 0$.

We shall see, however, that actually equation (1), which we will refer to as the " acceleration theory," implies more than is verified by experiment; moreover, presupposing an acceleration without any friction it implies a premature theory, the development of which has presented a hopelessly insoluble problem to mathematical physicists. Apparently a model was wanted which would explain that in its most stable state the supraconductor has always a persistent current. We shall give a formulation which is somewhat more restricted in this respect. On the other hand it includes one more important fact, namely, the experiment of Meissner and Ochsenfeld.‡ In this way we get a new description of the electromagnetic field in a supraconductor, which is consistent and, as it eliminates unnecessary statements, is in closer contact with experiment. This new description seems to provide an entirely new point of view for a theoretical explanation.

† Becker, Heller, and Sauter, ' Z. Physik,' vol. 85, p. 772 (1933); Braunbeck, ' Z. Physik,' vol. 87, p. 470 (1934); London, ' Nature,' vol. 133, p. 497 (1934).
‡ ' Naturw.,' vol. 21, p. 787 (1933).

§ 1—The Fundamental Equations

Although we intend to abandon the " acceleration equation " (1) we shall take this equation as a provisional basis in order to find out the point where it must be corrected. Taking the curl of (1) and using $\operatorname{curl} \mathbf{E} = -\frac{1}{c}\dot{\mathbf{H}}$ we obtain

$$\operatorname{curl} \Lambda \dot{\mathbf{J}} = -\frac{1}{c}\dot{\mathbf{H}}, \tag{2}$$

or since $\frac{1}{c}\mathbf{J} = \operatorname{curl} \mathbf{H}$ (neglecting the displacement current)

$$\operatorname{curl} \operatorname{curl} \Lambda \dot{\mathbf{H}} = -\frac{1}{c^2}\dot{\mathbf{H}}$$

and as $\operatorname{div} \mathbf{H} = 0$

$$\Lambda c^2 \nabla^2 \dot{\mathbf{H}} = \dot{\mathbf{H}}. \tag{3}$$

Here we can integrate with respect to time and obtain:

$$\Lambda c^2 \nabla^2 (\mathbf{H} - \mathbf{H}_0) = \mathbf{H} - \mathbf{H}_0. \tag{4}$$

(4) is a nonhomogeneous equation for $\mathbf{H}$. $\mathbf{H}_0$ denotes the magnetic field at the time $t = 0$. The general solution of (4) follows by superposition of any particular solution on the general solution of the homogeneous equation

$$\Lambda c^2 \nabla^2 \mathbf{H} = \mathbf{H}. \tag{5}$$

The solutions of (5) which behave regularly inside the supraconductor decrease exponentially very quickly as one recedes from the surface, where they are fitted into the values of the external field. $\Lambda c^2 = mc^2/ne^2$ is of the order of magnitude 10^{-11} cm^2. A particular solution of (4) may be written down immediately, namely,

$$\mathbf{H} = \mathbf{H}_0.$$

Now the general solution of (4) follows by superposition of this solution (*i.e.*, of the original field $\mathbf{H}_0$) on our general solution of the homogeneous equation, which is not appreciably different from zero except near the surface.

The general solution means, therefore, that practically the original field persists for ever in the supraconductor. Only in a layer of the order 10^{-5} cm below the surface all disturbances take place reversibly, provided the threshold value is not exceeded. The field $\mathbf{H}_0$ is to be regarded as " frozen in " and represents a permanent memory of the field which existed when the metal was last cooled below the transition temperature.

Until recently the existence of " frozen in " magnetic fields in supraconductors was believed to be proved theoretically and experimentally. By Meissner's experiment,† however, it has been shown that this point of view cannot be maintained. It results clearly from the thermodynamic discussion of Gorter‡ that at the transition to the supraconducting state any magnetic field which may have existed before in the conductor is pushed out of it so that experiments which seemed to show that magnetic fields are frozen in are to be explained by the existence of non-supraconducting inclusions, in which the magnetic lines of force are pressed together.

Since magnetic fields under no circumstances can be found in the supraconducting phase, one is tempted to give the integration constant $\mathbf{H}_0$ of (3) in (4) the value zero. But if not all solutions of a differential equation exist in reality, the equation gives too general a description. One should not use a differential equation like (1) which contains too many possibilities, as it gives nature more freedom than it wants. If in reality $\mathbf{H}_0$ is always confined to the value zero, then this means that

$$\Lambda c^2 \nabla^2 \mathbf{H} = \mathbf{H}$$

is to be considered as a fundamental law and not to be treated as a particular integral of a differential equation in consequence of (1). Hence we abandon (1). Since $\operatorname{curl} \mathbf{H} = \frac{1}{c} \mathbf{J}$ we can write (5) in the form

$$\operatorname{curl} \Lambda \mathbf{J} = -\frac{1}{c} \mathbf{H}. \tag{6}$$

This we postulate as the fundamental equation which replaces Ohm's law in supraconductors.

