APPLIED
MATH
201

COMPLEX
VARIABLES

APPLIED MATH
201

APPLIED MATH
201

APPLIED MATHEMATICS 201

APPLIED FUNCTION THEORY

NOTES

Professor: Sidney Goldstein
Room: Pierce 110, MWF at 9 AM

LECTURE I

Recommended Reading List:

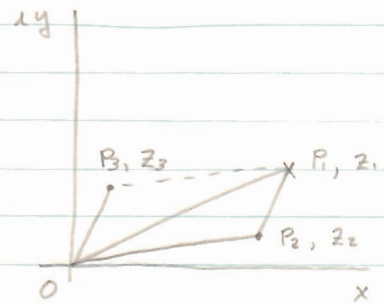
1. Copson, E.T., *Functions of a Complex Variable*, Oxford U. Press
2. Phillip, E.G., " Interscience
3. Churchill, R.V., *Complex Variables*
4. Titchmarch, E.C., *Theory of Functions*, Oxford U. Press
5. Rother, Ollendorf, et al, *Theory of Functions as Applied to Engineering Problems*, MIT Technology Press
6. McLaughlin, N.W., *Complex Variables Theory & Transform Calculus*, 2nd Ed., Cambridge U. Press
7. Van der Pol, et al, *Operational Calculus*, Cambridge U. Press
8. Tranter, C.J., *Integral Transforms in Mathematical Physics*, Methuen.
9. Sneddon, *Fourier Transforms*, McGH
10. Churchill, R.V., *Operational Mathematics*, 2nd Ed., McGH

Titschmarch - Hard
Churchill - Easy } Copson is standard text

First lecture on basic definitions of complex numbers and variables and substantiation of the usual rules of operation with them.

LECTURE II 9/28/60

ARGAND DIAGRAM:



$\overrightarrow{OP_3}$: $z_3 = z_1 - z_2$

(1) $z = x + iy = r(\cos\theta + i\sin\theta)$

(2) $e^{it} \equiv \cos t + i\sin t$; $e^{i2\pi n} = 1$

(3) $(e^{it})^n = \cos nt + i\sin nt$

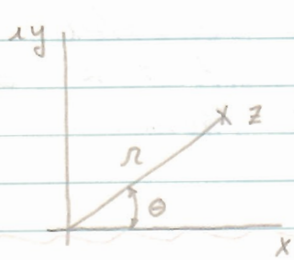
(4) $\overline{e^{it}} = e^{-it}$, $|e^{it}| = 1$

(5) $e^{s+it} = e^s e^{it}$

(6) $e^{(a+ib)t} = e^{at}(\cos bt + i\sin bt)$

(7) $\frac{d}{dt} e^{(a+ib)t} = (a+ib)e^{(a+ib)t}$

(9) $|e^{s+it}| = e^s$, $\arg(e^{s+it}) = t + 2\pi n$



(10) $z = re^{i\theta}$, $z = re^{i(\theta + 2\pi n)}$

$n = 0, \pm 1, \pm 2, \pm 3, \ldots$

(11) $|z_1 z_2 \cdots z_n| = |z_1||z_2| \cdots |z_n|$

(12) $\arg \prod_{i=1}^{n} z_i = \sum_{i=1}^{n} \arg z_i$

(13) $$z^{1/n} = r^{1/n}\left(\cos\frac{\theta + 2k\pi}{n} + i \sin\frac{\theta+2k\pi}{n}\right)$$

$$k = 0, \ldots, n-1$$

Mathematical Terms and Definitions From Copson;

Neighbourhood: By a neighborhood of a point z_0 in the Argand plane we mean the set of all points z such that $|z - z_0| < \epsilon$; we call ϵ the radius of this neighbourhood. p. 13.

Contour: A contour consists of a finite number of regular arcs which comprise a Jordan arc. p. 54.

Domain: A set of points in the Argand plane is said to be <u>connex</u> if every pair of its points can be joined by a polygonal arc which consists of only points of the set. An open connex set of points is called a domain. p. 15.

Simple Jordan Curve: A continuous arc without multiple point is called a Jordan arc. A simple example is the polygonal arc which consists of a finite chain of straight segments. p. 14.

Functions of a Complex Variable:

(14) $z = x + iy$

(15) $w = u + iv$

(16) $w = f(z)$

Continuity: $|f(z) - f(z_0)| < \epsilon$

because $|z - z_0| < \delta$, see Copson, p. 32.

LECTURE III 9-30-60

Derivatives of Complex Variables:

Given ϵ, $\exists\, \delta$ such that

(1) $\left| \dfrac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \epsilon$ for all z in a

domain D $|z - z_0| < \delta$

Now:

(2) $$f'(z) = \lim_{\zeta \to 0} \frac{f(z + \zeta) - f(z)}{\zeta}$$

For example, take $f(z) = z^2$

(3) $$f'(z) = \lim_{\zeta \to 0} \frac{(z + \zeta)^2 - z^2}{\zeta} = \lim_{\zeta \to 0} 2z + \zeta = 2z$$

Now take $f(z) = |z|^2 = z\bar{z}$

(4) $$\lim_{\zeta \to 0} \frac{(z + \zeta)(\bar{z} + \bar{\zeta}) - z\bar{z}}{\zeta} = \lim_{\zeta \to 0} \bar{z} + z\frac{\bar{\zeta}}{\zeta} + \bar{\zeta}$$

We make the substitutions: $\zeta = \rho e^{i\varphi}$, $\bar{\zeta} = \rho e^{-i\varphi}$

Thus,

(5) $\quad \bar{z} + z e^{-2i\varphi} + \bar{\zeta}$

does not have a unique limit because of the $e^{-2i\varphi}$ and $|z|^2 = f(z)$ is not defined as a function.

Regular, analytic, and holomorphic in general denote the same things about a function, i.e., single valued in domain D and differentiable except at singular points.

The necessary condition for $f(z)$ to be analytic is as follows:

(6) $\quad f(z) = u(x,y) + i\,v(x,y)$

(7) $\quad \lim\limits_{\Delta z \to 0} \dfrac{f(z+\Delta z) - f(z)}{\Delta z}$ must exist, be unique,

and be independent of direction. That is: it must be independent of γ.

(8) $\quad \Delta z = \Delta x + i\,\Delta y$, $\Delta f(z) = \Delta u + i\,\Delta v$

(9) $\quad \lim\limits_{\Delta z \to 0} \dfrac{\Delta f}{\Delta z} = \lim\limits_{\Delta z \to 0} \dfrac{\Delta u + i\,\Delta v}{\Delta x + i\,\Delta y}$

$$= \lim_{\Delta x \to 0} \frac{\Delta u + i\,\Delta v}{\Delta x} = \lim_{\Delta y \to 0} \frac{\Delta u + i\,\Delta v}{i\,\Delta y}$$

$$= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Therefore:

(10) $\quad \left.\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}\right\}$ Cauchy-Riemann equations

Are these conditions sufficient? Yes, if they exist and are continuous.

Another way to show this:

(11) $\bar{z} \equiv x - iy, \therefore \quad x = \frac{1}{2}(z + \bar{z})$

$$y = -\frac{i}{2}(z - \bar{z})$$

Assume u, v, have continuous first partials, in the function $w = u + iv$. We must show that

(12) $\dfrac{\partial w}{\partial \bar{z}} = 0$; Now:

(13) $$\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

(14) $$\frac{\partial}{\partial z} = \frac{\partial}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial}{\partial y}\frac{\partial y}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

Now:

(15) $$\frac{\partial w}{\partial \bar{z}} = \frac{1}{2}\left[\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial x} - \frac{\partial v}{\partial y}\right] = 0$$

by the Cauchy-Riemann equations or can be used to find the C-R equations.

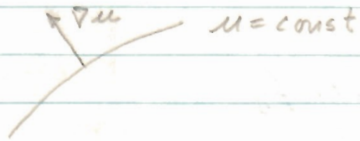
If $w = f(z) = u + iv$, then u and v are conjugate functions, that is,

(16) $$\frac{\partial^2 u}{\partial x\, \partial y} = \frac{\partial^2 u}{\partial y\, \partial x} \quad ; \quad \frac{\partial^2 v}{\partial x\, \partial y} = \frac{\partial^2 v}{\partial y\, \partial x}$$

It follows from the C-R equations:

(17) $$\left.\begin{aligned} u_{xx} + u_{yy} &= 0 \\ v_{xx} + v_{yy} &= 0 \end{aligned}\right\}$$ know as Laplace's equation in two dimensions.

We know ∇u gives two vectors, $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$. What relation does the C-R equations have to the gradient?



Now, $\nabla v \rightarrow \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$

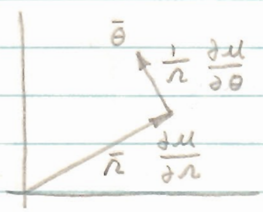
equal to by C-R: $\frac{\partial u}{\partial x}, -\frac{\partial u}{\partial y}$, or

(18) $\nabla u \rightarrow \bar{i}\,\frac{\partial u}{\partial x} + \bar{j}\,\frac{\partial u}{\partial y}$

$$\nabla v = \bar{i}\,\frac{\partial v}{\partial x} + \bar{j}\,\frac{\partial v}{\partial y} = -\bar{i}\,\frac{\partial u}{\partial y} + \bar{j}\,\frac{\partial u}{\partial x}$$

(19) $\nabla u \cdot \nabla v = -\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y \partial x} = 0$; therefore ∇u and ∇v are at right angles to each other and lines of constant u and constant v are orthogonal.

In polar form, we have:



The C-R equations in polar form are:

(20) $$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$\frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}$$

which may be found by taking $z = re^{i\theta}$ and $\Delta z = e^{i\theta}\Delta r + ire^{i\theta}\Delta\theta$ and proceeding with the limit process.

Important identities for doing Prob. 5 of Set I:

$$u_{xx} + u_{yy} = \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right)$$

$$\frac{\partial}{\partial \bar{z}} = \tfrac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

LECTURE IV 10 - 3 - 60

Given:

(1) $f(z) = u(x,y) + i v(x,y)$

the real and imaginary parts must satisfy:

(2) $u_{xx} + u_{yy} = 0$

$v_{xx} + v_{yy} = 0$

Take $u(x,y) = e^x (x \cos y - y \sin y)$ and it can be shown that this will give (2).

Now $f'(z) = u_x + i v_x = u_x - i u_y = v_y + i v_x$

$= u_y + i v_y = u_y + i u_x = -v_x + i v_y$

Take $e^z = e^{x+iy} = e^x (\cos y + i \sin y)$

and $-i (\cos y + i \sin y) = \sin y - i \cos y$. Given:

(3) $f'(z) = u_x - i u_y = e^x \{ x (\cos y + i \sin y)$

$- y (\sin y - i \cos y) + \cos y + i \sin y \}$

$= e^x \{ x e^{iy} + i y e^{iy} + e^{iy} \} = z e^z + e^z$

$= e^z (z+1)$

(4) $\therefore \; f(z) = z e^z + A + i B$

At $y = 0$:

(5) $f(x) = x e^x + A + i B = u(x,0) + i v(x,0)$

$= x e^x + i v(x,0)$ with $A = 0$

Another Way to Find $f(z)$ from $f'(z)$:

Define:

$$(6) \quad u_x(x,y) = \varphi_1(x,y)$$
$$u_y(x,y) = \varphi_2(x,y)$$

$$v_x(x,y) = \psi_1(x,y)$$
$$v_y(x,y) = \psi_2(x,y)$$

(7) Now $f'(z) = \varphi_1(x,y) - i\,\varphi_2(x,y)$

and $f'(z) = -\psi_1(x,y) + i\,\psi_2(x,y)$

$$= \varphi_1\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) - i\,\varphi_2\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right), \text{ etc.}$$

If the symmetry of $z, \bar{z}$ is thrown away, then the right hand side is independent of $\bar{z}$. Make $z = \bar{z}$: Then we get:

$$(8) \quad f'(z) = \varphi_1(z,0) - i\,\varphi_2(z,0)$$
$$= u_x(z,0) - i\,u_y(z,0) = u_x - i\,u_y$$

$$= -\psi_1(z,0) + i\,\psi_2(z,0)$$
$$= -v_x(z,0) + i\,v_y(z,0) = -v_x + i\,v_y$$

By putting $x = z,\ y = 0$ in (3) we have

$$(9) \quad f'(z) = e^z(z+1)$$

It is seen that there is really no way to tell which part (u or v) equation (3) really represents.

Positive Integral Powers; Polynomials:

The usual derivative formulas hold for ordinary functions z^n, e^z, etc.

Fundamentals Theorem of Complex Algebra:

$$a_0 + a_1 z + \dots + a_n z^n = a_n (z - z_1)^{\lambda} (z - z_2)^{\mu} \dots \dots (z - z_n)^{\omega}$$

where $\lambda + \mu + \dots + \omega = n$, has n roots.

Power Series: $\sum_{n=0}^{\infty} a_n z^n$ or $\sum_{n=0}^{\infty} a_n (z - z_0)^n$

Convergency: $S_n = \sum_{1}^{n} a_n z^n$ tends to limit with n.

If $\sum_{1}^{n} |a_n z^n|$ converges, it converges absolutely.

Write $|z| = r$, then $\sum_1 |a_n z^n| = \sum_1 |a_n| r^n$

Tests for Convergence:

Cauchy's Test: If $\lim_{n \to \infty} |a_n|^{1/n}$ exists, then

$\sum_1 \alpha_n$ converges absolutely if $\lim |\alpha_n|^{1/n} < 1$
$\sum \alpha_n$ diverges if $\lim |\alpha_n|^{1/n} > 1$

This will do theoretically, but is not practical.

Ratio Test: D'Alembert's Test: $\lim_{n \to \infty} \left| \frac{\alpha_{n+1}}{\alpha_n} \right|$

$\sum_1 \alpha_n$ converges absolutely if $\lim < 1$
$\sum \alpha_n$ diverges if $\lim > 1$

If $\lim = 1$, must use another test.

Raabe's Test: Use if $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$

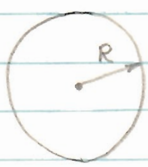
$\sum_i \alpha_n$ converges absolutely if:

$$\lim n \left\{ \left|\frac{\alpha_{n+1}}{\alpha_n}\right| - 1 \right\} < -1$$

LECTURE V 10-5-60

Power Series:

Define R as the radius of convergence of a power series, and $r = |z| = R$ when in the Argand diagram.



If $R = 0$, $z = 0$
If $R = \infty$, series converges for all z
If R is constant, points outside diverge, inside converge.

Theorem: If $f(z) = \sum_0^\infty a_n z^n$ is inside the circle of convergence, then $f(z)$ is a regular point inside the circle and can be differentiated term by term, viz., $f'(z) = \sum_0^\infty n\, a_n z^{n-1}$

Proof: Test $\sum n\, a_n z^{n-1}$ by the Cauchy Test:

(1) $\lim\limits_{n\to\infty} \left|(n a_n)^{1/n}\right| = \lim\limits_{n\to\infty} \left|n^{1/n}\right| \left|a_n^{1/n}\right| = \lim\limits_{n\to\infty} |a_n|^{1/n}$

Thus the radius of convergence of $f'(z)$ is the same as $f(z)$. Now set $\varphi(z) = \sum_i n\, a_n z^{n-1}$. We want to prove $\varphi(z) = f'(z)$. Set up:

(2) $$\left| \frac{f(z+h) - f(z)}{h} - \varphi(z) \right|$$

Expanding and working with the nth term of $f(z)$

(3) $(z+h)^n = z^n + n z^{n-1} h + \dots + {}_nC_m\, z^{n-m} h^m + \dots + h^n$

Now:

(4) $\dfrac{(z+h)^n - z^n}{h} - n z^{n-1} = \dfrac{n(n-1) h z^{n-1}}{2!} + \dots + {}_nC_m\, z^{n-m} h^{m-1}$

$+ \dots + h^{n-1}$; put $|h| = \eta$ and use the fact that the modulus of the sum is less than the sum of the modulii.

(5) $\left| \dfrac{(z+h)^n - z^n}{h} - n z^{n-1} \right| \le n(n-1) h r^{n-2} + \dots + {}_nC_m\, r^{n-m} \eta^{m-1} + \dots + \eta^{n-1}$

$= \dfrac{(r+\eta)^n - r^n}{\eta} - n r^{n-1}$

Now:

(6) $\dfrac{f(z+h) - f(z)}{h} - \varphi(z) = \sum_0^\infty a_n \left\{ \dfrac{(z+h)^n - z^n}{h} - n z^{n-1} \right\}$

If α_n is complex and $s = \sum \alpha_n$ converges absolutely, then $|s| \le \sum |\alpha_n|$ and:

(7) $\sum \left| \dfrac{f(z+h) - f(z)}{h} - \varphi(z) \right| \le \sum |a_n| \left\{ \dfrac{(r+\eta)^n - r^n}{\eta} - n r^{n-1} \right\}$

Suppose $\rho < R$, $\therefore \sum |a_n| \rho^n$ converges, $\exists$ (therefore there is) K, modulus of nth term, $|a_n| \rho^n < K$, and

(8) $|a_n| < \dfrac{K}{\rho^n}$

Substituting in (7),

$$(9)\quad \sum \left| \frac{f(z+h)-f(z)}{h} - \varphi(z) \right| \le \sum |a_n| \left\{ \frac{(r+\eta)^n - r^n}{\eta} - n r^{n-1} \right\}$$

$$\le K \sum_0^\infty \left\{ \frac{1}{\eta} \left[\left(\frac{r+\eta}{\rho}\right)^n - \left(\frac{r}{\rho}\right)^n \right] - n \frac{r^{n-1}}{\rho^n} \right\}$$

Make $r < \rho < R$, with $\sum \left(\frac{r}{\rho}\right)^n = \frac{\rho}{\rho - r}$,

$\sum \frac{n r^{n-1}}{\rho^n} = \frac{1}{\rho} \frac{1}{(1-\frac{r}{\rho})^2} = \frac{\rho}{(\rho - r)^2}$, the right side of (9) becomes:

$$(10)\quad K \left[\frac{1}{\eta} \left\{ \frac{\rho}{\rho - r - \eta} - \frac{\rho}{\rho - r} \right\} - \frac{\rho}{(\rho - r)^2} \right] = \frac{K \rho \eta}{(\rho - r)^2 (\rho - r - \eta)}$$

$\rightarrow 0$ as $\eta \rightarrow 0$, thus $\varphi(z) = f'(z)$

For any $r < \rho < R$, $f(z)$ is regular, $f'(z) = \sum n a_n z^{n-1}$. It follows that the power series may be differentiated term by term indefinitely inside the circle of convergence.

Exponential Function:

Define: $\exp z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots$

and we immediately see that $R = \infty$. Usual exponential function operations follow.

Define: $a^z = \exp(\ln a^z) = \exp(z \ln a)$

Any power series whose $R = \infty$ represents an integral function (entire function).

LECTURE VI   10-7-60

Exponential Functions:

$$e^{z_1} e^{z_2} = e^{z_1+z_2}$$
$$e^{z} = e^{x+iy} = e^{x} e^{iy}$$
$$e^{iy} = \cos y + i \sin y$$
$$e^{z} = e^{x}(\cos y + i \sin y)$$
$$e^{n\pi i} = (-1)^n \text{, where } n \text{ is an integer}$$
$$e^{\frac{1}{2}\pi i} = i$$
$$|e^{iy}| = 1$$
$$|e^{z}| = e^{x}, \quad \arg z = y + 2n\pi$$
$$e^{z+2n\pi i} = e^{z}$$

If y is changed by 2π, $\sin y$, $\cos y$, and e^{z} remain unchanged

In the Argand Plane

Let; $e^{z} = e^{\zeta}$, with $\zeta = \xi + i\eta$, then

$$e^{z-\zeta} = 1, \quad e^{x-\xi}\left[\cos(y-\eta) + i \sin(y-\eta)\right] = 1$$
$$x - \xi = 0, \quad x = \xi$$
$$\cos(y-\eta) = 1, \quad \sin(y-\eta) = 0$$
$$\therefore \eta = y + 2n\pi$$ as shown above.

Poles and Zeroes

Take e^{z}: there are no zeroes

Proof: $e^{z_1} = 0$, then $e^{-z_1} = \frac{1}{e^{z_1}}$

$|e^{-z_1}| = e^{-x_1}$

$\therefore$ for zeroes, x_1 must be infinite

Trigonometric Functions:

Definition:

$$\sin z = \sum_0^\infty (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

$$\cos z = \sum_0^\infty (-1)^n \frac{z^{2n}}{(2n)!}$$

The ratio test gives $R = \infty$ for both $\cos z$ and $\sin z$, therefore they are integral functions. Everything can and must be found from the series definition.

$$\cos(-z) = \cos z$$

$$\sin(-z) = -\sin z$$

Derivatives are found by differentiating term by term.

$\tan z = \dfrac{\sin z}{\cos z}$, $\cot z = \dfrac{1}{\tan z}$, $\sec z = \dfrac{1}{\cos z}$, $\sec z = \dfrac{1}{\sin z}$

$$\left.\begin{aligned} \cos z + i \sin z &= e^{iz} \\ \cos z - i \sin z &= e^{-iz} \end{aligned}\right\} \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

All the usual trigonometric identities follow.

Hyperbolic Functions:

Definition:

$$\sinh z = \frac{e^{z} - e^{-z}}{2} \; ; \quad \cosh z = \frac{e^{z} + e^{-z}}{2}$$

with $\cosh z = \displaystyle\sum_0^\infty \frac{z^{2n}}{(2n)!}$; $\sinh z = \displaystyle\sum_0^\infty \frac{z^{2n+1}}{(2n+1)!}$

Both are integral functions. It follows that $\tanh z = \dfrac{\sinh z}{\cosh z}$, etc. Also, derivatives follow.

$$\cosh z + \sinh z = e^{z}$$

$$\cosh z - \sinh z = e^{-z}$$

$$\cosh^2 z - \sinh^2 z = 1$$

Now: $\sinh iz = i\sin z$, $\cosh iz = \cos z$

$\sin iz = i \sinh z$, $\cos iz = \cosh z$

$$\sinh(\alpha \pm \beta) = \frac{1}{i}\sin(i\alpha \pm i\beta) = \frac{1}{i}(\sin i\alpha \cos i\beta \pm \cos i\alpha \sin i\beta)$$

$$= \sinh\alpha \cosh\beta \pm \cosh\alpha \sinh\beta$$

$$\cosh(\alpha \pm \beta) = \cosh\alpha \cosh\beta \pm \sinh\alpha \sinh\beta$$

$$\sin z = \sin(x+iy) = \sin x \cos iy + \cos x \sin iy$$
$$= \sin x \cosh y + i \cos x \sinh y$$

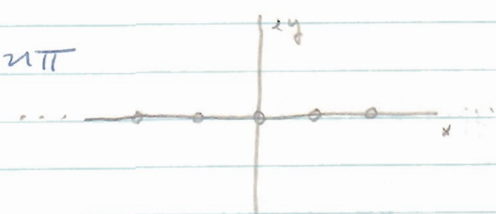
$$\cos z = \cos x \cosh y - i \sin x \sinh y$$

$$\sinh z = \sinh(x+iy) = \sinh x \cos y + i \cosh x \sin y$$

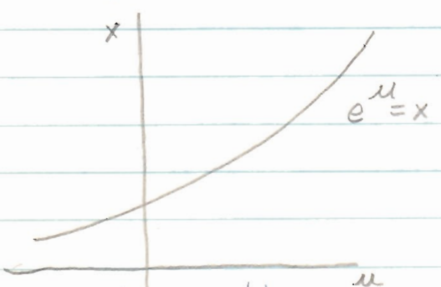
Zeroes of $\sin z$:

$\sin z = 0$ when $\underbrace{\sin x \cosh y = 0}_{\substack{\sin x = 0 \\ \cosh y \neq 0 \\ x = n\pi, \\ n = 0, \pm 1, \pm 2, \dots}}$, $\underbrace{\cos x \sinh y = 0}_{\substack{\sinh y = 0 \\ y = 0}}$

Thus the zeroes are at $y = 0$, $x = n\pi$



Logarithmic Functions:



Real Variable

$e^w = z$

$w = u + iv$

$e^u(\cos v + i \sin v) = z$

$e^u = |z|$, $u = \log|z|$, $v = \arg z$

$v = \arg z = \theta + 2\pi n$, $n = 0, \pm 1, \pm 2, \dots$

$$\therefore \; w = \operatorname{Log} z = \log|z| + i(\theta + 2\pi n)$$

Thus, $w = \text{Log}\, z$ is a many-valued function, each n giving a branch function.

For principle values:

$$\log z = \log|z| + i\theta$$

Also: $\text{Log}\, x = \log x + 2n\pi i$

For derivatives: $\frac{dz}{dw} = e^{w} = z$, $\frac{dw}{dz} = \frac{1}{z}$

$\therefore\ w = \text{Log}\, z$

$$\left.\begin{aligned} z_1 &= e^{w_1} \\ z_2 &= e^{w_2} \end{aligned}\right\} \log z_1 + \log z_2 \text{ is one value of } \text{Log}\, z_1 z_2$$

$\log z_1 = \log|z_1| + i\theta_1$, to be continued.

LECTURE VII 10-10-60

Powers of z: Take $z^p = \exp\left[p\, \text{Log}\, z\right]$

$\text{Log}\, z = \log z + n\pi i$, with $np \equiv m$, then:

$$z^p = \exp\left(p \log z + 2mpi\right) \qquad (e^{2m\pi i} = 1$$

$$= \exp\left(p \log z\right) = z^p$$

$$w = z^{p/q} = \exp\left(\tfrac{p}{q} \text{Log}\, z\right) = \exp\left(\tfrac{p}{q} \log z\right) \text{ (principle value)}$$

$$= \exp\left[\tfrac{p}{q} \log z + \tfrac{2np}{q}\pi i\right] = \exp\left(\tfrac{p}{q} \log z\right) e^{\frac{2np}{q}\pi i}$$

$$= z^{p/q}\, e^{\frac{2np}{q}\pi i}$$

Now: $w^q = z^p = z^p e^{2k\pi i}$

$$w = z^{p/q}\, e^{\frac{2k\pi i}{q}}, \quad k = 0, 1, 2, \ldots q-1$$

Thus there are q branches

$$z^{\alpha} = \exp(\alpha \log z) \quad \text{(principle value)}$$

$$= \exp(\alpha \operatorname{Log} z) \quad \text{(subsidiary values)}$$

$$= \exp(\alpha \log z)\, e^{2n\alpha\pi i}, \quad n = 0, \pm 1, \pm 2, \ldots$$

$$i^{i} = \exp\left(i \log e^{\frac{1}{2}\pi i}\right) = \exp\left(-\tfrac{1}{2}\pi\right) = e^{-\frac{1}{2}\pi} \quad \text{(principle value)}$$

$$= e^{-\frac{1}{2}\pi}\, e^{2\pi n i^2} = e^{-(2n+\frac{1}{2})\pi}, \quad n = 0, \pm 1, \pm 2, \ldots$$

Given $z_1^{\alpha} z_2^{\alpha}$, $(z_1 z_2)^{\alpha}$ is not necessarily the principle value if $z_1^{\alpha}, z_2^{\alpha}$ are principle values.

Many Valued Functions:

These are not regular functions.

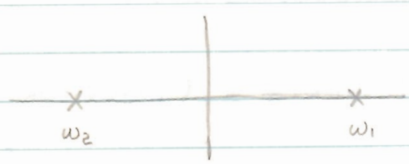
$$w = z^{1/2} = \exp\left(\tfrac{1}{2} \log z\right)$$

Let θ be the principle value of $\arg z$, $-\pi < \theta \leq \pi$, $|z| = r$.

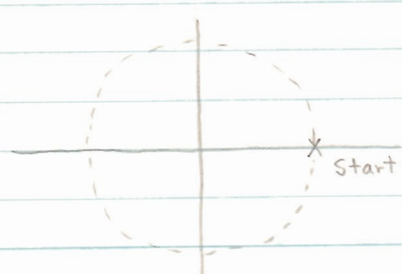
$$\log z = \log r + i\theta$$

$$w_1 = \exp\left(\tfrac{1}{2}\log r + \tfrac{1}{2} i\theta\right) = r^{1/2} e^{\frac{1}{2} i\theta}$$

$$w_2 = r^{1/2} e^{\frac{1}{2} i\theta} e^{\pi i} = -r^{1/2} e^{\frac{1}{2} i\theta}$$



when crossing negative real axis, w_1 will change discontinuously into w_2 and vice-versa



As z changes, let θ' change continuously.

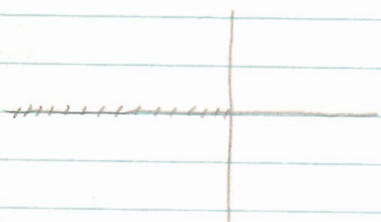
$$w_1' = r^{1/2} e^{\frac{1}{2}\theta' i}$$

$$w_2' = -r^{1/2} e^{\frac{1}{2}\theta' i}$$

When going around the origin once θ' changes by 2π, w_1' goes to w_2' and vice-versa.

Nevertheless, there is a single value of the equation $\omega^2 = z$; has two branches with origin as branch point.

Let us make a "cut" along the negative real axis, this will prevent complete encirclement of the origin.



$$\omega^2 = \frac{1}{z} = r^{-1} e^{-i\theta}\ (+2n\pi i)$$

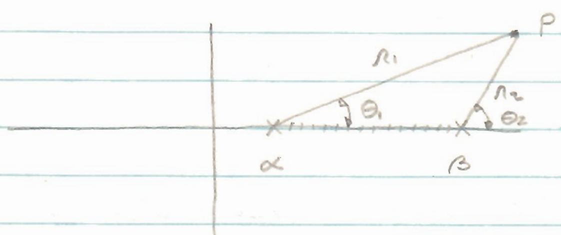
The point at ∞ is a branch point.

Should try $\omega^2 = z$

In General: $\omega^q = z^p$ defines a many valued function with q branches. Origin and point at ∞ are branch points. Each branch is single-valued in its cut. Branches are denoted by $\omega_1', \omega_2', \ldots, \omega_q'$. In principle, cut may be from origin to infinity along any direction.

Consider: $\omega^2 = z - a$, now $z = a$ is a branch point and so is infinity.
Make the cut from a to $-\infty$ thus getting single value in cut.

Consider: $\omega^2 = (z-\alpha)(z-\beta)$



$$z_1 = z - \alpha = r_1 e^{i\theta_1}$$
$$z_2 = z - \beta = r_2 e^{i\theta_2}$$

$$\omega^2 = r_1 r_2 e^{i(\theta_1+\theta_2)} e^{2n\pi i}$$
$$\omega_1 = (r_1 r_2)^{1/2} e^{1/2\, i(\theta_1+\theta_2)}$$
$$\omega_2 = -(r_1 r_2)^{1/2} e^{1/2\, i(\theta_1+\theta_2)}$$

If P encloses α once, θ_1 changes by 2π, θ_2 comes back to starting point. $\omega_1 \to \omega_2$, $\omega_2 \to \omega_1$, for odd enclosures. Nothing changes for even enclosures. Enclosing α and β any number of times will cause both ω_1 and ω_2 to return to original positions. Only cut needed is between α and β. Study Chapt. 2 in Phillips: ¶ 10, 11, 12.

LECTURE VIII 10-14-60

Many Valued Functions:

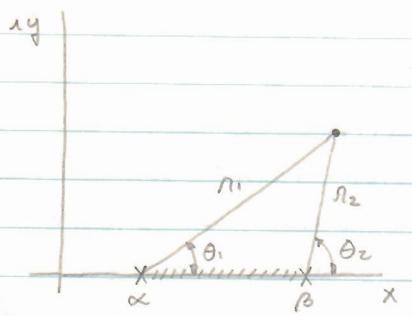
$$w^2 = (z-\alpha)(z-\beta)$$

$$z-\alpha = r_1 e^{i\theta_1}$$

$$z-\beta = r_2 e^{i\theta_2}$$

$$w_1' = (r_1 r_2)^{1/2} e^{\frac{i}{2}(\theta_1' + \theta_2')}$$

$$w_2' = -(r_1 r_2)^{1/2} e^{\frac{i}{2}(\theta_1' + \theta_2')}$$



α and β are branch points.
Investigate ∞ for branch point.
Put $\zeta = \frac{1}{z}$

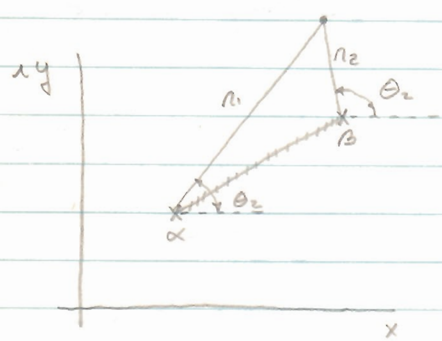
$$w^2 = (1-\alpha\zeta)(1-\beta\zeta)$$

$$= \frac{\alpha\beta}{\zeta^2}\left(\zeta - \frac{1}{\alpha}\right)\left(\zeta - \frac{1}{\beta}\right)$$

$$w = \frac{1}{\zeta}\left[\alpha\beta\left(\zeta - \frac{1}{\alpha}\right)\left(\zeta - \frac{1}{\beta}\right)\right]^{1/2}$$

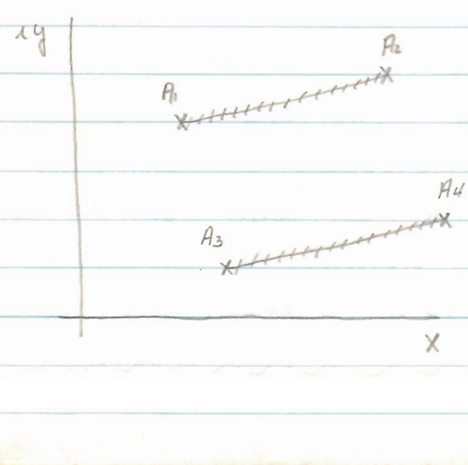
Therefore $\zeta = 0$ $(z = \infty)$ is not a branch point but a pole.

In general, where α and β are complex:



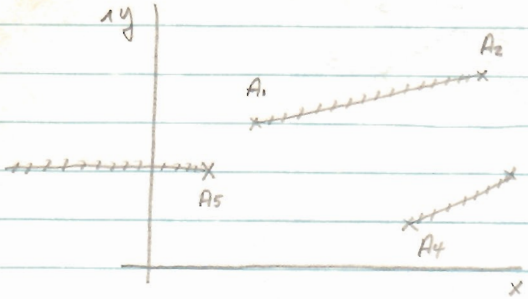
The cut is still along the line between α and β

If we have $w^2 = (z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_{2m})$, even number of factors.



Make the cuts to prevent the encirclement of an odd number of points or any point an odd number of times. Making the cut from A_1 to A_3 and from A_2 to A_4 would also suffice, or from A_1 to A_4 and A_2 to A_3.

For an odd number of factors:

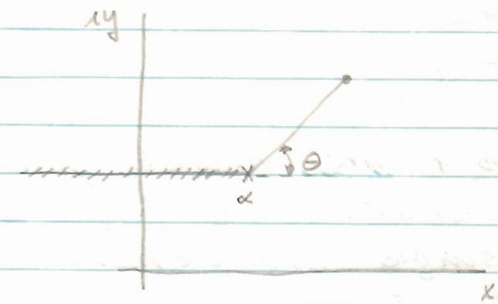


Now $z = \infty$ is a branch point. However, any cuts that satisfy the aforementioned conditions will suffice.

Let us now examine ω^3 types of many-valued functions:

I: $\omega^3 = z - \alpha$

$$z - \alpha = r e^{i\theta}$$
$$\omega_1' = r^{1/3} e^{1/3 i \theta'}$$
$$\omega_2' = r^{1/3} e^{1/3 i \theta'} e^{2/3 i \pi}$$
$$\omega_3' = r^{1/3} e^{1/3 i \theta'} e^{4/3 i \pi}$$



α: $\theta' \to \theta' + 2\pi$
$\omega_1' \to \omega_2'$
$\omega_2' \to \omega_3'$
$\omega_3' \to \omega_1'$

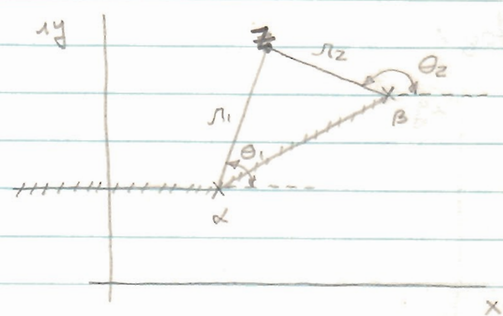
α: $\theta' \to \theta' + 4\pi$
$\omega_1' \to \omega_3'$
$\omega_2' \to \omega_1'$
$\omega_3' \to \omega_2'$

α; $\theta' \to \theta' + 6\pi$
$\omega_1' \to \omega_1'$
$\omega_2' \to \omega_2'$
$\omega_3' \to \omega_3'$

Thus we see that α and ∞ are branch points.

II: $\omega^3 = (z-\alpha)(z-\beta)$

$$z - \alpha = r_1 e^{i\theta_1}$$
$$z - \beta = r_2 e^{i\theta_2}$$
$$\omega_1' = (r_1 r_2)^{1/3} e^{1/3 i (\theta_1' + \theta_2')}$$
$$\omega_2' = (r_1 r_2)^{1/3} e^{1/3 i (\theta_1' + \theta_2')} e^{i 2/3 \pi}$$
$$\omega_3' = (r_1 r_2)^{1/3} e^{1/3 i (\theta_1' + \theta_2')} e^{i 4/3 \pi}$$



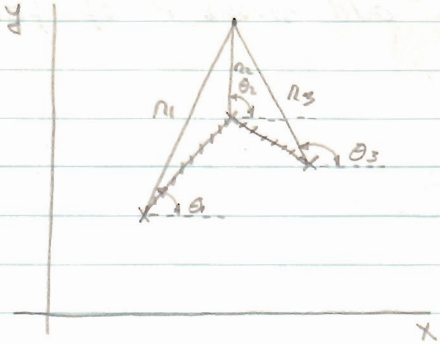
α: $\theta_1' \to \theta_1' + 2\pi$, $\theta_2' \to \theta_2'$
$\omega_1' \to \omega_2'$
$\omega_2' \to \omega_3'$
$\omega_3' \to \omega_1'$

α, β: $\theta_1' \to \theta_1' + 2\pi$, $\theta_2' \to \theta_2' + 2\pi$
$\omega_1' \to \omega_3'$
$\omega_2' \to \omega_1'$
$\omega_3' \to \omega_2'$

Thus α, β and ∞ are branch points.