Equation (6) says more than (2), so far as it includes Meissner's effect. Proceeding from (6) to (2) by differentiating with respect to the time we

† Meissner and Ochsenfeld, ' Naturw.,' vol. 21, p. 787 (1933); ' Z. ges. Kälte-industr.,' vol. 11, p. 125 (1934) ' Phys. Z., vol. 35, p. 954 (1934); de Haas and Casimir, ' Physica,' vol. 1, p. 291 (1934); Mendelssohn and Babbit, ' Nature,' vol. 133, p. 459 (1934); Mendelssohn and Moore, ' Nature,' vol. 133, p. 413 (1934). Schubnikoff and Rjalinin, ' Phys. Z. Sowjet,' vol. 5, p. 671 (1934); ' Nature,' vol. 134, p. 286 (1934). Keeley and Mendelssohn, ' Nature,' vol. 134 p. 773 (1934).

‡ ' Arch. Musée Teyler,' ser. III, vol. 7, p. 378 (1933); ' Nature,' vol. 132, p. 931 (1933). Gorter and Casimir, ' Physica,' vol. 1, p. 306 (1934), *see also* Ehrenfest and Rutgers, ' Comm. Phys. Lab., Leiden,' Supp. 75b. (Nachtrag).

lose this content. The logical relation between the three propositions (1), (2) and (6) may be represented by the following scheme:

$$\begin{tikzcd}[column sep=small] (1)\quad \Lambda\dot{\mathbf{J}} = \mathbf{E} \arrow[dr] & & (6)\quad \operatorname{curl} \Lambda\mathbf{J} = -\frac{1}{c}\mathbf{H} \arrow[dl] \\ & (2)\quad \operatorname{curl} \Lambda\dot{\mathbf{J}} = -\frac{1}{c}\dot{\mathbf{H}}. & \end{tikzcd}$$

The propositions (1) and (6) possess, so to speak, the same degree of generality. Assuming (6) instead of (1) we comprehend more in *one* respect, namely, Meissner's Effect, but less in another respect, for we cannot deduce (1) from (6); but we obtain from (2) the weaker statement:

$$\operatorname{curl}(\Lambda\dot{\mathbf{J}} - \mathbf{E}) = 0.$$

Inasmuch as (1) says more than (2) it expresses a prejudice which, in our opinion, is not tested by experience. We are only enabled to conclude, that $\Lambda\dot{\mathbf{J}} - \mathbf{E}$ can be represented as the gradient of a quantity μ:

$$\Lambda\dot{\mathbf{J}} - \mathbf{E} = \operatorname{grad} \mu. \tag{7}$$

Now the question arises whether μ is merely an integration constant or whether it represents a real physical quantity. Comparing (6) with (7) (which may be written in the form $\Lambda(\dot{\mathbf{J}} - \operatorname{grad} \mu/\Lambda) = \mathbf{E}$) we see that we can put together these six equations in the form of an equation for an antisymmetrical tensor:

$$\Lambda\left(\frac{\partial \mathrm{J}_i}{\partial x_k} - \frac{\partial \mathrm{J}_k}{\partial x_i}\right) = \frac{1}{c} f_{ik}. \tag{8}$$

Here we have named the field strengths E_x, E_y, E_z, H_x, H_y, H_z as usual by if_{14}, if_{24}, if_{34}, f_{23}, f_{31}, f_{12} and the co-ordinates x, y, z, ict by x_1, x_2, x_3, x_4. Then the quantity μ/Λ has to be regarded as the time-like supplement of the current density $\mathbf{J}$. As is well known from ordinary electrodynamics, this is the density of charge ρ.

Therefore the relativistic covariance would require

$$\mathrm{J}_4 = \frac{\mu}{ic\Lambda} = ic\rho. \tag{9}$$

This interpretation by the principle of covariance† now gives to equation (7) the quality of an *independent physical statement*

$$\Lambda\,(\dot{\mathbf{J}} + c^2 \operatorname{grad} \rho) = \mathbf{E}, \tag{10}$$

where ρ is connected with $\mathbf{E}$ by

$$\rho = \operatorname{div} \mathbf{E}. \tag{11}$$

Here for the sake of simplicity we have taken the value of the dielectric constant ε equal to one as we do not know anything about it. This may subsequently have to be corrected.

Putting (11) in (10) we get

$$\Lambda\,(\dot{\mathbf{J}} + c^2 \operatorname{grad} \operatorname{div} \mathbf{E}) = \mathbf{E},$$

or since $\operatorname{grad} \operatorname{div} \mathbf{E} = \nabla^2 \mathbf{E} + \operatorname{curl} \operatorname{curl} \mathbf{E} = \nabla^2 \mathbf{E} - \frac{1}{c^2} \dot{\mathbf{J}}$,

$$\Lambda c^2\, \nabla^2 \mathbf{E} = \mathbf{E} \tag{12}$$

and by taking the divergence we get

$$\Lambda c^2\, \nabla^2 \rho = \rho. \tag{13}$$

Thus we see the 10 quantities $\mathbf{E}$, $\mathbf{H}$, $\mathbf{J}$, ρ obey the same equation.

So far we have neglected throughout the displacement current. If one considers it, these equations follow:

$$\left.\begin{aligned} \nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} &= \frac{1}{\Lambda c^2}\mathbf{E} \\ \nabla^2 \mathbf{H} - \frac{1}{c^2}\frac{\partial^2 \mathbf{H}}{\partial t^2} &= \frac{1}{\Lambda c^2}\mathbf{H} \\ \nabla^2 \mathbf{J} - \frac{1}{c^2}\frac{\partial^2 \mathbf{J}}{\partial t^2} &= \frac{1}{\Lambda c^2}\mathbf{J} \\ \nabla^2 \rho - \frac{1}{c^2}\frac{\partial^2 \rho}{\partial t^2} &= \frac{1}{\Lambda c^2}\rho \end{aligned}\right\}. \tag{14}$$

The field strength $\mathbf{E}$ and $\mathbf{H}$ may be derived as usual from a scalar potential ϕ and a vector potential A by

$$\left.\begin{aligned} \mathbf{E} &= -\operatorname{grad} \phi - \frac{1}{c}\dot{A} \\ \mathbf{H} &= \operatorname{curl} A \end{aligned}\right\}. \tag{15}$$

† It is very remarkable that equation (6) has a four-dimensional supplement, which does not explicitly contain the velocity of the supraconductor. Of course (10) is not the only possible form of a covariant supplement, but it is apparently distinguished by its simplicity and there is no reason to consider more complicated ones.

Comparing (15) with (10) and (6) we see, that it is possible† to choose the potentials—which are not absolutely uniquely determined by the field—so that they become proportional to the density of current and charge:

$$\left.\begin{aligned} \Lambda c\mathbf{J} &= -\,\mathrm{A} \\ \Lambda c^2 \rho &= -\,\phi \end{aligned}\right\}. \tag{16}$$

These equations, which could also be employed as the fundamental equations of the theory do not contain any dynamics as equation (1) nor any other explicit statements about temporal variations. Like the specific resistance in Ohm's law Λ is a constant depending on the material. There is no particular reason now, for attributing to it the value given in (1) although for the atomistic explanation no very different interpretation is to be expected (see equation 32)).

The additional condition for the vector potential

$$\operatorname{div} \mathrm{A} + \frac{1}{c}\,\dot{\phi} = 0$$

corresponds, because of (16), to the equation of continuity:

$$\operatorname{div} \mathbf{J} + \dot{\rho} = 0.$$

Putting the second equation (16) in (10), which expresses the modified equation of acceleration, the latter becomes

$$\Lambda\dot{\mathbf{J}} = \mathbf{E} + \operatorname{grad}\,\phi. \tag{17}$$

We learn from it that the acceleration of $\mathbf{J}$ is only due to that part of $\mathbf{E}$ which remains when the potential part ($-$ grad ϕ) has been subtracted. When $\mathbf{E}$ is merely a potential field it has therefore no influence on the supraconducting current.

§ 2—The Law of Conservation of Energy, Production of Heat, Boundary Conditions

As usual the law of conservation of energy follows from Maxwell's equations. But now in the equation

$$\operatorname{div} c\,[\mathbf{EH}] = -\frac{\partial}{\partial t}\{\tfrac{1}{2}(\mathbf{H}^2 + \mathbf{E}^2)\} - (\mathbf{JE}),$$

† Here we consider only simply connected supraconductors. As to the generalization for multiply connected supraconductors see a paper in "Physica."

the term (**JE**) has not simply the significance of Joule heat. From the equation $\mathbf{E} = \Lambda\,(\dot{\mathbf{J}} + c^2 \operatorname{grad} \rho)$ we find:

$$(\mathbf{JE}) = \Lambda\,\{\mathbf{J}\dot{\mathbf{J}} + c^2\,(\mathbf{J} \operatorname{grad} \rho)\}$$
$$= \Lambda\left\{\frac{d}{dt}\left(\frac{\mathbf{J}^2}{2}\right) + c^2 \operatorname{div}(\rho\mathbf{J}) - c^2\rho \operatorname{div} \mathbf{J}\right\}$$
$$= \frac{d}{dt}\left(\frac{\Lambda}{2}\,\mathbf{J}^2\right) + c^2\Lambda\left\{\operatorname{div}(\rho\mathbf{J}) + \frac{d}{dt}\left(\frac{\rho^2}{2}\right)\right\}.$$

Therefore the energy principle may be written in the following form:

$$\operatorname{div} c\,[\mathbf{EH}] = -\frac{\partial}{\partial t}\left\{\tfrac{1}{2}\,(\mathbf{H}^2 + \mathbf{E}^2) + \frac{\Lambda}{2}\,(\mathbf{J}^2 + c^2\rho^2)\right\} - \Lambda c^2 \operatorname{div} \rho\mathbf{J}. \quad (18)$$

In addition to the usual terms for the magnetic and the electric energy there is a term $\frac{\Lambda}{2}\,\mathbf{J}^2$, a kind of kinetic energy of the persistent currents and another term $\frac{\Lambda}{2}\,c^2\rho^2$, which may be interpreted as an additional potential energy connected with the density of electric charges in the supraconductor.