III: $\omega^3 = (z-\alpha)(z-\beta)(z-\gamma)$

$$z - \alpha = r_1 e^{i\theta_1}$$
$$z - \beta = r_2 e^{i\theta_2}$$
$$z - \gamma = r_3 e^{i\theta_3}$$
$$\omega_1' = (r_1 r_2 r_3)^{1/3} e^{i\,1/3(\theta_1' + \theta_2' + \theta_3')}$$
$$\omega_2' = (r_1 r_2 r_3)^{1/3} e^{1/3\, i(\theta_1' + \theta_2' + \theta_3')} e^{i\frac{2}{3}\pi}$$
$$\omega_3' = (r_1 r_2 r_3)^{1/3} e^{1/3\, i(\theta_1' + \theta_2' + \theta_3')} e^{i\frac{4}{3}\pi}$$



α: $\theta_1' \to \theta_1' + 2\pi$, $\theta_2' \to \theta_2'$, $\theta_3' \to \theta_3'$, $\omega_1' \to \omega_2'$, $\omega_2' \to \omega_3'$, $\omega_3' \to \omega_1'$

α, β: $\theta_1' \to \theta_1' + 2\pi$, $\theta_2' \to \theta_2' + 2\pi$, $\theta_3' \to \theta_3'$, $\omega_1' \to \omega_3'$, $\omega_2' \to \omega_1'$, $\omega_3' \to \omega_2'$

α, β, γ: $\theta_1' \to \theta_1' + 2\pi$, $\theta_2' \to \theta_2' + 2\pi$, $\theta_3' \to \theta_3' + 2\pi$, $\omega_1' \to \omega_1'$, $\omega_2' \to \omega_2'$, $\omega_3' \to \omega_3'$

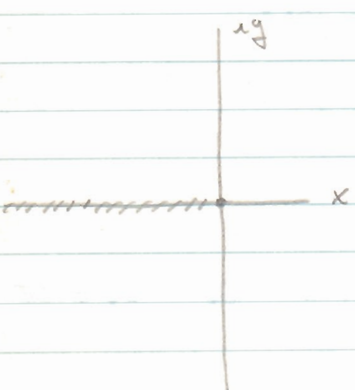
Thus α, β, γ are branch points. Now the general case becomes clear. Only triple encirclements of a point or a single encirclement of three points is allowable. If we have $\omega^3 = (z-\alpha)^2$, there will be only one branch cut from α to $-\infty$.

Logarithmic Functions:

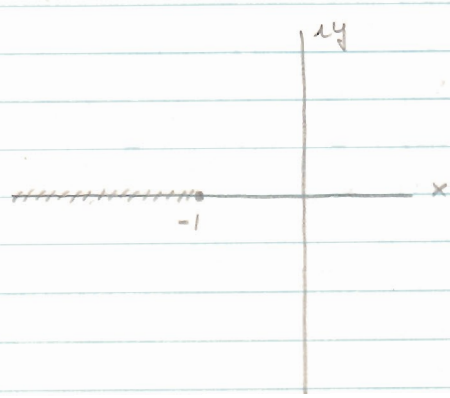
$$\omega = \operatorname{Log} z = \log z + 2n\pi i = \log r + i\theta + 2n\pi i$$

There are an infinite number of branches, with the origin and ∞ as branch points

$\omega = \operatorname{Log} z$



$\omega = \operatorname{Log}(z+1)$



Let us start with:

$$1 - z + z^2 - z^3 + \dots$$

Multiply by $1+z$ and get unity. Thus we may deduce that

$$\frac{1}{z+1} = 1 - z + z^2 - z^3 + \dots$$

Now consider:

$$f(z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \dots , \quad |z|<1$$

$$f'(z) = 1 - z + z^2 - z^3 + \dots = \frac{1}{1+z} = \frac{d}{dz} \log(z+1)$$

$$\therefore \quad f(z) - \log(1+z) = \text{constant for } |z|<1$$

for $z=0$, constant $=0$

Then $\log(1+z) = z - \dfrac{z^2}{2} + \dfrac{z^3}{3} - \dfrac{z^4}{4} + \dots \qquad |z|<1$

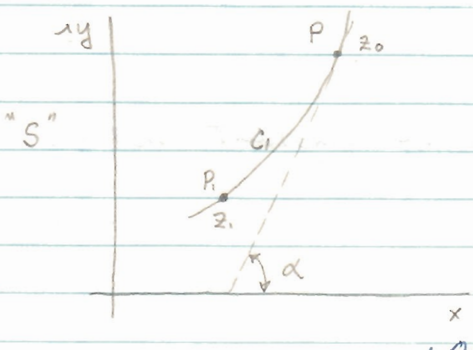
LECTURE IX 10-17-60

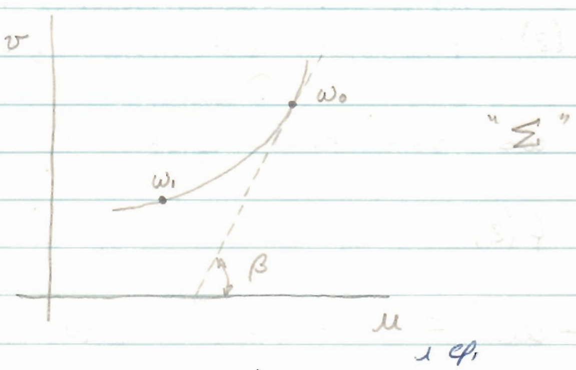
Read Chapter II in Phillips

One to one mapping, single valued in both Σ and S planes;

Conformal Mapping:

If $f'(z) \neq 0$ at any interior point of S, then the mapping is conformal.





$z_1 - z_0 = r e^{i\theta_1}$ $\qquad$ $w_1 - w_0 = \rho e^{i\varphi_1}$

Now: $f'(z_0) = \lim \frac{w_1 - w_0}{z_1 - z_0} = \lim \frac{\rho e^{i\varphi_1}}{r e^{i\theta_1}}$

$$= \lim \frac{\rho}{r} \lim e^{i(\varphi_1 - \theta_1)}$$

$$= \lim \frac{\rho}{r} e^{i \lim (\varphi_1 - \theta_1)} = \lim \frac{\rho}{r} e^{i(\beta - \alpha)}$$

and; $\lim \frac{\rho}{r} = |f'(z_0)|$

$$\beta - \alpha = \arg f'(z_0) = \lambda \ , \quad \beta = \alpha + \lambda$$

If the curve in the "s" plane is intersected by a curve, it can be shown that the resulting intersection of curves in the "w" plane has an angle of intersection equal to the angle of intersection in the "s" plane, and the angles are in the same sense.

Linear Magnification:

$$dw = \frac{dw}{dz} dz \ ; \quad |dw| = \left|\frac{dw}{dz}\right| |dz| = |f'(z)| \, |dz|$$

$$\arg dw = \arg dz + \arg f'(z)$$

If $f'(z) = 0$ at z_0, it is said to have zero order.
We can write:

$$f'(z) = (z - z_0)^n \text{ times a regular function}$$

or; $f(z) - f(z_0) = (z - z_0)^{n+1} g(z)$

where we see that $g(z) = A$, a complex constant, not $\neq 0$

$$w = f(z)$$

$$w_1 - w_0 = (z_1 - z_0)^{n+1} g(z_1)$$

$$\rho e^{i\varphi_1} = r^{n+1} e^{i(n+1)i\theta_1} g(z_1)$$

Let $z_1 \to z_0$, then $g(z_1) \to g(z_0) = A \neq 0$.

$$\rho \approx |A|\, r^{n+1}$$

$$\beta \approx (n+1)\alpha + \arg A$$

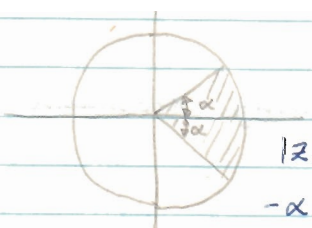
Now $\beta_1 - \beta_2 = (n+1)(\alpha_1 - \alpha_2)$

or $\delta = (n+1)\delta$ since $\beta_1 - \beta_2 = \alpha_1 - \alpha_2$
and the mapping is not conformal.

Consider the mapping of the function $w = z^2$.
We have:

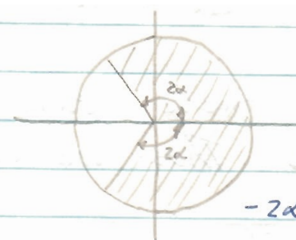
$$|w| = |z|^2 \;;\quad \arg w = 2 \arg z$$

"z" Plane "w" Plane



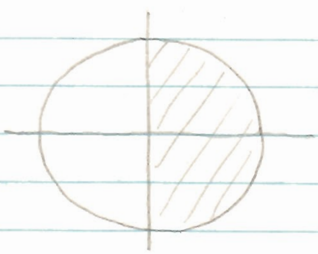
$|z| < R$

$-\alpha \le \arg z \le \alpha$

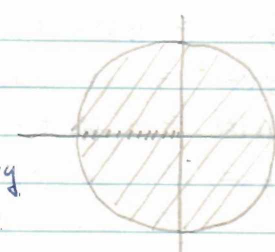


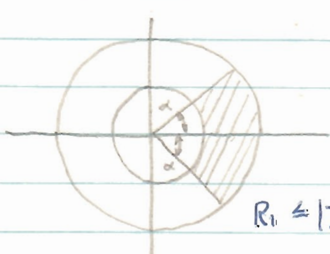
$|w| \le R^2$

$-2\alpha \le \arg w \le 2\alpha$

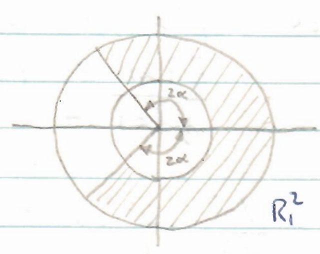


As can be seen, if $\alpha > \pi/2$, there will be overlap and it is necessary to make a cut in the "w" plane.





$R_1 \le |z| \le R_0$



$R_1^2 \le |w| \le R_0^2$

Conclusions: The whole of "w" plane comes from ½ of the "z" plane.

If $w = f(z)$, $f'(z) \neq 0$, mapping is conformal in both directions.

(1) $u_x = v_y$, $u_y = -v_x$; w is a regular function of z, $w = u + iv = f_1(z)$

(2) $u_x = -v_y$, $u_y = v_x$; $\bar{w} = u - iv = f_2(z)$ which is the same as w except reflected around the u axis.

Suppose $w = u + iv = f(z)$ and regular
$\bar{w} = u - iv = \overline{f(z)}$

then $\lim \dfrac{\Delta u + i\Delta v}{\Delta x + i\Delta y}$ for $w = f(z)$ independent of $\dfrac{\Delta x}{\Delta y}$

$\lim \dfrac{\Delta u - i\Delta v}{\Delta x - i\Delta y}$ for $\bar{w} = \overline{f(z)}$

Both of these cases lead to idendical Cauchy-Riemann equations. Thus we can write:

$$\bar{w} = \overline{f(z)} = \bar{f}(\bar{z})$$

Suppose
$$f(x) = g(x) + i h(x)$$
$$g(z) = u_1 + i v_1$$
$$h(z) = u_2 + i v_2$$
$$f(z) = u_1 + i v_1 + i u_2 - v_2$$
$$= u_1 - v_2 + i(u_2 + v_1)$$
$$\therefore \overline{f(x)} = \bar{f}(x)$$

Suppose
$$f(\bar{z}) = \alpha \bar{z} = (a + bi)(x - iy) = f(z)$$
$$\bar{f}(z) = (a - ib)(x + iy)$$
$$\overline{\bar{f}(z)} = f(\bar{z})$$

Also: $\dfrac{d\bar{w}}{d\bar{z}} = u_x - i v_x = \overline{u_x + i v_x}$

Then: $\dfrac{\overline{d f(z)}}{dz} = \dfrac{d \bar{f}(\bar{z})}{d\bar{z}}$

Harmonic Functions:

Given $V(x,y)$ satisfies $V_{xy} = V_{yx}$, taking $w = f(z) = u + iv$ in D, $f'(z) \neq 0$.

See # 13 - 18 in Phillips. By 10-23-60, write in questions for discussion.

LECTURE X 10-19-60

Beginning with: $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 4 \frac{\partial^2 V}{\partial z \partial \bar{z}}$, then:

$$\frac{\partial V}{\partial z} = \frac{\partial V}{\partial w} \frac{dw}{dz} \quad ; \quad \frac{\partial V}{\partial \bar{z}} = \frac{\partial V}{\partial \bar{w}} \frac{d\bar{w}}{d\bar{z}}$$

$$\frac{\partial^2 V}{\partial z \partial \bar{z}} = \frac{d\bar{w}}{d\bar{z}} \frac{\partial}{\partial z}\left(\frac{\partial V}{\partial \bar{w}}\right) = \frac{d\bar{w}}{d\bar{z}} \frac{dw}{dz} \frac{\partial}{\partial w}\left(\frac{\partial V}{\partial \bar{w}}\right)$$

$$= \frac{dw}{dz} \overline{\frac{dw}{dz}} \frac{\partial^2 V}{\partial w \partial \bar{w}} = |f'(z)|^2 \frac{\partial^2 V}{\partial w \partial \bar{w}}$$

$$\therefore \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 4 |f'(z)|^2 \frac{\partial^2 V}{\partial w \partial \bar{w}}$$

Now: $\frac{\partial^2 V}{\partial u^2} + \frac{\partial^2 V}{\partial v^2} = 4 \frac{\partial^2 V}{\partial w \partial \bar{w}}$

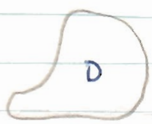
Then: $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = |f'(z)|^2 \left(\frac{\partial^2 V}{\partial u^2} + \frac{\partial^2 V}{\partial v^2}\right)$

with $f(z) = u + iv$

If $V_{xx} + V_{yy} = 0$, $V_{uu} + V_{vv} = 0$

Superficial Magnification:

"z" Plane   "w" Plane



maps into →

Now the area of the region Δ is:

$$A = \iint_\Delta du\,dv = \iint_D |f'(z)|^2\,dx\,dy$$

because: $A = \iint_\Delta du\,dv = \iint_D |J|\,dx\,dy$

$$J = \frac{\partial(u,v)}{\partial(x,y)} = u_x v_y - u_y v_x = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

$$= u_x^2 + v_x^2 = |u_x + i v_x|^2 = |f'(z)|^2$$

Suppose $w = f(z)$ transforms conformally S to Σ in a one to one manner, then $f(z)$ is a simple (univalent) function is S.

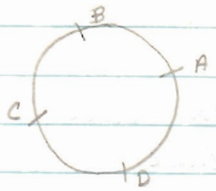
Take a contour C: If $f(z)$ is regular on and within C and simple (schlicht) on boundary, then it is simple inside the region C.

Riemann Theorem on Conformal Transformations:

Suppose a simple closed Jordan curve C and S is an open region bounded by C. There exists one and only one $f(z)$ that is regular in S that maps conformally into the region defined by the unit circle $|w| < 1$ in the "w" plane, and transforms a given direction in S into the real axis in w.

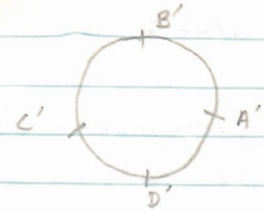
Correspondence of Domains:

"z" Plane



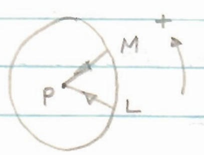
Suppose this transformation

"w" Plane

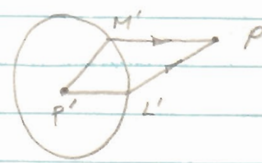


Does inside in "z" go into the inside in "w" or vice-versa? Examining the points 0 and ∞ is one way to tell this. Another is to determine the sense of direction around the curve as we go from one plane to another. If the same, inside to inside. If not, inside to outside and vice-versa. Suppose:

"z"



Rotatate $\overrightarrow{LP}$ to $\overrightarrow{MP}$



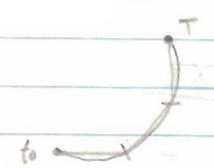
"w"

We can see that if P' is outside, the rotation must be in the opposite sense.

Integrals of $f(z)$ in the "z" Plane:

Line analog: Line integral analog:

Given: $x(t)$, $y(t)$



$$z = x(t) + i y(t)$$

$$z_s = x(t_s) + i y(t_s)$$

$$t_0 < t_1 < t_1 < \dots < t_n$$

$$t_0 \leq t \leq T = t_n$$

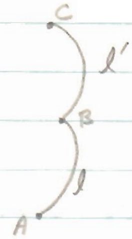
$$\sum_1 (\text{secants}) = \sum_{s=1}^{n} |z_s - z_{s-1}| = \sum_{s=1}^{n} \left\{ (x_s - x_{s-1})^2 + (y_s - y_{s-1})^2 \right\}^{1/2}$$

Take the limit as $n \to \infty$. If this limit exists, the curve is rectifiable.

For a Regular Jordan Curve: $x'(t)$, $y'(t)$ are continuous in the interval, $t_0 \leq t \leq T$

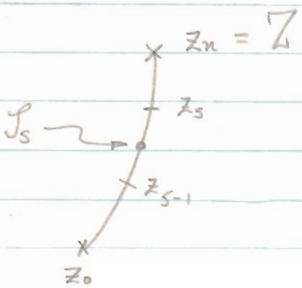
The length is then: $\int_{t_0}^{T} \left\{ [x'(t)]^2 + [y'(t)]^2 \right\}^{1/2} dt$

Take two arcs:



these are Jordan arcs. The total length can be taken as the sum of the individual lengths computed by line integrals.

Take arc Γ and $\int f(z)\, dz$ along it with $z = x(t) + i y(t)$, $t_0 \leq t \leq T$



$$\Sigma\,(\text{secants}) = \sum_{s=1}^{n} f(\zeta_s)(z_s - z_{s-1})$$

Take: $\left.\begin{array}{c} \lim\limits_{\substack{n \to \infty \\ |t_s - t_{s-1}|_{max} \to 0}} \Sigma \end{array}\right\} I = \int\limits_{\Gamma(z_0 \text{ to } Z)} f(z)\, dz$

$f(z)$ is not necessarily regular, but it must be existing and continuous along the curve.

LECTURE XI 10-21-60

Contour Integration:

Given:

$$\int_{\Gamma,\ t=0}^{t=T} f(z)\, dz \,, \quad \Gamma \text{ is described by: } \quad \begin{array}{l} x = x(t) \\ y = y(t) \\ f(z) = u + iv \\ u(x,y) = u[x(t), y(t)] \end{array}$$

$$= \int_{t_0}^{T} (u + iv)(x' + iy')\, dt$$

$$= \int_{t_0}^{T} (ux' - y'v)\, dt + i \int_{t_0}^{T} (vx' + uy')\, dt = \int_{t_0}^{T} F(t)\,(x' + iy')\, dt$$

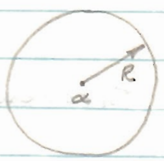
Suppose a point in $f(z)$ with $f(z) = F(t)$ ← complex function

Let the point move on the circle $|z| = 1$, then $z = e^{i\theta}$, $\theta_1 \le \theta \le \theta_2$, with $f(z) = f(e^{i\theta}) = F(\theta)$ and $dz = i e^{i\theta} d\theta$

$$\int_\Pi f(z)\,dz = \int_{\theta_1}^{\theta_2} F(\theta)\, i e^{i\theta}\, d\theta$$

Examples:

Find $\int dz$, $\int \frac{dz}{z-\alpha}$ around the circle $|z-\alpha| = R$



$$z - \alpha = Re^{i\theta}$$
$$dz = iRe^{i\theta} d\theta$$

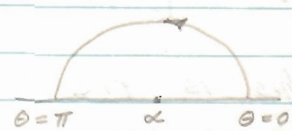
then $$\int_C dz = \int_{-\pi}^{\pi} iRe^{i\theta} d\theta = R\left[e^{i\theta}\right]_{-\pi}^{\pi} = 0$$

and $$\int_C \frac{dz}{z-\alpha} = \int_{-\pi}^{\pi} \frac{iRe^{i\theta} d\theta}{Re^{i\theta}} = 2\pi i$$

Now consider $\int_C (z-\alpha)^n dz$:

$$\int_C (z-\alpha)^n dz = \int_{-\pi}^{\pi} i R^{n+1} e^{i(n+1)\theta} d\theta = 0 \quad \text{for } n \neq -1$$

Now take the path of integration to be:



$$\int_0^\pi dz = -2R$$

Suppose $f(z)$ is continuous and bounded, $|f(z)| \le M$, on a contour of length l:

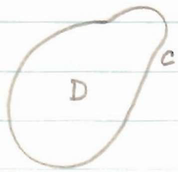
Theorem: $$\left| \int_\Pi f(z)\,dz \right| \le Ml$$

Proof: $$\int f(z)\,dz = \int_{t_0}^{T} \underbrace{F(t)\,(x' + iy')}_{\varphi(t)}\,dt = \lim_{n\to\infty} \sum_{s=1}^{n} \varphi(t_s)(t_s - t_{s-1})$$

$$\le \lim_{n\to\infty} \sum_{s=1}^{n} |\varphi(t_s)|\,(t_s - t_{s-1}) = \int_{t_0}^{T} |\varphi(t)|\,dt = \int_{t_0}^{T} |F| \sqrt{x'^2 + y'^2}\,dt \le \int_{t_0}^{T} M\sqrt{x'^2+y'^2}\,dt \le Ml$$

Cauchy's Theorem:

Given: $f(z)$ is a regular function at every point in the domain enclosed by the contour C and at every point on the closed contour C.



$$\int_C f(z)\,dz = \int_C (u+iv)(dx+i\,dy)$$

$$= \int_C u\,dx - v\,dy + i\int_C v\,dx + u\,dy$$

Suppose $P(x,y)$, $Q(x,y)$ are continuous and $\frac{\partial P}{\partial y}$; $\frac{\partial Q}{\partial x}$ exist and are continuous in D. Then:

$$\int P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$ by the divergence theorem.

Let: $\left.\begin{array}{l} P \to u \\ Q \to -v \end{array}\right\}$ $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = -\dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y} = 0$

$\therefore \int_C u\,dx + (-v)\,dy = 0$

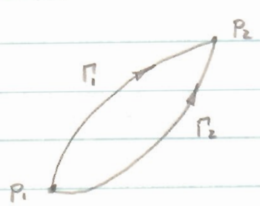
Let: $\left.\begin{array}{l} P \to v \\ Q \to u \end{array}\right\}$ $\dfrac{\partial u}{\partial x} - \dfrac{\partial v}{\partial y} = 0$ from $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}$

$\therefore \int_C v\,dx + u\,dy = 0$

$\therefore \int_C f(z)\,dz = 0$

This is the "weak" form of Cauchy's Theorem.
Strong form: $f(z)$ is a regular function inside the domain enclosed by the closed contour C and continuous on C.

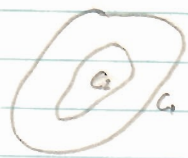
Let Γ_1 and Γ_2 be two regular Jordan arcs:

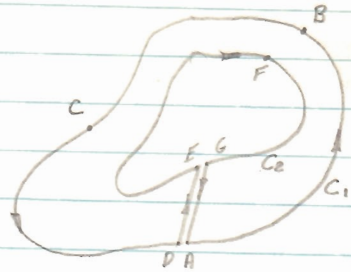


It is obvious from Cauchy's Theorem:

$$\int_{\Gamma_1} f(z)\,dz = \int_{\Gamma_2} f(z)\,dz$$

Let $f(z)$ be regular on and between C_1 and C_2:





We can show that $\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$

$$\int_{ABCDEFGA} f(z)\,dz = 0 = \int_{ABCD} f(z)\,dz - \int_{GFE} f(z)\,dz + \underbrace{\int_{DE} f(z)\,dz - \int_{AG} f(z)\,dz}$$

These two integrals are equal because $f(z)$ is single valued.

LECTURE XVIII   11-7-60

Point at Infinity:

Given:

$$\sum_{0}^{\infty} a_n z^n + \sum_{1}^{\infty} b_n z^{-n}$$

For $a_n \neq 0$, $a_n = 0$ for $n \geq m+1$, we have pole of order m at ∞.

Examine by substitution $z = \frac{1}{\zeta}$

Rational Functions:

Take: $f(z) = \dfrac{F(z)}{G(z)}$

$$F(z) = a_0 + a_1 z + a_2 z^2 + \dots$$
$$G(z) = b_0 + b_1 z + b_2 z^2 + \dots$$

If $f(z)$ has poles at $z_1, z_2, \dots$, $f(\frac{1}{z})$ has poles at $\frac{1}{z_1}, \frac{1}{z_2}, \dots$

If infinite number of singularities outside $z > R$, then $f(\frac{1}{z})$ would have essential singularity at the origin.

If finite number of poles $\alpha, \beta, \dots$

then $g(z) = (z-\alpha)^a (z-\beta)^b \dots$ has no singularities except possibly at the origin.

$g(\frac{1}{z}) = \sum_{0}^{\infty} \dfrac{a_n}{z^n}$ must terminate or $g(z)$ has essential singularity at ∞.

Picard's Theorem for Integral Functions:

An integral function that is not constant takes every possible value with one possible exception.

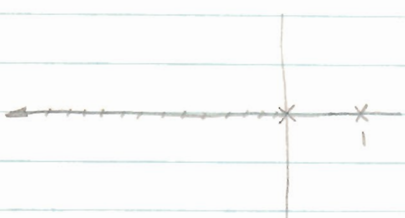
Therefore, an integral function that does not have the two values a, b, is a constant.

A meromorphic function is one that must take on three different values to be non-constant.

Branch Point: BP are singularities. Consider $z^{1/2}$, $\log z$
If P varies continuously around the point z_0 and does not return to its original value, then z_0 is a branch point. Once plane is cut, function is regular within the cut. However, cannot write Taylor series about branch point.

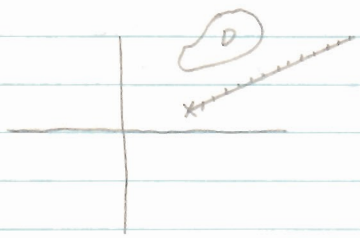
Consider: $\dfrac{1}{\log z}$; $z = re^{i\theta}$

$\log z = \log r + i(\theta + 2n\pi)$, $n = 0, \pm 1, \ldots$



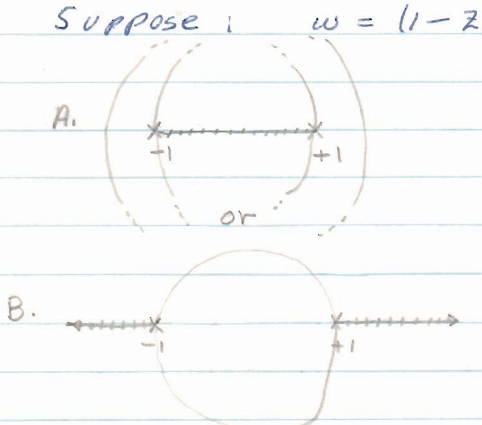
For $\log r = 0$ at $z = 1$, $f(z)$ has pole at $z = 1$. Therefore, $f(z)$ has a singularity in the cut plane.

Suppose:



All the usual theorems can be applied to points in the domain D, as long as cut does not pass through domain.

Suppose: $w = (1 - z^2)^{1/2}$

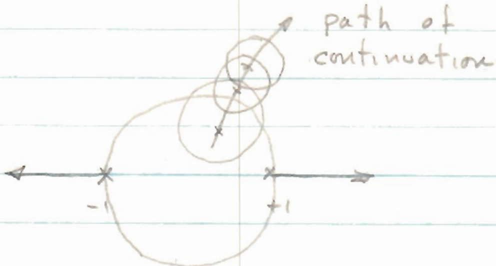


Can expand around the origin with taylor series, and get B. Expand by binomial expansion.

For annulus outside ± 1 as in A, $w = (-z^2)^{1/2}\left(1 - \frac{1}{z^2}\right)^{1/2}$

$= \pm iz\left(1 - \frac{1}{z^2}\right)^{1/2}$ and expand by binomial theorem.

To continue the function, one must specify the path taken with respect to the branch points, viz:



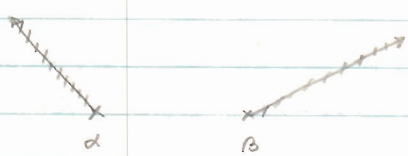
Take n th roots of unity: $\Omega_0^{(n)}, \Omega_1^{(n)}, \ldots \Omega_{n-1}^{(n)}$

with $\Omega_s^{(n)} = e^{\frac{2 s \pi i}{n}}$, thus showing $\left[\{f(z)\}^n\right]^{1/n}$ is a many valued function.

If $w^2 = (z-\alpha)(z-\beta)(z-\gamma) \ldots \ldots |\kappa|$

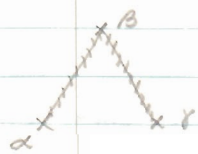
with $|z| < |\kappa|$ we may not expand in a Laurent series for an odd number of branch points because of cut to ∞.

Consider: $w^3 = (z-\alpha)(z-\beta)$



Can expand in Taylor series in $|z| < |\alpha|, |\beta|$, but there is no Laurent series

Now take $w^3 = (z-\alpha)(z-\beta)(z-\gamma)$



Can now expand in Laurent series for $|z| > |\alpha|, |\beta|, |\gamma|$

What is case for $z^{1/3}(1-z)^{1/2}$? Also for $w^3 = \frac{1}{(z-\alpha)(z-\beta)(z-\gamma)}$

LECTURE XIX   11-14-60

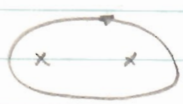
Continuation of Branch Points and Cuts:

$\omega^3 = (z-\alpha)(z-\beta)(z-\gamma)\cdots$, ∞ is a branch point if number of factors is not multiple of 3.

$\omega^n = (z-\alpha)(z-\beta)\cdots$, ∞ is branch point if number of factors is nm, $m = 1, 2, 3, \ldots$

Take $z^{1/3}(1-z)^{1/2} = \omega$. Branch points at $0, 1$.

Point at ∞ is a branch point as seen by making $\zeta^{-1} = z$.



encircling $0, 1$, once, $\arg \omega$ increases by $5\pi/6$

or

No series representation is possible either in ascending or descending powers of z, because cannot encircle limit points without passing through cuts.

If $\zeta = f(z)$ is multiple valued, usually so is $f(\zeta)$ but cannot be sure.

Assume $f(z)$, $g(z)$ with f single valued and g multiple valued, $f(z) + g(z)$ is multiple valued with same branch points as $g(z)$. Example: $iz + (1-z^2)^{1/2}$

All the functions we have been discussing are known as algebraic functions, i.e., general solution of:

$$g_0(z)\,\omega^n + g_1(z)\,\omega^{n-1} + \cdots + g_n(z) = 0$$

Another Example: $w^n = (z-\alpha)^a (z-\beta)^b$

For single cut:  , $n = m(a+b)$, $m = 1, 2, 3, \ldots$

$$\arg w = \frac{a}{n}\left\{\arg(z-\alpha) + 2m\pi\right\} + \frac{b}{m}\left\{\arg(z-\beta) + 2N\pi\right\}$$

On each encirclement $\arg w$ increases by $2\pi\frac{(a+b)}{n}$

Inverse Trig Functions: Study $\arcsin z = \operatorname{Log}\left\{iz + (1-z^2)^{1/2}\right\}$

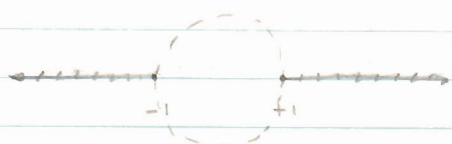
Also: $\left\{\operatorname{Log}\frac{1}{1-z}\right\}^{1/2}$ and singularities of $\operatorname{Log}\operatorname{Log} z$

and $$\frac{1}{\operatorname{Log}\left\{iz + (1-z^2)^{1/2}\right\}}$$

LECTURE XX 11-16-60

Inverse Trig. Functions:

$g^2 - 2igz - 1 = 0$; $g = iz + (1-z^2)^{1/2}$



$\operatorname{Log}\left\{iz + \sqrt{1-z^2}\right\}$; $z = 0$, $2n\pi i$ are branch points also $z = \infty$

$g = 0$ means $iz = -\sqrt{1-z^2}$, $-z^2 = 1-z^2$

same cuts as above. No Laurent expansions outside circle.

Take $f_1, f_2 = \sqrt{1-z^2}$, $f_1 = -f_2$, $f_1^2 = f_2^2 = 1-z^2$

$f_1 f_2 = -(1-z^2)$

$$\operatorname{Log}(iz+f_1) + \operatorname{Log}(iz+f_2) = \operatorname{Log}(iz+f_1)(iz+f_2) + 2n\pi$$

$$\operatorname{Log}(iz+f_1)(iz+f_2) = \operatorname{Log}\left\{-z^2 + iz(f_1+f_2) + f_1 f_2\right\}$$

$$= \operatorname{Log}(-1) = (2p+1)\pi$$

$$\operatorname{Log}(iz + J_2) = -\operatorname{Log}(iz + J_1) + (2m+1)\pi$$

If g_1 is one branch of $\operatorname{Log}(iz + \sqrt{1-z^2})$, other branches are $g_1 + 2n\pi i$ and $-g_1 + (2m+1)\pi i$

$\dfrac{1}{\operatorname{Log}[iz + \sqrt{1-z^2}]}$ gives same BP's. A pole at origin at branch point $z = 0$.

Expanding near the origin: $\dfrac{1}{\log(iz + 1 - \frac{1}{2}z^2 + \ldots)}$

$= -\frac{i}{z} + \sum a_n z^n$ so a pole exists in this branch with residue $-i$.

Reference: Titschmarch, p. 144

Look at $w = \arcsin z$, $z = \sin w$, $\cos w = \sqrt{1-z^2}$

$$e^{iw} = \cos w + i \sin w = iz + \sqrt{1-z^2}$$

$$iw = \operatorname{Log}\{iz + \sqrt{1-z^2}\}$$

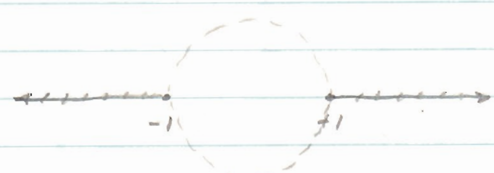
$$w = -i \operatorname{Log}\{iz + \sqrt{1-z^2}\}$$, same as before.

BP's at $\pm 1, \infty$

If w_1 is one branch

$2n\pi + w_1$

$(2n+1)\pi - w_1$



Now $\dfrac{dz}{dw} = \cos w = (1-z^2)^{1/2}$, $\dfrac{dw}{dz} = (1-z^2)^{-1/2}$

Return to this later.

Consider: $z = \sin w_1 = \sin w_2$

$\sin w_2 - \sin w_1 = 0$

$$2 \sin \frac{w_2 - w_1}{2} \cos \frac{w_2 + w_1}{2} = 0$$

$\cos \frac{\omega_2+\omega_1}{2} = 0$, $\omega_2+\omega_1 - (2m+1)\pi$

$\sin \frac{\omega_2-\omega_1}{2} = 0$, $\omega_2 \mp \omega_1 + 2n\pi$

Back to $\frac{dw}{dz} = \pm \left(1 + \frac{1}{2}z^2 + \frac{\frac{1}{2}\cdot\frac{3}{2}}{2!} z^4 + \dots\right)$

$$= \pm \left\{ 1 + \frac{z^2}{2} + \frac{1\cdot 3}{2\cdot 4} z^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6} z^6 + \dots \right\}$$

Integrating term by term:

$$w = \text{constant} \pm \left\{ z + \frac{1}{2}\frac{z^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{z^5}{5} + \dots \right\}$$

Take branch which is zero at origin. $\cos w = (1-z^2)^{1/2}$
$= +1$ at $z=0$,
so + sign is used and constant must be zero, for $w=0$ at $z=0$.

We can do similar things for inverse hyperbolic functions.
$\arctan z = \frac{1}{2i} \operatorname{Log} \frac{1+iz}{1-iz}$, $\tanh^{-1} z = \frac{1}{2} \operatorname{Log} \frac{1+z}{1-z}$

Cauchy's Residue Theorem: If $f(z)$ is regular on and inside C with a finite number of singularities within each with residue R_p, then:

$$\int_C f(z)\,dz = 2\pi i \sum R_p \ , p = 1, \dots n.$$

A meromorphic function is one that has only poles for singularities. See Phillips, for proof of Residue Theorem, p. 115-166.

One must use caution in examining residues at ∞. Sometimes a function that is regular at ∞ will have a residue by the definition. Residue at ∞ for Laurent Series is $-b_1$

LECTURE XXI 11-18-60

Zeroes:

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = \overset{\overset{\text{zeroes}}{\uparrow}}{N} - \overset{\overset{\text{poles}}{\uparrow}}{P}$$

, $f(z)$ is meromorphic inside C and has no zeroes on C.

Proof:

Apply Cauchy's Theorem to $F(z) = \frac{f'(z)}{f(z)}$.

$z=\alpha$ has zeroes of order a, $f(z) = \varphi(z)(z-\alpha)^a$; $\varphi(z)$ regular $\neq 0$. $f'(z) = a(z-\alpha)^{a-1}\varphi(z) + (z-\alpha)^a\varphi'(z)$.

$\therefore$ $F(z) = \frac{a}{z-\alpha} + \frac{\varphi'}{\varphi}$, $\frac{\varphi'}{\varphi}$ regular. $F(z)$ has simple pole at α with residue a.

Suppose $z=\beta$ is pole of order b. $f(z) = (z-\beta)^{-b}\psi(z)$
$\psi(z)$ regular near b $\neq 0$

$\therefore$ $F(z) = \frac{-b}{z-\beta} + \frac{\psi'}{\psi}$, $\frac{\psi'}{\psi}$ is regular

Then: $\frac{1}{2\pi i}\int_C \frac{f'}{f}\,dz = \Sigma a - \Sigma b = N - P$

If f is regular inside C; $P=0$,

$$N = \frac{1}{2\pi i}\int \frac{f'}{f}\,dz = \frac{1}{2\pi i}\Delta_C \operatorname{Log} f(z)$$

Now: $\operatorname{Log} f(z) = \log|f(z)| + i\arg f(z) + 2n\pi i$,
Then:

$$N = \frac{1}{2\pi}\Delta_C \arg f(z)$$

Rouches Theorem: Phillips, p. 108

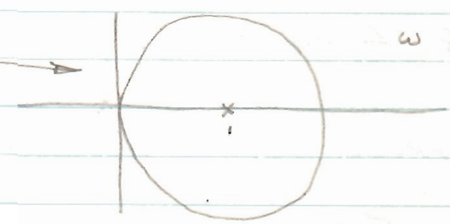
Given $f(z)$, $g(z)$, $|f(z)| > |g(z)|$ on c, then $f(z)$, $f(z) + g(z)$ have the same number of zeroes inside c.