The last term ($-\,\Lambda c^2 \operatorname{div} \rho\mathbf{J}$) is the most interesting one. Its significance may become clearer by integrating over the whole of the supraconducting phase and transforming the integrals of the divergences into surface integrals:

$$c\int[\mathbf{EH}]_n\,d\sigma = -\frac{\partial}{\partial t}\int\left[\tfrac{1}{2}\,\mathbf{H}^2 + \mathbf{E}^2) + \frac{\Lambda}{2}(\mathbf{J}^2 + \rho^2 c^2)\right]dS - \Lambda c^2\int \rho\mathbf{J}_n\,d\sigma.$$

In consequence of the continuity of the tangential components of **E** and **H** we can substitute on the left-hand side the components of the field outside the supraconductor, where $c\,[\mathbf{EH}]$ is known to be the Poynting vector, *i.e.*, the flow of electromagnetic energy. The term

$$Q = \Lambda c^2 \int \rho\mathbf{J}_n\,d\sigma, \quad (19)$$

therefore, must be an amount of energy, which balances the flow of electromagnetic energy particularly for stationary processes ($\partial/\partial t\,\ldots = 0$), where the state of the supraconductor does not change. Q must be a non-electromagnetic form of energy and as such heat is the only form of energy which comes into question.

In the interior of the supraconductor the law of conservation of energy can be written in the form

$$\operatorname{div} \mathrm{S} + \frac{\partial \mathrm{W}}{\partial t} = 0. \tag{20}$$

Here the quantities

$$\mathrm{W} = \tfrac{1}{2}(\mathbf{H}^2 + \mathbf{E}^2) + \frac{\Lambda}{2}(\mathbf{J}^2 + c^2\rho^2) \tag{21}$$

and

$$\mathrm{S} = c\,[\mathbf{EH}] + \Lambda c^2 \rho \mathbf{J} \tag{22}$$

are to be interpreted as the density and as the flow of the total energy in the supraconductor.

Equation (20) states that *inside* the supraconductor no energy disappears, *i.e.*, changes into heat. The production of heat is therefore to be localized exactly on the surface of the supraconductor. There the flow of energy, changing discontinuously from $\Lambda c^2\rho\mathbf{J} + c[\mathbf{EH}]$ to $c[\mathbf{EH}]$, has a surface divergence. (19) shows that the production of heat occurs where the electric current has a normal component to the surface and meets there an electric density. It can easily be shown that the heat Q produced, where the current I enters and leaves the supraconductor, is always positive and is exactly equal to VI, V being the difference of voltage through which the current passes. Joule's law, therefore, is fulfilled, at least for the supraconductor as a whole.

For the sake of completeness we announce the components of the Maxwell stresses T_{ik} in the supraconductor:

$$\mathrm{T}_{ik} = \mathrm{T}_{ik}(\mathrm{E}) + \mathrm{T}_{ik}(\mathrm{H}) - \mathrm{T}_{ik}(\mathrm{J}) - \frac{\delta_{ik}}{2}\Lambda c^2\rho^2.$$

Here $$\mathrm{T}_{ik}(\mathrm{E}) = \mathrm{E}_i\mathrm{E}_k - \tfrac{1}{2}\delta_{ik}\mathrm{E}^2, \text{ etc.}$$

$$\delta_{ik} = \begin{cases} 1 & \text{for } i = k \\ 0 & \text{for } i \neq k. \end{cases}$$

Then the energy impulse tensor Θ_{ik} comprehending as usual the stresses, the flow and the density of energy in its four-dimensional scheme may be written in the form:

$$\Theta_{ik} = \sum_{r=1}^{4} f_{ir} f_{kr} - \tfrac{1}{4}\delta_{ik} \sum_{r,s=1}^{4} f_{rs}^2 - \Lambda\left(\mathrm{J}_i\mathrm{J}_k - \frac{\delta_{ik}}{2}\sum_{r=1}^{4}\mathrm{J}_r^2\right).$$

Here we use the same notation as in equation (8). The four-dimensional divergence of this tensor vanishes identically:

$$\sum_{k=1}^{4} \frac{\partial \Theta_{ik}}{\partial x_k} = 0.$$

That means that no *ponderomotiv* volume forces are acting inside the supraconductor, even when charges and currents are present in its interior.

It remains to establish the *boundary conditions* for the transition from the supraconductor to the adjacent insulator or normal conductor respectively. Of course, as always in Maxwell's theory, the tangential components of $\mathbf{E}$ and the normal components of $\mathbf{H}$ and of $\mathbf{J} + \dot{\mathbf{E}}$ (or $\mathbf{J} + \dot{\mathbf{D}}$) must be continuous. Further, we think it reasonable to postulate continuity of the normal component of $\mathbf{E}$ (or $\mathbf{D}$) and of the tangential component of $\mathbf{H}$ in order to get a unique solution. Discontinuity of these components would mean the possibility of mathematical surface charges of an arbitrary amount. As the space charges lie always so near the surface that they appear macroscopically to be superficial ones it does not seem physically plausible to assume mathematical surface densities in addition. In contrast to normal conductors surface currents and surface charges of a finite amount would lead to an infinite value of the additional energy term $\frac{\Lambda}{2}(\mathbf{J}^2 + c^2\rho^2)$ in the supraconductor and therefore they are excluded by the theory itself. But applying these boundary conditions (continuity of *all* components of $\mathbf{E}$ and $\mathbf{H}$) there is some consideration necessary, as the smallest normal conducting layer on the surface would give rise to surface charges according to Maxwell's theory. A special question will be the boundary conditions at the boundary between supraconducting and non-supraconducting phase in the same metal. We shall treat this in § 4.

§ 3—Supraconducting Sphere on a Homogeneous Electric Field

As a simple application of this theory we consider a supraconducting sphere in a homogeneous electric field. Let R be the radius of the sphere and $r_0\vartheta_0\psi$ spherical polar co-ordinates with the axis coinciding with the direction of the field.

(1) *The adjacent medium is an insulator*—Then we have the equations

$$\nabla^2\phi = 0 \text{ outside the sphere,}$$

$$\nabla^2\phi = \beta^2\phi \qquad \beta^2 = \frac{1}{\Lambda c^2} \text{ inside the sphere.}\dagger$$

† In the supraconductor ϕ can always be chosen so that it becomes proportional to ρ (see equation (16)) and then, like the latter, obeys equation (13).

The potential is completely determined from its asymptotic values at great distances by the postulate that ϕ and $\partial\phi/\partial r$ shall be continuous on the surface of the sphere. We obtain

$$\phi = \begin{cases} \left(\dfrac{A}{r^2} - rE\right)\cos\vartheta & \text{or } r \geqslant R \\ \dfrac{B}{\beta r}\left(\cosh\beta r - \dfrac{\sinh\beta r}{\beta r}\right)\cos\vartheta & \text{for } r \leqslant R \end{cases}$$

E is the asymptotic field strength and

$$A = ER^3\left[1 - \frac{3}{\beta R}\left(ctgh\,\beta R - \frac{1}{\beta R}\right)\right]$$

$$B = -\frac{3RE}{\sinh\beta R}.$$

For $r \leqslant R$ the potential gives immediately the electric charges according to $\rho = -\dfrac{1}{\Lambda c^2}\phi$. It is easy to see that the distribution of these charges near to the surface is practically the same as the distribution of charges induced on the surface of a normal conducting sphere in a homogeneous electric field. The lines of the electric field strength end on the charges below the surface.

(2) *Now the adjacent medium may be a conductor* of the conductivity σ. Then the equations and boundary conditions of ϕ are exactly the same as before and we obtain the same electric field and charges. But now outside the sphere a current **J** is connected with the field by $\mathbf{J} = -\sigma\,\mathrm{grad}\,\phi$. Since div $\mathbf{J} = 0$ we have to postulate the continuity of J_r on the surface of the sphere. Now the current **J** inside is uniquely determined by its normal component on the surface and by the equations

$$\mathrm{curl}\,\mathrm{curl}\,\mathbf{J} + \beta^2\mathbf{J} = 0 \quad \text{and} \quad \mathrm{div}\,\mathbf{J} = 0.$$

We obtain

$$J_r = -\frac{k\cos\vartheta}{\beta r^2}\left(\cosh\beta r - \frac{\sinh\beta r}{\beta r}\right)$$

$$J_\vartheta = \frac{k\sin\vartheta}{2r}\left[\left(1 + \frac{1}{\beta^2 r^2}\right)\sinh\beta r - \frac{1}{\beta r}\cosh\beta r\right]$$

$$k = \frac{3\sigma ER}{\sinh\beta R}\left[2 + \frac{\beta^2R^2}{1 - \beta R\,ctgh\,\beta R}\right].$$

In the supraconductor the distribution of current is not parallel to the electric field. The streamlines of **J** are broken at the surface in contrast to the electric lines of force, which are continuous.

§ 4—Supraconducting Wire. The Transition Curve

As another application we shall consider the problem of the distribution of a given current in a circular supraconducting wire of infinite length.

We use cylindrical co-ordinates z, r, ϑ with the z-axis coinciding with the axis of the wire. Let a be the radius of the wire and R that of the boundary surface between the supraconducting and the normally conducting phase.

1—As long as $I < I_T = 2\pi ac H_T$ (H_T is the threshold value) only the supraconducting phase exists. With the notation $\beta^2 = \frac{1}{\Lambda c^2}$ the equation for the current density **J** may be written

$$\text{curl curl } \mathbf{J} + \beta^2 \mathbf{J} = 0. \tag{23}$$

From reasons of symmetry **J** has only a z-component, which can only depend on r. Then $J = J_z(r)$ obeys the equation

$$\frac{\partial^2 J}{\partial r^2} + \frac{1}{r}\frac{\partial J}{\partial r} - \beta^2 J = 0.$$

This is the Bessel differential equation for $J_0(i\beta r)$.† This solution has still to be normalized. This gives

$$J = I \frac{i\beta}{2\pi a} \frac{J_0(i\beta r)}{J_1(i\beta a)}. \tag{24}$$

For H we get

$$H = H_\phi = -\frac{1}{\beta^2 c} \text{curl}_\phi \, \mathbf{J} = \frac{I}{2\pi ac} \frac{J_1(i\beta r)}{J_1(i\beta a)}. \tag{25}$$

Current and field are near the surface in a layer of the thickness $\frac{1}{\beta} \sim 10^{-5}$ cm. If $r \gg 10^{-5}$ cm we may replace the Bessel functions by exponential functions and we get

$$\left.\begin{aligned} J = J_z &= I \frac{\beta}{2\pi\sqrt{ar}} e^{\beta(r-a)} \\ H = H_\phi &= I \frac{1}{2\pi c\sqrt{ar}} e^{\beta(r-a)} \end{aligned}\right\} \text{ for } r \leqslant a.$$

Outside the wire **H** is given, as usual, by div **H** = 0, rot **H** = 0 and the postulate of continuity on the surface.