Then if N is zeros of $f(z)$, N' of $f(z)+g(z)$,

$$N = \frac{1}{2\pi i} \Delta_c \arg f(z) \quad , \quad N' = \frac{1}{2\pi i} \Delta_c \arg (f+g)$$
$$= \frac{1}{2\pi i} \Delta_c \arg f + \frac{1}{2\pi i} \Delta_c \left(1 + \frac{g}{f}\right)$$

$$N' - N = \frac{1}{2\pi i} \Delta_c \arg\left(1 + \frac{g}{f}\right) \quad , \quad w = 1 + g/f$$

$$|w - 1| = \frac{|g|}{|f|} < 1$$



$$\therefore N' - N = \frac{1}{2\pi} \Delta_c \arg w = 0$$

Now suppose $f = a_n z^n$, $g = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1}$

$f + g = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + a_n z^n$, leads to fundamental theorem of algebra.

Now $|g| \leq |a_0| + |a_1| R + |a_2| R^2 + \cdots + |a_{n-1}| R^{n-1}$

$$< R^{n-1} \left\{ |a_0| + |a_1| + \cdots + |a_{n-1}| \right\}$$

where $R = |z| > 1$; then $\dfrac{|g|}{|f|} < \dfrac{\frac{1}{R} \sum_{s=0}^{n-1} |a_s|}{|a_n|}$

Define $K = \dfrac{|a_0| + |a_1| + \cdots + |a_{n-1}|}{|a_n|}$, $\dfrac{|g|}{|f|} < \dfrac{K}{R} < 1$, if $R > K$

and K gives N zeros by Rouches' theorem.

Consider: $\varphi(z) = \dfrac{a_0 + a_1 z + \cdots + a_n z^n}{(z - z_1)(z - z_2) \cdots (z - z_n)} \rightarrow a_n$ as $z \rightarrow \infty$

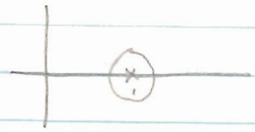
then $a_0 + \cdots + a_n z^n = c_n (z - z_1) \cdots (z - z_n)$

Suppose $F(z) = a_0 + a_1 z + \dots + a_n z^n = (z - z_1)\varphi(z)$

Proceeding as before $\omega = 1 + g/f$, $|g/f| < \frac{K}{R}$

$|\omega - 1| < \frac{K}{R}$

Let $R \to \infty$:



$$\Delta \arg(f+g) = \Delta \arg f + \Delta \arg \omega = \Delta \arg f + O\left(\tfrac{1}{R}\right)$$

If $\Delta \arg z = \alpha$, $\Delta \arg f = n\alpha$, $\Delta \arg(f+g) = n\alpha + O\left(\frac{1}{R}\right)$

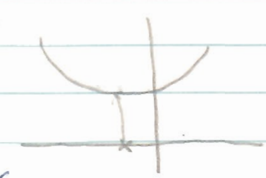
Suppose all coefficients of $F(z)$ are real, $\overline{F(z)} = F(\bar{z})$

$F(z) = u(x,y) + iv(x,y)$; $F(\bar{z}) = u(x,y) - iv(x,y)$.

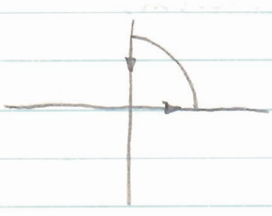
Example: What are roots of $F(z) = u + iv = z^4 + z^3 + 1$
$a_0 = 1$, $a_1 = a_2 = 0$, $a_3 = 1$, $a_4 = 1$, $K = 2$, all roots inside $C = 2$, no real roots. Find turning points.

$$F(x) = z^4 + z^3 + 1 \Rightarrow F(x) = x^4 + x^3 + 1$$
$$F'(x) = 4z^3 + 3z^2 \Rightarrow F'(x) = x^2(4x + 3)$$
$$F'(0) = 0$$
$$F''(0) = 0$$
$$F'''(0) \neq 0$$

Turning point at $F(-3/4)$, $1 - \frac{1}{4}\left(\frac{3}{4}\right)^3 > 0$



$F(iy) = 1 + y^4 - iy^3 = 0$; $1 + y^4 = 0$, $y^3 = 0$

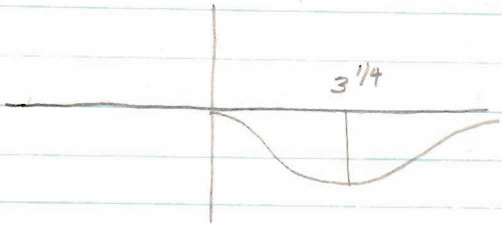


$\Delta \arg F = 0$ on the real axis

On quadrant $\Delta \arg F = \Delta \arg z^4 + O\left(\frac{1}{R}\right)$
$= 2\pi + O\left(\frac{1}{R}\right)$

On imaginary axis: $\arg F = \tan^{-1} \dfrac{-y^3}{1+y^4}$

$$\frac{d}{dy}\left(\frac{y^3}{1+y^4}\right) = \frac{3y^2 - y}{(1+y^4)^2} = 0 \quad \text{at } y^4 = 3$$



$\therefore \quad \Delta \arg F = 0$

Δ ar F in 1st quadrant $= 2\pi$

LECTURE <u>XXII</u> 11-21-60

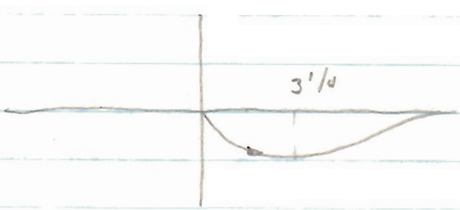
Find roots of $F(z) = z^4 + z^3 + 1 = 0$ (Phillips, p. 109

what is modulus of zero in first quadrant

On real axis: $\Delta \arg F = 0$

On imaginary axis: $1 + y^4 - iy^3$

$\arg F = \text{arc tan}\left(\frac{-y^3}{1+y^4}\right)$



$\therefore$ on imaginary axis: $\Delta \arg F = \tan^{-1} \frac{1}{2}$

On unit circle: $z = e^{i\theta}$; $F = u + iv = e^{4i\theta} + e^{3i\theta} + 1$

$u = 1 + \cos 3\theta + \cos 4\theta$, $v = \sin 3\theta + \sin 4\theta$

$\arg F = \tan^{-1} \frac{v}{u}$, $\theta = 0$, $u = 3$, $v = 0$, $\arg F = 0$

$\theta = \pi/2$, $u = 2$, $v = -1$,

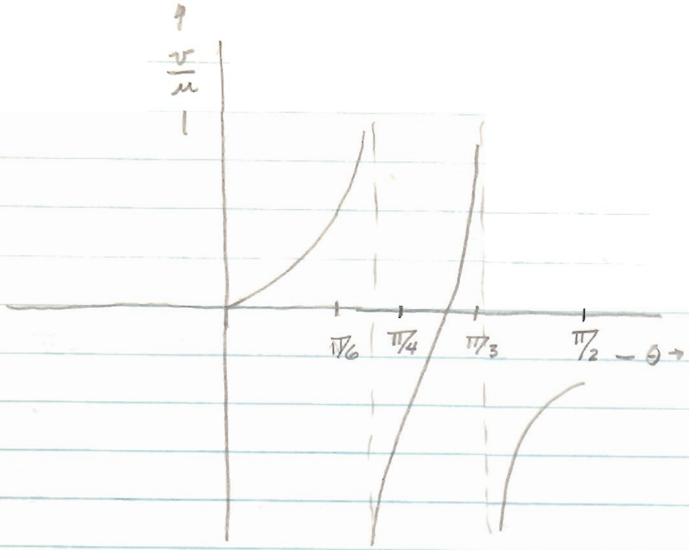
$\arg F = n\pi - \tan^{-1} \frac{1}{2}$

$n = 0$ or 2

If $n = 0$, modulus of zero in first quadrant is greater than 1.

If $n = 2$, less than one.

We want to find which n holds.



θ	u	v
0	$+3$	0
$\pi/6$	$+1/2$	$1+\sqrt{3}/2$
$\pi/4$	$-1/\sqrt{2}$	$1/\sqrt{2}$
$\pi/3$	$-1/2$	$-\sqrt{3}/2$
$\pi/2$	2	-1

arc tan $\frac{v}{u}$, goes 0 to $\pi/2$, $\pi/2$ to $3\pi/2$, $3\pi/2$ to $2\pi - \tan^{-1} 1/2$

Therefore altogether, θ goes from θ to $2\pi - \tan^{-1} 1/2$ and $n = 2$ and modulus of zero is less than one.

Maximum Modulus Theorem:

$f(z)$ is regular in and on C;

then: $|f(\alpha)| \leq M$ where M is the greatest value of $|f(z)|$ on C.

Also, if $|f(z)| \leq M$ on C, then $|f(z)| < M$ at all points within C unless $f(z)$ is constant and equal to M.

See Proofs: Phillips § 42
Copson § 7.3

If $f(z) = u + iv$, u and v are harmonic functions. Consider $\exp f(z)$ which attains its maximum on the boundary. $\exp f(z) = \exp[u + iv]$, $|\exp f(z)| = e^{u}$ therefore u has its maximum on the boundary.

Analytic Continuation:

Recall theorem relating $f(z)$ and $g(z)$ in single domain.

Regular functions defined by definite integrals.

$$f(z) = \int_a^b F(z,t)\, dt$$

If we can find the derivatives by differentiating under the integral sign, then:

$$f^{(n)}(z) = \int_a^b F_z^{(n)}(z,t)\, dt$$

See Copson, Ch. 5, ¶ 5.4

Infinite integrals: $\lim_{T\to\infty} \int_a^T F(z,t)\,dt \equiv \int_a^\infty F(z,t)\,dt = f(z)$

which can be differentiated under the integral sign if the integral exists and is uniformly convergent in some domain D_1 within C. Suppose this true of every domain within C. Then $f(z)$ is a regular function of z within C and can be differentiated under the integral sign.

Uniform Convergence: If $\int_a^\infty F(z,t)\,dt$ converges to $f(z)$

then $\left| f(z) - \int_a^T F(z,t)\,dt \right| < \epsilon$ if $T > T_0(\epsilon, z)$

If we can find T_0 that depends on ϵ only then uniformly convergent. That is, limit is independent of the order in which the sum is taken.

If $\int_a^\infty |F(z,t)|\,dt$ converges, $\int_a^\infty F(z,t)\,dt$ is absolutely convergent.

See Copson; ¶ 5.52

Wierstrass M-Test.

Given: $F(z,t)$ regular, $t \geq a$,

$|F(z,t)| \leq M(t)$ for all z in D_1.

$\int_a^\infty M(t)\,dt$ converges

Examples: $\displaystyle\int_0^\pi \frac{dt}{1 - z\cos t}$ $\displaystyle\int_0^\infty e^{-zt}\,dt$

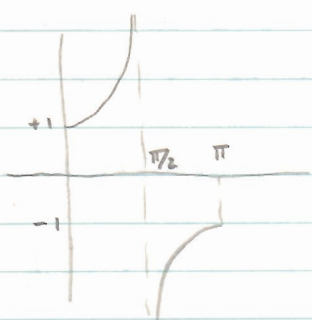
LECTURE XXIII 11-23-60

Functions Defined by Integrals:

Consider: $$\int_0^\pi \frac{dt}{1 - z\cos t} = F(z)$$

The integrand is continuous in z and t except for $z = \sec t$.

$0 \leq t \leq \pi$



In the cut plane, the integrand is regular.

Take $z = x$ between $-1 \leq x \leq 1$ along the real axis. Make substitution: $\tau = \tan\frac{1}{2}t$,

$$\cos t = \frac{1-\tau^2}{1+\tau^2}, \quad dt = \frac{2\,d\tau}{1+\tau^2}$$

$$\therefore F(x) = \frac{2}{1-x}\int_0^\infty \frac{d\tau}{1 + \frac{1+x}{1-x}\tau^2} \Rightarrow \frac{\pi}{\sqrt{1-x^2}}$$

Thus $F(z) = \dfrac{\pi}{(1-z^2)^{1/2}}$ in the cut plane.

Consider: $F(z) = \int_0^\infty e^{-zt}\,dt$

Integrand is regular everywhere. If $x = R(z) > \delta > 0$,

then $|e^{-zt}| = e^{-xt} \leq e^{-\delta t}$ for $t \geq 0$

and $\int_0^\infty e^{-\delta t}\,dt$ converges uniformly.

Then for $Re(z) > 0$, the integral defines a regular function of z, and

$$\int_0^\infty e^{-xt}\,dt = \frac{1}{x} \qquad \text{then in } R(z) > \delta > 0$$

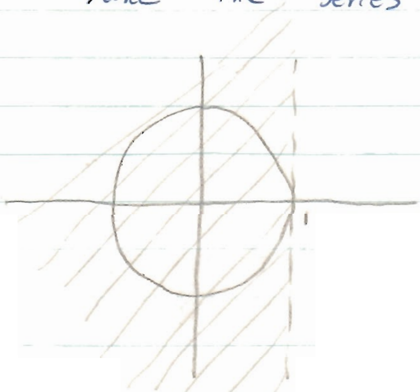
$$\int_0^\infty e^{-zt}\,dt = \frac{1}{z}$$

Taking real and imaginary parts:

$$\int_0^\infty e^{-xt}\cos yt\,dt - i\int_0^\infty e^{-xt}\sin yt\,dt = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} = \frac{1}{z}$$

Consider: $F(z) = \int_0^\infty e^{-(1-z)t}\,dt$, $R(1-z) > 0$, $R(z) < 1$ ①

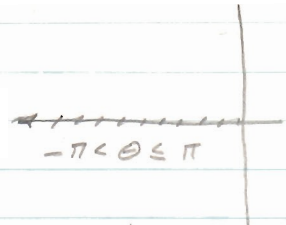
Take the series: $1 + z + z^2 + \cdots + z^n + \cdots$ $|z| < 1$ ②
$= (1-z)^{-1}$



Both ① and ② are defined in the unit circle. A process by which one moves from the region of first definition is called <u>analytic continuation</u>.

Consider the double-valued function $w = z^{1/2}$ with $-\pi < \theta \leq \pi$,

$w_1 = r^{1/2} e^{i\frac{1}{2}\theta}$, $w_2 = -w_1$



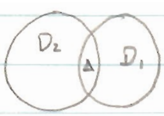
Another way: $0 < \theta \leq 2\pi$
must cut along real axis

$w_3 = r^{1/2} e^{i\frac{1}{2}\theta}$, $w_4 = -w_3$
For $0 < \theta \leq \pi$, the definitions overlap, $w_1 = w_3$

When trying to continue a many valued function, it is better not to use but one branch.

Analytic Continuation:

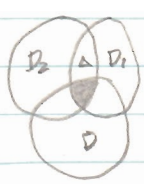
Consider $f_1(z)$, $f_2(z)$, regular in D_1, D_2.



$\Delta = (D_1, D_2)$ which is the common part with $f_1(z) = f_2(z)$. Can consider a new domain defined by the outside boundary of D_1, D_2, in which:

$\varphi(z) = f_1(z)$, $\varphi(z) = f_2(z)$

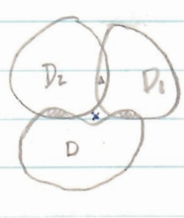
Consider continuation:



$f(z)$ regular in D
$f_1(z)$ " " D_1
$f_2(z)$ " " D_2

$f_1 = f_2 = \Delta$ and then $f_1 = f_2 = f$ in Δ.

Now Consider:



$f_1 = f$, $f_2 = f$,

however f_1 is not necessarily f_2 in Δ since x may be a singularity (branch point)

If x is a branch point, f cannot be continued without passing thru cut.

Definition of Analytic Function: Analytic where originally defined and all its continuations.

Region of existence is the region into which continuation is possible. Boundary of this region is called a natural boundary.

Take $\sum_{n=0}^{\infty} z^{2n} = f(z)$, $|z|<1$

Notice $f(z) = z + f(z^2)$, singularity $z^2=1$
$f(z^2) = z^2 + f(z^4)$, " $z^4=1$
or in general when $z^{2n}=1$, $z = e^{2p\pi i/m}$

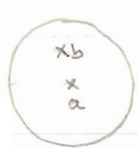
$z=1$, $z = \exp\left[\frac{2p\pi i}{2n}\right]$, $p = 0, 1, 2, 3, \cdots$ for all n

We get unlimited number of singularities or non-isolated essential singularities so circle is natural boundary $|z|<1$.

LECTURE XXIV 11-25-60

Analytic Continuation:

Wierstrass' continuation by power series.



Start with $f(z) = \sum \alpha_n (z-a)^n$, $|z-a|<R$

Now expand around b. Might only be valid inside original circle, but also could be valid outside, $\therefore$ analytic continuation.

The process can be continued over again. This method is perfectly general, but seldom used.

The largest possible circle must have a singularity on the boundary; otherwise, one could continue indefinitely.

A singular point of an analytic function lies on the boundary of a circle of continuation and not within a circle. See Titchmarsh ¶ 4.2.

A regular function must be one-valued in a region and have a unique derivative.

An analytic may be many valued.

Examples of Analytic Continuation:

I. $$f(z) = 1 + z + z^2 + \cdots + z_n + \cdots = \frac{1}{1-z}, \quad |z| < 1$$

$\alpha \neq 1$

$$f_2(z) = \frac{1}{1-\alpha} + \frac{z-\alpha}{(1-\alpha)^2} + \frac{(z-\alpha)^2}{(1-\alpha)^3} + \cdots$$

$$= \frac{1}{1-\alpha} \sum_{n=0}^{\infty} \left(\frac{z-\alpha}{1-\alpha}\right)^n ; \quad \left|\frac{z-\alpha}{1-\alpha}\right| < 1$$

Radius: $|1-\alpha|$, circle always passes thru one.

$f_2(z)$ sums to $\dfrac{1}{1 - \frac{z-\alpha}{1-\alpha}} = \dfrac{1-\alpha}{1-z}$

f_2 overlaps f, thus it is a continuation. If α is real and less than one, the new circle is completely within the old.

II. $f_1(z) = \sum_{n=0}^{\infty} \beta_n z^n$, $|z| < 1$

$$\beta_1 = \frac{1}{2} \; ; \; \beta_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n}$$

This is really the binomial expansion of $\frac{1}{(1-z)^{1/2}}$ so that branch point is present.

Take $f_2(z) = \frac{1}{(1-\alpha)^{1/2}} \sum_{n=0}^{\infty} \beta_n \left(\frac{z-\alpha}{1-\alpha}\right)^n$; $\alpha \neq 1$

$$= \frac{1}{\left(1 - \frac{z-\alpha}{1-\alpha}\right)^{1/2}} = \left(\frac{1-\alpha}{1-z}\right)^{1/2}$$

We want to choose proper sign of the square root of $f_2(z)$ in order to continue $f_1(z)$.

Let $\alpha = a + ib$; $1-\alpha = 1-a-ib = Re^{i\varphi}$, $-\pi < \varphi \leq \pi$

then $\frac{1}{(1-\alpha)^{1/2}} = \frac{1}{R^{1/2} e^{i 1/2 \varphi}}$ or $-\frac{1}{R^{1/2} e^{i 1/2 \varphi}}$

will alway mean this for + root.

Now: $\cos\varphi = \frac{1-a}{R}$, $\sin\varphi = -\frac{b}{R}$

For $b > 0$; $-\pi < \varphi < 0$
$b < 0$; $0 < \varphi < \pi$
$b = 0$, $a < 1$: $\varphi = 0$
$b = 0$, $a > 1$: $\varphi = \pi$

We can also reason this out by drawing vectors between 1 and α.

Consider case of $b > 0$:

We define in terms of the cut what the square root is:

$$f_3(z) = |1-z|^{1/2} \exp\left\{-\tfrac{1}{2} \arg(1-z)\right\}$$

$$\pi/2 > \arg(1-z) > -3\pi/2$$

At origin, $f_3(z) = f_1(z)$

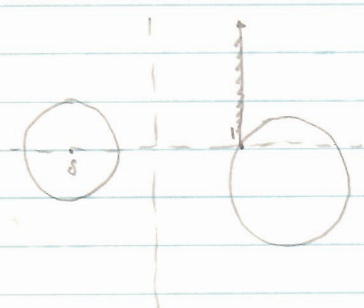
$$f_3(\alpha) = |1-\alpha|^{1/2} \exp\left\{-\tfrac{1}{2} i \arg(1-\alpha)\right\}$$

$$\pi/2 < \arg(1-\alpha) < -3\pi/2 \quad , \quad -\pi < \varphi < 0 \quad , \quad \varphi = 0$$

$f_3(z)$ will be continuation of $f_1(z)$.
Check for $b < 0$.

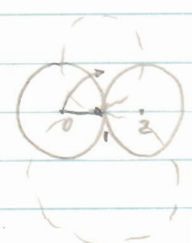
LECTURE XXV 11-28-60

Case of $b < 0$:



We must define f_3 correctly to connect f_1 and f_2.

case of $b = 0$:



$\alpha = 2$; need another circle for continuation

$$f_2(z) = e^{-i\pi/2}\left[1 - \beta_1(z-2) + \beta_2(z-2)^2 + \cdots + (-1)^n (z-2)^n + \cdots\right]$$

$$-f_2(z) = e^{i\pi/2}\left[\quad\right], \quad |z-2| < 1$$

which converges to $\mp i(z-1)^{1/2}$

Which sign to use? If path is as in diagram, $\arg(1-z)$ decreases, $\arg(1-z)^{-1/2}$ increases, thus $-f_2(z)$ is the proper series. If we continue thru the lower circle, $f_2(z)$ is the proper series.

Riemann - Schwartz Principle of Reflection:

Recall $w = f(z)$, $\bar{w} = \overline{f(z)} = \bar{f}(\bar{z}) \overset{\text{in general}}{\neq} f(\bar{z})$

If $f(z)$ is Taylor series about origin with real coefficients, $f = \bar{f}$.

Suppose $f(z)$ is regular in the following domain:

(1) If $\bar{w} = f(\bar{z})$
then $f(x)$ is real

$$w = f(z) = u(x,y) + iv(x,y)$$

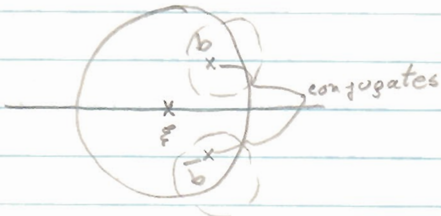
$$\bar{w} = u(x,y) - iv(x,y)$$

$\bar{w} = f(\bar{z}) = u(x,-y) + iv(x,-y)$; $v(x,y) = -v(x,-y)$
$v(x,0) = 0$

(2) If w is real on the real axis, then $\bar{w} = f(\bar{z})$

$w = a_0 + a_1(z-\xi) + \cdots + a_n(z-\xi)^n + \cdots$

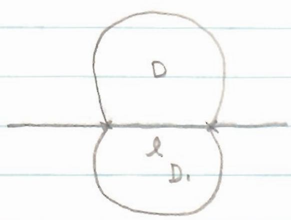
Then, if the coefficients are real, $\bar{w} = f(\bar{z})$ inside the circle. We now must continue this over the rest of the domain.



$b: \; w = a_0 + a_1(z-\zeta) + \cdots$
$\bar{b}: \; w = \bar{a}_0 + \bar{a}_1(z-\bar{\zeta}) + \cdots$

We can keep continuing until original domain is covered.

Suppose we have the following domain:



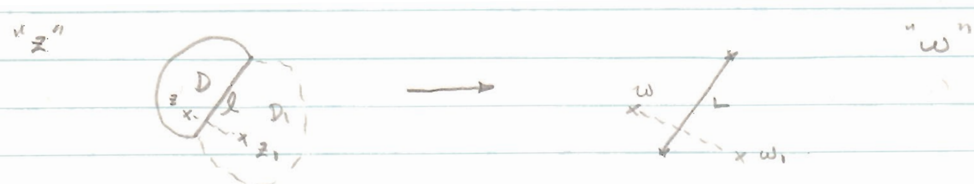
$w = f(z)$ regular in D and continuous on l, and real when z is real

Now $\bar{w} = \overline{f(z)} = \bar{f}(\bar{z})$. Then $\bar{f}(z)$ is the analytic continuation into D_1.

Define: $\left.\begin{aligned} \varphi(z) &= f(z) \text{ in } D \\ &= \bar{f}(z) \text{ in } D_1 \end{aligned}\right\}$ then $\varphi(z)$ is regular in $D + D_1$

Reference: ¶ 8.47 Copson

More general:



See Titchmarsh ¶ 4.51

Borel's Method of Continuation:

Suppose: $f(z) = \sum_0^\infty a_n z^n$, $|z| < 1$

Define: $\varphi(z) = \sum_0^\infty \frac{a_n}{n!} z^n$

Also: $F(z) = \int_0^\infty e^{-t} \varphi(zt)\, dt$

Then: $\varphi(zt) = \sum \frac{a_n}{n!} z^n t^n$

and $F(z) = \int_0^\infty e^{-t} \sum \frac{a_n z^n}{n!} t^n \, dt$

$= \sum \frac{a_n z^n}{n!} \int_0^\infty e^{-t} t^n \, dt = \sum a_n z^n = f(z)$

LECTURE XXVI 11-30-60

Borel's Sum of Divergent (or asymptotic) Series:

Consider: $\sum_1^\infty u_n$ which may or may not converge

If $u(x) = \sum_1^\infty \frac{u_n x^n}{n!}$; $\int_0^\infty e^{-x} u(x)\, dx = \sum u_n$ whenever $\sum u_n$ converges

If $\sum u_n$ diverges; define $\sum \frac{u_n z^n}{n!} = u(z)$.

Consider: $1 - \frac{1!}{a} + \frac{2!}{a^2} - \frac{3!}{a^3} + \cdots$ $a > 0$

then $u_n = (-1)^n \frac{n!}{a^n}$ and $u(z) = \frac{1}{1 + \frac{z}{a}}$, $|z| < a$

which can be continued along the positive real axis, and we can write:

$$\int_0^\infty \frac{e^{-t}}{1 + t/a}\, dt = a \int_0^\infty \frac{e^{-t}\, dt}{a + t} = a \int_0^\infty \frac{e^{-av}}{1+v}\, dv$$

$$= a e^a \int_a^\infty \frac{e^{-w}}{w}\, dw \sim 1 - \frac{1}{a} + \frac{2!}{a^2} - \frac{3!}{a^3} + \cdots$$

which is the asymptotic series. If we stop sum at $(-1)^n \dfrac{n!}{a^n}$ error is less than ratio of last term in alternating series.

Evaluation of Integrals by the Residue Theorem:

The contour we shall use will be the unit circle.

Consider: $F(\sin\theta, \cos\theta) = f(\theta)$

$$\int_0^{2\pi} f(\theta)\,d\theta = \int_{-\pi}^{\pi} f(\theta)\,d\theta = \int_{-\pi+a}^{\pi+a} f(\theta)\,d\theta$$ whenever $f(\theta)$ is defined over an interval of 2π

Symmetry about $\pi/2$:

If $f(\theta)$ is even about $\pi/2$, $\int_0^{\pi} f(\theta)\,d\theta = 2\int_0^{\pi/2} f(\theta)\,d\theta$

Now make substitutions: $z = e^{i\theta}$

$$dz = i e^{i\theta}\,d\theta, \quad d\theta = \frac{dz}{iz}$$

and $\cos\theta = \frac{1}{2}(z + z^{-1})$; $\sin\theta = \frac{1}{2i}(z - z^{-1})$.

Almost always can express integrals as real part of transformed integral in z.

References: Phillips ¶ 44, Copson, ¶ 6.3, 6.4.

Evaluation of $\dfrac{F(z)}{G(z)} = \dfrac{R_1}{z-\alpha} + \psi(z)$ for simple pole

when $G(z) = (z-\alpha)\,\varphi(z)$

$$\text{Residue} = \left[\frac{(z-\alpha)\,F(z)}{G(z)}\right]_{\alpha} = \frac{F(\alpha)}{G'(\alpha)} = \frac{F(\alpha)}{\varphi(\alpha)}$$

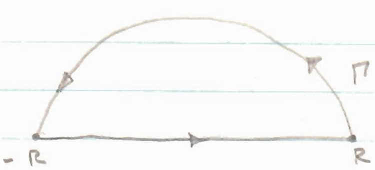
Consider $\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx$

Now $\int^{\infty}$, $\int_{-\infty}$ may not converge while above may, then we say:

$$\lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx = P\int_{-\infty}^{\infty} f(x)\,dx \quad \text{called the principle value.}$$

$$P\int_{-\infty}^{\infty} f(x)\,dx = 0 \quad \text{if } f(x) \text{ odd}$$

Define the contour:



C is whole contour.

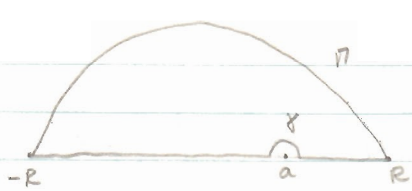
$0 \le \arg z \le \pi$

Then $\int_C f(z)\,dz = 2\pi i \sum R$ (if no singularities on c)

$$= \int_{-R}^{R} f(x)\,dx + \int_{\Gamma} f(z)\,dz$$

As R goes to ∞, $\int_\Gamma$ will disappear

Now suppose $f(z)$ has a pole at $z = a$, we indent:



Then: $\int_C f(z)\,dz = \int_{-R}^{a-\epsilon} f(x)\,dx + \int_{a+\epsilon}^{R} f(x)\,dx + \int_{\gamma} f(z)\,dz + \int_{\Gamma} f(z)\,dz$

Now $f(z) = \dfrac{K}{z-a} + \varphi(z)$

Then: $\displaystyle\int_{-R}^{a-\epsilon} \frac{dx}{x-a} = \ln \epsilon - \ln(R+a)$

$\displaystyle\int_{a+\epsilon}^{R} \frac{dx}{x-a} = \ln(R-a) - \ln \epsilon$

Both of these converge as $\epsilon \to 0$, however their sum does not.

$$\text{Sum} = \ln \frac{R+a}{R-a}$$

thus we get the principle value of the integral, when we let $\epsilon \to 0$, and $R \to \infty$

What about $\displaystyle\int_\gamma f(z)\,dz = K \int_{-\pi}^{0} \frac{dz}{z-a} = -i K \pi$

LECTURE XXVII 12-2-60

Contour Integration:

Refer to previous lecture.

Result: $\displaystyle\int_c f(z)\,dz = P\int_{-R}^{R} f(x)\,dx - \pi K i + \int_\Gamma f(z)\,dz$

Now on Γ, if $|f(z)| \leq M$, $\left|\int_\Gamma f(z)\,dz\right| < \pi M R$

We want to show $\int_\Gamma f(z)\,dz \to 0$ as $R \to \infty$.

Can do if $|f(z)| \leq \dfrac{A}{R^p}$, $p > 1$

If as $z f(z) \to 0$ uniformly in θ for $0 \leq \theta < \pi$, Then $R|f(z)| = |z f(z)| < \epsilon$ for $R > R_0$, $|f(z)| < \frac{\epsilon}{R}$,

and $|\int_\Gamma f(z)dz| < \pi\epsilon$

Consider case of $f(z) = \frac{F(z)}{G(z)}$

$$F = a_0 + a_1 z + \cdots + a_n z^n$$
$$G = b_0 + b_1 z + \cdots + b_m z^m$$
$$= b_m (z-\alpha_1)(z-\alpha_2) \cdots (z-\alpha_m)$$

$\int_\Gamma \to 0$, if $m > n+1$

Proof: $|f(z)| = \frac{|F|}{|G|} \leq \frac{|a_0| + |a_1|R + \cdots + |a_n|R^n}{|G|}$

$$\leq \frac{KR^n}{|G|} \leq \frac{KR^n}{|b_n|(R-|\alpha_1|)(R-|\alpha_2|)\cdots(R-|\alpha_m|)}$$

with $K = \sum_{s=0}^{n} |a_s|$

There is always some number L such that

$R - |\alpha_i| \geq LR$

Then: $|f(z)| \leq \frac{K_1 R^n}{R^m}$ and $m > n+1$ for $|f(z)| \to 0$ as $R \to \infty$

Consider: $f(z) = e^{miz} g(z)$

On Γ: $z = Re^{i\varphi}$, $|f(z)| = |e^{miR(\cos\varphi + i\sin\varphi)}||g(z)|$

$$= e^{-mR\sin\varphi}|g(z)| \leq M(R)\, e^{-mR\sin\varphi}$$

Now: $|\int_\Gamma f(z)dz| = |i\int_0^\pi f(Re^{i\varphi})\, Re^{i\varphi}\, d\varphi| \leq \int_0^\pi R|f|\, d\varphi$

$$\leq R\, M(R) \int_0^\pi e^{-mR\sin\varphi}\, d\varphi$$

Now $\sin\varphi$ is even about $\pi/2$, so:

$$\left| \int_\Gamma f(z)\, dz \right| \leq 2RM(R) \int_0^{\pi/2} e^{-mR\sin\varphi}\, d\varphi$$

For $0 \leq \varphi \leq \pi/2$, $\frac{\sin\varphi}{\varphi}$ decreases as φ increases

and $1 \geq \frac{\sin\varphi}{\varphi} \geq \frac{2}{\pi}$

Thus $\sin\varphi \geq \frac{2}{\pi}\varphi$, and $e^{-mR\sin\varphi} \leq e^{-\frac{2mR\varphi}{\pi}}$

and $\int_0^{\pi/2} e^{-mR\sin\varphi}\, d\varphi \leq \int_0^{\pi/2} e^{-\frac{2mR\varphi}{\pi}}\, d\varphi$

$$= \frac{\pi}{2mR}\left(1 - e^{-mR}\right) \leq \frac{\pi}{2mR}$$

Therefore $\left| \int_\Gamma f(z)\, dz \right| \leq \frac{\pi}{m} M(R)$

and $\int_\Gamma e^{miz} g(z)\, dz \to 0$, providing $|g(z)| \leq \frac{A}{R^q}$, $q > 0$

or if as $R \to \infty$, $g(z) \to 0$ uniformly in $\arg z$

for $0 \leq \arg z \leq \pi$.

This result is called Jordan's Lemma

We can now evaluate:

$$\int_{-\infty}^{\infty} g(x)\, e^{imx}\, dx, \quad \int_{-\infty}^{\infty} g(x) \cos mx\, dx, \quad \int_{-\infty}^{\infty} g(x) \sin mx\, dx$$

These converge if $g(x) \to 0$, as $x \to \infty$.
Some identities:

$$\int_{-R}^{R} f(x)\, dx = \int_0^R \left[f(x) + f(-x) \right] dx$$

Read: Phillips, ¶ 45, 46
Copson: ¶ 6.5
6.51
6.52

LECTURE XXVIII   12-5-60

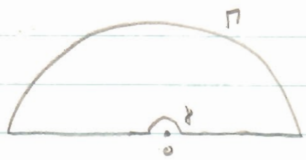
Evaluation of Real Integrals:

Consider: $I = \int_0^\infty \frac{\sin m\xi}{\xi}\, d\xi \quad (m > 0)$

Let $m\xi = x$; $\frac{d\xi}{\xi} = \frac{dx}{x}$

$\therefore \quad I = \int_0^\infty \frac{\sin x}{x}\, dx$

Cannot integrate around infinite semicircle as e^y becomes unbounded. Use $\sin x = \operatorname{Im} e^{ix}$



$\int_\Gamma \to 0$ Jordan's Lemma

$\int_\gamma \to -\pi i$

$e^{iz} \to 1$ as $z \to 0$

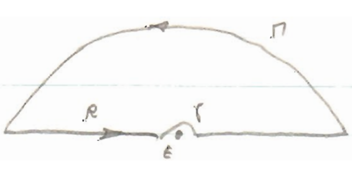
Then $P\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\, dx - \pi i = 0$; $\underbrace{P\int_{-\infty}^{\infty} \frac{\cos x}{x}\, dx}_{=0} + iP\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx - \pi i = 0$

since integrand odd

$P\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx - \pi = 2\int_0^\infty \frac{\sin x}{x}\, dx - \pi = 0$; $\therefore \int_0^\infty \frac{\sin x}{x}\, dx = \pi/2$

Consider: $\int_0^\infty x^{a-1} g(x)\, dx$; $f(z) = z^{a-1} g(z)$

$\arg z = 0$ on the + real axis ; $0 \le \arg z \le \pi$



On the negative real axis $z = re^{i\pi}$

$\int_R^\varepsilon (re^{i\pi})\, g(re^{i\pi})\, e^{i\pi}\, dr$

$+ \int_\varepsilon^R r^{a-1} g(r)\, dr + \int_\gamma z^{a-1} g(z)\, dz + \int_\Gamma z^{a-1} g(z)\, dz = \oint_C f(z)\, dz$

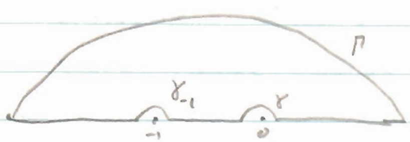
$z f(z) = z^a g(z) \to 0$ uniformly in θ for $0 \le \theta \le \pi$, as $r \to 0$

For small r, $|g(z)| \le \frac{A}{r^n}$, $\int_\gamma f(z)\,dz \to 0$, if $m < a$.

If $z f(z) = z^a g(z) \to 0$ uniformly in θ, $0 \le \theta \le \pi$,
as $r \to \infty$, $\int_\Gamma f(z)\,dz \to 0$. For large r $|g(z)| \le \frac{B}{r^n}$,

then for $n > a$, $\int_\Gamma f(z)\,dz \to 0$

If $g(z) = \frac{1}{1+z}$; $\int_{\Gamma,\gamma} \frac{z^{a-1}}{1+z}\,dz \to 0$ if $0 < a < 1$



The residue at $z = -1$ of $\frac{z^{a-1}}{z+1} = e^{i\pi}$

$$\lim_{z \to e^{i\pi}} z^{a-1} = e^{i\pi(a-1)} = -e^{i\pi a}$$

$$\therefore \int_{\gamma_{-1}} = \pi i\, e^{i\pi a}$$

and $e^{i\pi a} P\int_R^\epsilon x^{a-1} g(-x)\,dx + \int_\epsilon^R x^{a-1} g(x)\,dx + \pi i\, e^{i\pi a} = 0$

with the limit taken as $\epsilon \to 0$, $R \to \infty$.
Then

$$-e^{i\pi a} P\int_0^\infty \frac{x^{a-1}}{1+(-x)}\,dx + \int_0^\infty \frac{x^{a-1}}{1+x}\,dx + i\pi e^{i\pi a} = 0$$

Take imaginary part:

$$-\sin \pi a \quad P\int_0^\infty \frac{x^{a-1}}{1-x}\,dx + \pi \cos \pi a$$

$$\text{or} \quad P\int_0^\infty \frac{x^{a-1}}{1-x}\,dx = \pi \cot \pi a$$

For the other part

$$- \sin \pi a \int_0^\infty \frac{x^{a-1}}{1+x} dx + \pi = 0 \quad ; \quad \int_0^\infty \frac{x^{a-1}}{1+x} dx = \pi \operatorname{cosec} \pi a$$

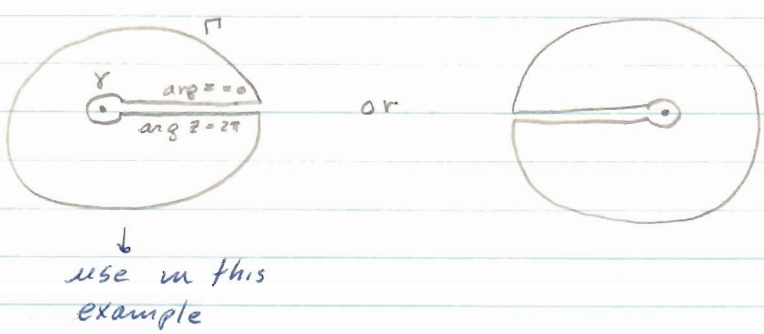
If $g(x) = \frac{1}{(1+x)^2}$ which has a double pole at $x = -1$.

No principle value exists because no limit as $\epsilon \to 0$. That is

$$\int_A^{a-\epsilon} + \int_{a+\epsilon}^B \frac{dx}{(x-a)^2} = \frac{2}{\epsilon} - \frac{1}{a-A} - \frac{1}{B-a}$$

which is not independent of ϵ.

Take new contours:



Therefore: $\int f(z)\, dz = \int \frac{z^{a-1}}{(1+z)^2} dz$

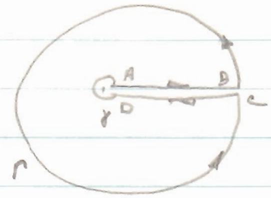
on the contour $0 \leq \arg z \leq 2\pi$

Now $\frac{z^{a-1}}{(1+z)^2} \to 0$ uniformly for $0 \leq \theta \leq 2\pi$ as $r \to 0$ if $a > 0$.

Same argument as $r \to \infty$ if $a < 2$
Therefore $0 < a < 2$.

LECTURE XXIX   12-7-60

Consider $f(z) = \dfrac{z^{a-1}}{(1+z)^2}$, $0 < a < 2$



$\int_\Gamma + \int_\gamma \to 0$ as $R, \rho \to \infty, 0$

we are left with:

$$\int_0^\infty \frac{r^{a-1}}{(1+r)^2}\,dr - \int_0^\infty \frac{(re^{2\pi i})^{a-1}}{(1+r)^2}\,dr$$

$$= \int = 2\pi i \; R(z=-1)$$

Let $\zeta = z+1$, $z = \zeta - 1 = e^{i\pi}(1-\zeta)$

$$f = \frac{e^{i\pi(a-1)}(1-\zeta)^{a-1}}{\zeta^2} = -\frac{e^{i\pi a}}{\zeta^2}\left[1-(a-1)\zeta + \cdots\right]$$

$\therefore \quad R = (a-1)e^{i\pi a}$

Now: $\int \to (e^{-i\pi a} - e^{i\pi a}) \int \dfrac{x^{a-1}}{(x+1)^2}\,dx = 2\pi i\,(a-1)\,e^{i\pi a}$

$$\therefore \quad \int_0^\infty \frac{x^{a-1}}{(x+1)^2}\,dx = \pi\,\frac{1-a}{\sin\pi a} \quad , \; a \neq 1$$

Now $|r^{a-1}| = |e^{(a-1)\log r}| = |e^{(\mathrm{R}a-1)\log r}| = |r^{\mathrm{R}a-1}|$

$\dfrac{1-a}{\sin\pi a}$ has poles in the "a" plane at $0, -1, \pm 2, \pm 3, \ldots$

$0 < \mathrm{R}a < 2$

$0 < \epsilon < \mathrm{R}a < \delta < 2$

For $r < 1$, $|r^{a-1}| < r^{\epsilon-1}$

For $r > 1$, $|r^{a-1}| < r^{\delta-1}$

then $\int_0^\infty \frac{r^{\epsilon-1}}{(r+1)^2}\,dr$ converges for $\epsilon > 0$

$\int_0^\infty \frac{r^{\delta-1}}{(r+1)^2}\,dr$ converges for $\delta < 2$

then $\int_0^\infty \frac{r^{a-1}}{(r+1)^2}\,dr = \pi \frac{1-a}{\sin \pi a}$ $\quad 0 < R(a) < 2$

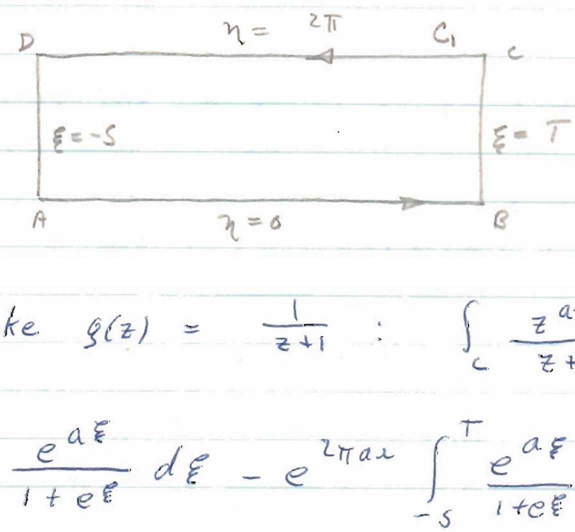
$\int_0^\infty \frac{dr}{(r+1)^2} = 1$

Consider other contours:

$z = e^\zeta, \quad z = x+iy, \quad \zeta = \xi + i\eta, \quad r = e^\xi, \quad \theta = \eta, \quad s = \log r$

$\gamma \to : \quad 0 \le \eta \le 2\pi, \quad s = \log \epsilon = -S$

$\Gamma \to : \quad 0 \le \eta \le 2\pi, \quad \xi = \log R = T$



$\int_C z^{a-1} g(z)\,dz = \int_{C_1} e^{(a-1)\zeta} g(e^\zeta) e^\zeta\,d\zeta$

Take $g(z) = \frac{1}{z+1}$: $\int_C \frac{z^{a-1}}{z+1}\,dz = \int_{C_1} \frac{e^{a\zeta}}{1+e^\zeta}\,d\zeta$

$$\int_{-S}^{T} \frac{e^{a\xi}}{1+e^\xi}\,d\xi - e^{2\pi a i} \int_{-S}^{T} \frac{e^{a\xi}}{1+e^\xi}\,d\xi$$

$$+ \int_0^{2\pi} \frac{e^{aT+ia\eta}}{1+e^{T+i\eta}}\,d\eta + \int_0^{2\pi} \frac{e^{-aS+ia\eta}}{1+e^{-S+i\eta}}\,d\eta = \int_{C_1} \frac{e^{a\zeta}}{1+e^\zeta}\,d\zeta$$

Residues at $e^\zeta = -1$, $\quad \zeta = (2\pi+1)\pi i$, $\quad w = \zeta - \pi i$

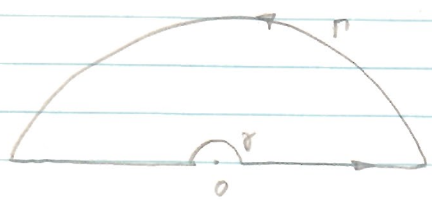
$$f(\zeta) = \frac{e^{a\zeta}}{1+e^\zeta} = \frac{e^{a\pi i} e^{aw}}{1+e^w e^{\pi i}} = \frac{e^{a\pi i}\,[1 + aw + \cdots]}{1 - e^w}$$

LECTURE XXX 12-9-60

Consider: $\int_C \log z \; g(z)\, dz$, $f(z) = \log z \; g(z)$, $0 < \arg z < \pi$

$\log z = \log r + i\theta$

Take for contour:



$z\, g(z) \log z$ should $\to 0$ as $R \to \infty$

Suppose $|g(z)| \leq \dfrac{A}{r^p}$, small r

then $|z\, g(z) \log(z)| \leq A r^{1-p} \left\{ (\log r)^2 + \pi^2 \right\}^{1/2} \leq \sqrt{2} A r^{1-p} (-\log r)$

If $p < 1$: $\left| \int_\gamma f(z)\, dz \right| \leq 2\sqrt{2}\, \pi A \epsilon^{1-p} |\log \epsilon| \to 0$ as $\epsilon \to \infty$.

Now suppose $|g(z)| \leq \dfrac{A}{r^q}$ for large r, then for

$q > 1$, $\int_\Gamma \to 0$ as $R \to \infty$

Example: $\displaystyle\int \frac{\log z}{z^2+1}\, dz$, $p = 0$, $q = 2$ which satisfies the above conditions.

Then: $$\int_0^\infty \log r \; g(r)\, dr + \underbrace{\int_0^\infty (\log r + i\pi)\, g(-r)\, dr}_{\int_{-\infty}^{0} (\log r)\, g(r)\, dr} = 2\pi i \sum R$$

Therefore: $$P\int_0^\infty \log r \left[g(r) + g(-r) \right] dr + i\pi \underbrace{\int_0^\infty g(-r)\, dr}_{\text{This gives a way to solve odd integrands}} = 2\pi i \sum R$$

The principle part notation applies only for the first integral as it diverges when the limit is not taken.

Consider: $\quad I = \displaystyle\int_0^{\frac{1}{2}\pi} \frac{a \sin 2\theta}{1 - 2a\cos 2\theta + a^2}\,\theta\,d\theta \quad$, a real $\nleq 1$

$$I = \int_0^{\infty} \frac{2a\,x\,\tan^{-1}x}{(1+x^2)\left\{(1-a)^2 + (1+a)^2 x^2\right\}}\,dx$$

using the substitution $x = \tan\theta$.

Now use: $\log(1+ix) = \log(1+x^2)^{1/2} + i\tan^{-1}x$

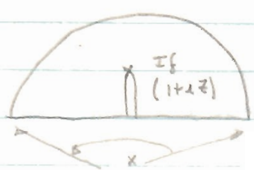
from: $\log(1+iz) = \log i + \log(z-i)$

However, it will be better to use:

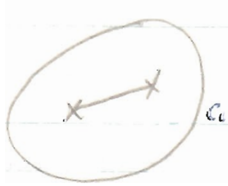
$\log(1-ix) = \log(1+x^2)^{1/2} - i\tan^{-1}x$

$\log(1-iz) = \log(-i) + \log(z+i)$ which puts the branch point outside the contour. Therefore, we integrate using for the integrand:

$$f(z) = \frac{2az\log(1-iz)}{(1+z^2)\left(\{1-a\}^2 + \{1+a\}^2 z^2\right)}$$



If we have two branch points inside contour we must cut plane



However, can not take ΣR inside C_1 except in special cases where derivatives may be single valued and integration by parts is possible.

Recall for homework reference:

$$\Gamma(a) = \int_0^\infty e^{-t}\, t^{a-1}\, dt$$

The point is that sometimes one comes to integrals in the calculation that are well-known and can be written immediately.

References: Phillips ¶ 48, 49
Copson ¶ 6.7, 6.8, 6.81, 6.82, 6.85
Titschmarsh ¶ 3.2, 3.21, 3.22, 3.23

Sometimes useful to take contour as quadrant.

LECTURE XXXI 12-14-60

Meromorphic and Integral Functions:

Meromorphic: only singularities are poles
Integral: only singularity at $z = \infty$

Example of integral function:

$$F(z) = F(0)\left(1 - \frac{z}{\alpha_1}\right)\left(1 - \frac{z}{\alpha_2}\right) \cdots \left(1 - \frac{z}{\alpha_n}\right)$$

$$\operatorname{Log} F(z) = \operatorname{Log} F(0) + \sum_{j=1}^{n} \operatorname{Log}\left(1 - \frac{z}{\alpha_j}\right)$$

Define $f(z) = \dfrac{F'(z)}{F(z)} = \displaystyle\sum_{j=1}^{n} \frac{1}{\left(1 - \frac{z}{\alpha_j}\right)}\left(-\frac{1}{\alpha_j}\right) = \sum_{j=1}^{n} \frac{1}{z - \alpha_j}$

which is meromorphic.

One is then led to believe that a general integral function can be expanded thus:

$$F(z) = \prod_{j=1}^{\infty}\left(1 - \frac{z}{\alpha_j}\right)$$

and a meromorphic function as:

$$f(z) = \sum_{i=1}^{\infty} \frac{c_i}{z - \alpha_i}$$

Suppose a meromorphic function with no pole at the origin and ordered in $0 < |a_1| < |a_2| < |a_3| \cdots$ with residues $b_1, b_2, b_3, \ldots$. Consider a contour enclosing more and more poles with $|f(z)| = o(R_n)$ such that $\frac{|f(z)|}{R_n} \to 0$ as $R_n \to \infty$. If these conditions are satisfied then we can expand this function as:

$$f(z) = f(0) + \sum_n b_n \left(\frac{1}{z - a_n} + \frac{1}{a_n} \right)$$

See Phillips for proof.

Now consider an integral function with simple zeroes not at origin:

$$F(z) = (z - a_j)\,\varphi(z) \quad ; \quad \operatorname{Log} F(z) = \operatorname{Log}(z - a_j) + \operatorname{Log} \varphi$$

Then $f = \frac{F'}{F} = \frac{1}{z - a_j} + \frac{\varphi'}{\varphi}$ which is now a meromorphic function with a simple pole.

$$\frac{d \operatorname{Log} F}{dz} = f = \frac{F'(0)}{F(0)} + \sum_n b_n \left(\frac{1}{z - a_n} + \frac{1}{a_n} \right)$$

and $\int_0^z = \operatorname{Log} F - \log F(0) = \frac{F'(0)}{F(0)} z + \sum_n b_n \left[\operatorname{Log}\left(1 - \frac{z}{a_n}\right) + \frac{z}{a_n} \right]$

with the result $F = F(0)\, e^{\frac{F'}{F} z} \prod \left(1 - \frac{z}{a_n}\right)$

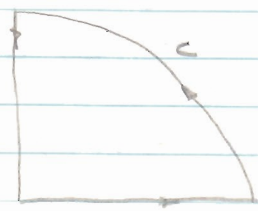
55, 56, 57 for Wednesday

LECTURE XXXII 12-16-60

Integration round a quadrant:

Consider: $\int_0^\infty \frac{\sin ax}{x^4+1}\,dx$, $a > 0$.

$$\int_C \frac{e^{iaz}}{z^4+1}\,dz$$



$$\int_0^\infty \frac{\sin ax\,dx}{x^4+1}\,dx = \int_0^\infty \frac{e^{-ay}}{y^4+1}\,dy - \frac{\pi}{2} e^{-\frac{a}{\sqrt{2}}} \cos\left(\frac{a}{\sqrt{2}} + \frac{\pi}{4}\right)$$

Expansion of Integral and Meromorphic Function.

$$F(z) = F(0)\left[1 - \frac{z}{\alpha_1}\right]\left[1 - \frac{z}{\alpha_2}\right] \cdots \left[1 - \frac{z}{\alpha_n}\right]$$

Take logarithmic derivative;

$$f(z) = \frac{d}{dz} \operatorname{Log} F(z) = \frac{F'(z)}{F(z)} = \sum -\frac{1}{\alpha_n}\left(\frac{1}{1 - \frac{z}{\alpha_n}}\right)$$

$$= \sum \left(\frac{1}{z - \alpha_n}\right)$$

For infinite expansion in partial fractions the result may not converge.

Example: $F(z) = \frac{\sin z}{z}$

$$f = \frac{d}{dz} \operatorname{Log} F = \cot z - \frac{1}{z} = \frac{z\cos z - \sin z}{z \sin z}$$

or $$f = \frac{z - \frac{z^3}{2!} + \cdots - \left(z - \frac{z^3}{3!} + \cdots\right)}{z^2 - \frac{z^4}{3!} + \cdots}$$

$$= \frac{-z^3/3}{z^2} \quad \cdots \quad = -\frac{z}{3} + \cdots$$

For $f(z) = 0$: $z = n\pi \quad (n = \pm 1, \pm 2, \ldots)$

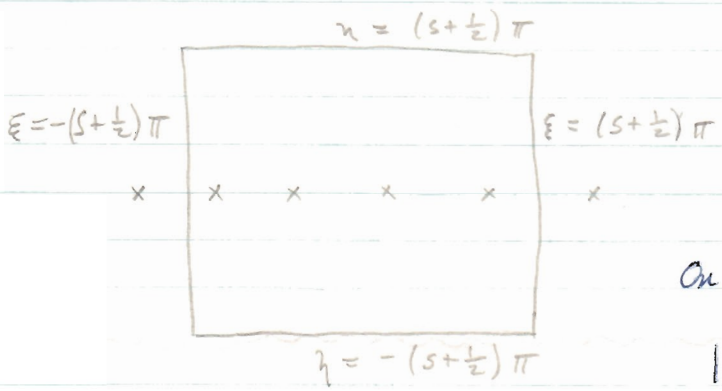
We can show that this cannot be expanded as partial fractions because of divergence.

Define: $g(\zeta) = \dfrac{f(\zeta)}{\zeta(\zeta - z)}$, works for $\cot z - \frac{1}{z}$

no pole at $\zeta = 0$ because $f(0) = 0$

poles at :	residues :
z	$\dfrac{f(z)}{z}$
$n\pi$	$\dfrac{1}{n\pi(n\pi - z)}$

Take the following contour:



Let s go to ∞ thru integral values so no poles will be hit.

On C:

$$|\zeta| \geq (s+\tfrac{1}{2})\pi$$

$$|\zeta - z| \geq |\zeta| - |z| \geq (s+\tfrac{1}{2})\pi - |z|$$

If $|f(z)| \le M$, independent of s; then $|g| \le \dfrac{M}{(s+\frac{1}{2})\pi\left[(s+\frac{1}{2})\pi - |z|\right]}$

and $\left|\oint\right| < \dfrac{2(2s+1)\pi M}{(s+\frac{1}{2})\pi\left[(s+\frac{1}{2})\pi - |z|\right]} \to 0$ as $s \to \infty$

Now show $|f(z)| \le M$ on C.

$f(z) = \cot z - \frac{1}{z}$, $|f| \le |\cot z| + \frac{1}{|z|}$

$y = (s+\frac{1}{2})\pi$; $|\cot z| = \left| i\,\dfrac{e^{iz}+e^{-iz}}{e^{iz}-e^{-iz}}\right| = \left|\dfrac{e^{2iz}+1}{e^{2iz}-1}\right|$

$\le \dfrac{1+|e^{2i(x+iy)}|}{1-|e^{2i(x+iy)}|} = \dfrac{1+e^{-2y}}{1-e^{-2y}} = \dfrac{1+e^{-(2s+1)\pi}}{1-e^{-(2s+1)\pi}}$

< 2 (say) if $e^{-(2s+1)\pi} < \frac{1}{3}$

Similarly for $y = -(s+\frac{1}{2})\pi$

$\cot z = \cot\left[iy \pm (s+\frac{1}{2})\pi\right]$

Result: $\frac{1}{2\pi i}\oint_C g(\zeta)\,d\zeta = \dfrac{f(z)}{z} + \sum_{n=-\infty}^{\infty} \dfrac{1}{n\pi(n\pi - z)} = 0$

Therefore:

$$\cot z - \frac{1}{z} = z \sum_{-\infty}^{\infty}{}' \frac{1}{n\pi(z-n\pi)}$$

$$= z \sum_{-\infty}^{\infty}{}' \left(\frac{1}{z-n\pi} + \frac{1}{n\pi}\right)$$

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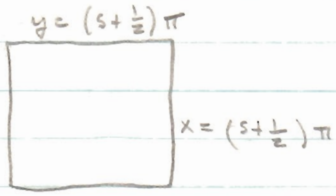
Consider $\oint_c f(z)\,dz$, $f(z) = \pi \cot \pi z \; g(z)$

where $g(z)$ is meromorphic with no coincident poles with $\cot \pi z$.

Also: $|f(z)| \leq M(s)$

with $\left| \oint_c f(z)\,dz \right| \leq M(s)\,L(s) \to 0$ in the limit as $s \to \infty$



$L(s) = (4s+2)\pi$

Suppose we find residues: $\to S$

then: $\sum_{-\infty}^{\infty} g(n) + S = 0$

with $g(0) + \sum_{n=1}^{\infty} \left[g(n) + g(-n) \right] = -S'$

We can also do $f(z) = \pi \csc \pi z \; g(z)$ and get same thing except alternating series.

Topics Left in Course:

Conformal Transformations with many valued functions
Applications to Potential Problems
Schwartz - Christoffel Transformations
Laplace and Fourier Transforms.

Conformal Transformations:

Notation: $z = x + iy = re^{i\theta}$
$\quad\quad\quad\; s = \xi + i\eta = \rho e^{i\varphi}$
$\quad\quad\quad\; A = a e^{i\alpha}$

Define: $A z^n = A r^n e^{in\theta}$ in $\lambda \leq \theta < 2\pi - \lambda$

For simplicity, take $A = 1$, and ζ in $0 \le \theta < 2\pi$

Opening of Sectors in ζ-plane:

Take $\zeta = z^{1/2}$, $z = \zeta^2$, $\rho = r^{1/2}$, $\varphi = \frac{1}{2}\theta$

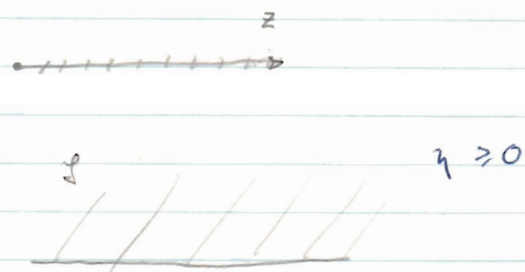
in $-\lambda \le \theta < 2\pi - \lambda$

$-\frac{1}{2}\lambda \le \varphi < \pi - \frac{1}{2}\lambda$

$\lambda = 0$, $0 \le \theta < 2\pi$, $0 \le \varphi < \pi$

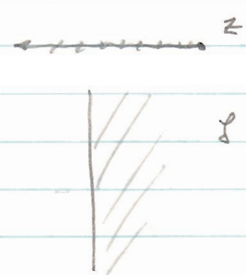
Case I. Cut z plane along + real axis
with $0 \le \theta \le 2\pi$

and $0 \le \varphi \le \pi$
so getting upper
half of ζ plane



Case II. Cut along − real axis:
$-\pi \le \theta \le \pi$

with $-\frac{1}{2}\pi \le \varphi \le \pi/2$

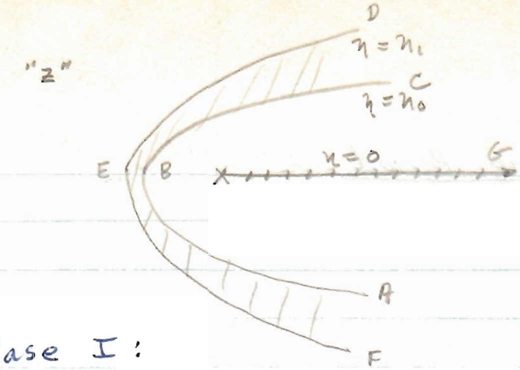


Now: $z = x + iy = \zeta^2 = (\xi + i\eta)^2 = \xi^2 - \eta^2 + 2i\xi\eta$

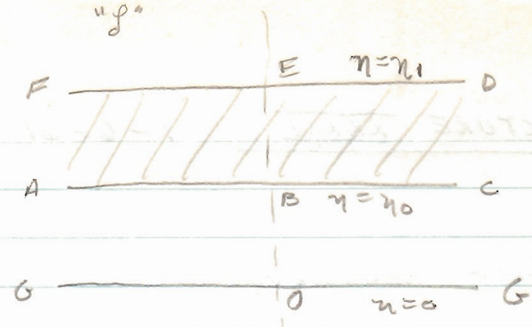
$x = \xi^2 - \eta^2$, $y = 2\xi\eta$

For $\eta > \eta_0$: $y^2 = 4\eta_0^2(x + \eta_0^2)$

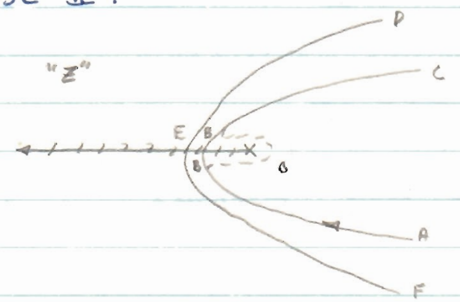
which is a set of confocal paraboli.

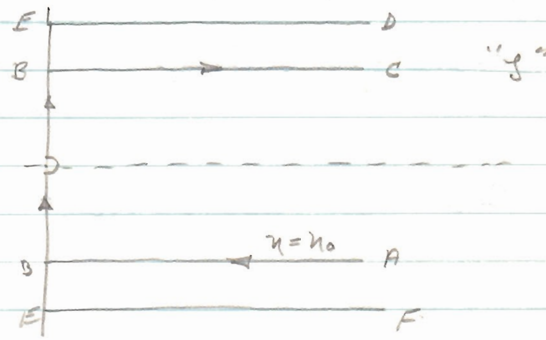


Case I:



Case II:

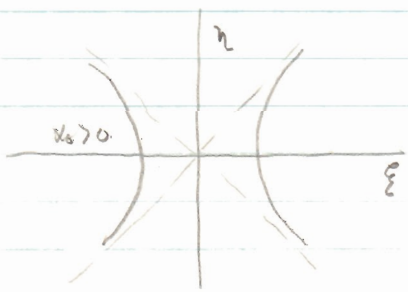


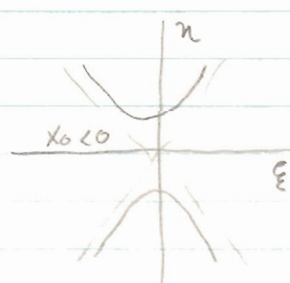


For $\xi = \xi_0$; $y^2 = 4\,\xi_0^2\,(\xi_0^2 - x)$

which are the confocal paraboli above except going to $-\infty$ instead of $+\infty$.

What about straight lines? $x = x_0$, $\xi^2 - \eta^2 = x_0$





LECTURE XXXIV 1-6-61 Reading Period

Further References:

Kellogg: Foundations of Potential Theory
N. W. McLaughlin: Complex Variable Theory and Transform Calculus
H. and B. S. Jeffreys: Methods of Mathematical Physics: ¶ 12.08 (p 389)

This is assignment for reading period.

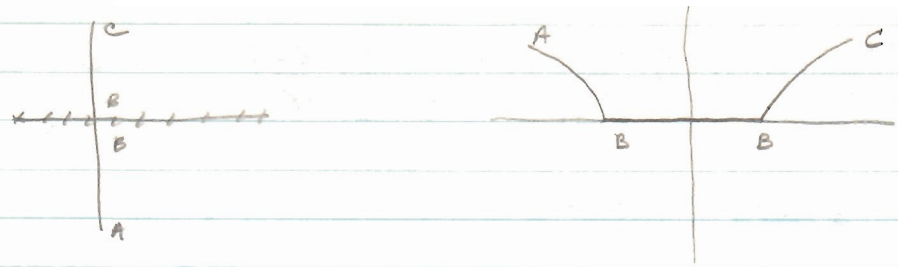
Continuation of Conformal Mapping:

$\zeta = z^{1/2}$, $z = \zeta^2$, $x = \xi^2 + \eta^2$, $y = 2\xi\eta$

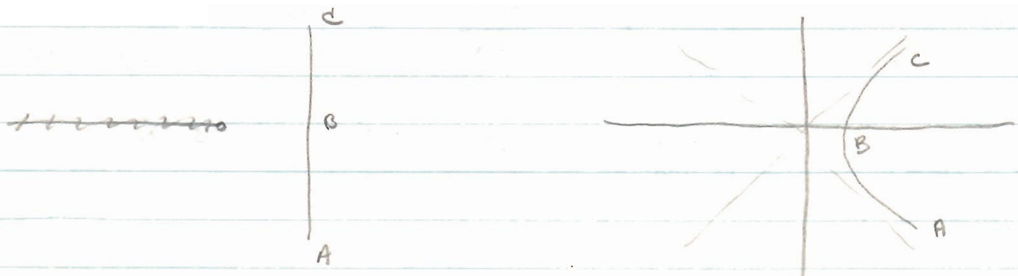
Case I: $x_0 < 0$

Case II: $x_0 > 0$

Case I: $x_0 > 0$



Case II: $x_0 < 0$



Applications in Potential Theory:

$w = \varphi + i\psi = f(z)$, $\nabla^2\varphi = 0$, $\nabla^2\psi = 0$, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

Suppose $\frac{\sin\theta}{r}$ near $r = 0$:

$w = \phi + i\psi = \frac{1}{z} = \frac{e^{-i\theta}}{r} = \frac{\cos\theta - i\sin\theta}{r}$

$\phi = \frac{\sin\theta}{r}$, separate out, $w = \frac{1}{z} + f(z)$

Suppose ψ, $2\pi K$, $\psi = K\theta$

$\phi + i\psi = K\log z = K(\log r + i\theta)$

separate $w = K\log z + f(z)$

Solutions are unique, either real or imaginary part of $f(z)$

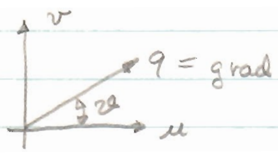
Uses of Conformal Mapping in Potential Theory:

Recall: $\dfrac{\partial^2 \phi}{\partial \xi^2} + \dfrac{\partial^2 \psi}{\partial \eta^2} = \left(\dfrac{\partial^2 \varphi}{\partial x^2} + \dfrac{\partial^2 \psi}{\partial y^2} \right) \left| \dfrac{dz}{d\zeta} \right|^2 = 0$

where $g(\zeta) = \varphi + i\psi$

Also: $u = \dfrac{\partial \varphi}{\partial x} = \dfrac{\partial \psi}{\partial y}$, $v = \dfrac{\partial \phi}{\partial y} = -\dfrac{\partial \psi}{\partial x}$

$$q e^{-i\vartheta} = u - iv = \frac{\partial \varphi}{\partial x} + i \frac{\partial \psi}{\partial x} = \frac{dw}{dz} = \frac{dw}{d\zeta}\frac{d\zeta}{dz}$$



$$= \left(\frac{\partial \varphi}{\partial \xi} - i \frac{\partial \varphi}{\partial \eta} \right) \left| \frac{d\zeta}{dz} \right| e^{i \arg \frac{d\zeta}{dz}}$$

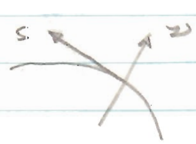
or $q e^{-i(\vartheta + \arg \frac{d\zeta}{dz})} = \left| \dfrac{d\zeta}{dz} \right| \left(\dfrac{\partial \varphi}{\partial \xi} - i \dfrac{\partial \varphi}{\partial \eta} \right)$

$|\nabla_u \varphi| = \left| \dfrac{d\zeta}{dz} \right| \left| \dfrac{dw}{d\zeta} \right|$, $h = \left| \dfrac{dz}{d\zeta} \right|$

Then $(dx)^2 + (dy)^2 = h^2 [(d\xi)^2 + (d\eta)^2]$

For ξ = constant, $ds = h\, d\eta$

Component



$$\frac{\partial \varphi}{\partial \nu} = \frac{\partial \psi}{\partial s}$$

Suppose: $\dfrac{\partial \varphi}{\partial \nu} = lU + mV$ as BC:

$$\frac{\partial \psi}{\partial s} = U \frac{dy}{ds} - V \frac{dx}{ds}$$

$$\psi = Uy - Vx + \text{const.}$$

is BC in ψ

Suppose $\dfrac{\partial \varphi}{\partial \nu} = -\Omega\,(ly - mx)$ as BC

$$\frac{\partial \psi}{\partial s} = \Omega\left(y\frac{dy}{ds} + x\frac{dx}{ds}\right)$$

$$\psi = -\tfrac{1}{2}\,\Omega\,(x^2+y^2) + \text{const.}$$

Suppose $u - iv = \dfrac{dw}{dz}$

Then $\log \dfrac{dw}{dz} = \log q - i\theta$

Consider the transformation: $z = \cosh \zeta$

Now: $\zeta = \cosh^{-1} z = \operatorname{Log}\left(z + (z^2-1)^{1/2}\right)$

Branch Points at $z = \pm 1, \infty$

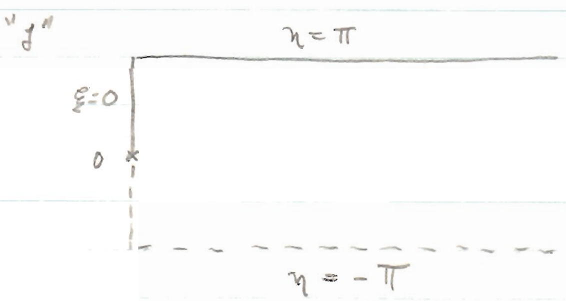
or

We will use:

Consider ζ_1, $2n\pi i \pm \zeta_1$, $a < \eta \le a + 2\pi$

we choose $-\pi < \eta \le \pi$ as the strip.

or $-\pi < \arg\left[z + (z^2-1)^{1/2}\right] \le \pi$, $\xi \ge 0$



more generally, $z = c \cosh \zeta$,

$$\xi \geq 0 \,, \quad -\pi \leq \eta \leq \pi$$

such that lower boundary of strip is included in region.