The well-known solution is

$$H = H_\phi = I \frac{1}{2\pi rc} \qquad \text{for } r \geqslant a.$$

† Confusion between current density J and the Bessel functions J should not arise since the latter have always numbers as indices.

In the supraconductor the electric field is not connected with the current; it is only determined by the equations $\nabla^2\mathbf{E} = \beta^2\mathbf{E}$ and curl $\mathbf{E} = 0$. If we assume continuity of all components of $\mathbf{E}$ on the boundaries these equations involve only a very short continuation of the external fields at the ends into a layer of the supraconductor about 10^{-5} cm thick. In this layer $\mathbf{E}$ decreases exponentially from the adjacent normal conducting leads, so that in practice one would find no *potential-difference in a supraconductor*, in agreement with the classical experiment of supraconductivity.

2—Suppose now $I > I_T = 2\pi ac H_T$.

From the centre to a radius R we have $H < H_T$. Here the equations of § 1 are valid, beyond R, where $H \geqslant H_T$, we will assume Ohm's law.†

Then the total current I is divided into

$$I = I^{(i)} + I^{(e)}.$$

On the surface of the inner current $I^{(i)}$ the magnetic field produced by it will have just the value H_T, therefore we get for $I^{(i)}$ the equation

$$I^{(i)} = 2\pi c R H_T. \tag{26}$$

The part $I^{(e)}$ flowing on the outside as an ordinary conduction current is necessarily accompanied by an electric field strength $E = E_z$ according to Ohm's law

$$\sigma E \,.\, \pi\,(a^2 - R^2) = I^{(e)} = I - 2\pi c R H_T = I - I_T \frac{R}{a},$$

or

$$E = \frac{I - I_T \dfrac{R}{a}}{\sigma\pi a^2 \left[1 - \left(\dfrac{R}{a}\right)^2\right]}. \tag{27}$$

This field strength $\mathbf{E}$ has to be continued through the boundary into the supraconducting part. If the equation of acceleration (1) were true, the current $I^{(i)}$ in the supraconducting phase would be accelerated by the electric fields. The magnetic field connected with the current would increase and rise above the threshold value H_T below the separating surface. The latter would, therefore, shrink inwards and the process would continue until no supraconducting phase would be left.

† This condition for the boundary between both phases of the supraconductor is to be regarded as quite provisional, as possibly the whole conception of these two separated phases is too simple.

But during this process in the non-supraconducting part the magnetic field would decrease. An elementary calculation shows that as soon as R is smaller than a certain value R_0 given by

$$R_0 = a\left(\frac{I}{I_T} - \sqrt{\left(\frac{I}{I_T}\right)^2 - 1}\right), \tag{28}$$

the magnetic field in the non-supraconducting phase would not everywhere exceed H_T. There supraconductivity should appear again, in contradiction with the mechanism we have described, provided that we accept the conception that supraconductivity appears where the magnetic field is smaller than the threshold value.

In contrast to that our equations do not imply any acceleration by an electrostatic field in a supraconductor. We will consider how far this enables us to avoid these difficulties.

For current density $\mathbf{J}^{(i)}$ and magnetic field $\mathbf{H}^{(i)}$ we can simply take the solution (24), (25) replacing a by R:

$$J = J_z = i\beta c H_T \frac{J_0(i\beta r)}{J_1(i\beta R)}$$

$$H = H_\phi = H_T \frac{J_1(i\beta r)}{J_1(i\beta R)}.$$

Since $\dot{H}^{(i)} = 0$ the electric field $\mathbf{E}^{(i)}$ has no curl and therefore it may be represented by a potential ϕ, the latter obeying the equation,

$$\nabla^2\phi - \beta^2\phi = 0,$$

and the boundary condition on the surface,

$$-\left(\frac{\partial\phi}{\partial z}\right)_{r=R} = E_z = E.$$

The only regular solution is

$$\phi = -Ez\frac{J_0(i\beta r)}{J_0(i\beta R)}. \tag{29}$$

Therefrom we derive the components of the electric field $E^{(i)}$

$$\left.\begin{aligned} E_z^{(i)} &= E\,.\,\frac{J_0(i\beta r)}{J_0(i\beta R)} \\ E_r^{(i)} &= i\beta z\,E\,\frac{J_1(i\beta r)}{J_0(i\beta R)} \end{aligned}\right\}. \tag{30}$$

As this field is derived from a potential it has no influence on the distribution of current calculated before. In contrast to the " theory of acceleration " we have now stationary conditions.

It seems important to emphasize that the field $\mathbf{E}^{(i)}$ could not be determined unless the normal component of $\mathbf{E}$ on the boundary surface between supraconducting phase and normal conducting phase is discontinuous. It is evident that the boundary between two phases of one substance is a quite different thing from the boundary between two substances and therefore the same boundary conditions may not be expected in both cases. If E_n were postulated to be continuous on the surface between both phases no solution would exist as long as we maintain the conception that $H < H_T$ or $H > H_T$ decides where supraconductivity occurs and where not.

It seems to be remarkable that our solution has charges linearly increasing along the wire. The order of magnitude of the electric fields belonging to them are, however, very small compared with the magnetic fields, at least for such lengths of supraconducting wire as have been investigated until now. For longer wires it may be necessary to consider an influence of these charges and fields on equation (26). On the other hand it would be interesting to calculate the conditions at the ends of a finite wire where the current enters. But here we shall defer the consideration of such details.

The radius R of the boundary surface is not yet determined. We see only that any radius $R > R_0$ (see equation (28)) would agree with our electrodynamics and could give us a stationary distribution of current with a certain field strength E. R functions here as a parameter and therefore cannot be determined by the differential equations of the problem. In order to determine R we need a further condition. Whether this condition can be derived from thermodynamics or whether the present theory requires still another supplement as to the electrodynamics at the boundary between both phases is a question which must be the subject of a more thorough and general consideration, with which we cannot deal in this paper. But one is tempted to guess that in our special case only the value $R = R_0$ (equation (28)) can come into question. It is distinguished from any other possible value of R. For instance, it could be characterized as giving the minimum of the surface charges or of the Joule-heat. Taking the Joule-heat,

$$Q = IE = \frac{1}{\sigma\pi a^2} \cdot I^2 \frac{1 - \frac{I}{I_T}\frac{R}{a}}{1 - \left(\frac{R}{a}\right)^2},$$

we get for its minimum the equation,

$$\left(\frac{R}{a}\right)^2 - 2\left(\frac{R}{a}\right)\left(\frac{I}{I_T}\right) + 1 = 0.$$

The only solution of this equation which comes into question ($R < a$) is $R = R_0$ (equation (28)).

Though this supposition requires further examination we will shortly consider its consequences.

Putting $R = R_0$ into (27) we get the resistance ω as a function of the current I $\left(\omega_0 = \frac{1}{\sigma\pi a^2}\right.$ being the resistance per cm length of the wire in the non-supraconducting state$\left.\right)$,

$$\omega = \frac{E}{I} = \frac{\omega_0}{I}\frac{I - I_T\frac{R_0}{a}}{1 - \frac{R_0^2}{a^2}}$$

$$= \frac{\omega_0}{2}\left(1 + \sqrt{1 - \left(\frac{I_T}{I}\right)^2}\right). \tag{31}$$

This can be used, of course, only for $I \geqslant I_T$. For $I = I_T$ this solution gives the value $\omega_0/2$. For $I < I_T$ we have the solution which was calculated at the beginning of this paragraph with $E = 0$; that means $\omega = 0$. Therefore the resistance drops discontinuously from its half value to zero.

Considered as a function of I_T (31) gives the transition curve as a function of temperature. Possibly Λ is to be regarded as dependent on temperature. But the result (31) is not dependent on the value of Λ. I_T is given by the curve of the threshold value of the critical magnetic field H_T which, near the critical temperature T_0, can be represented linearly in the form

$$H_T = k\,(T_0 - T).$$

Therefore analogously

$$I_T = 2\pi c a H_T = b\,(T_0 - T).$$

This gives

$$\omega = \frac{\omega_0}{2}\left\{1 + \sqrt{1 - b^2\left(\frac{T - T_0}{I}\right)^2}\right\}.$$

In the limit for very weak currents the resistance curve therefore drops absolutely vertically from its value at the temperature T_0 to the value zero (see fig. 1). The comparison with the measurements of de Haas and

Voogd† shows qualitative conformity, but it seems premature to discuss the experiments without considering further details.

Concluding Remarks

In considering the equations (16) one is very strongly reminded of Gordon's formulæ for electric current and charge in his relativistic formulation of Schrödinger's Theory:

$$\left.\begin{aligned} \mathbf{J} &= \frac{he}{4\pi im}(\psi \operatorname{grad} \psi^* - \psi^* \operatorname{grad} \psi) - \frac{e^2}{mc}\psi\psi^*\mathbf{A} \\ \rho &= \frac{he}{4\pi imc^2}\left(\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}\right) - \frac{e^2}{mc^2}\psi\psi^*\phi \end{aligned}\right\}. \quad (32)$$