Now: $x = c \cosh \xi \cos \eta$

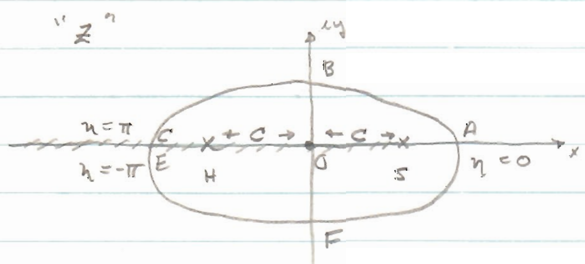
$y = c \sinh \xi \sin \eta$

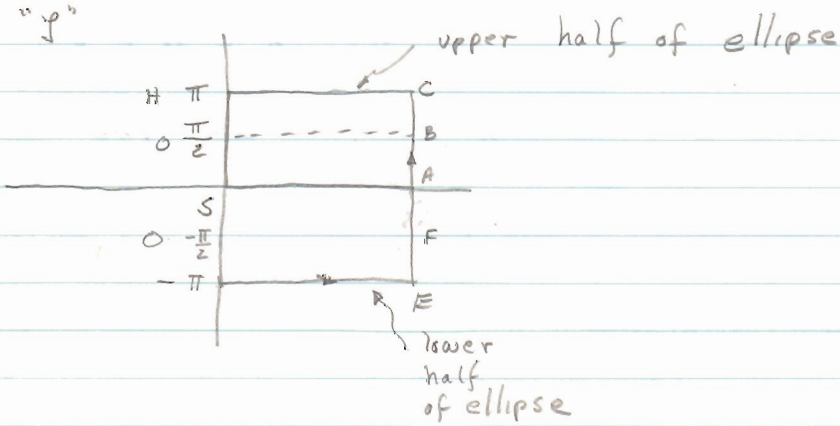
For $\xi = \xi_0$: $a = c \cosh \xi_0$, $b = c \sinh \xi_0$

$$x = a \cos \eta \,, \quad y = b \sin \eta$$

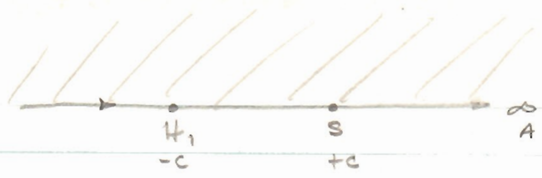
These equations are ellipses:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \,, \quad a^2 - b^2 = c^2 \quad \text{Confocal}$$

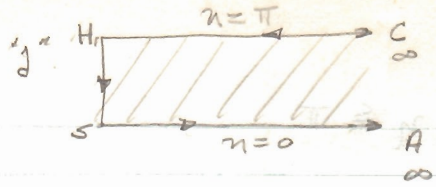


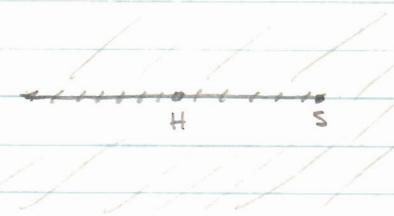


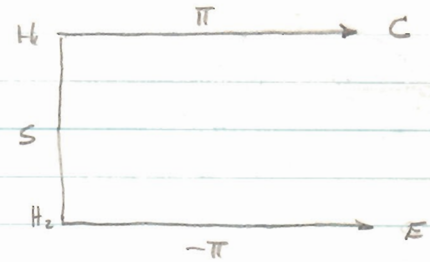
"z"



"ζ"







For $\eta = \eta_0$: $x = c \cos\eta_0 \cosh\xi$

$y = c \sin\eta_0 \sinh\xi$

$$\frac{x^2}{A^2} - \frac{y^2}{B^2} = 1 , \quad A = c\cos\eta_0 , \quad B = c\sin\eta_0 \qquad A^2 + B^2 = c^2$$

$\eta_0 = 0$, $y = 0$, $x = c\cosh\xi$, $0 \le \xi \le \infty$, SA , A at ∞

$\eta_0 = \pm\pi$, $y = 0$, $x = -c\cosh\xi$, $0 \le \xi \le \infty$, HC , C at ∞

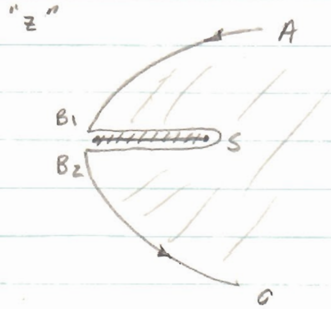
$\eta_0 = \pm\frac{1}{2}\pi$, $x = 0$, $y = \pm c\sinh\xi$, $\eta_0 = +\frac{1}{2}\pi$

$= -\frac{1}{2}\pi$

$-\frac{1}{2}\pi < \eta_0 < \frac{1}{2}\pi$, $x > 0$

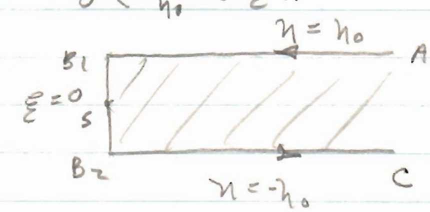
$|\eta_0| > \frac{1}{2}\pi$, $x < 0$

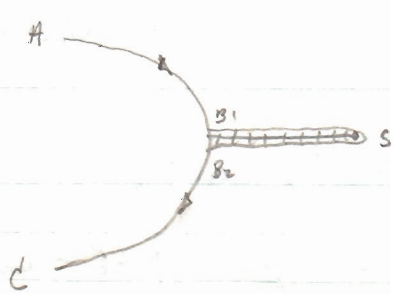
"z"

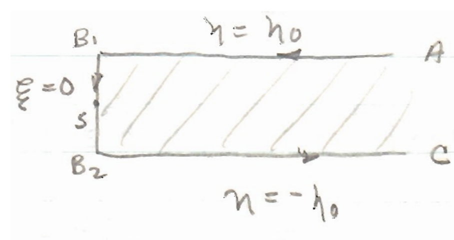


$0 < \eta_0 < \frac{1}{2}\pi$

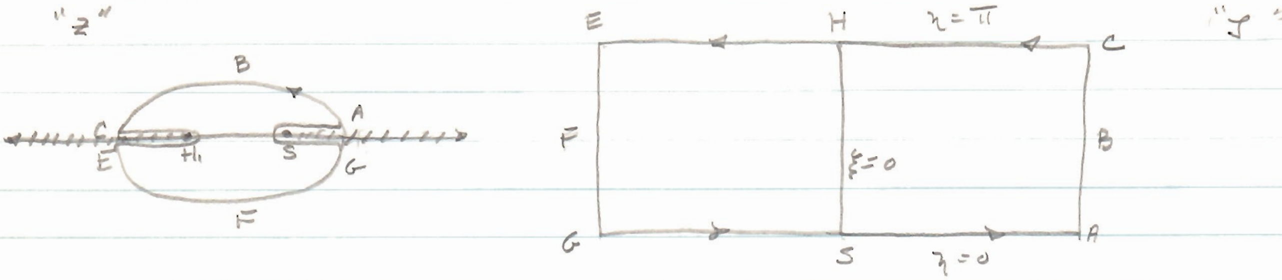
"ζ"



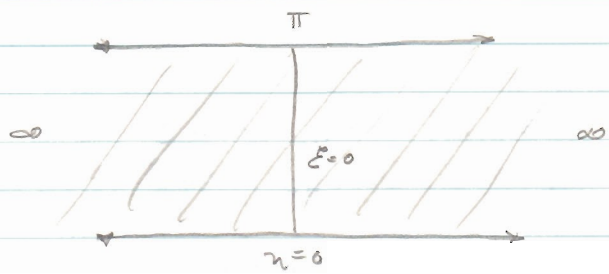




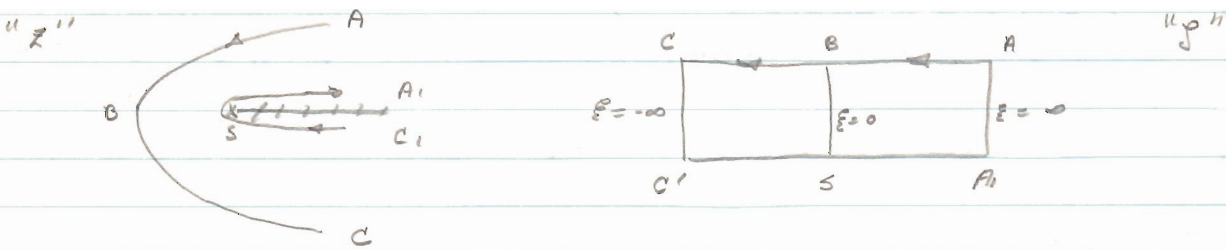
$0 \le \eta \le \pi$

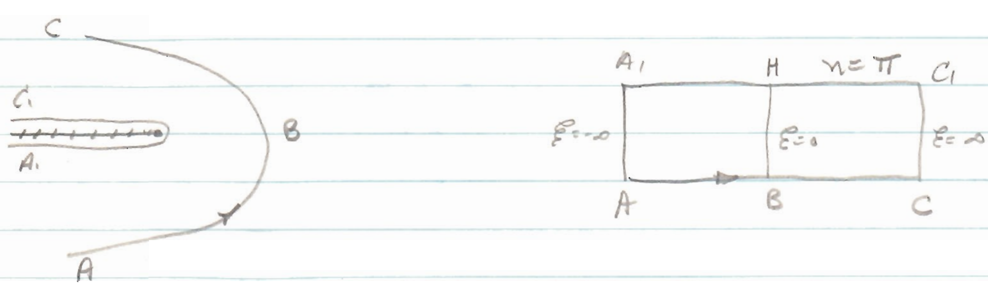


As $\xi_0 \to \infty$:



$\eta_0 < \frac{1}{2}\pi$





Again write: $z = c \cosh \zeta$, $\quad \sinh(\xi + i\eta) = \sinh \xi \cos \eta + i \cosh \xi \sin \eta$

$\dfrac{dz}{d\zeta} = c \sinh \zeta$,

$$|\sinh(\xi + i\eta)|^2 = (\cosh^2 \xi - 1)\cos^2 \eta + \cosh^2 \xi \sin^2 \eta$$

$$= \cosh^2 \xi - \cos^2 \eta = \tfrac{1}{2}(\cosh 2\xi - \cos 2\eta)$$

$$|\cosh \zeta|^2 = \cosh^2 \xi - \sin^2 \eta = \tfrac{1}{2}(\cosh 2\xi + \cos 2\eta)$$

$$\xi \to \infty, \quad \xi > 0, \quad z = \frac{c}{2}(e^{\zeta} + e^{-\zeta}), \quad e^{\zeta} = e^{\xi + i\eta}$$

$$\xi \to \infty, \quad z \sim \frac{c}{2} e^{\zeta}, \quad r \sim \frac{c}{2} e^{\xi}, \quad \xi \sim \log \frac{2r}{c}$$

$$z = c \sin \zeta, \quad \cosh \zeta = \sin(\tfrac{1}{2}\pi - i\zeta_1)$$

Electrostatic Potential:

$$w = \varphi + i\psi, \quad \varphi \text{ is potential}$$

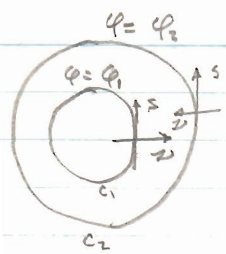
$$E = -\nabla \varphi, \quad E_x = -\frac{\partial \varphi}{\partial x} = -\frac{\partial \psi}{\partial y}$$

$$E_y = -\frac{\partial \varphi}{\partial y} = +\frac{\partial \psi}{\partial x}$$

Take φ constant on conductor surface:

$$\sigma = \frac{1}{4\pi} E_n = -\frac{1}{4\pi}\frac{\partial \varphi}{\partial n}$$

However, if surface is ∞ cylinder, only the radial component of E exist.



$$\sigma = -\frac{1}{4\pi}\frac{\partial \varphi}{\partial n} = -\frac{1}{4\pi}\frac{\partial \psi}{\partial s}$$

On C_1: $Q_1 = -\frac{1}{4\pi}\int \frac{\partial \psi}{\partial s}\, ds = -\frac{1}{4\pi}[\psi]$ change in ψ

On C_2: $\sigma = -\frac{1}{4\pi}\frac{\partial \varphi}{\partial n} = +\frac{1}{4\pi}\frac{\partial \psi}{\partial s}$

$$Q_2 = \frac{1}{4\pi}[\psi] = -Q_1$$

ψ is multiple-valued.

By definition: $$C = \frac{Q_1}{\varphi_1 - \varphi_2} = \frac{[\psi]}{4\pi(\varphi_2 - \varphi_1)}$$

Now suppose: $\zeta = f(z) = \xi + i\eta$

We want: $\xi = \xi_1$, $\xi = \xi_2$

φ must be harmonic: $\varphi = A\xi + B$

$$w = A\zeta + B$$
$$\psi = A\eta$$

Then $[\psi] = A[\eta]$

$$\varphi_2 - \varphi_1 = A(\xi_2 - \xi_1)$$

Suppose ξ's are circles of radius a, b; $b > a$

$$w = A \log z + B$$
$$\varphi = A \log r + B \quad (\text{harmonic})$$
$$\psi = A\vartheta$$

Then: $C = \dfrac{2\pi}{4\pi \log b/a} = \dfrac{1}{2 \log b/a}$

Suppose conductors are confocal ellipses:

Choose: $z = c \cosh \zeta$, given ξ_0, ξ_1 as axis.

$$w = A\zeta + B \quad + iG \text{ not needed}$$
$$\varphi = A\xi + B$$
$$\psi = A\eta + G$$

Now $C = \dfrac{2\pi}{4\pi(\xi_1 - \xi_0)}$

For $\xi = \xi_0$, $x = a \cos \eta$, $y = b \sin \eta$

$$\frac{a}{c} = \cosh \xi_0, \quad \frac{b}{c} = \sinh \xi_0$$

$$\xi_0 = \frac{a+b}{c} = \left(\frac{a+b}{a-b}\right)^{1/2} = \log\left(\frac{a+b}{a-b}\right)^{1/2}$$

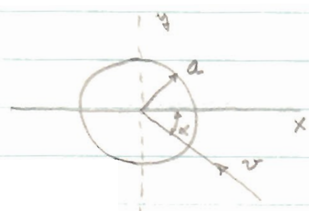
$$\xi_1 = \log \frac{a_1+b_1}{c}$$ 59, 62, 63 for Fri.

$$\xi_0 = \log \frac{a_0+b_0}{c}$$

$$\xi_1 - \xi_0 = \log \frac{a_1+b_1}{a_0+b_0}$$

Then: $$C = \frac{1}{2\log \frac{a_1+b_1}{a_0+b_0}}$$

LECTURE XXXV 1-9-61



$$\left.\begin{aligned} u &= \frac{\partial \varphi}{\partial x} \\ v &= \frac{\partial \varphi}{\partial y} \end{aligned}\right\}$$ $$\nabla^2 \varphi = 0$$

$$w = \varphi + i\psi = f(z)$$

$$u - iv = \frac{dw}{dz}$$

ψ is constant on internal boundary.

Undisturbed flow: $u = -U\cos\alpha$

$v = U\sin\alpha$

$u - iv = -U(\cos\alpha + i\sin\alpha) = -Ue^{i\alpha}$

$$w = -Uze^{i\alpha}$$

$w = w' - Uze^{i\alpha}$, w' is disturbed flow.

To begin with, although incorrect, choose:

$\varphi', \psi' = O\left(\frac{1}{r}\right)$ at ∞ to start with.

$\alpha = 0$: $w = w' - Uz$

$\psi = \psi' - Ur\sin\theta$

Then: $w = \frac{A}{z} - Uz \quad , \quad \frac{1}{z} = \frac{e^{-i\theta}}{r}$

$$\psi = \frac{-A}{r}\sin\theta - Ur\sin\theta$$

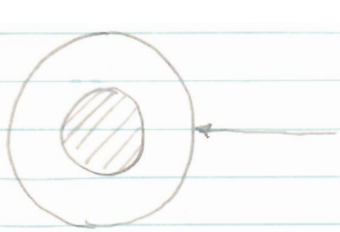
$$A = -Ua^2$$

$$w = -U\left[z + \frac{a^2}{z^2}\right]$$

$$\left.\begin{aligned} \psi &= -U\left[r - \frac{a^2}{r}\right]\sin\theta \\ \varphi &= -U\left[r + \frac{a^2}{r}\right]\cos\theta \end{aligned}\right\} \quad \begin{aligned} q e^{i\vartheta} &= u - iv \\ &= \frac{dw}{dz} = -U\left[1 - \frac{a^2}{z^2}\right] \end{aligned}$$

For $\alpha \neq 0$, rotate axis, that is, write $z = z e^{i\alpha}$. Can choose conditions on velocity such that:

$$\left.\begin{aligned} u &+ U\cos\alpha \\ v &- U\sin\alpha \end{aligned}\right\} = O\left(\frac{1}{r}\right) \quad , \text{ potential is logarithmic}$$



$$\int \frac{\partial\varphi}{\partial n}\,ds = \int \frac{\partial\psi}{\partial s}\,ds = [\psi]$$

for incompressible fluid.

$\int \frac{\partial\varphi}{\partial s}\,ds = 0 \quad , \quad \int q_s\,ds = K \quad ,$ many-valued,

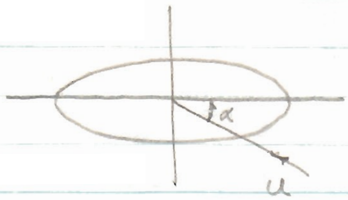
called circulation, will have branch points inside cylinder:

$$\varphi = \frac{K\theta}{2\pi} \quad , \quad \psi = -\frac{K}{2\pi}\log\frac{r}{a}$$

$$w = -\frac{iK}{2\pi}\log\frac{z}{a}$$

must add this to $w = -U\left[z e^{i\alpha} + \frac{a^2}{z}e^{-i\alpha}\right]$

Ellipsoidal Case:



$$\xi = \xi_0, \quad z = c \cosh \zeta$$

$$x = A \cos \eta, \quad y = A \sin \eta$$

$$A = c \cosh \xi_0, \quad B = c \sinh \xi_0$$

$$\left.\begin{aligned} c e^{\xi_0} &= A + B \\ c e^{-\xi_0} &= A - B \end{aligned}\right\} \quad \begin{aligned} c &= \sqrt{A^2 - B^2} \\ e^{\xi_0} &= \sqrt{\frac{A+B}{A-B}} \end{aligned}$$

$$w = -U z e^{i\alpha} = -U c e^{i\alpha} \cosh \zeta \text{ at } \infty$$

$$\psi = \psi_0 = -Uc \left[\cos\alpha \sinh \xi \sin \eta + \sin \alpha \cosh \xi \cos \eta \right]$$

at ∞.

$$e^{-\zeta} = e^{-\xi} \cos \eta + i e^{-\xi} \sin \eta$$
$$i e^{-\zeta} = i e^{-\xi} \cos \eta - e^{-\xi} \sin \eta$$

$$|e^{-\zeta}| = e^{-\xi} = O\left(\tfrac{1}{r}\right) \text{ as } \xi \to \infty$$

$$u - iv = \frac{dw}{dz} = \frac{dw}{d\zeta} \Big/ \frac{dz}{d\zeta}$$

Then:

$$\psi = \psi_0 + e^{-\xi} \left[D \sin \eta + E \cos \eta \right]$$

$$D = Uc \cos \alpha \, e^{\xi_0} \sinh \xi_0 = UB \sqrt{\frac{A+B}{A-B}} \cos \alpha$$

$$E = Uc \sin \alpha \, e^{\xi_0} \cosh \xi_0 = UA \sqrt{\frac{A+B}{A-B}} \sin \alpha$$

$$w = -Uc\, e^{i\alpha} \cosh \zeta - U \sqrt{\frac{A+B}{A-B}} \left\{ B \cos \alpha - i A \sin \alpha \right] e^{-\zeta}$$

We can choose: $\varphi = \frac{K}{2\pi} \eta$, $\quad w = -\frac{iK}{2\pi} \zeta + \text{constant}$

Is this all right at ∞? Recall: $e^{\zeta} \sim \frac{2z}{c}$,

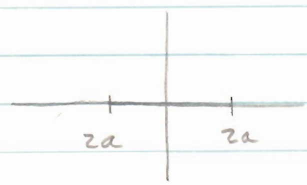
$\zeta \sim \log z + \text{constant}$, so OK.

Going back to circle:

Take $\alpha = -1$, then $w = z + \frac{a^2}{z}$

Circle in z plane, $z = a e^{i\theta}$, $\varphi = a e^{i\theta} + a e^{-i\theta}$
$= 2a \cos\theta$

goes to line in "w" plane:

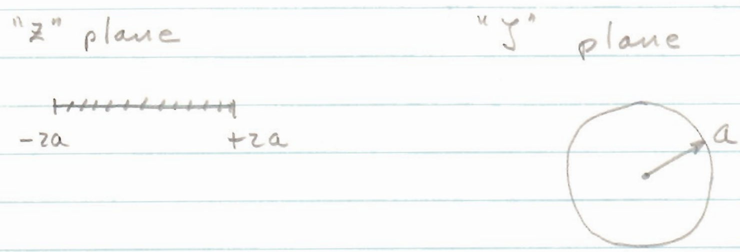


We now interchange roles of z and w:

$z = \zeta + \frac{a^2}{\zeta}$, $\zeta^2 - \zeta z + a^2 = 0$

$\zeta = \frac{1}{2}\left\{ z + (z^2 - 4a^2)^{1/2} \right\}$, branch points at $z = \pm 2a$

Then, cut z plane in $-2a$ to $+2a$.



There are two branches of ζ. Consider first, branch outside $+2a$:

$\zeta = a e^{i\varphi}$, $dz = a(e^{i\varphi} + e^{-i\varphi})\,d\varphi = 2a \cos\varphi \, d\varphi$

then $x = 2a\cos\varphi$, $y = 0$

$\zeta = \frac{1}{2} z \left\{ 1 + \left[1 - \frac{4a^2}{z^2} \right]^{1/2} \right\}$, $|z| > 2a$

$= \frac{1}{2} z \left\{ 1 + 1 - \frac{2a^2}{z^2} + \dots \right\} = -z\left(1 - \frac{a^2}{z^2} + \dots\right)$

For the other branch:

$$\zeta = \frac{1}{2} z \left\{ 1 - \left(1 - \frac{2a^2}{z^2} + \cdots \right) \right\} = \frac{a^2}{z} + \cdots$$

Take circle on which $\zeta = Re^{i\varphi}$

$$z = Re^{i\varphi} + \frac{a^2}{R} e^{-i\varphi}$$

$$x = \left(R + \frac{a^2}{R} \right) \cos\varphi$$

$$y = \left(R - \frac{a^2}{R} \right) \sin\varphi$$

$$A = R + \frac{a^2}{R}, \quad B = R - \frac{a^2}{R}, \quad \begin{aligned} x &= A\cos\varphi \\ y &= B\sin\varphi \end{aligned}$$

Thus circle goes into ellipse:

All confocal ellipses.

Recapitulation:

$$z = \zeta + \frac{a^2}{\zeta} \qquad (a \text{ real})$$

$|\zeta| = R$, $R > a$ gives $\left.\begin{aligned} x &= \left(R + \frac{a^2}{R}\right)\cos\varphi \\ y &= \left(R - \frac{a^2}{R}\right)\sin\varphi \end{aligned}\right\}$ ellipse

$R > a$: $A = R + \dfrac{a^2}{R}$, $B = R - \dfrac{a^2}{R}$

$$x = A\cos\varphi, \quad y = B\sin\varphi$$

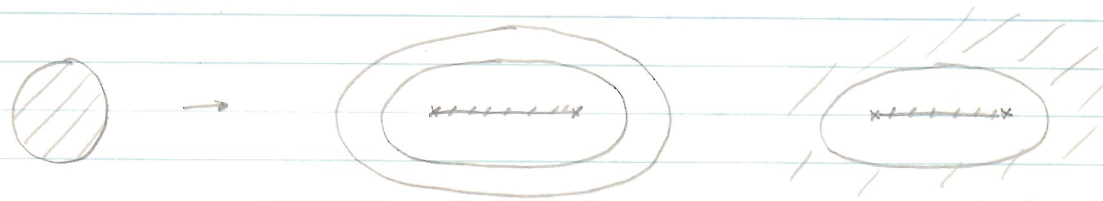
64 for Monday

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$$

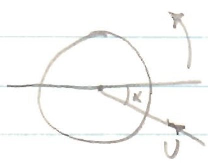
$$R < a: \quad A_1 = R + \frac{a^2}{R} > A$$

$$B_1 = \frac{a^2}{R} - R \; (>0) = B$$

$$\left.\begin{array}{l} x = A_1 \cos\varphi \\ y = -B_1 \sin\varphi \end{array}\right\} \quad \frac{x^2}{A_1^2} + \frac{y^2}{B_1^2} = 0$$



Application:



We require:

$$w = -Uz e^{i\alpha} - \frac{iK}{2\pi} \log z + O\left(\frac{1}{|z|}\right) \text{ at } \infty$$

$$\frac{dw}{dz} = -Ue^{i\alpha} - \frac{iK}{2\pi z} + O\left(\frac{1}{|z|^2}\right)$$

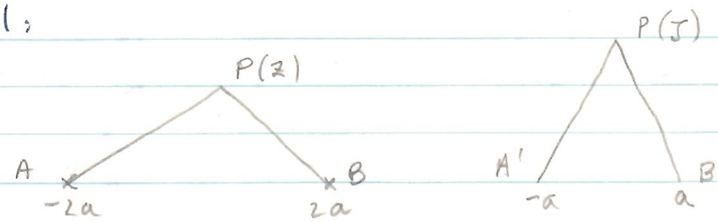
Use this in problem 64.

Take $|\zeta| = a$, $\quad z = \zeta + \frac{a^2}{\zeta}$, $\quad \frac{z}{2a} = \frac{\zeta^2 + a^2}{2a\zeta}$

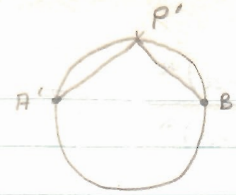
$$\frac{z - 2a}{z + 2a} = \left(\frac{\zeta - a}{\zeta + a}\right)^2$$

$$\arg(z - 2a) - \arg(z + 2a) = 2\left[\arg(\zeta - a) - \arg(\zeta + a)\right]$$

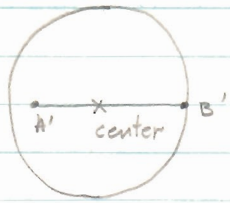
Recall:



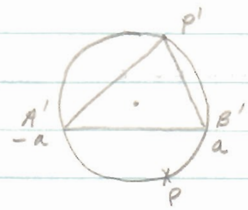
$\angle APB = 2 \angle A'P'B'$

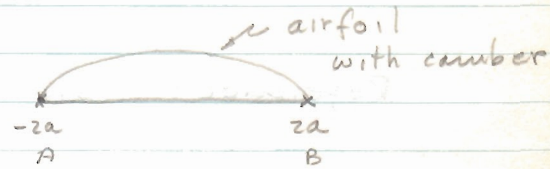


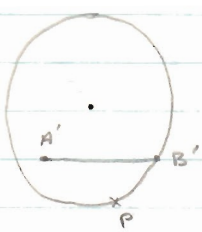
To constuct airfoil: move circle:



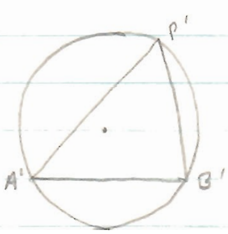
Move again:

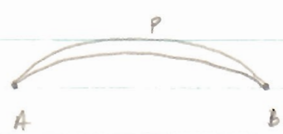




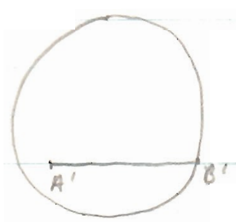


To remove cusp; use $\dfrac{z - na}{z + na} = \left(\dfrac{\zeta - a}{\zeta + a} \right)^n$





Two skeletons because n slightly less than 2



$$\frac{z}{na} = \frac{(\zeta + a)^n + (\zeta - a)^n}{(\zeta + a)^n - (\zeta - a)^n} = \frac{\left(1 + \frac{a}{\zeta}\right)^n + \left(1 - \frac{a}{\zeta}\right)^n}{\left(1 + \frac{a}{\zeta}\right)^n - \left(1 - \frac{a}{\zeta}\right)^n}$$

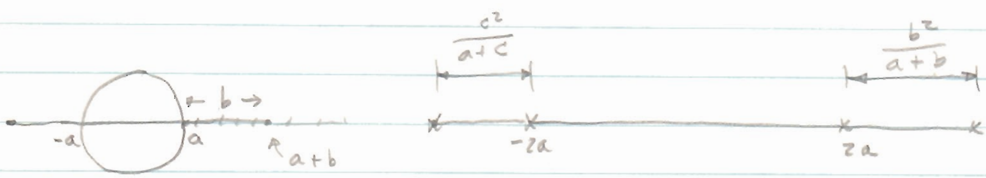
$$= \frac{1 + 1 + \frac{n(n-1)}{2}\frac{a^2}{\zeta^2} + \cdots}{n\frac{a}{\zeta} + \frac{n(n-1)(n-2)}{6}\frac{a^3}{\zeta^3} + \cdots}$$

$$z = \zeta + \frac{n^2 - 1}{3}\frac{a}{\zeta} + \cdots$$

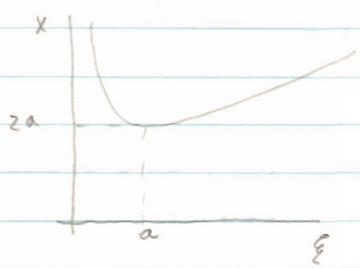
Reference: Glauert Areofoil and Airscrew Theory

Another Application:

$$z = \zeta + \frac{a^2}{\zeta}, \quad \text{real } \zeta = \xi, \quad x = \xi + \frac{a^2}{\xi}$$



$$a + b + \frac{a^2}{(a+b)} = 2a + \frac{b^2}{a+b}$$

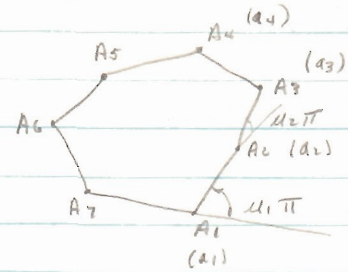


This is called Joukowski's Transformation.

Schwartz - Christoffel Transformation :

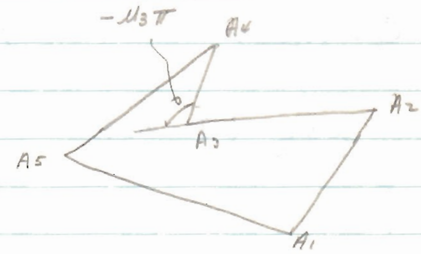
Reference : Kellogg

Take Polygon :

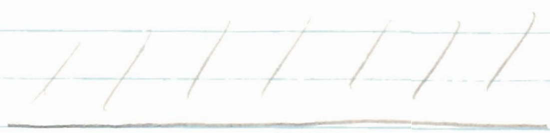


sides cannot intersect.

Inside goes to upper half "z" of J plane



"J"

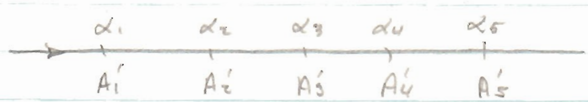


$\sum_s \mu_s = 2$

Find the proper transformation $z = F(J)$

"J"

$\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 \ldots < \alpha_n$



F must be singular at the vertices.

$$\arg F'(J) = \arg dz - \arg dJ$$

However $\arg dJ = 0$, as seen from above.

on $A_n A_1$, $\arg dz = \lambda$

$A_1 A_2$, " $= \lambda + \mu_1 \pi$

$A_2 A_3$, " $= \lambda + (\mu_1 + \mu_2) \pi$

$A_{s-1} A_s$, " $= \lambda + \left(\sum_1^{s-1} \mu_s \right) \pi$

LECTURE XXXVI 1-11-60

Continuation of Last Time.

Take $G(\zeta) = e^{i\lambda} F'(\zeta) H(\zeta)$

Consider: $(\zeta - \alpha_s)^{\mu_s}$, $0 < \arg(\zeta - \alpha_s) < \pi$

$\arg(\zeta - \alpha_s)^{\mu_s} = 0$ for real $\zeta > \alpha_s$
$= \mu_s \pi$ " " $\zeta < \alpha_s$

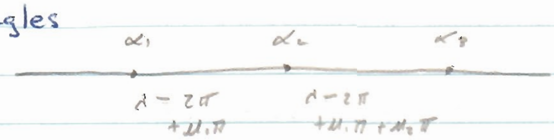
$$H(\zeta) = (\zeta - \alpha_1)^{\mu_1} (\zeta - \alpha_2)^{\mu_2} \dots (\zeta - \alpha_n)^{\mu_n}$$

In ζ plane, must go round infinite semicircle to get back to A_1.

$$H(\zeta) = \zeta^2 \left(1 - \frac{\alpha_1}{\zeta}\right)^{\mu_1} \dots \left(1 - \frac{\alpha_n}{\zeta}\right)^{\mu_n}$$

$$= \zeta^2 \left(1 - \frac{\sum \alpha_s \mu_s}{\zeta} + \dots\right)$$

Interior angles $= (1-\mu_s)\pi$



at ∞, $\arg H(\zeta)$ increases by 2π

Near α_s, $F(\zeta) = \alpha_s + \text{const}\, (\zeta - \alpha_s)^{1-\mu_s} \varphi(\zeta)$ must be the form of the function.

Then $F'(\zeta) = (\zeta - \alpha_s)^{-\mu_s} \psi(\zeta)$

If regular at ∞, $F \sim c_0 + \frac{c_1}{\zeta} + \dots$
$F' \sim -\frac{c_1}{\zeta^2} + \dots$

Now $H \sim \zeta^2 + \dots$
then $F'H$ is bounded.

Pick $Ae^{i\lambda}$, $F'(\mathcal{J}) = \dfrac{A}{H(\mathcal{J})}$

Then $$F(\mathcal{J}) = A \int_{\mathcal{J}_0}^{\mathcal{J}} \frac{d\mathcal{J}}{\prod_{s=1}^{n} (\mathcal{J}-\alpha_s)^{-\mu_s}} + B$$

Change branches by changing A

Recall Bilinear Transformation that interchanges 3 points on the real axis into other 3 point set. Then need $n-3$ other points to fix constants.

$|A|$ gives sides
$\arg A$ gives orientation.

Look at ∞ again:

$$F' = \frac{A}{\mathcal{J}^2}\left(1 + \frac{\sum \alpha_s \mu_s}{\mathcal{J}} + \dots\right.$$

$$F(\mathcal{J}) = z_B + \frac{A}{\mathcal{J}} + O\left(\frac{1}{\mathcal{J}^2}\right)$$

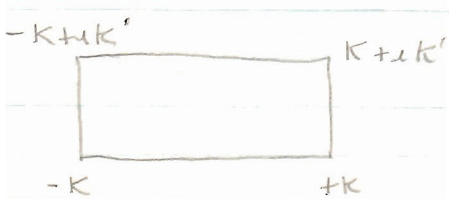
$$\arg(z - z_B) = \text{constant} - \arg \mathcal{J}$$

$$z - z_B \sim -\frac{A}{\mathcal{J}}$$

Example:

Take rectangle: $\mu_1 = \mu_2 = \mu_3 = \mu_4 = \frac{1}{2}$
Sides $\parallel$ to axis. Take α's as ± 1, and $\pm$ constant $= \pm \frac{1}{k}$ for $0 < k < 1$.

$$F(\mathcal{J}) = \int_0^{\mathcal{J}} \frac{d\mathcal{J}}{\sqrt{(1-\mathcal{J}^2)(1-k^2\mathcal{J}^2)}}$$ gives elliptic integral.



$$K' = \int_1^{1/k} \frac{dt}{\sqrt{(t^2-1)(1-k^2t^2)}}$$

See Copson ¶ 8.52

Use bilinear transformation $\zeta' = \dfrac{1}{\zeta - \alpha_n}$

$\zeta = \alpha_n + \dfrac{1}{\zeta'}$, $\zeta - \alpha_s = \alpha_n - \alpha_s + \dfrac{1}{\zeta'}$

$$= \frac{1}{\zeta'} - \frac{1}{\alpha_s'} = \frac{1}{\alpha_s' \zeta'} (\zeta' - \alpha_s') , \quad s \neq n$$

and $d\zeta = - \dfrac{d\zeta'}{\zeta'^2}$

Then: $$z = \text{const} \int^{\zeta'} \frac{d\zeta'}{\zeta'^2 \prod_{s=1}^{n-1} (\zeta' - \alpha_s') \, \zeta'^{\,1 - \Sigma \mu}} + \text{constant}$$

$$= \text{const} \int^{\zeta'} \frac{d\zeta'}{\prod_{s=1}^{n-1} (\zeta' - \alpha_s')^{\mu_s}}$$

Infinity causes branch point at one of the vertices, and is no longer a regular point.

$$H(\zeta) = (\zeta - \alpha_1)^{\mu_1} \cdots (\zeta - \alpha_{n-1})^{\mu_{n-1}}$$

$$\sim \zeta^{\mu_1 + \cdots + \mu_{n-1}} = \zeta^{2 - \mu_n}$$

$$H(\zeta) = \zeta^{2 - \mu_n} \varphi(\zeta)$$

at ∞ , $\arg F'(\zeta)$ decreases by $(2 - \mu_n)\pi$
$H(\zeta)$ increases " "

Open Polygon: One vertex (A) at ∞.

Take A at α_n

$\mu_n = 1$ $\therefore$ $\sum_{s=1}^{n-1} \mu_s = 1$

Near α_n , $F'(\zeta) = \dfrac{\varphi(\zeta)}{\zeta - \alpha_n}$

$$= \frac{\text{const}}{\zeta - \alpha_n} + \psi(\zeta)$$

Infinity is now a pole instead of branch point.
because:

$$F(\zeta) = \text{const.} \log(\zeta - \alpha_n) + \chi(\zeta)$$

Next try infinite open rectangle.

$$F(\zeta) = K \log(\zeta - \alpha_n) + \chi(\zeta)$$

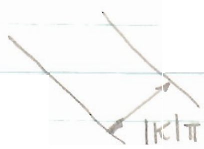
↑ regular at (near) α_n

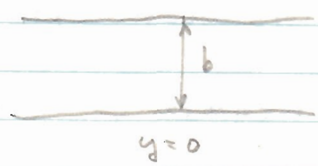
$\log(\zeta - \alpha_n)$ decreases discontinuously by $i\pi$

∴ z increases discontinuously by $-K\pi i$

↑ real

If K not real, strip ∦ x axis

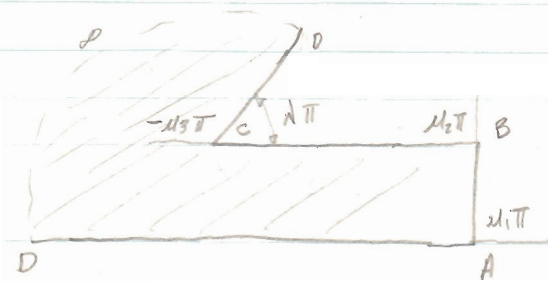




$K = -\frac{b}{\pi}$ leads to $-\frac{b}{\pi} \log(\zeta - \alpha_n)$

If we put α_n at ∞, we get $+\frac{b}{\pi} = K$

Open sides not parallel:



$\mu_3\pi = -(1-\lambda)\pi$

$\mu_4 = 2 - \lambda$

$\mu_1 = 1/2$

$\mu_2 = 1/2$

$\mu_3 = -1 + \lambda$

$\mu_4 = 2 - \lambda$

$$H(\zeta) = (\zeta-\alpha_1)^{\mu_1}(\zeta-\alpha_2)^{\mu_2}(\zeta-\alpha_3)^{\mu_3}(\zeta-\alpha_4)^{\mu_4}$$

$\arg(\zeta-\alpha_4)$ decreases by π

We want $\arg z$ increase by $\pi(1-\lambda) = \pi(\mu_4 - 1)$
$\arg \frac{1}{z}$ decreases by $\pi(\mu_4-1)$

Then: $\frac{1}{z} = (\zeta-\alpha_4)^{\mu_4-1}\varphi(\zeta)$, $\varphi(\alpha_4) \neq 0$

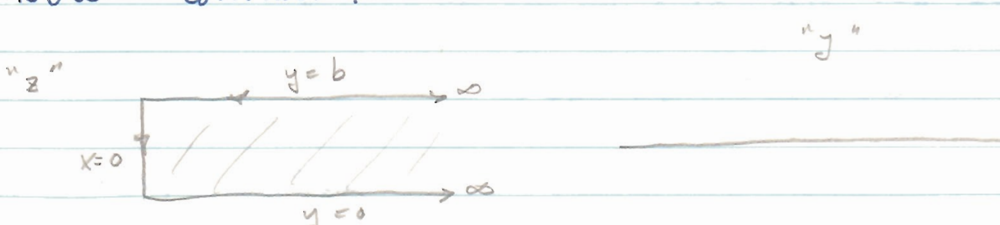
$$F(\zeta) = (\zeta-\alpha_4)^{1-\mu_4}\psi(\zeta)$$

$$F'(\zeta) = (\zeta-\alpha_4)^{-\mu_4}\chi(\zeta)$$

This makes $F'H$ regular near α_4.

Now let A, B go to ∞ and become one point, $\alpha_1 = \alpha_2$. However, AB is not the same point as D.
Apply previous results to breadth; $K \log \zeta$

Now consider:



$\mu_1 = \mu_2 = \frac{1}{2}$, $\mu_3 = 1$, $\alpha_1 = -1$, $\alpha_2 = 1$

with $F = \int \frac{d\zeta}{(\zeta^2-1)^{1/2}}$

$\zeta = \cosh t$
$(\zeta^2-1)^{1/2} = \sinh t$
$d\zeta = \sinh t \, dt$

Get $F(\zeta) = \cosh^{-1}\zeta + \text{const}$

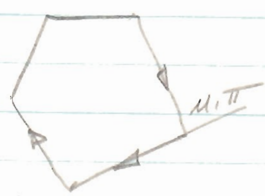
$z = \cosh^{-1}\zeta$
$\zeta = \cosh z$

we don't have breadth yet, which is given by constants. From before:

$$F = A \int \frac{d\zeta}{(\zeta^2 - 1)^{1/2}} \pm B$$

We get $\zeta = \frac{b}{\pi} \cosh \frac{\pi z}{b}$

What about outside of Polygon?



Go round in opposite direction.
(1) Sign of each μ is changed

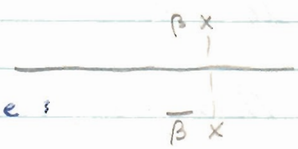
(2) Need point to correspond to $z = \infty$, say β
so pole at β.