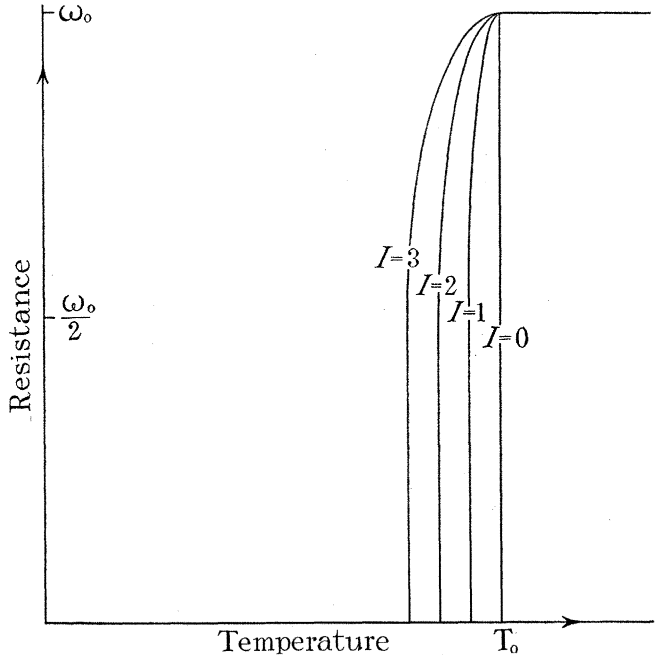

Fig. 1—The transition curve for different values of the current I. I in arbitrary units

Let ψ be the wave function of a single electron in the self-consistent field of the others; then in Gordon's formulation $\psi\psi^*$ gives at least approximately the value of the statistical expectation for an electron at every point of the space. Summing over all electrons $\Sigma\psi\psi^*$ gives the number of electrons per cm^3, varying about its average only in very small spaces.

† 'Comm. Phys. Lab. Leiden,' No. 214c.

Therefore macroscopically, since $\Lambda = \frac{m}{ne^2}$, the last terms in (32) are equal to $\frac{1}{\Lambda c}\mathbf{A}$ and $\frac{1}{\Lambda c^2}\phi$ respectively.

Using the original eigenfunctions of the free electrons in the metal the terms in brackets in (32) would vanish by summing over all electrons and (32) would become exactly identical with (16). This follows for $\mathbf{J}$ from reasons of symmetry, for ρ from the compensating presence of the positive ions. But actually the eigenfunctions of the electrons in the metal are disturbed by the magnetic field and therefore the terms in brackets in (32) do not vanish. Moreover, they compensate almost completely the terms containing the potentials and only a very small diamagnetism results, the so-called Landau-Peierls diamagnetism.†

But now suppose the electrons to be coupled by some form of interaction. Then the lowest state of the electrons may be separated by a finite distance from the excited ones and the disturbing influence of the field on the eigenfunctions can only be appreciable if it is of the same order of magnitude as the coupling forces. As long as the magnetic field is sufficiently weak there should not be more than a negligible disturbance of the eigenfunctions, and therefore equations (32) would be approximately identical with (16). With increasing magnetic field the very highly degenerate excited states, which are partly paramagnetic, split up. Some of them decrease and, being already at a lower temperature than would be possible without field, suddenly become occupied and supraconductivity disappears. Of course these last remarks are to be taken as indicating roughly a programme which requires a detailed quantum mechanical investigation.

In conclusion we should like to express our thanks to Professor F. A. Lindemann, F.R.S., for his kind hospitality at the Clarendon Laboratory and for his interest in our work. We should also like to thank Imperial Chemical Industries whose generous assistance to one of us has enabled us to undertake this work.

Summary

A new formulation of the dependence of current on field in supraconductors is established.

$$\mathbf{E} = \Lambda(\dot{\mathbf{J}} + c^2 \operatorname{grad} \rho) \qquad (10)$$

$$\mathbf{H} = -\Lambda c \operatorname{curl} \mathbf{J}. \qquad (6)$$

† Landau, 'Z. Physik,' vol. 64, p. 629 (1930); Peierls, 'Z. Physik,' vol. 80, p. 763 (1933).

A characteristic feature of this formulation is the possibility of electrostatic fields existing in supraconductors. In contrast to the customary conception that in a supraconductor a current may persist without being maintained by an electric or magnetic field, the current is characterized as a kind of diamagnetic volume current, the existence of which is necessarily dependent upon the presence of a magnetic field. That magnetic field itself may be produced reciprocally by the current (§ 1).

The law of conservation of energy is discussed. The production of Joule-heat is localized on the surface of the supraconductor, where the current enters and leaves it (§ 2).

As examples the field, boundary surfaces, and distribution of currents in a supraconducting sphere and a wire are treated and the transition curve is calculated (§§ 3, 4).

Shear Waves through the Earth's Core

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1—Growth of the Core Theory

Seismological evidence of a central core to the earth was first pointed out by Oldham in 1906.* From his analysis of travel-time data regarding longitudinal (P) and transverse (S) waves observed at great distances from earthquake epicentres, he concluded that at a depth equal to about three-fifths of the radius there occurs a transition to material possessing radically different physical properties from that external to this boundary.

With the aid of more extensive data assembled by Turner† and others, the problem was later re-examined independently by Knott‡ and by Gutenberg.§ The latter concluded that at a depth of 2900 km the

* 'Quart. J. Geol. Soc.,' vol. 62, p. 456 (1906).

† "The Large Earthquakes of 1913," 'Rep. Brit. Ass.' (1917).

‡ 'Proc. Roy. Soc. Edin.,' vol. 39, p. 157 (1919).

§ 'Nachr. Ges. Wiss. Göttingen,' p. 1 (1914). This paper was probably not available to Knott in 1918. It has long been out of print, and no copy is available in N.Z. I am indebted to Professor Gutenberg for the loan of his own copy.