Use $e^{i\lambda}$ to make real on real axis

$$H(\zeta) = \Pi (\zeta - \alpha_s)^{-\mu_s}$$

$$G(\zeta) = F'(\zeta)\, H(\zeta)$$

We continue across real axis by reflection



∞ is regular point in ζ plane:

$F = c_0 + \frac{c_1}{\zeta} + \ldots$, $F' = O\left(\frac{1}{\zeta^2}\right)$, $H = O\left(\frac{1}{\zeta^2}\right)$

Then $G = O\left(\frac{1}{\zeta^4}\right)$

$(\zeta - \beta)^2 (\zeta - \bar{\beta})^2\, G(\zeta)$ is regular everywhere and is bounded and therefore constant.

Then we can write: $F'(\zeta) = A \dfrac{\Pi (\zeta - \alpha_s)^{\mu_s}}{(\zeta - \beta)^2 (\zeta - \bar{\beta})^2}$

There is no $\log(\zeta - \beta)$ in F
so no $\frac{1}{(\zeta - \beta)}$ in F'

That is to say: Residue at $\zeta = \beta$ of $F'(\zeta)$ must be zero.

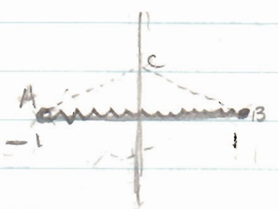
$$F' = A \frac{\prod (\omega + \beta_1 - \alpha_s)^{\mu_s}}{\omega^2 (\omega + \beta - \bar{\beta})^2}$$

$$= \frac{A \prod (\beta - \alpha_s)^{\mu_s}}{\omega^2} \left\{ \frac{\prod \left(1 + \frac{\omega}{\beta - \alpha_s}\right)^{\mu_s}}{\left(1 + \frac{\omega}{\beta - \bar{\beta}}\right)^2} \right\}$$

This gives necessary condition that:

$$\sum_s \frac{\mu_s}{\beta - \alpha_s} - \frac{2}{\beta - \bar{\beta}} = 0$$

Example: Cut "z" plane



Transform region outside cut in upper half of ζ plane

$\mu_1 = \mu_2 = 1$, Take $\zeta = i$, $z = \infty$

$\alpha_1 = -1$, $\alpha_2 = +1$

This is right choice if:

$$\frac{1}{1+i} + \frac{1}{i-1} = \frac{2}{2i}$$

$$\frac{2i}{i^2 - 1} = \frac{1}{i} \quad 2i^2 = i^2 - 1 \quad \text{ok}$$

ω

$\frac{dz}{d\omega} = A(\omega+1)^{-1}(\omega-1)^{-1}$

$\frac{A}{(\omega+1)(\omega-1)} + B$

$= \frac{A}{2}\left[\frac{1}{\omega-1} - \frac{1}{\omega+1}\right]$

$$F'(\zeta) = A \frac{\zeta^2 - 1}{(\zeta^2+1)^2} = A \left\{ \frac{2\zeta^2 - (\zeta^2+1)}{(\zeta^2+1)^2} \right\}$$

$$F(\zeta) = -\frac{A\zeta}{\zeta^2+1} + B$$

$Z = \frac{A}{2} \log\left(\frac{\omega-1}{\omega+1}\right) + B$

$$z = \frac{2\zeta}{\zeta^2 + 1}$$

Is this right?

Transform from z to unit circle in w-plane;

$$w = -i\,\frac{z+i}{z-i}$$

"ω"

∞ is same point in z and w planes.

$$z = i\,\frac{iw+1}{iw-1}, \qquad z^2+1 = -\frac{4iw}{(iw-1)^2}$$

$$Z = 2i\,\frac{iw+1}{iw-1}\;\frac{(iw-1)^2}{-4iw}$$

$$= -\frac{1}{2w}(iw+1)(iw-1) = \frac{1}{2w}(1-iw)(1+iw)$$

$$= \frac{1}{2}\left(w+\frac{1}{w}\right)$$ Jouchowski Transformation.

Laplace Transforms:

p-multiplied Laplace Transforms or Heaviside Operators.

Define operator $\frac{1}{p}$

Operation on $f(t)$: $\quad \frac{1}{p}f(t) \equiv \int_0^t f(\xi)\,d\xi \quad , \quad t \geq 0$

Example: $\quad \frac{dx}{dt} - \alpha x = F(t)$

$$x = x_0 \text{ at } t = 0$$

$$x - x_0 - \frac{\alpha}{p}x = \frac{1}{p}F(t)$$

or $\quad px - \alpha x = p x_0 + \underset{\varphi(p)}{\underset{\parallel}{F(t)}}$ ← means $\mathcal{L}\,F(t) = \varphi(p)$

$$x = x_0\,\frac{p}{p-\alpha} + \frac{1}{p-\alpha}\,\varphi(p)$$

Comparing with known solution:

$$\frac{p}{p-\alpha} = e^{\alpha t}, \quad \frac{1}{p-\alpha}$$

$$\frac{1}{p-\alpha} F(t) = e^{\alpha t} \int_0^t e^{-\alpha t} f(t)\, dt$$

LECTURE <u>XXXVII</u>

Recap:

$$\frac{p}{p-\alpha} = e^{\alpha t}$$

$$\varphi(p) = f(t), \quad \frac{1}{p-\alpha}\varphi(p) = e^{\alpha t}\int_0^t e^{-\alpha\xi} f(\xi)\, d\xi$$

$$\frac{p}{(p-\alpha)^2} = t e^{\alpha t}, \quad \frac{p}{(p-\alpha)^3} = \frac{t^2}{2!} e^{\alpha t},$$

$$\frac{p}{(p-\alpha)^n} = \frac{t^{n-1}}{(n-1)!} e^{\alpha t}$$

$$a_n \frac{d^n x}{dt^n} + a_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + \cdots + a_1 \frac{dx}{dt} + a_0$$

$$+ b_1 \int_0^t x\, dt + b_2 \int_0^t dt \int_0^t x\, dt + \cdots + b_m \left(\int_0^t\right)^m x\, dt = f(t)$$

$$\frac{d^r x}{dt^r} = x_r \quad \text{at} \quad t = 0 \quad \text{for } r = 0, 1, 2, \cdots n-1$$

Then $$x(p) = \frac{F(p) + C(p)}{L(p)}$$

$$F(p) = f(t), \quad L(p) = a_n p^n + a_{n-1} p^{n-1} + \cdots + a_1 p$$

$$+ a_0 + \frac{b_1}{p} + \frac{b_2}{p^2} + \cdots + \frac{b_m}{p^m}$$

and with:

$C(p) =$ sum of positive powers in $L(p)\,B(p)$

where $B(p) = x_0 + \frac{x_1}{p} + \frac{x_2}{p^2} + \dots + \frac{x_{n-1}}{p^{n-1}}$

Then:

$$C(p) = x_0\,(a_n p^n + a_{n-1} p^{n-1} + \dots + a_1 p)$$
$$+ x_1\,(a_n p^{n-1} + \dots + a_2 p)$$
$$+ x_2\,(a_n p^{n-2} + \dots + a_3 p)$$

This theorem can be shown. $f(t)$ is arbitrary.
However, if:

$$f(t) = \sum A_{n,\alpha} \frac{t^{n-1}}{\alpha(n-1)!} e^{\alpha t}$$

$$F(p) = \sum A_{n,\alpha} \frac{p}{(p-\alpha)^n}$$

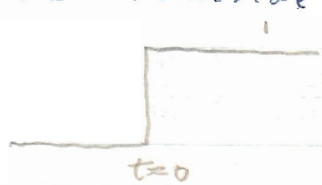
Solutions obtained by partial fraction expansions.
Suppose α complex:

$$\frac{p}{p-\beta-i\gamma} = e^{\beta t} e^{i\gamma t}$$

$$\frac{p(p-\beta)}{(p-\beta)^2+\gamma^2} = e^{\beta t}\ \text{constant}$$

$$\frac{p}{(p-\beta)^2-\gamma^2} = \frac{1}{\gamma} e^{\beta t} \sin\gamma t$$

We deal only with $t > 0$, then p operates only on the Heaviside unit function, $\varphi(p)\,H(t)$



$$F(p) = f(t) \quad , \quad F(p)\,H(t) = 0 \;, \; t < 0$$
$$= f(t) \;, \; t > 0$$

Duhamel's Integral:

$$a_n \frac{d^n x}{dt^n} + \dots + a_0 x = f(t)$$

Take $x_r = 0$

Then: $x = \dfrac{F(p)}{L(p)}$

Now, take $F(t) = H(t)$, then $U(t) = \dfrac{1}{L(p)}$

$$x = \frac{F(p)}{L(p)} = f(0)\,U(t) + \int_0^t f'(\tau)\,U(t-\tau)\,d\tau$$

$$\text{or} = f(0)\,U(t) + \int_0^t U(\tau)\,f'(t-\tau)\,d\tau$$

$$\int_0^t U(\tau)\,f'(t-\tau)\,d\tau = \underset{\overset{\shortparallel}{0}}{U(0)} + \int_0^t U'(t-\tau)\,f(\tau)\,d\tau$$

$$= \int_0^t U'(\tau)\,f(t-\tau)\,d\tau$$

We know $\dfrac{1}{p^n} = \dfrac{t^n}{n!}$:

$$\varphi(\mu) = \sum_{n=0}^{\infty} \frac{a_n}{\mu^n} \quad (1) \quad , \quad f(t) = \sum_{n=0}^{\infty} \frac{a_n}{n!} t^n \quad (2)$$

Use Borel sum formula

$$\varphi(\mu) = \mu \int_0^\infty e^{-\mu t} \left(\sum_{n=0}^{\infty} \frac{a_n}{n!} t^n \right) dt = \mu \int_0^\infty e^{-\mu t} f(t)\,dt$$

Using:

$$\frac{a_n}{n!} \int_0^\infty e^{-\mu t}\, t^n\, dt = \frac{a_n}{n!} \left(\frac{n!}{\mu^{n+1}} \right) = \frac{a_n}{\mu^{n+1}}$$

Then, we can form new definition:

$$\varphi(p) = p \int_0^\infty e^{-pt} f(t)\, dt$$

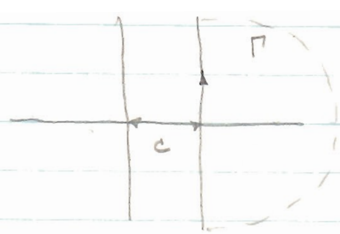
Take $\varphi(\mu) = \mu \int_0^\infty e^{-\mu t} f(t)\, dt$ ③

A Theorem is; for $\mu > \alpha > 0$

Form $\frac{\varphi(\lambda)}{\lambda}$, $R(\lambda) > \alpha$, Then:

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{\lambda t} \frac{\varphi(\lambda)}{\lambda}\, d\lambda, \quad c > \alpha \qquad ④$$

This is inverse of ③.
Contour is



We now show that ③ is solution of ④. Take $\mu > c$

$$\oint_\Gamma \frac{\varphi(\lambda)}{\lambda(\mu-\lambda)}\, d\lambda \to 0 \quad \text{as } \lambda \to \infty$$

Change order of integration:

$$\frac{1}{2\pi i} \int_0^\infty e^{-\mu t} \int_{c-i\infty}^{c+i\infty} e^{\lambda t} \frac{\varphi(\lambda)}{\lambda} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\varphi(\lambda)}{\lambda}\, d\lambda \int_0^\infty e^{-(\mu-\lambda)t} dt$$

$$= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\varphi(\lambda)}{\lambda(\mu-\lambda)}\, d\lambda = \frac{\varphi(\mu)}{\mu}$$

Thus if $\varphi(\mu)$ is given by ① then $f(t)$ is given by ②.

Convolution Integral:

$$\varphi_1(p) = f_1(t)$$
$$\varphi_2(p) = f_2(t)$$

Then: $$\varphi_1(p)\,\varphi_2(p)\,\frac{1}{p} = \int_0^t f_1(\xi)\, f_2(t-\xi)\, d\xi$$

$$= \int_0^t f_2(\xi)\, f_1(t-\xi)\, d\xi$$

To show this, we must prove:

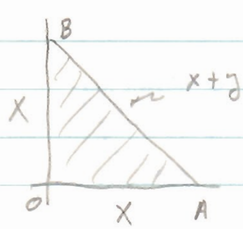
if $$\varphi_1(\mu) = \mu \int_0^\infty e^{-\mu t} f_1(t)\, dt$$

$$\varphi_2(\mu) = \mu \int_0^\infty e^{-\mu t} f_2(t)\, dt$$

Then: $$\varphi_1(\mu)\,\varphi_2(\mu)\,\frac{1}{\mu} = \mu \int_0^\infty \left[e^{-\mu t} \int_0^t f_1(t-\xi) f_2(\xi)\, d\xi \right] dt$$

or $$\varphi_1(\mu)\,\varphi_2(\mu)\,\frac{1}{\mu} = \mu \int_0^\infty e^{-\mu x} f_1(x)\, dx \int_0^\infty e^{-\mu y} f_2(y)\, dy$$

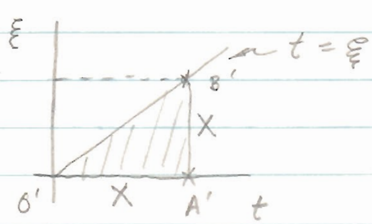
$$= \mu \int_0^\infty\!\!\int_0^\infty e^{-\mu(x+y)} f_1(x)\, f_2(y)\, dx\, dy$$



$$= \mu \lim_{X \to \infty} \iint e^{-\mu(x+y)} f_1(x)\, f_2(y)\, dx\, dy$$

Let, $x + y = t$

$y = \xi$, $x = t - \xi$



Jacobian of transformation:

$$\frac{\partial x}{\partial t} = 1, \quad \frac{\partial x}{\partial \xi} = -1, \quad \frac{\partial y}{\partial t} = 0$$

$$\frac{\partial y}{\partial \xi} = 0$$

$$\therefore \quad J = \frac{\partial(x,y)}{\partial(t\,\xi)} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1 \quad ; \; dx\,dy = dt\,d\xi$$

Then: $\displaystyle\iint_{OAB} e^{-u(x+y)} f_1(x)\, f_2(y)\, dx\, dy$

$$= \iint_{O'A'B'} e^{-ut} f_1(t-\xi)\, f_2(\xi)\, d\xi\, dt$$

$$= \int_0^X \left[e^{-ut} \int_0^t f_1(t-\xi)\, f(\xi)\, d\xi \right] dt$$

Take lim as $X \to \infty$ as we get what we started with.

Solution of Equations with p operators:

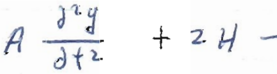
Consider: $\dfrac{\partial^2 y}{\partial t^2} = \dfrac{\partial^2 y}{\partial x^2}$, $y(0,x) = y_0(x)$, $\dot{y}(0,x) = y_1(x)$

$$p^2 y = \frac{\partial^2 y}{\partial x^2} + p^2 y_0(x) + p\, y_1(x)$$

Actually $\quad p^2 Y = \dfrac{\partial^2 Y}{\partial x^2} + p^2 y_0(x) + p\, y_1(x)$

Need boundary conditions on $Y(p,x)$

Take $y = 0$ at $x = 0$, $x = l$



Can solve for $Y(p,x)$ and then $y(t,x)$

Now consider:

$$A \frac{\partial^2 y}{\partial t^2} + 2H \frac{\partial^2 y}{\partial x\, \partial t} + B \frac{\partial^2 y}{\partial x^2} + \dots$$

Hyperbolic : $H^2 - AB > 0$

Parabolic : $H^2 - AB = 0$

Elliptic : $H^2 - AB < 0$

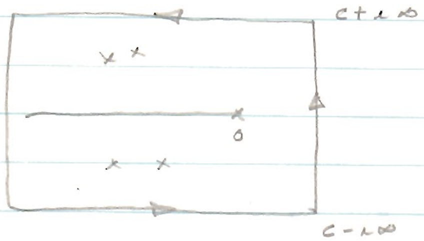
Take heat conduction: $\frac{\partial^2 y}{\partial x^2} = \frac{\partial y}{\partial t}$ (parabolic)

Laplace's Equation: $\frac{\partial^2 y}{\partial t^2} + \frac{\partial^2 y}{\partial x^2} = 0$ (elliptic)

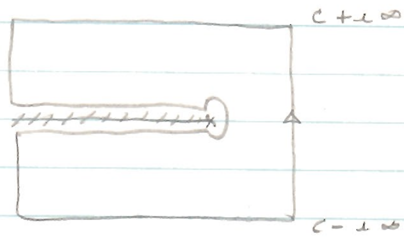
Cannot treat elliptic equations with p operators

To find inverse transforms, use:

$$f(t) = \frac{1}{2\pi i} \int_L e^{\lambda t} \frac{\varphi(\lambda)}{\lambda} \, d\lambda$$



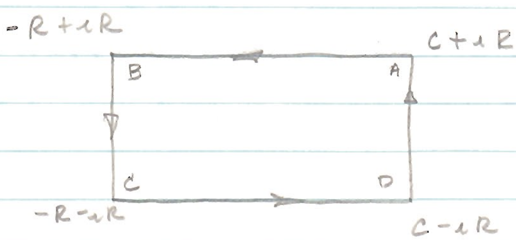
single-valued

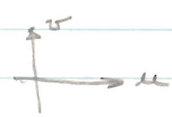


multiple valued

For initial conditions, compute integral first then let $t \to 0$.

Choose Contour:





If $\varphi(\lambda)$ is bounded, $\left| \frac{\varphi(\lambda)}{\lambda} \right| \leq M(R) \to 0$

uniformly as $R \to \infty$

$$\left| \int \right| \leq M(R) \int |e^{\lambda t}| \, d\lambda = M(R) \int_c^{-R} e^{ut} \, du$$

$$= \left| \frac{M}{t} \right| e^{ut} \Big]_c^{-R} = \frac{M}{t} \left(e^{ct} - e^{-Rt} \right) < \frac{M}{t} e^{ct} \to 0, \quad R \to \infty$$

On BC: $\lambda = -R + iv$, $|e^{\lambda t}| = e^{-Rt}$

$$\left| \int_{BC} e^{\lambda t} \, d\lambda \right| \leq e^{-Rt} \cdot 2R \to 0 \text{ as } R \to \infty$$

Now suppose $\varphi(\lambda)$ has only finite number of poles, like a rational fraction.
Then:

$f(t) = \varphi(0) +$ the sum of the residues of $\dfrac{e^{\lambda t}\varphi(\lambda)}{\lambda}$ at the poles of $\varphi(\lambda)$

Take $\varphi(\lambda) = \dfrac{H(\lambda)}{G(\lambda)}$

For simple poles, the residues of the pole $\lambda = \alpha_s$ is

$$\lim_{\lambda \to \alpha_s} \frac{(\lambda - \alpha_s) e^{\lambda t} H(\lambda)}{\lambda \, G(\lambda)} = \frac{e^{\alpha_s t} H(\alpha_s)}{\alpha_s \, G'(\alpha_s)}$$

Also, no pole (or zero of $G(\lambda)$) at $\lambda = 0$
Then:

$$f(t) = \frac{H(0)}{G(0)} + \sum_s \frac{H(\alpha_s)}{\alpha_s \, G'(\alpha_s)} e^{\alpha_s t}$$

Partial fraction expansion.

For $G(\lambda)$ an integral function, $\sum_s$ is an infinite sum.

Consider: $\dfrac{\partial^2 V}{\partial x^2} = \dfrac{\partial V}{\partial t}$; $\left.\begin{array}{l} V = 0 \text{ at } x = 0 \\ V = 1 \text{ at } x = 1 \end{array}\right\} t > 0$

and $V = 0$, at $t = 0$, $0 \leq x \leq 1$

What we want to find is $\left(\frac{\partial V}{\partial x}\right)_{x=0}$.

Now: $\frac{\partial^2 V}{\partial x^2} = pV$, define $p = q^2$, $q = p^{1/2}$

$V = A \sinh qx + B \cosh qx$

$B = 0$, $A = \frac{1}{\sinh q}$

$V = \frac{\sinh qx}{\sinh q}$, $\frac{\partial V}{\partial x} = \frac{q \cosh qx}{\sinh q}$

Then $\left(\frac{\partial V}{\partial x}\right)_{x=0} = \frac{q}{\sinh q} = \frac{1}{\frac{\sinh q}{q}}$

Now $\frac{\sinh q}{q} = 1 + \frac{p}{3!} + \frac{p^2}{5!} + \cdots$

The zeroes at at $q = p^{1/2} = \pm i n\pi$, $n = 1, 2, 3, \cdots$

$p = -n^2\pi^2$

Now take $H(p) = 1$, $G = \frac{\sinh q}{q}$, $G(0) = 1$

$G(p) = \xi(q) = \frac{\sinh q}{q}$

$\frac{dG}{dp} = \frac{d\xi}{dq} \Big/ \frac{dp}{dq} = \frac{1}{2q}\left\{\frac{\cosh q}{q} - \frac{\sinh q}{q^2}\right\}$

When $G = 0$, $\sinh q = 0$, $G' = \frac{\cosh q}{2q^2}$

$= -\frac{\cos n\pi}{2n^2\pi^2} = \frac{(-1)^{n+1}}{2n^2\pi^2} = G'(\alpha_n)$

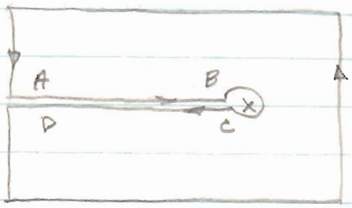
$\alpha_n = -n^2\pi$

Now: $\quad V = \dfrac{H(0)}{G(0)} + \sum_n \dfrac{H(\alpha_n)}{\alpha_n G'(\alpha_n)} e^{\alpha_n t}$

$$= 1 + 2 \sum_{n=1}^{\infty} (-1)^n e^{-n^2 \pi^2 t}$$

which a good series for large t.

Now condider a branch point at the origin:



$$\frac{1}{2\pi i} \int_L = \text{sum of residues of } \frac{e^{\lambda t}\varphi(\lambda)}{\lambda} \text{ at poles of } \varphi$$

$$+ \frac{1}{2\pi i} \int_M$$

$\lambda = r e^{i\theta}$, $\quad \dfrac{d\lambda}{\lambda} = i\, d\theta$, $\quad e^{\lambda t} \to 1$ as $\gamma \to 0$

$\varphi \to \varphi_0$ as $\gamma \to 0$

$$\frac{1}{2\pi i} \int_\gamma \to \frac{1}{2\pi i} \int_\gamma i \varphi_0 \, d\theta = -\varphi_0$$

$$\left.\begin{array}{l} \text{On } AB: \ \lambda = R e^{i\pi} \\ \quad\ CD: \ \lambda = R e^{-i\pi} \end{array}\right\} \quad \frac{d\lambda}{\lambda} = \frac{dR}{R}$$

$$+ \varphi_0 + \frac{1}{2\pi i} \left[\int_0^\infty e^{-Rt} \left[\varphi(Re^{i\pi}) - G(Re^{-i\pi}\right] \frac{dR}{R} \right]$$

$$\frac{1}{2\pi i} \int_0^\infty e^{-Rt} R^{\frac{1}{2}n - 1} \underbrace{\left(e^{\frac{1}{2} n\pi i} - e^{-\frac{1}{2} n \pi i} \right)}_{2i \sin \frac{n\pi}{2}} dR$$

$= 0$ neven

$= 2i(-1)^{n-2}$

for $n = 2m+1$

$$\frac{(-1)^m}{\pi}\int_0^\infty e^{-Rt} R^{m-1/2}\, dR = \frac{(-1)^m}{\sqrt{\pi}} \frac{2(2n)}{n!\,(2t^{1/2})^{2n+1}}$$

Contribution of term q^{2n+1}

LECTURE XXXVIII   1-16-61

Recall:

integration denoted by:

$$\oint_{-\infty}^{0^+}$$

Integral will vanish at ∞ if $\varphi(\lambda)$ is bounded.

Suppose $\varphi(p) \sim \psi(p^{1/2})$, then we have branch point and pole at origin.

$$\frac{1}{2\pi i}\int_L = \frac{1}{2\pi i}\int_M = \varphi_0 + \frac{1}{2\pi i}\int_{-\infty}^{0} + \frac{1}{2\pi i}\int_0^{-\infty}$$

Possibly can expand in powers of $p^{1/2}$ and integrate term by term:

$$\psi(\lambda^{1/2}) = \sum a_m \lambda^{m/2}$$

Examine $\lambda^{m/2}$; integral will vanish unless m odd. Result is:

$$(-1)^m \frac{2}{\sqrt{\pi}} \frac{(2m)!}{m!}\left(\frac{1}{2\sqrt{t}}\right)^{2m+1}$$

Now look at $\varphi(p) = e^{-a p^{1/2}} = e^{-aq}$, $a > 0$

where $p^{1/2} = q$

$$\left| e^{-a\lambda^{1/2}} \right| = \exp\left(-a \operatorname{Re} \lambda^{1/2}\right)$$

Now λ is in $-\pi \leq \arg \lambda \leq \pi$
so that $-\pi/2 \leq \arg \lambda^{1/2} \leq \pi/2$

So $e^{-a\lambda^{1/2}}$ is bounded.

We can expand in a power series:

$$e^{-a\lambda^{1/2}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} a^n (\lambda^{1/2})^n$$

Integrating term by term:

$$f(t) = 1 - \frac{2}{\sqrt{\pi}} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} \frac{(2m)!}{m!} \left(\frac{a}{2\sqrt{t}}\right)^{2m+1}$$

$$f(t) = 1 - \frac{2}{\sqrt{\pi}} \sum_{m=0}^{\infty} (-1)^m \frac{1}{(2m+1)} \frac{1}{m!} y^{2m+1}$$

where $y = \frac{a}{2\sqrt{t}}$

We see that: $e^{-w^2} = \sum_{m=0}^{\infty} (-1)^m \frac{w^{2m}}{m!}$

$$\int_0^y e^{-w^2} dw = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{y^{2m+1}}{2m+1}$$

or we have: $\frac{2}{\sqrt{\pi}} \int_0^y e^{-w^2} dw = \operatorname{erf} y$

$\operatorname{erf} y \to 1$ as $y \to \infty$

$$e^{-ap^{1/2}} = 1 - \operatorname{erf} \frac{a}{2\sqrt{t}} = \int_y^{\infty} e^{-w^2} dw = \operatorname{erfc} y$$

Now consider: $\varphi = q e^{-aq}$, $q = p^{1/2}$, $\varphi(0) = 0$

$$\left| \frac{\lambda^{1/2} e^{-a\lambda^{1/2}}}{\lambda} \right| \to 0 \quad \text{as} \quad |\lambda| \to \infty$$

The series is: $\sum \frac{1}{(2m)!} a^{2m} q^{2m+1}$

$$f(t) = \frac{q}{\sqrt{\pi}} \sum_{m=0}^{\infty} (-1)^m \frac{(2m)!}{(2m)!\, m!} \frac{a^{2m}}{(2\sqrt{t})^{2m+1}}$$

$$= \frac{1}{\sqrt{\pi t}} \sum_{n=0}^{\infty} \frac{(-1)^m}{m!} \left(\frac{a}{2\sqrt{t}}\right)^{2m}$$

$$= \frac{1}{\sqrt{\pi t}} e^{-a^2/4t}$$

Recall interpretation of $\frac{q}{\sinh q}$ and convergence of $e^{n^2\pi^2 t}$

Write $\frac{q}{\sinh q} = \frac{2q}{e^q - e^{-q}} = 2q\, e^{-q}\left(1 - e^{-2q}\right)^{-1}$

$$= 2q \sum_{n=0}^{\infty} e^{-(2n+1)q}$$

Then $f(t) = \frac{2}{\sqrt{\pi t}} \sum_{n=0}^{\infty} e^{-\frac{(2n+1)^2}{4t}}$ which follows a rapidly decaying exponential for small t.

Check: $\frac{1}{2\pi i} \int_L e^{\lambda t} \frac{\varphi(\lambda)}{\lambda} d\lambda$

$$= \frac{1}{2\pi i} \int_L \frac{e^{\lambda t}}{\lambda} 2\lambda^{1/2} \sum_{n=0}^{\infty} e^{-(2n+1)\lambda^{1/2}} d\lambda$$

On L: $-\pi/2 \le \arg \lambda \le \pi/2$

$-\pi/4 \le \arg \lambda^{1/2} \le \pi/4$

Then $f(t) = \frac{2}{\sqrt{\pi t}} \sum_{n=0}^{\infty} e^{-\frac{(2n+1)^2}{4t}}$

If we had just $\frac{1}{\sinh q}$, we would get $\sum_i \operatorname{erfc} \frac{(2n+1)}{2\sqrt{t}}$

$$\operatorname{erfc} x = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-w^2} dw = \frac{1}{\sqrt{\pi}} \int_{x^2}^\infty e^{-u} \frac{du}{u^{1/2}}, \quad w = u^{1/2}$$
$$w^2 = u$$

$$\sqrt{\pi} \operatorname{erfc} x = \left[\frac{-e^{-u}}{u^{1/2}} \right]_{x^2}^\infty - \frac{1}{2} \int_{x^2}^\infty \frac{e^{-u}}{u^{3/2}} du$$

Get:
$$\sqrt{\pi} \operatorname{erfc} x = e^{-x^2} \left[\frac{1}{x} - \frac{1}{2x^3} + \frac{1 \cdot 3}{2^2} \frac{1}{x^5} + \cdots + \frac{1 \cdot 3 \cdots (2n-1)}{2^n} \frac{1}{x^{2n+1}} \right] + R_n$$

This is asymptotic expansion.

$$R_n = (-1)^{n+1} \frac{1 \cdot 3 \cdots (2n+1)}{2^{n+1}} \int_{x^2}^\infty \frac{e^{-u}}{u^{1/2(2n+3)}} du$$

$$\int_{x^2}^\infty \frac{e^{-u}}{u^{1/2(2n+3)}} du = 2 \int_x^\infty \frac{e^{-w^2}}{e^{2w+2}} dw < 2 \int_x^\infty \frac{dw}{w^{2n+2}}$$

$$= \frac{2}{2n+1} \frac{1}{x^{2n+1}}$$

Then $|R_n| < \frac{1 \cdot 3 \cdots (2n-1)}{2^n} \frac{1}{x^{2n+1}}$

All this has resulted from diffusion equation $\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$

For wave equation, $\frac{\partial^2 V}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 V}{\partial t^2}$

get $e^{-p/c}$

Recall: $\frac{\varphi(p)}{p} = \int_0^\infty e^{-\lambda t} f(t)\,dt$

$$f(t) = \frac{1}{2\pi i}\int_{c-i\omega}^{c+i\omega} e^{\lambda t}\frac{\varphi(\lambda)}{\lambda}\,d\lambda$$

This will show connection with Fourier transforms.

$$h(t) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega t + b_n \sin n\omega t)$$

$$a_n = \frac{\omega}{\pi}\int_{-\pi/\omega}^{\pi/\omega} h(t)\cos n\omega t\,dt$$

$$b_n = \frac{\omega}{\pi}\int_{-\pi/\omega}^{\pi/\omega} h(t)\sin n\omega t\,dt$$

$$c_n = \tfrac{1}{2}(a_n + i b_n) = \frac{\omega}{2\pi}\int_{-\pi/\omega}^{\pi/\omega} e^{-in\omega t}\,h(t)\,dt$$

Can then write:

$$h(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega t}$$

where $c_n = \frac{\omega}{2\pi}\int_{-\pi/\omega}^{\pi/\omega} e^{-in\omega t}\,h(t)\,dt$

with $\int_{-\pi/\omega}^{\pi/\omega} e^{in\omega t}\,dt = \omega$

Then $h(t) = \frac{\omega}{2\pi}\sum_{n=-\infty}^{\infty} e^{+in\omega t}\int_{-\pi/\omega}^{\pi/\omega} e^{-in\omega\tau}\,h(\tau)\,d\tau$

Let $\omega \to 0$ then $\omega \to d\omega$, $n\omega = \Omega$

Then: $h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\Omega t}\,d\omega \int_{-\infty}^{\infty} h(\tau)\,e^{-i\Omega\tau}\,d\tau$

This is Fourier's integral theorem.
We now write the transforms.

$$g(\Omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\Omega\tau}\, h(\tau)\, d\tau$$

$$h(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\Omega t}\, g(\Omega)\, d\Omega$$

Now write $\Omega = ic + \Omega'$, $\Omega' = \Omega - ic$

$-ic$

Consider: $h_1(t) = e^{ct}\, h(t)$

$$g(\Omega' + ic) = g_1(\Omega')$$

Then $g_1(\Omega') = \frac{1}{\sqrt{2}\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-i\Omega\tau}\, h_1(\tau)\, d\tau$

and $h_1(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty - ic}^{\infty - ic} e^{i\Omega t'}\, g_1(\Omega')\, d\Omega'$

$\operatorname{Im}(\Omega') = -c$, $\operatorname{Im}(t) = 0$

Put $\lambda = i\Omega' + c$, $\operatorname{Re}(\lambda) = c$

$\sqrt{2\pi}\, g_1\left(\frac{\lambda}{i}\right) = \frac{\varphi(\lambda)}{\lambda}$, $h_1(t) = f(t)$

Then $\operatorname{Re}(\lambda) = c$

and $\frac{\varphi(p)}{p} = \int_{-\infty}^{\infty} e^{-p\tau}\, f(\tau)\, d\tau$

with $f(t) = \frac{1}{2\pi i} \int_{c - i\infty}^{c + i\infty} e^{\lambda t}\, \frac{\varphi(\lambda)}{\lambda}\, d\lambda$

$\operatorname{Re}(p) = c$, $\operatorname{Im}(t) = 0$

This concludes the course.

PROBLEM SET #1

APPLIED MATHEMATICS 201

Applied Function Theory

Fall 1955/56

PROBLEMS:

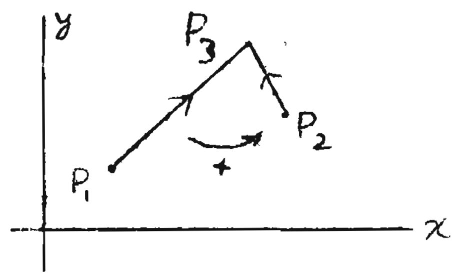
1. (a). Prove that $|Z_1 - Z_2|^2 + |Z_1 + Z_2|^2 = 2|Z_1|^2 + 2|Z_2|^2$

   (b). Interpret the above result geometrically.

   (c). Deduce from (a) that $|\alpha + \sqrt{\alpha^2 - \beta^2}| + |\alpha - \sqrt{\alpha^2 - \beta^2}| = |\alpha - \beta| + |\alpha + \beta|$

   (Z_1, Z_2, α, β complex numbers).

2. Let the points P_1, P_2, P_3 correspond in the Argand diagram to Z_1, Z_2, Z_3, respectively. Let $\measuredangle P_1 P_3 P_2 \equiv \theta$, where θ lies in the range $-\pi < \theta \leq \pi$, and is taken to be positive when the direction of rotation from the vector $\overline{P_1 P_3}$ to the vector $\overline{P_2 P_3}$ is positive, as shown. Show that



$$\arg \frac{Z_3 - Z_2}{Z_3 - Z_1} = \theta,$$

when the principal value is taken.

3. If the points P_1, P_2, P_3, P_4 correspond to Z_1, Z_2, Z_3, Z_4, respectively, prove that, if

$$\arg \frac{Z_3 - Z_2}{Z_3 - Z_1} = \arg \frac{Z_4 - Z_2}{Z_4 - Z_1} \qquad \text{................(1)},$$

then the points P_3 and P_4 are on the same side of the line $P_1 P_2$ and P_1, P_2, P_3, P_4 are concyclic.

Conversely, prove that if P_3 and P_4 lie on the same side of $P_1 P_2$, and P_1, P_2, P_3, P_4 are concyclic, then equation (1) is satisfied.

4. Show that the five points corresponding to the roots of the equation

$$32Z^5 = (Z + 1)^5$$

are concyclic.

APPLIED MATHEMATICS 201 Fall 1955/56

Applied Function Theory

5. If $Z = x + i y$, $\bar{Z} = x - i y$; and if $u(x,y)$ has differentiable first-order partial derivatives, and is equal to $V(Z,\bar{Z})$ when expressed in terms of Z and $\bar{Z}$, express

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

in terms of Z and $\bar{Z}$.

6. Prove that

$$u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$$

satisfies Laplace's equation, and find $f(Z)$, if $f(Z)$ is a regular function, whose real part is equal to u.

Check the result by finding the real and imaginary parts of your value of $f(Z)$.

7. Use the result of Ex. 5 (and with u as defined therein) to express $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$ in terms of polar coordinates r and Θ, when $x = r\cos\Theta$, $y = r\sin\Theta$.

8. Prove that $u = \sin x \cosh y + \partial \cos x \sinh y + x^2 - y^2 + 4xy$ Satisfies Laplace's equation, and find $f(Z)$, if $f(Z)$ is a regular function, whose real part is equal to u.

Check the result by finding the real and imaginary parts of your value of $f(Z)$.

PROBLEM SET # 2

9. (a). Find the zeros of (i) $\cos Z$; (ii) $\sinh Z$; (iii) $\cosh Z$

(b). Prove that

(i) $(Z - n\pi)\ \mathrm{cosec}\ Z \to (-1)^n$, as $Z \to n\pi$

(ii) $(Z - n\pi)\ \cot Z \to 1$, as $Z \to n\pi$

(iii) $\{Z - (n + 1/2)\pi\}\ \sec Z \to (-1)^{n+1}$, as $Z \to (n + 1/2)\pi$

(iv) $\{Z - (n + 1/2)\pi\}\ \tan Z \to -1$, as $Z \to (n + 1/2)\pi$

what are the corresponding results for the hyperbolic functions?

10. (a). If $Z = \tanh w$, show that

$$w = 1/2\ \mathrm{Log}\ \frac{1 + Z}{1 - Z}$$

(b). If $Z = \tan w$, show that

$$w = 1/2i\ \mathrm{Log}\ \frac{1 + iZ}{1 - iZ}$$

11. [Phillips, Examples 1, No. 12] If $w = \sqrt{(1-Z)(1 + Z^2)}$, A is the point (2,o), and P (the point corresponding to Z) describes a path in the first quadrant ($x \geq o$; $y \geq o$), prove that, if $w = 1$ when $Z = o$, the value of w at A is $-\sqrt{5}\,i$.

Problems

12. (Phillips, Examples I, No. 15) If

$$f(z) = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} z^n ,$$

then prove

a.) that $f(z)$ is regular, when $|z|<1$

b.) that the derivative of $f(z)$ is $\frac{\alpha f(z)}{(1+z)}$

c.) that the derivative of $(1+z)^{-\alpha} f(z)$ is zero.

Deduce that $f(z) = (1+z)^{\alpha}$

13. Use the result that if $z = x+iy$, $\zeta = \xi + i\eta = f(z)$ is a regular function of z in a domain D in which $f'(z) \neq 0$, and V is a function of x and y with differentiable first-order partial derivatives, then

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = |f'(z)|^2 \left\{ \frac{\partial^2 V}{\partial \xi^2} + \frac{\partial^2 V}{\partial \eta^2} \right\},$$

to find the left side of the above equation in terms of polar coordinates (r,θ). $[r \neq 0]$

14. (Phillips, Examples I, No. 8) If $f(z)$ is a regular function of z, prove that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \cdot |f(z)|^2 = 4\,|f'(z)|^2$$

15. (Phillips, Examples I, No. 9) If $f(z)$ is a regular function of z, such that $f'(z) \neq 0$, prove that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \cdot \log |f'(z)| = 0$$

If $|f'(z)|$ is the product of a function of x and a function of y show that

$$f'(z) = \exp\,(az^2 + \beta z + \gamma),$$

where a is a real constant, and β and γ are complex constants.

16. (Phillips, Examples II, No. 10) Find the transformation which maps the outside of the circle $|z|=1$ on the half-plane $\mathcal{R}w \geq 0$, so that the points $z=1, -i, -1$ correspond to $w=i, o, -i$ respectively. What corresponds in the w- plane to

(i) the lines $\arg z = \text{const}$; $|z| \geq 1$

(ii) the concentric circles $|z| = r$, $(r > 1)$?

17. (Phillips, Examples II, No. 11) Prove that

$$w = \frac{1+iz}{i+z}$$

maps the part of the real axis between $z=1$ and $z=-1$ on a semicircle in the w- plane.

Find all the figures that can be obtained from the originally selected part of the axis of x by successive applications of this transformation.

18. Take C to be the circle $|z| = 1$. By using Cauchy's Theorem and considering

$$\int_C \frac{dz}{z+2},$$

prove that

$$\int_0^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\, d\theta = 0$$

19. (Phillips, Ex. IV, No. 3). The function $f(z)$ is regular when $|z| < R'$, and C is the circle $|z| = R$ $(R < R')$. Prove that, if $|\alpha| < R$,

$$f(\alpha) = \frac{1}{2\pi i} \int_C \frac{R^2 - \alpha\bar{\alpha}}{(z-\alpha)(R^2 - \bar{\alpha} z)} f(z)\, dz,$$

and deduce that for $0 < r < R$,

$$f(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \phi) + r^2} f(Re^{i\phi})\, d\phi$$

20. (Phillips, Ex. IV, No. 1). The function $f(z)$ is regular in $|z - \alpha| < R$, and $0 < r < R$; also C is the circle $|z - \alpha| = r$. Use Cauchy's Theorem for $f(z)$ and Cauchy's formula for $f'(z)$ (with the closed contour C) to prove that

$$f'(\alpha) = \frac{1}{\pi r} \int_{-\pi}^{\pi} P(\theta)\, e^{-i\theta}\, d\theta$$

where $P(\theta)$ is the real part of $f(\alpha + re^{i\theta})$.

21. Prove that, if $f(z)$ is regular within and on a closed contour C, and α and β are two distinct points within C

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-\alpha)(z-\beta)}\,dz = \frac{f(\alpha)}{\alpha-\beta} + \frac{f(\beta)}{\beta-\alpha}$$

Deduce Liouville's theorem from this result, that an integral function of z bounded in the whole z - plane must be a constant.

22. (Phillips Ex. IV, No. 7). Prove that

$$\cosh\left(z+\frac{1}{z}\right) = a_0 + \sum_{n=1}^{\infty} a_n\left(z^n+\frac{1}{z^n}\right), \text{ where}$$

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos n\theta \cosh(2\cos\theta)\,d\theta$$

23. (Phillips Ex. IV, No. 8). Find the Taylor and Laurent series which represent the function

$$\frac{z^2-1}{(z+2)(z+3)}, \text{ in:}$$

(i) $|z|<2$ (ii) $2<|z|<3$ (iii) $|z|>3$.

24. (Phillips. Ex. I, No. 13). Given that $w^2 = z^2 - 2z + 2$, and that $w = +\sqrt{2}$ at $z = 0$, that the point P represents z in the Argand diagram, and that P describes, in the positive sense, a circle with center at $z = 1+i$ and of radius $\sqrt{2}$, starting at 0 ($z = 0$, where $w = \sqrt{2}$), and that the value of w at P varies continuously as P moves round the circle, find the value of w at P: (i) where P crosses the axis of y (ii) where P returns to 0.

25. The function $f(z)$ is regular where $|z| < R'$, and C is the circle $|z| = R$ ($R < R'$). Prove that, if $0 < |\alpha| < R$

$$f(\alpha) = \frac{1}{2\pi i}\int_C \left\{ \frac{R^2\alpha - z^2\bar{\alpha}}{z(z-\alpha)(R^2 - z\bar{\alpha})} + \frac{1}{z} \right\} f(z)\, dz$$

Deduce that for

$$0 < r < R, \quad f(re^{i\theta}) = \frac{i}{\pi}\int_C \left\{ \frac{Rr\sin(\theta-\phi)}{R^2 + r^2 - 2Rr\cos(\theta-\phi)} - \frac{i}{2} \right\} f(Re^{i\phi})\, d\phi$$

26. Prove that $\exp\left\{ \frac{1}{2} z \left(w - \frac{1}{w} \right) \right\}$ is regular in the w-plane, except at the origin. If it is expanded as a Laurent series

$$\sum_{n=-\infty}^{\infty} w^n J_n(z)$$

prove that

(i) $J_{-n}(z) = (-1)^n J_n(z)$

(ii) $J_n(z) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - z\sin\theta)\, d\theta$

(iii) $J_n(z) = \sum_{m=0}^{\infty} \frac{(-1)^m (\frac{1}{2}z)^{n+2m}}{m!\,(n+m)!} \qquad (n \geq 0)$

27. Describe the positions of the zeros of the function $e^{1/z} - c$, where c is a non-zero complex constant, equal to $Ke^{i\gamma}$ (K, γ real, $-\pi < \gamma \leq \pi$), and show that, whatever be the value of c (other than zero), there is a zero in ~~which~~ $0 < |z| < r$, where r is any positive number, no matter how small.

28. What are the positions and natures of the singularities of the following functions in the z-plane (excluding the point at infinity)?

(i) $f_1(z) = \dfrac{\cot \pi z}{(z-1)^2}$; (ii) $f_2(z) = z \operatorname{cosec} z$; (iii) $f_3(z) = \dfrac{z^4}{(c^2+z^2)^4}$, c real

(iv) $f_4(z) = \dfrac{1}{z(e^z - 1)}$.

29. Find the residues at the poles of the four functions in Ex. 28.

30. What is the nature of the point at infinity for each of the four functions in Ex. 28?

31. Find the order of the pole at the origin and the residue there for each of the three functions: (i) $\cot z$ (ii) $\operatorname{cosec}^2 z \log(1-z)$ (iii) $z/(\sin z - \tan z)$.

32. (Phillips, Ex. IV, No. 10). The only singularities in the entire z - plane (including the point at infinity) of a single-valued function $f(z)$ are poles of orders 1 and 2 at $z = -1$ and $z = 2$, with residues at these poles 1 and 2 respectively. Given that $f(0) = 7/4$, $f(1) = 5/2$, determine the function and expand it in a Laurent series valid in $1 < |z| < 2$.

33. Except for poles b_k of order r_k ($k=1,2, \ldots, n$) within a closed contour C, $f(z)$ is regular within and on C. Also, $f(z)$ has no zeros on C, and its zeros within C are at a_p of order s_p ($p=1,2,\ldots, \cancel{n}\, m$).

Prove that

$$\frac{1}{2\pi i}\int_C \frac{z f'(z)}{f(z)}\, dz = \sum_{\rho=1}^{m} s_\rho a_\rho - \sum_{k=1}^{n} r_k b_k$$

34. Show that the equation

$$z^4 + z^3 + 4z^2 + 2z + 3 = 0$$

has no real roots and no purely imaginary roots, and determine in which quadrants the roots lie.

Problem No. 35.

If $w = \cosh^{-1}z$, show that w is the many-valued function Log $\{z+(z^2-1)^{\frac{1}{2}}\}$, and that if w_1 is one branch, the other values are given by $2n\pi i \pm w_1$. State the positions of the branch points, and suitable cuts in the z-plane. In which regions of the z-plane is each branch regular? Define the values of one branch inside $|z|=1$ by fixing the value at $z=0$, and find a Taylor series for this branch valid in $|z|<1$.

No. 36.

(Phillips, Ex. IV, No. 12) Show that, if b is real, the series

$$\tfrac{1}{2}\log(1+b^2)+i\cdot\arctan b+\frac{z-ib}{1+ib}-\frac{1}{2}\left(\frac{z-ib}{1+ib}\right)^2+\frac{1}{3}\left(\frac{z-ib}{1+ib}\right)^3-\ldots. \qquad (1)$$

is an analytic continuation of the function defined by the series

$$z-\tfrac{1}{2}z^2+\tfrac{1}{3}z^3-\ldots.. \qquad (2)$$

No. 37.

(Phillips Ex. IV, No. 13). The power series

$$z+\tfrac{1}{2}z^2+\tfrac{1}{3}z^3+\ldots\ldots \qquad (1)$$

and

$$i\pi-(z-2)+\tfrac{1}{2}(z-2)^2-\tfrac{1}{3}(z-2)^3+\ldots\ldots \qquad (2)$$

have no common region of convergence. Prove that these are analytic contributions of the same function.

No. 38.

Use the method of contour integration to prove the result in 38 and the following results.

(Phillips Ex. V, No. 2). $$\int_0^{\pi}\frac{\cos^2 3\theta\, d\theta}{1-2p\cos 2\theta+p^2}=\frac{\pi}{2}\,\frac{1-p+p^2}{1-p} \quad (0<p<1).$$

E/E

Problem No. 39.

$$\int_0^{2\pi} \cos^n\theta \, d\theta = 0 \quad \text{if n is odd}$$

$$= \frac{1.3.5.\cdots(n-1)}{2.4.6.\cdots n} 2\pi \quad \text{if n is even.}$$

40. (Phillips, Ex. V, No. 5).

$$\int_0^\infty \frac{x^6 \, dx}{(a^4+x^4)^2} = \frac{3\sqrt{2}\,\pi}{16a} \quad (a>0)$$

41.

$$\int_0^\infty \frac{x^3 \sin mx}{x^4+a^4} \, dx = \frac{1}{2}\pi e^{-ma/\sqrt{2}} \cos\frac{ma}{\sqrt{2}}$$

$$(m>0, a>0)$$

42.

$$\text{(I)} \int_0^\infty \frac{(1+x^2)\cos ax}{1+x^2+x^4} \, dx = \frac{\pi}{\sqrt{3}} e^{-\sqrt{3}a/2} \cos\frac{a}{2}$$

$$\text{(II)} \int_0^\infty \frac{x \sin ax}{1+x^2+x^4} \, dx = \frac{\pi}{\sqrt{3}} e^{-\sqrt{3}a/2} \sin\frac{a}{2}$$

$(a>0)$

B/c

No. 43.

$$\text{(I)} \quad \int_0^\infty \frac{x^{a-1}}{1+x+x^2}\,dx = \frac{2\pi}{\sqrt{3}} \; \frac{\cos\left(\frac{\pi+2\pi a}{6}\right)}{\sin \pi a}$$

$$\text{(II)} \quad \int_0^\infty \frac{x^{a-1}\,dx}{1-x+x^2} = \frac{2\pi}{\sqrt{3}} \; \sin\left(\frac{\pi+2\pi a}{3}\right) \Big/ \sin \pi a$$

$(0 < a < 2)$

$B_{/E}$

Problems

44. (Phillips, Ex. V, No. 15)

$$P\int_0^\infty \frac{x^4}{x^6-1}\,dx = \frac{\sqrt{3}\,\pi}{6}$$

45. (Copson, Chap. 6, No. 30)

$$\int_0^\infty \frac{\sinh ax}{\sinh \pi x}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sinh ax}{\sinh \pi x}\,dx = \frac{1}{2}\tan\frac{1}{2}a\,; \quad (-\pi < a < \pi)$$

(Integrate $e^{az}/(\sinh \pi z)$ round the rectangle of sides $y = 0$, $y = 1$, $x = \pm R$, indented at $z = 0$ and $z = i$.)

46.
$$\frac{1}{\pi}P\int_0^\infty \frac{x^b - x^{-b}}{x^2-1}\,dx = \tan\frac{1}{2}\pi b\,; \quad (-1 < b < 1)$$

Deduce the result of Ex. 45.

47. (Phillips, Ex. V, No. 12)

$$\int_0^\pi \frac{x\sin x}{1-2a\cos x + a^2}\,dx = \frac{\pi}{a}\log\frac{1+a}{a} \qquad a > 1$$

(Integrate $z/(a - e^{-iz})$ round the rectangle of sides $y = 0$, $y = N$, $x = \pm\pi$.)

48. (1)
$$P\int_0^\infty \frac{dx}{1-x^5} = \frac{2\pi}{25}\left(\sin\frac{\pi}{5} + 3\sin\frac{2\pi}{5}\right)$$

(11)
$$P\int_0^\infty \frac{\log x}{x^{10}-1}\,dx = \frac{\pi^2}{25}\left(\frac{5}{2} - \cos\frac{\pi}{5} + 3\cos\frac{2\pi}{5}\right)$$

49.
$$\int_0^{\frac{1}{2}\pi} \frac{a\sin 2\theta}{1-2a\cos 2\theta + a^2}\,\theta\,d\theta = \frac{\pi}{4}\log(1+a), \text{ if } -1 < a < 1$$
$$= \frac{\pi}{4}\log\left(1+\frac{1}{a}\right), \text{ if } |a| > 1$$

(substitute $x = \tan\theta$, and integrate $2az\log(1 - iz)/\{1+z^2\}\{(1-a)^2 + (1+a)^2 z^2\}$ round an appropriate contour.)

50. (Phillips. Ex. V, No. 22)

$$\oint_C z^2 \log\frac{z+1}{z-1}\,dz = \frac{4}{3}\pi i$$

where C is the circle $|z| = 2$.

-be - 11-29-55

Problems

51. (Phillips, Ex. V, No. 11)

Evaluate $\oint_C f(z)\,dz$, where $f(z) = (1+z^4)^{-1}$ and C is the ellipse $x^2 - xy + y^2 + x + y = 0$.

52. $$\int_0^\infty \frac{\cos x}{\sqrt{x}}\,dx = \int_0^\infty \frac{\sin x}{\sqrt{x}}\,dx = \sqrt{\frac{\pi}{2}}$$

(Integrate $e^{iz}/\sqrt{z}$ round a suitable contour.)

53. $$\int_0^\infty \frac{\sin ax}{e^{2\pi x}-1}\,dx = \frac{1}{4}\coth\tfrac{1}{2}a - \frac{1}{2a} \qquad (a>0)$$

(Integrate $e^{iaz}/(e^{2\pi z}-1)$ round the rectangle of sides $x = 0$, $x = R$, $y = 0$, $y = 1$, indented at $z = 0$ and i, and take imaginary parts.)

Due Monday

52, 53, 54

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Problem No. 54.

Show that

(i) $\frac{z}{e^z-1}+\frac{1}{2}z$ is equal to $\frac{1}{2}z \coth \frac{1}{2}z$, and is an even function of z;

(ii) For $|z| < 2\pi$, $z/(e^z - 1)$ can be expanded in a Maclaurin series of the form

$$\frac{z}{e^z-1} = 1 - \frac{1}{2}z + \sum (-1)^{n-1} B_n \frac{z^{2n}}{(2n)!},$$

and find B_1.

(iii) $\frac{z}{e^z-1} = 1 - \frac{1}{2}z + 2z^2 \sum_{p=1}^{\infty} \frac{1}{z^2+4p^2\pi^2} \cdot \ (z \neq \pm 2p\pi i).$

(iv) $\frac{B_n}{(2n)!} = \frac{2}{(2\pi)^{2n}} S_{2n}$, where $S_k = \sum_{p=1}^{\infty} \frac{1}{p^k} \ (k > 1).$

(Use $\frac{x}{1+x} = x - x^2 + x^3 - \cdots + (-1)^{m-1} x^m + \frac{(-1)^m x^{m+1}}{1+x}.$)

Problem No. 55.

Values of the Bernoullian numbers B_n (defined in Ex. 54) may be computed from the series in 54(ii), or from any of the series in 56. Such methods are very laborious, and a less laborious method may be found as follows: Take z as real and equal to x, and write

$$\frac{x}{e^x-1} = 1 + A_1 x + A_2 \frac{x^2}{2!} + \cdots + \frac{A_n x^n}{n!} + \cdots,$$

so that

$$A_1 = -\tfrac{1}{2}, \quad A_3 = A_5 = \cdots = A_{2n+1} = \cdots = 0, \quad A_2 = B_1, \ldots,$$
$$A_{2n} = (-1)^{n-1} B_n, \ldots .$$

Now if $f(z)$ is a polynomial or integral function of z, let us agree to write symbolically $f(A)$ on the understanding that we shall expand in powers of A and replace A^n by A_n. Then we must avoid multiplying together two "powers" of A, since $A_{m+n} \neq A_m . A_n$; but if we keep to operations which are linear in A_n we may proceed to apply the rules of algebra in a formal way. Thus

$$e^{\alpha y} e^{Ay} = e^{(\alpha + A)y}$$

Also $e^{(\alpha+A)y}$ may be expanded in powers of y before A^n is replaced by A_n. Then

$$\frac{x}{e^x - 1} = e^{Ax} \quad ; \quad x = e^{(A+1)x} - e^{Ax}$$

and we may equate coefficients of x, if we first expand in powers of x and then replace A^n by A_n. Hence, we have

$$A + 1 - A = 1,$$

$$(A + 1)^2 - A^2 = 0,$$

and $\quad (A + 1)^n - A^n = 0 \quad (n > 1),$

if A^n is replaced by A_n. Thus $2A_1 + 1 = 0$, $A_1 = -\frac{1}{2}$.

In this way we obtain a recurrence relation for the A_n, but the formula for A_n contains all A with subscripts less than n, and a more convenient formula for computation is obtained as follows. If $f(z)$ is a polynomial in z containing no power of z less than the second, since $(A+1)^n - A^n = 0$ for $n > 1$, $f(A + 1) - f(A) = 0$. Take $f(z) = z^n (z - 1)^{n+1}$; $n > 1$.

Using the methods above, find the value of B_1, prove that for $n > 1$,

$$(2n+1)B_n - \frac{2n-1}{3!} n(n-1) B_{n-1} + \dots + (-1)^m \frac{(2n-2m+1)}{(2m+1)!} n(n-1)\cdots(n-2m+1) B_{n-m} + \dots = 0 \quad (n>1)$$

the last term being a multiple of $B_{\frac{1}{2}(n+1)}$ or $B_{\frac{1}{2}n}$ according as n is odd or even, and compute the values of B_2, B_3, B_4, B_5. Prove that

$$\sum_{p=1}^{\infty} \frac{1}{p^2} = \frac{\pi^2}{6}; \quad \sum_{p=1}^{\infty} \frac{1}{p^4} = \frac{\pi^4}{90}, \quad \sum_{p=1}^{\infty} \frac{1}{p^6} = \frac{\pi^6}{945}; \quad \sum_{p=1}^{\infty} \frac{1}{p^8} = \frac{\pi^8}{9450}$$

Problem No. 56. Obtain the expansions,

$$x \cot x = 1 - \sum_{n=1}^{\infty} \frac{2^{2n} B_n}{(2n)!} x^{2n} \qquad |x| < \pi$$

$$\tan x = \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1) B_n}{(2n)!} x^{2n-1}, \quad |x| < \frac{\pi}{2}$$

(continued on next page)

Problem No. 56, continued.

$$x \operatorname{cosec} x = 1 + \sum_{n=1}^{\infty} \frac{(2^{2n}-2) B_n}{(2n)!} x^{2n} \; ; \; |x| < \pi$$

$$\log \frac{\sin x}{x} = - \sum_{n=1}^{\infty} \frac{2^{2n-1} B_n}{n (2n)!} x^{2n} \; ; \; |x| < \pi$$

Problem No. 57.

By integrating $\pi z^{-2m} \cot \pi z \quad (m \geq 1) \longrightarrow$ round a suitable contour, prove that

$$\sum_{m=1}^{\infty} \frac{1}{n^{2m}} = \frac{(2\pi)^{2m} B_m}{2(2m)!}$$

where m is integral and B_m is the m-th Bermoullian number.

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APPLIED MATHEMATICS 201

Problem No. 58.

(a). $\phi = \beta x^{\frac{1}{2}}$ on ~~$x = 0$, $y \geq 0$~~ $y = 0,\ x \geq 0$, (where β is a constant), and at all other points in the z-plane ϕ has no singularities and

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

Also grad $\phi \longrightarrow 0$ at an infinite distance from the positive real axix.

Find a solution for ϕ.

(b). At all points on the parabola $y^2 = 4\eta_0^{\,2}(x + \eta_0^{\,2})$, $\psi = Uy$, where U is a constant. At every point outside the parabola

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0$$

and ψ has no singularities outside the parabola. Also grad $\psi \longrightarrow 0$ at an infinite distance from the parabola in the region outside the parabola.

Find a solution for ψ .

APPLIED MATHEMATICS 201

Problem No. 59.

Study the transformation, $z = e^{\zeta}$; $\zeta = \operatorname{Log} z$.

What regions correspond in the z-plane to the rectangles bounded by

(1) $\eta = \eta_1$; $\eta = \eta_2$; $\xi = \xi_1$; $\xi = \xi_2$ (2) $\eta = 0$; $\eta = \pi$; $\xi = \xi_1$; $\xi = \xi_2$

and to the open rectangle $\eta = 0$, $\eta = \pi$, $\xi \geq \xi_1$)

and to the strip $\eta = 0$, $\eta = \pi$, $-\infty \leq \xi \leq \infty$,

and their interiors?

What precautions are necessary in connection with the branch chosen of Log z?

If $\operatorname{Log} z = \log r + i\theta$, and $-\pi \leq \theta \leq \pi$, what corresponds with the open rectangle $\eta = \pi$, $\eta = -\pi$, $\log r_1 \geq \xi \geq -\infty$, and its interior? If $0 \leq \theta \leq 2\pi$, what corresponds with the open rectangle $\eta = 0$, $\eta = 2\pi$, $\log r_1 \leq \xi \leq \infty$, and its interior.

What deduction can be drawn from the transformations of the open rectangle $\eta = 0$, $\eta = \pi$, $0 \leq \xi \leq \infty$ under the transformations $z = e^{\zeta}$, $z_1 = \cosh \zeta$?

Problem No. 60.

Show that the capacity per unit length of a long straight circular cable of radius a parallel to the ground with its central line at a height h above the ground is

$$\left[2\cosh^{-1}\frac{h}{a}\right]^{-1}$$

if $K = 1$, and that if a/h is small, this is approximately

$$\left[2\log\frac{2h}{a} - \frac{a^2}{4h^2}\right]^{-1}$$

[Take $x = 0$ on the ground, and the circle as $(x - h)^2 + y^2 = a^2$, $\phi = 0$ on $x = 0$, $\phi = \phi_1 (>0)$ on the circle, $\nabla^2\phi = 0$, ϕ has no singularities and $r\phi$ is bounded on $x \geq 0$ outside the circle. If $w = \phi + i\psi$, the capacity is $-\frac{1}{4\pi}[\psi]/\phi_1$, where $[\psi]$ is the change in ψ for a complete circuit of the circle. Consider $w = \phi + i\psi = 2q\log\frac{z+\lambda}{z-\lambda}$ for real λ.]

Problem No. 61.

Two right circular cylinders of radii a and b ($b > a$) are one inside the other, with parallel axes at a distance d apart. Show that the capacity of the condenser thus formed (for $K = 1$) is

$$\left[2\cosh^{-1}\left[\frac{a^2+b^2-d^2}{2ab}\right]\right]^{-1}.$$

Take the line joining the centers of the circles as axis of x, and consider

$$w = 2q\log\frac{z+\lambda}{z-\lambda}.$$

Problem No. 62.

An earthed conducting circular cylinder of radius a is placed in a uniform electrostatic field of intensity A parallel to the x-axis. If ϕ is the potential outside the cylinder, and $w = \phi + i\psi$, find w, ϕ, ψ.

[$\phi = 0$ on the circle; outside the circle $\nabla^2\phi = 0$ and ϕ has no singularities; $\phi = -Ax + O\left(\frac{1}{r}\right)$ at ∞.]

Problem No. 63.

If $z = c\cosh\zeta$, $\zeta = \xi + i\eta$, then $\xi = \xi_0$ is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with $a = c\cosh\xi_0$, $b = c\sinh\xi_0$.

An earthed conducting cylinder with this section is placed in a uniform field of intensity A parallel to the x-axis. Prove that

(cont. on next page)

Problem No. 63, continued.

$$w = \phi + i\psi = -Ac \cosh \zeta + Be^{-\zeta},$$

and find B, ϕ and ψ.

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Problem No. 64.

Use the conformal transformation

$$\zeta = Z + \frac{a^2}{Z} \quad (a < R)$$

to deduce the complex potential $w = \phi + i\psi$ for the irrotational flow of an ideal fluid past the elliptic cylinder

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1 \quad (A > B)$$

from the flow past a circular cylinder of radius R. Find R and $\underline{a}$ in terms of A and B. [Take the undisturbed velocity to have components — $U \cos\alpha$, + $U \sin\alpha$ along the axes of x and y, the circulation about the cylinder to be + K.

Verify that the expression for w is the same as that obtained by the use of elliptic coordinates, ξ, η, where $z = c \cosh \zeta$, $\zeta = \xi + i\eta$, and c is suitably chosen.

Problem No. 65.

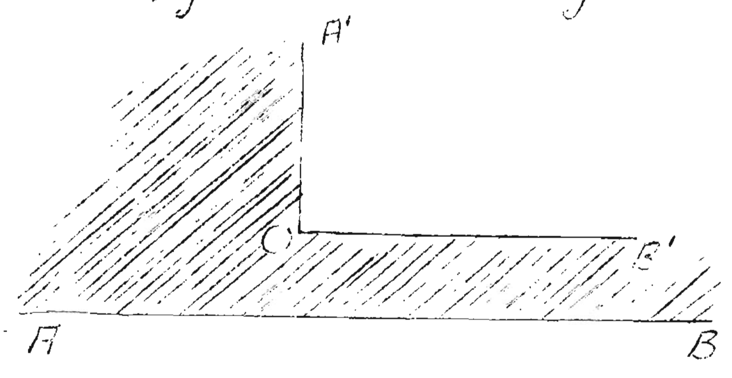
Use the general theory of the mapping of a polygon on a half-plane to show that the function

$$z = f(\zeta) = -\frac{b}{\pi}\log\zeta + \frac{2bi}{\pi}\{(\zeta-1)^{\frac{1}{2}} - i\log[1-i(\zeta-1)^{\frac{1}{2}}]\}$$

$$= \frac{2bi}{\pi}\{(\zeta-1)^{\frac{1}{2}} - \arctan(\zeta-1)^{\frac{1}{2}}\},$$

with the branches properly chosen, maps on to the upper half-plane of ζ the region in the z-plane between A'OB' and AB, where A'O is $x = 0$, $y \geq 0$, OB' is $y = 0$, $x \geq 0$, and AB is $y = -b$.

[$\zeta = \infty$ at A and A', $\zeta = 0$ at B and B', $\zeta = 1$ at O.]



Problem No. 66.

The function

Im

$$w = -i\,\frac{\zeta - \bar{\beta}}{\zeta - \beta},$$

with $I(\beta) > 0$, maps the upper half-plane of ζ on the exterior of the unit circle $|w| = 1$, $\zeta = \beta$ corresponding to $w = \infty$. Use this result, and the general formula for mapping the infinite region outside a closed polygon on a half-plane, to derive the formula

$$z = A \int_{w_0}^{w} \frac{\prod (w - \alpha_s)^{\mu_s}}{w^2}\, dw$$

for mapping the region outside a closed polygon on the exterior of the unit circle, the points at infinity corresponding with each other. In order that the mapping be conformal at infinity, $\sum \mu_s \alpha_s$ must be zero.

Illustrate the formulae by determining from them the functions

$$(a)\ z = \frac{2\zeta}{1+\zeta^2} \qquad (b)\ z = \tfrac{1}{2}\left(w + \tfrac{1}{w}\right)$$

to map the z-plane cut along the real azis from -1 to + 1 on to(a) the upper half-plane of ζ, with $\zeta = i$ corresponding to $z = \infty$; (b) the region outside the unit circle $|w| = 1$, with the points at infinity corresponding in the w- and z planes.

$[\alpha_1 = -1,\ \alpha_2 = +1.]$

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APPLIED MATHEMATICS 201

1.
(a) Prove $|(Z_1 + Z_2)|^2 + |(Z_1 + Z_2)|^2 = 2|(Z_1)|^2 + 2|(Z_2)|^2$

(b) Interpret the result geometrically.

(c) Deduce from (a) that $\left|\alpha + \sqrt{\alpha^2 - \beta^2}\right| + \left|\alpha - \sqrt{\alpha^2 - \beta^2}\right| = |\alpha - \beta| + |\alpha + \beta|$.

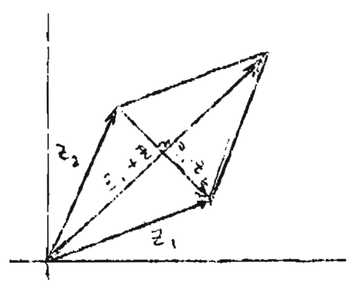
[(Z_1 , Z_2, α, β) Complex Nos.]

SOLUTIONS:

(a). Since $|(Z)|^2 = Z \cdot \bar{Z}$, and $(\overline{Z_i + Z_j}) = (\bar{Z}_i + \bar{Z}_j)$, we have

$$|(Z_1 - Z_2)|^2 + |(Z_1 + Z_2)|^2 = (Z_1 - Z_2)(\bar{Z}_1 - \bar{Z}_2) + (Z_1 + Z_2)(\bar{Z}_1 + \bar{Z}_2)$$
$$= 2(Z_1\bar{Z}_1 + Z_2\bar{Z}_2)$$
$$= 2|(Z_1)|^2 + 2|(Z_2)|^2$$

(b).



The sum of the squares of the two diagonals of a parallelogram is equal to the sum of the squares of all of the sides.

Many geometrical proofs can be invented to show this.

(c). Define z and z_2 by:

$$\left.\begin{aligned} \alpha + \beta &= 2Z_1^2 \\ \alpha - \beta &= 2Z_2^2 \end{aligned}\right\}$$

From which $\alpha = Z_1^2 + Z_2^2$; $\sqrt{\alpha^2 - \beta^2} = 2Z_1Z_2$,

and $\left|\alpha + \sqrt{\alpha^2 - \beta^2}\right| + \left|\alpha - \sqrt{\alpha^2 - \beta^2}\right| = |\alpha + \beta| + |\alpha - \beta|$

becomes $|(Z_1 + Z_2)|^2 + |(Z_1 - Z_2)|^2 = 2|Z_1|^2 + 2|Z_2|^2$

so the result follows from (a). [N.B. $|z^2| = |z|^2$]

APPLIED MATHEMATICS 201

Applied Function Theory
Fall 1955/56

Solutions

2. We know

$$\arg \frac{z_3 - z_2}{z_3 - z_1} = \arg (z_3 - z_2) - \arg (z_3 - z_1) + 2n\pi$$

$$(n = 0, \pm 1, \pm 2, \ldots)$$

and for the principal value we choose that value of n which makes

$$\pi < \arg \frac{z_3 - z_2}{z_3 - z_1} \leq \pi \text{ .}$$

Let $z_3 - z_2$ be represented by the vector $\overrightarrow{P_2 P_3}$, and $z_3 - z_1$ by the vector $\overrightarrow{P_1 P_3}$. Through the origin O, draw $\overrightarrow{OX}$ along the positive real axis, $\overrightarrow{OQ_1}$ parallel to, and in the same sense as $\overrightarrow{P_1 P_3}$, and $\overrightarrow{OQ_2}$ parallel to, and in the same sense as $\overrightarrow{P_2 P_3}$.

When $-\pi <$ angle $XOQ_1 \leq \pi$, let the angle XOQ_1 be called positive when the direction of rotation from $\overrightarrow{OX}$ to $\overrightarrow{OQ_1}$ is positive (Figs. 1,2,3,5), and negative otherwise (Fig. 4). Similarly for $\measuredangle XOQ_2$ (positive in Figs. 1, 2, 4; negative in Figs. 3, 5). Then

$$\arg (z_3 - z_2) = \measuredangle XOQ_2 \ ; \quad \arg (z_3 - z_1) = \measuredangle XOQ_1 \ .$$

$$\arg (z_3 - z_2) - \arg (z_3 - z_1) = \measuredangle XOQ_2 - \measuredangle XOQ_1 = \measuredangle Q_1OQ_2$$

measured positively or negatively according as the direction of rotation from $\overrightarrow{OQ_1}$ to $\overrightarrow{OQ_2}$ is positive or negative (without any condition on the magnitude of the angle), as marked in the figures.

In Figs. 1, 2, 3 $\quad -\pi < \measuredangle Q_1 O Q_2 \leq \pi$, so n = 0.

$$\arg \frac{z_3 - z_2}{z_3 - z_1} = \measuredangle Q_1 O Q_2 = \measuredangle P_1 P_3 P_2 \quad \text{(as defined).}$$

In Fig. 4, $\measuredangle Q_1 O Q_2$ is positive and $> \pi$, so we take n =-1 , and

$$\arg \frac{z_3 - z_2}{z_3 - z_1} = -\left[2\pi - \measuredangle Q_1 O Q_2\right] = \measuredangle P_1 P_3 P_2$$

(as defined).

In Fig. 5, $\measuredangle Q_1 O Q_2$ is negative and $< -\pi$, so we take n = 1, and

$$\arg \frac{z_3 - z_2}{z_3 - z_1} = 2\pi + \measuredangle Q_1 O Q_2 = \measuredangle P_1 P_3 P_2 \quad \text{(as defined)}$$

The figures illustrate various cases that may arise. In these, and in any other cases (each of which is only trivially different from one of those shown), the result stated holds.

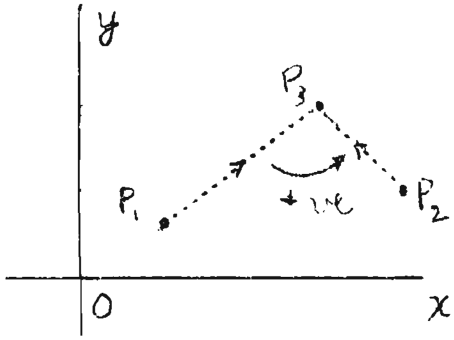


Fig 1a

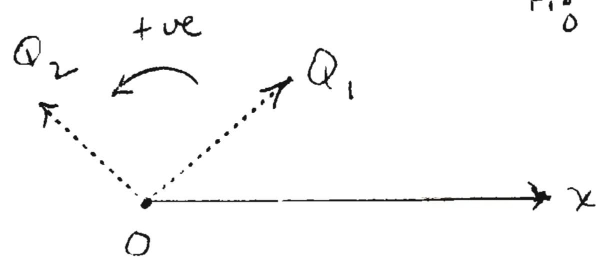


Fig 1b

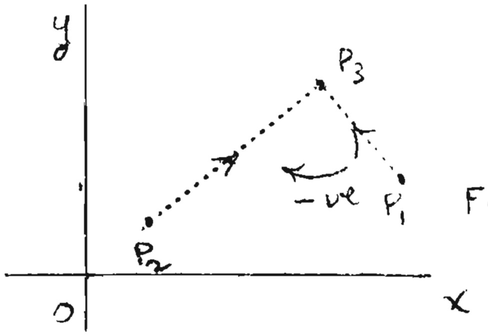


Fig 2a

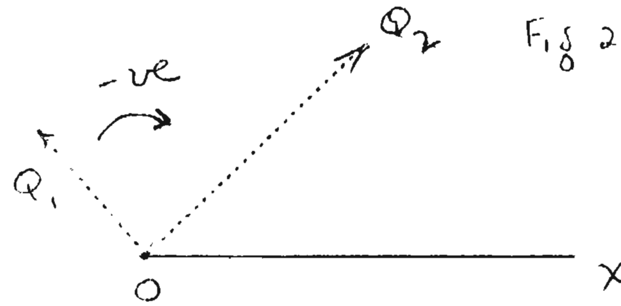


Fig 2b

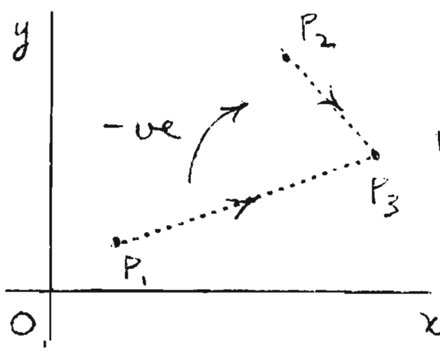
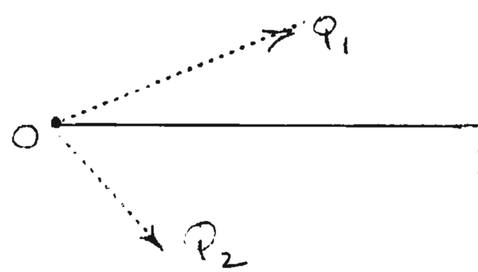


Fig 3a

Fig 3b



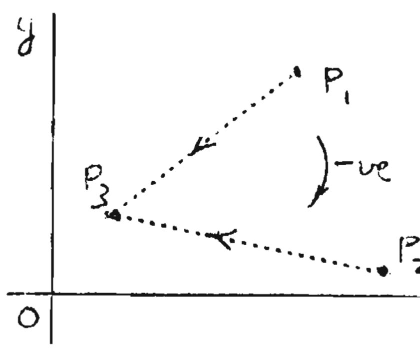
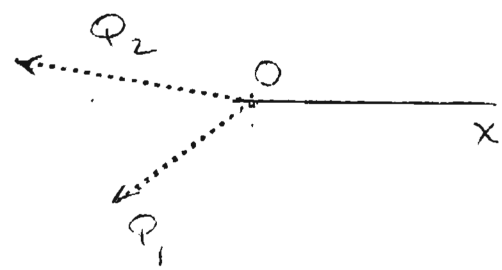


Fig 4a

Fig 4b



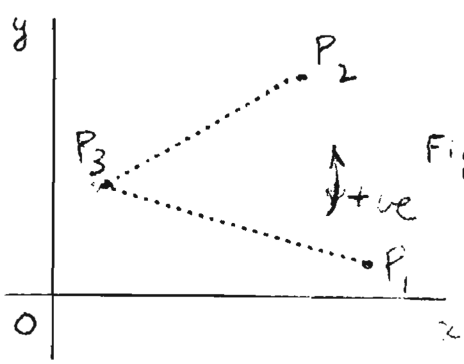
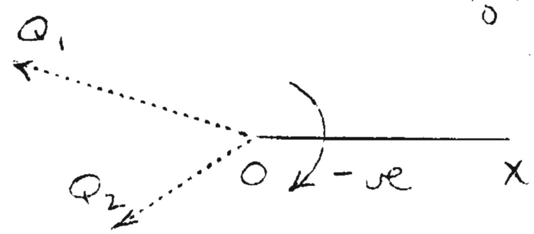


Fig 5a

Fig 5b



Solutions

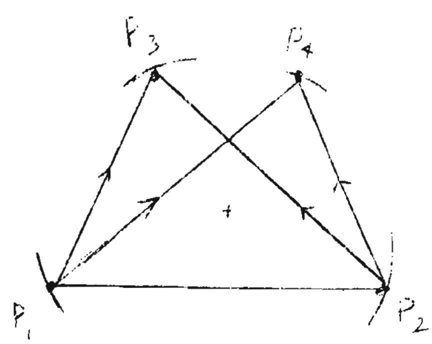
3. From Ex. 2, and with the conventions given therein, if

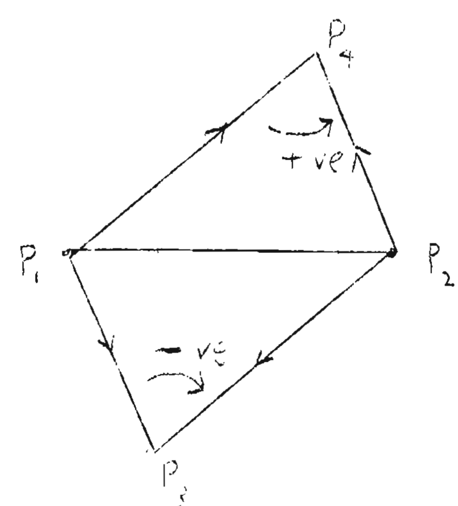
$$\arg \frac{z_3 - z_2}{z_3 - z_1} = \arg \frac{z_4 - z_2}{z_4 - z_1} \quad \ldots\ldots \quad (1)$$

then, $$\measuredangle P_1P_3P_2 = \measuredangle P_1P_4P_2 \quad \ldots\ldots\ldots\ldots \quad (2)$$

If P_3, P_4 were on opposite sides of the line P_1P_2, the angles $P_1P_3P_2$ and $P_1P_4P_2$ would have opposite signs. Hence P_3 and P_4 are on the same side of P_1P_2. Eq. (2) is, from plane geometrical considerations, the condition that P_1, P_2, P_3, P_4 be concyclic, and is satisfied.

Conversely, if P_3, P_4 are on the same side of P_1P_2, and (2) is satisfied, then (1) is satisfied, by Ex. 2.





Solutions

4. Let $w = \dfrac{2z}{z+1}$, so that $z = \dfrac{w}{2-w} = \dfrac{2}{2-w} - 1$, and consider the equation $w^5 = 1$; the five distinct solutions are

$$w = \exp\left(\tfrac{2}{5} m\pi i\right), \quad m = 0, \pm 1, \pm 2; \qquad \ldots\ldots (1)$$

and the five corresponding points are concyclic, lying on the unit circle whose center is at the origin.

Now, if we set $\zeta_1 = -w$, each point is changed into the point symmetrically placed with respect to the origin, and the unit circle is unchanged.

Next, let $\zeta_2 = \zeta_1 + 2$ $(= 2 - w)$; every point is moved by 2 units parallel to the real axis, and the circle becomes the unit circle with its center at (2,0) in the ζ_2 plane, cutting the real axis at (1,0) and (3,0).

If we put $\zeta_3 = \dfrac{1}{\zeta_2}$ $\left(= \dfrac{1}{2-w}\right)$ and if
$\zeta_3 = r_3 e^{i\theta_3}$; $\zeta_2 = r_2 e^{i\theta_2}$, then
$r_3 e^{i\theta_3} = 1/r_2\, e^{-i\theta_2}$; $r_3 = \dfrac{1}{r_2}$ and $\theta_3 = -\theta_2$.

This, then, inverts the circle with respect to the origin (unit radius of inversion), and then reflects across the real axis. Another circle is thus obtained, still symmetrical about the real axis, and cutting the real axis at (1,0) and $(\frac{1}{3}, 0)$; its center is at $(\frac{2}{3}, 0)$ and its radius is $\frac{1}{3}$.

(next page)

The reflection in the real axis leaves this circle unchanged.

Setting $\zeta_4 = 2\zeta_3$ $\left(= \frac{2}{2-w}\right)$, then all linear dimensions are multiplied by 2, and the resulting figure is similar. The circle which results has center at $\left(\frac{4}{3}, 0\right)$ and radius $\frac{2}{3}$.

Finally, putting $z = \zeta_4 - 1$ $\left(= \left[\frac{2}{2-w}\right] - 1\right)$, the figure moves a distance of -1 parallel to the real axis, and becomes the circle with center at $\left(\frac{1}{3}, 0\right)$, and radius $\frac{2}{3}$.

The five values of z corresponding to the values of w in (1) are the roots of

$$\left(\frac{2z}{z+1}\right)^5 = 1,$$ and the representative points

lie on the circle with center at $\left(\frac{1}{3}, 0\right)$, and radius $\frac{2}{3}$.

In a word, then, the bilinear transformation

$$z = \frac{w}{2-w} \qquad \left(\alpha = 1,\ \beta = 0,\ \gamma = -1,\ \delta = 2;\ w = \frac{\alpha z + \beta}{\gamma z + \delta}\right)$$

transforms circles into circles, and points on a circle into points on a corresponding circle.

Solutions

5. Method I: Since $u(x,y)$ has differentiable first-order partial derivatives

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}, \text{ and}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right)$$

But $2x = z + \bar{z}$; $2y = -i(z - \bar{z})$

So $$\frac{\partial}{\partial \bar{z}} = \frac{\partial x}{\partial \bar{z}}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \bar{z}}\frac{\partial}{\partial y} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

Similarly $$\frac{\partial}{\partial z} = \frac{\partial x}{\partial z}\frac{\partial}{\partial x} + \frac{\partial y}{\partial z}\frac{\partial}{\partial y} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

thus $$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 4\frac{\partial^2 U}{\partial z \partial \bar{z}}}.$$

Method II: $$\frac{\partial}{\partial x} = \frac{\partial z}{\partial x}\frac{\partial}{\partial z} + \frac{\partial \bar{z}}{\partial x}\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}$$

$$\frac{\partial}{\partial y} = \ldots\ldots = i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right)$$

and continue for $\frac{\partial^2}{\partial x^2}$; $\frac{\partial^2}{\partial y^2}$, ... etc.

6.

First Method:

$$u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$$

$$\begin{cases} u_x = 3x^2 - 3y^2 + 6x \\ u_y = -6xy - 6y \end{cases}$$

(over)

Now $f(z)=u+iv$. Since f(z) is regular, f'(z) has a value which is independent of the direction in which the derivative is taken. So keep y constant. We obtain

$$f'(z)=u_x+iv_x,$$ and by C-R conditions

for analytic functions

$$f'(z) = u_x - iu_y = 3x^2-3y^2+6x+6xyi+6yi$$

Now, as shown in class, we may replace x by z, and y by 0, thus leaving f'(z) unchanged.

$$f'(z) = f'(x+iy) \longrightarrow f'(z+i0) = f'(z)$$

(If you don't believe this, try it in a few cases).

So we have:

$$f'(z) = 3z^2+6z$$

(Note that we might equally as well have replaced x by 0, y by -iz).

Now integrate:

$$f(z) = z^3+3z^2+A+iB$$ → (A,B real constants)

When $y=0$, $z=x$, and

$$f(z) = x^3+3x^2+A+iB$$

But

$$\mathcal{R}\left[f(z)\big|_{y=0}\right] = u\big|_{y=0} \equiv x^3+3x^2+A = x^3+3x^2+1$$

So A = 1. B is undetermined.

And

$$\boxed{f(z) = z^3+3z^2+1+iB}$$

(over)

Solutions

7. From Ex. 5

$$\nabla^2 u = 4\frac{\partial^2 U}{\partial z \partial \bar{z}} = 4\frac{\partial}{\partial \bar{z}}\left(\frac{\partial U}{\partial z}\right) = 4\left[\left(\frac{\partial}{\partial r}\frac{\partial r}{\partial \bar{z}} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial \bar{z}}\right)\left\{\frac{\partial}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial z}\right\}\right]$$

now $\quad z = re^{i\theta} \; ; \quad \bar{z} = re^{-i\theta}$

$$z\bar{z} = r^2 \; ; \quad \frac{z}{\bar{z}} = e^{2i\theta}$$

so that
$$\left\{\begin{array}{ll} \dfrac{\partial r}{\partial z} = \dfrac{\bar{z}}{2r} = \dfrac{1}{2}e^{-i\theta} \; ; & \dfrac{\partial \theta}{\partial z} = -\dfrac{ie^{-i\theta}}{2r} \\ \dfrac{\partial r}{\partial \bar{z}} = \dfrac{z}{2r} = \dfrac{1}{2}e^{i\theta} \; ; & \dfrac{\partial \theta}{\partial \bar{z}} = \dfrac{ie^{i\theta}}{2r} \end{array}\right\}$$

then
$$\nabla^2 u = 4\left[\frac{1}{2}e^{i\theta}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial \theta}\right)\left\{\frac{e^{-i\theta}}{2}\left(\frac{\partial U}{\partial r} - \frac{i}{r}\frac{\partial U}{\partial \theta}\right)\right\}\right]$$

$$= \ldots = \boxed{\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}} \; ,$$

since $\quad i^3 = -i \; ; \quad i^2 = -1 \; ; \quad \dfrac{\partial^2 U}{\partial r \partial \theta} = \dfrac{\partial^2 U}{\partial \theta \partial r}$.

(An easier way of obtaining this result will be given later.)

8. Following the method of problem 6,

$$u = \sin x \cosh y + 2\cos x \sinh y + x^2 - y^2 + 4xy$$

$$u_x = \cos x \cosh y - 2\sin x \sinh y + 2x + 4y$$

$$u_y = \sin x \sinh y + 2\cos x \cosh y - 2y + 4x$$

$$x \to z \, ; \; y \to 0$$

and $$f(z) = (1-2i)(\sin z + z^2) + A + iB$$

since

$$\Re\left[f(x+i0)\right] = u\Big|_{y=0} ,$$ we have $A = 0$

(over) →

B remains undetermined.

So

$$f(z) = (1-2i)(\sin z + z^2) + iB$$

Since

$$\sin z = \sin x \cosh y + i \cos x \sinh y$$

and

$$z^2 = x^2 - y^2 + 2ixy$$

we have

$$u = \Re[f(z)] = \sin x \cosh y + 2\cos x \sinh y + x^2 - y^2 + 4xy$$

$$v = \Im[f(z)] = \cos x \sinh y - 2\sin x \cosh y + 2xy - 2(x^2 - y^2) + B$$

where B is an arbitrary, real constant.

Solutions

9a.

i) $\cos z = \cos(x+iy) \quad \cos x \cosh y - i \sin x \sinh y = 0$

$\cos x \cosh y = 0 \;;\; \sin x \sinh y = 0$

But $\cosh y > 0$, so $\cos x = 0$

And $x = (n + 1/2)\pi \;;\; n = 0, \pm 1, \pm 2, \ldots$

Also, if x is as stated, $\sin x \neq 0$

So $\sinh y = 0$; $y = 0$.

Finally

$$z = (n + 1/2)\pi \quad \text{are the roots of } \cos z$$

where $n = 0, \pm 1, \pm 2, \ldots$

ii) $\sinh z = \sinh(x+iy) = -i \sin(-y + ix) = 0$

or $0 = \sin y \cosh x + i \cos y \sinh x$

As above, $\sin y \cosh x = 0 \rightarrow \sin y = 0;\; y = n\pi \;;\; n = 0, \pm 1, \ldots$

$\cos y \sinh x = 0 \rightarrow \sinh x = 0;\; x = 0$

So that

$z = n\pi i$, where $n = 0, \pm 1, \pm 2, \ldots$ are the zeros of $\sinh z$.

iii) $\cosh z = \cos(iz) = \cos y \cosh x + i \sin y \sinh x = 0$

or $\cos y \cosh x = 0 \rightarrow \cos y = 0;\; y = (n+\frac{1}{2})\pi \;;\; n = 0, \pm 1, \pm 2, \ldots$

$\sin y \sinh x = 0 \rightarrow \sinh x = 0;\; x = 0$

So that

$z = (n+\frac{1}{2})\pi i$, where $n = 0, \pm 1, \pm 2, \ldots$ are the zeros of $\cosh z$.

b.) i) $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots \quad (-1)^{n-1} \frac{z^{2n+1}}{(2n+1)!} + \ldots$

$$\frac{\sin z}{z} = 1 - \frac{z^z}{3!} + \ldots \quad (-1)^{n-1} \frac{z^{2n}}{(2n+1)!} + \ldots$$

Call $f(z) = 1 - \frac{\sin z}{z}$. It is an integral function; it is continuous at the origin, and $\lim_{z\to 0} f(z) = f(o) = 0$. We may invert the two limiting processes of summing the series and taking $\lim_{z\to 0}$. Doing so, we find

$$\frac{\sin z}{z} \to 1 \quad \text{as } z \to 0.$$

Hence $$\frac{\sin(z-n\pi)}{z-n\pi} \to 1 \quad \text{as } z \to n\pi.$$

Now $\dfrac{\sin z}{z-n\pi} = \dfrac{\sin[(z-n\pi)+n\pi]}{z-n\pi} = \dfrac{\sin(z-n\pi)}{z-n\pi}\cos n\pi + \dfrac{\cos(z-n\pi)}{z-n\pi}\sin n\pi$

and $\dfrac{\sin z}{z-n\pi} \to \cos n\pi = (-1)^n$, as $z \to n\pi$

then $(z-n\pi)\operatorname{cosec} z = \left[\dfrac{\sin z}{z-n\pi}\right]^{-1} = (-1)^n$. Q.E.D.

---

ii) $(z-n\pi)\cot z = (z-n\pi)\operatorname{cosec} z \cos z \to (-1)^n \cos n\pi = 1$

iii) $\dfrac{\cos z}{z-(n+\frac{1}{2})\pi} = (-1)^{n+1}\ \dfrac{\sin[z-(n+\frac{1}{2})\pi]}{z-(n+\frac{1}{2})\pi} \to (-1)^{n\ 1}$, as $z \to (n+\frac{1}{2})\pi$.

so $\left\{z-(n+\frac{1}{2})\pi\right\}\sec z \to (-1)^{n+1}$, as $z \to (n+\frac{1}{2})\pi$.

iv) $\left\{z-(n+\frac{1}{2})\pi\right\}\tan z = \left\{z-(n+\frac{1}{2})\pi\right\}\sec z \sin z \to (-1)^{n+1}\sin(n+\frac{1}{2})\pi = -1$,

as $z \to (n+\frac{1}{2})\pi$

The corresponding results for the hyperbolic functions may be obtained from these as follows (or may be obtained directly by similar methods.):

since $i \sin \frac{z}{i} = \sinh z$; $\cos\frac{z}{i} = \cosh z$

$\operatorname{cosec}\frac{z}{i} = i \operatorname{cosech} z$; $\sec\frac{z}{i} = \operatorname{sech} z$

$\cot\frac{z}{i} = i \coth z$; $\tan\frac{z}{i} = -\, i \tanh z$

from i) $(\frac{z}{i} - n\pi)\operatorname{cosec}\frac{z}{i} \to (-1)^n$, as $\frac{z}{i} \to n\pi$.

similarly, from ii), iii), iv)

$$(\tfrac{z}{i} - n\pi)\cot\tfrac{z}{i} = (z-n\pi i)\coth z \to 1, \text{ as } z \to n\pi i$$

$$i\left\{\tfrac{z}{i} - (n+\tfrac{1}{2})\pi\right\}\sec\tfrac{z}{i} = \left\{z-(n+\tfrac{1}{2})\pi i\right\}\operatorname{sech} z \to (-1)^{n\ 1}\ i,$$

as $z \to (n+\frac{1}{2})\pi i$,

$$-\left\{\tfrac{z}{i} - (n+\tfrac{1}{2})\pi\right\}\tan\tfrac{z}{i} = \left\{z-(n+\tfrac{1}{2})\pi i\right\}\tanh z \to 1,$$

as $z \to (n+\frac{1}{2})\pi i$

10a. If $z = \tanh w = \dfrac{\sinh w}{\cosh w} = \dfrac{e^{w} - e^{-w}}{e^{w} + e^{-w}}$

So $\dfrac{1 + z}{1 - z} = \dfrac{2e^{w}}{2e^{-w}} = e^{2w}$

And $w = 1/2 \, \mathrm{Log}\left[\dfrac{1 + z}{1 - z}\right]$

b.) Similarly, if $z = \tan w$, $iz = \dfrac{i \sin w}{\cos w} = \dfrac{e^{iw} - e^{-iw}}{e^{iw} + e^{-iw}}$

And $\dfrac{1 + iz}{1 - iz} = e^{2iw}$

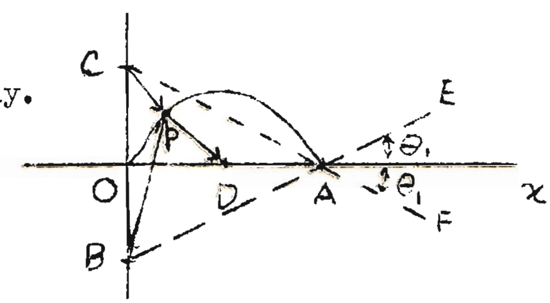
So $w = \arctan z = \dfrac{1}{2i} \, \mathrm{Log}\left[\dfrac{1 + iz}{1 - iz}\right]$

Alternatively, if $z = \tan w$, then $iz = \tanh iw$, so the second result follows from the first.

In each case, w is an infinitely many-valued function. When $w = \tan^{-1} z$,

$\dfrac{dw}{dz} = \dfrac{1}{1 + z^2}$, for all branches. When $w = \tanh^{-1} z$, $\dfrac{dw}{dz} = \dfrac{1}{1 - z^2}$, for all branches

11. Let B, C, D, be the points representing $- i$, $+i$, $+ 1$, respectively. Let BA, extended to E, make an angle θ_1 with the positive direction (AX) of the X- axis. Then, if CA is extended to F, $\angle FAX = \angle XAE = \theta_1$.



$1 - z$, $z + i$, $z - i$ are represented, respectively, by the vectors $\vec{PD}$, $\vec{BP}$, $\vec{CP}$, and the angles which these vectors make with the positive real axis may be taken to be the arguments of $1 - z$, $z + i$, $z - i$ (the angles being counted positively or negatively, according as the sense of rotation from the positive real axis to the vector in question is positive or negative). The angles are taken to lie in the range $-\pi < \theta \leq \pi$, when P is at O, and as P moves, the angles are taken to vary continuously. The plane is not specified as being cut so the angles <u>may not</u> continue to be the principal values of the arguments.

With these definitions of the arguments, consider the situation when P, the representative point, is at the origin O:

$\arg(z - i) = -\pi/2$; $\arg(z + i) = \pi/2$, $\arg(1 - z) = 0$

Thus, the sum of these is zero. Also $|(1 - z)(1 + z^2)| = 1$. So the two branches of w have values

$$w_1 = \left\{ |(1-z)(1+z^2)| \right\}^{1/2} \exp 1/2 \left\{ \arg(1-z) + \arg(z-i) + \arg(z+i) \right\}$$

$$w_2 = \left\{ |(1-z)(1+z^2)| \right\}^{1/2} \exp 1/2 \left\{ \arg(1-z) + \arg(z-i) + \arg(z+i) + 2\pi i \right\}$$

$$= -w_1$$

Since $w=+1$ at $z=o$, we are considering the first branch, w_1.

Now, at A, $|(1-z)(1+z^2)| = 5$, so $|w_1| = \sqrt{5}$

The Vector $\overrightarrow{CP}$ turns in the positive direction as P moves from O to A along the curve in the first quadrant; when P is at A, it makes an angle $-\theta_1$ with the positive real axis, so arg $(z-i)$ increases to $-\theta_1$.

Similarly, arg $(z+i)$ decreases to $+\theta_1$. The Vector $\overrightarrow{PD}$ turns always in the negative sense as P moves from O to A on the curve, so arg $(1-z)$ decreases from o to $-\pi$.

At A, arg $(1-z)$ + arg $(z-i)$ + arg $(z+i)$ $= -\pi - \theta_1 + \theta_1 = -\pi$ and $w_1 = \sqrt{5}\, e^{-\frac{i\pi}{2}} = -\sqrt{5}\, i$.

12. Consider the series $\sum \beta_n$, where

$$\beta_n = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} z^n$$

Then $\left|\dfrac{\beta_{n+1}}{\beta_n}\right| = \left|\dfrac{(\alpha-n)}{(n+1)} z\right| \to |z|$, as $n \to \infty$

Hence, the series converges absolutely for $|z| < 1$, $f(z)$ is regular when $|z| < 1$, and

$$\frac{f'(z)}{\alpha} = 1 + (\alpha-1)z + \frac{(\alpha-1)(\alpha-2)}{2!} z^2 + \ldots + \frac{(\alpha-1)(\alpha-2)\ldots(\alpha-n)}{n!} z^n + \ldots$$

So by multiplication of (absolutely convergent) series

$$\frac{(1+z)\, f'(z)}{\alpha} = 1 + (\alpha-1)z + \frac{(\alpha-1)(\alpha-2)}{2!} z^2 + \ldots + \frac{(\alpha-1)(\alpha-2)\ldots(\alpha-n)}{n!} z^n + \ldots$$

$$+ z + (\alpha-1)z^2 + \ldots + \frac{(\alpha-1)(\alpha-2)\ldots(\alpha-n+1)}{(n-1)!} z^n +$$

The coefficient of z^n in the sum is

$$\frac{(\alpha-1)(\alpha-2)\ldots(\alpha-n+1)}{n!} \left\{ (\alpha-n) + n \right\} = \alpha \frac{(\alpha-1)\ldots(\alpha-n+1)}{n!}$$

and

$$\frac{(1+z)\, f'(z)}{\alpha} = 1 + \alpha z + \frac{\alpha(\alpha-1)}{2!} z^2 + \ldots + \alpha \frac{(\alpha-1)\ldots(\alpha-n+1)}{n!} z^n + \ldots$$

$$= f(z)$$

So $f'(z) = \frac{\alpha f(z)}{1+z}$ $(|z| < 1)$.

Thus $\frac{d}{dz}\left\{(1+z)^{-\alpha} f(z)\right\} = -\alpha(1+z)^{-1-\alpha} f(z) + (1+z)^{-\alpha} f'(z)$
$= -\alpha(1+z)^{-1-\alpha} f(z) + \alpha(1+z)^{-1-\alpha} f(z) = 0$

$(1+z)^{-\alpha}$ is regular in the plane with a cut from -1 to $-\infty$ along the negative real axis, so for $|z| < 1$, $(1+z)^{-\alpha} f(z)$ is a regular function with a zero derivative, and is therefore a constant. At $z=0$, $f(z)=1$, so the constant is 1, i.e.

$$(1+z)^{-\alpha} f(z) = 1, \quad f(z) = (1+z)^{\alpha} \qquad (|z| < 1)$$

No. 13.

Let $\zeta = \log z = \log r + i\theta$, $\xi = \log r$, $\eta = \theta$.

If we make <u>any</u> cut in the z-plane from the origin to infinity, ζ is a regular function in the cut plane. Also $\frac{d}{dz}(\log z) = \frac{1}{z}$, and is not zero for any finite r.

Since $\left|\frac{d}{dz}\log z\right| = \frac{1}{r}$, if $r \neq 0$ or ∞

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{1}{r^2}\left(\frac{\partial^2 V}{\partial \xi^2} + \frac{\partial^2 V}{\partial \theta^2}\right)$$

Now

$$\frac{\partial V}{\partial \xi} = \frac{\partial V}{\partial r} \Big/ \frac{d}{dr}(\log r) = r\frac{\partial V}{\partial r}$$

and

$$\frac{\partial^2 V}{\partial \xi^2} = r\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) = r^2\frac{\partial^2 V}{\partial r^2} + r\frac{\partial V}{\partial r}$$

Hence

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2}$$

No. 14

Let $w = u + iv = f(z)$.

Then $|f(z)|^2 = |w|^2 = w\bar{w}$. $\Rightarrow \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = |f'(z)|^2 \cdot \left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right)|f(z)$

But
$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z\,\partial \bar{z}}$$
(Example No. 5)

Similarly
$$\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2} = 4\frac{\partial^2}{\partial w\,\partial \bar{w}}$$

and
$$\frac{\partial^2}{\partial w\,\partial \bar{w}}(w\bar{w}) = \frac{\partial}{\partial w}w = 1$$

Hence
$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2\frac{\partial^2}{\partial w\,\partial \bar{w}}(w\bar{w})$$
$$= 4|f'(z)|^2.$$

Solution No. 24.

$$z^2 - 2z + 2 = (z-1)^2 + 1 = (z - 1)^2 - i^2 = (z - 1 - i)(z - 1 + i) =$$

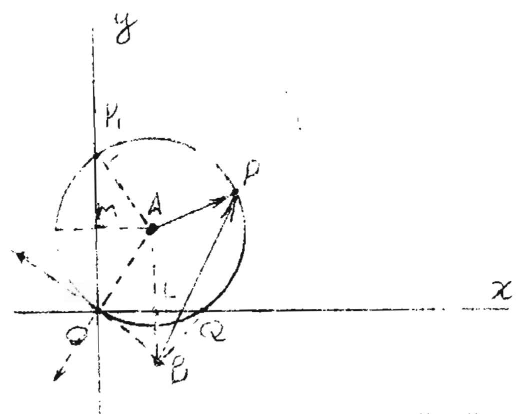
$(z - z_1)(z - z_2)$, where $z_1 = 1 + i$, $z_2 = 1 - i$

Let A and B represent the points z_1, z_2 respectively. In the (x, y) plane, A and B have coordinates (1, 1) and (1, -1) respectively. $z - z_1$ and $z - z_2$ are represented by the vector $\overrightarrow{AP}$ and $\overrightarrow{BP}$ respectively.

There are two branches of w, say w_1 and w_2; and

$$|w_1| = |w_2| = |z - z_1|^{\frac{1}{2}} \; |z - z_2|^{\frac{1}{2}}$$

$\arg w_1 = \frac{1}{2}\{\arg(z - z_1) + \arg(z - z_2)\}$, $\arg w_2 = \arg w_1 + \pi$.
$\{$or $w_2 = - w_1\}$. At O, $|z - z_1| = \sqrt{2}$ $|z - z_2| = \sqrt{2}$, so $|w_1| = \sqrt{2}$.



[Since OA $= \sqrt{2}$, O lies on the circle.]

At O, $\overrightarrow{AO}$ and $\overrightarrow{BO}$ make angles $-\frac{3\pi}{4}$ and $+\frac{3\pi}{4}$ with OX (the axis of x), so $\arg(z - z_1) = -\frac{3\pi}{4}$, $\arg(z - z_2) = +\frac{3\pi}{4}$. Hence $\arg w_1 = 0$, $w_1 = \sqrt{2}$, and we must consider w_1. [At O, $w_2 = -\sqrt{2}$].

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Draw AM and AL perpendicular to OY and OX, respectively. Since A is the point (1, 1), $\angle$ OAM $= 45^\circ$, OM $=$ MA $= 1$, and if the axis of y met the circle again in P_1, then by symmetry, $OP_1 = 2$, and P_1 is the point 2i. (If P_1 is this point, it is easily checked that $AP_1 = \sqrt{2}$, so P_1 lies on the circle).

Also, $\angle$ OAB $= \angle$ OBA $= 45^\circ$, so $\angle$ BOA $= 90^\circ$, and BO is a tangent to the circle. If the axis of X meets the circle again in Q, then it easily seems that Q is at $z = 2$ and that BQ is the second tangent from B to the circle.

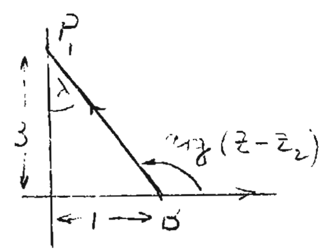
As P moves around the circle in the positive sense, the vector $\overrightarrow{AP}$ rotates continually in the positive sense, and therefore $\arg(z - z_1)$ continually increases. At P_1,

$z - z_1 = 2i - (1+i) = -1 + i$, and $\arg(z - z_1) = \frac{3\pi}{4}$. When P returns to O, $\arg(z - z_1) = \frac{5\pi}{4}$. Also, both at P_1 and at O, $|z - z_1| = \sqrt{2}$.

Solution No. 24 Continued.

As P moves round the circle in the positive sense, the vector BP rotates in the negative sense until P reaches Q, and then rotates in the negative sense, so arg $(z - z_2)$ decreases until P reaches Q (when arg $(z - z_2) = \frac{\pi}{4}$) and then increases. When P is at P_1, $z - z_2 = 2i - 1 + i = -1 + 3i$, so arg $(z - z_2) = \frac{1}{2}\pi + \lambda$, (see where λ is an acute angle $(0 < \lambda < \frac{1}{2}\pi)$, and

$$\tan\lambda = \frac{1}{3}$$



Also at P_1, $|z - z_2| = \sqrt{10}$.

At O, arg $(z - z_2) = \frac{3\pi}{4}$, $|z - z_2| = \sqrt{2}$.

Hence at P_1, $|w_1|^2 = \sqrt{2}\sqrt{10} = \sqrt{2u}$, $|w_1| = 20^{\frac{1}{4}}$

$$2 \arg w_1 \quad \frac{3\pi}{4} + \frac{\pi}{2} + \lambda = \frac{5\pi}{4} + \lambda$$

and $w = w_1 = \boxed{20^{\frac{1}{4}} \exp\left(\frac{5\pi}{8} + \frac{\lambda}{2}\right)i}$, where $\tan\lambda = \frac{1}{3}$, $0 < \lambda < \frac{1}{2}\pi$.

When P returns to O, $|w_1|^2 = 2$, $2 \arg w_1 = \frac{5\pi}{4} + \frac{3\pi}{4} = 2\pi$.

Hence $w = w_1 = \sqrt{2}\, e^{\pi i} = \boxed{-\sqrt{2}}$

[When P returns to O it has completed a closed circuit which does not include the branch point $z = z_2$ of $(z - z_2)^{\frac{1}{2}}$, and which does include the branch point $z = z_1$ of $(z - z_1)^{\frac{1}{2}}$, once. So $(z - z_2)^{\frac{1}{2}}$ has returned to its original value, but $(z - z_1)^{\frac{1}{2}}$ takes its original value with the sign changed. Hence w takes its original value with the sign changed.]

18.

$\frac{1}{z+2}$ is a regular function within and on $|z| = 1$; its only singularity is at $z = -2$. Hence $\int_C \frac{dz}{z+2} = 0$.

On c_1, $z = e^{i\theta} = \cos\theta + i\sin\theta$, $dz = i\cdot e^{i\theta}d\theta = i(\cos\theta + i\sin\theta)\,d\theta$

$$\frac{1}{z+2} = \frac{\bar{z}+2}{(z+2)(\bar{z}+2)} = \frac{\bar{z}+2}{z\bar{z}+2(z+\bar{z})+4},$$ and on C, $z\bar{z} = 1$, $z + \bar{z} = 2\cos\theta$

Hence on C, $\frac{1}{z+2} = \frac{(2+\cos\theta) - i\sin\theta}{5+4\cos\theta}$

Hence $$\int_{-\pi}^{\pi} \frac{(2+\cos\theta - i\sin\theta)(\cos\theta + i\sin\theta)}{5+4\cos\theta}\,d\theta = 0$$

The real and imaginary parts must vanish separately. [The imaginary part is an odd function of θ , so the result for the imaginary part is obvious.] For the real part, we obtain

$$\int_{-\pi}^{\pi} \frac{2\cos\theta + \cos^2\theta + \sin^2\theta}{5+4\cos\theta} = \int_{-\pi}^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta} = 0.$$

19. Since $|\alpha| < R$, the point α is within C, and since $\left|R^2/\bar{\alpha}\right| = \left|R^2/\alpha\right| > R$, R^2/α is outside C.

Then
$$\frac{R^2-\alpha\bar{\alpha}}{(z-\alpha)(R^2-z\bar{\alpha})} = \frac{1}{z-\alpha} + \frac{\bar{\alpha}}{R^2-z\bar{\alpha}} = \frac{1}{z-\alpha} - \frac{1}{z-R^2/\bar{\alpha}},$$
$$\frac{1}{2\pi i}\int_C \frac{R^2-\alpha\bar{\alpha}}{(z-\alpha)(R^2-z\bar{\alpha})} f(z)\,dz = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z-\alpha} - \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z-R^2/\bar{\alpha}}$$
since by Cauchy's Theorem
$$\frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z-R^2/\bar{\alpha}} = 0$$
$f(z)/[z-R^2/\bar{\alpha}] \longrightarrow$ being regular on and within C, while by Cauchy's Integral Theorem $\frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z-\alpha} = f(\alpha)$

Let $\alpha = re^{i\theta}$, and on C put $z = Re^{i\phi}$, $R^2/z = \bar{z} = Re^{-i\phi}$, $dz = iRe^{i\phi}d\phi$, $\frac{dz}{z} = i\,d\phi$.

Then on C, $(z-\alpha)(R^2/z-\bar{\alpha})$ $(Re^{-i\phi} - re^{i\theta})(Re^{-\phi} - re^{-i\theta})$
$$R^2 - Rr\left(e^{i(\theta-\phi)} \quad e^{-i(\theta-\phi)}\right) + r^2 = R^2 - 2Rr\cos(\theta-\phi) + r^2$$

Hence
$$\frac{1}{2\pi i}\int_C \frac{R^2-\alpha\bar{\alpha}}{(z-\alpha)(R^2-z\bar{\alpha})} f(z)\,dz = \frac{1}{2\pi i}\int_C \frac{R^2-\alpha\bar{\alpha}}{(z-\alpha)(R^2/z-\bar{\alpha})} f(z)\frac{dz}{z}$$
$$= \frac{1}{2\pi i}\int_{-\pi}^{\pi} \frac{i[R^2-r^2]\, f(Re^{i\phi})\,d\phi}{R^2-2Rr\cos(\theta-\phi)+r^2}$$
$$= f(\alpha) = f(re^{i\theta})$$

If u and v are the real and imaginary parts of f, and u and v are expressed as functions of R and ϕ when $z = Re^{i\phi}$, i.e., $f(Re^{i\phi}) = u(R,\phi) + iv(R,\phi)$; $f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$, then on taking the real part, we have
$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{R^2-r^2}{R^2-2Rr\cos(\theta-\phi)+r^2}\, u(R,\phi)\,d\phi = u(r,\theta) \text{.......}(1)$$
This formula, known as Poisson's integral formula, expresses u at any point inside the circle in terms of its values on the circle, when u is the real part of a regular function of z (and therefore when u is a two-dimensional harmonic function).

There is a similar formula for the imaginary part v.

No. 20.

On C, let $z - a = re^{i\theta}$. Then $z = a + re^{i\theta}$, $f(z)$ $= f(a+re^{i\theta})$. Let $f(a+re^{i\theta}) = P+iQ$, (P, Q real). f(z) is regular within and on C, so

$$\frac{1}{2\pi i}\int_C f(z)\,dz = 0$$

On C, $dz = ire^{i\theta}d\theta$, and r is a constant, so

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}(P+iQ)e^{i\theta}\,d\theta = 0.$$

The real and imaginary PARTS vanish separately, and the conjugate must also vanish, i.e.,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}(P-iQ)e^{-i\theta}\,d\theta = 0 \quad \ldots\ldots\ldots (1)$$

Also, $$f'(\alpha) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-\alpha)^2}\,dz = \frac{1}{2\pi i}\int_{-\pi}^{\pi}\frac{P+iQ}{(re^{i\theta})^2}\,ire^{i\theta}d\theta$$

i.e., $$\frac{1}{2\pi r}\int_{-\pi}^{\pi}(P+iQ)e^{-i\theta}\,d\theta = f'(\alpha) \quad \ldots\ldots (2)$$

Divide (1) by r; add to (2). Thus,

$$\frac{1}{\pi r}\int_{-\pi}^{\pi}Pe^{-i\theta}\,d\theta = f'(\alpha)$$

16. Let the required transformation be

$$w = \frac{\alpha z + \beta}{\gamma z + \delta}.$$

Since $z = -i, 1, -1$ correspond to $w = 0, i, -i$ respectively

$$\therefore \quad \beta = i\alpha; \quad i = \alpha\frac{i+1}{\gamma+\delta}; \quad -i = \alpha\frac{i-1}{\delta-\gamma}$$

i.e. $$i(\gamma+\delta) = \alpha + i\alpha$$
$$i(\gamma-\delta) = -\alpha + i\alpha$$

Add and subtract. Hence, $\gamma = \alpha$; $i\delta = \alpha$; $\delta = -i\alpha$

and

$$w = \alpha\frac{z+i}{\alpha(z-i)} = \frac{z+i}{z-i}$$

is the required transformation. Since the circle $|z| = 1$ transforms into a circle or a straight line, and three points on the circle transform into three points on the straight line $\therefore$ $Rw > 0$, the circle transforms into that st. line. It remains to show that the outside of the circle transforms into $R(w) > 0$.

Note that $z = i\,\dfrac{w+1}{w-1}$,

and the following pairs of points correspond:

A, A': $z = 1$, $w = i$

B, B': $z = -i$, $w = 0$

C, C': $z = -1$, $w = -i$

D, D': $z = +i$, $w = \infty$

O, O': $z = 0$, $w = -1$,

L, L': $z = \infty$, $w = +1$.

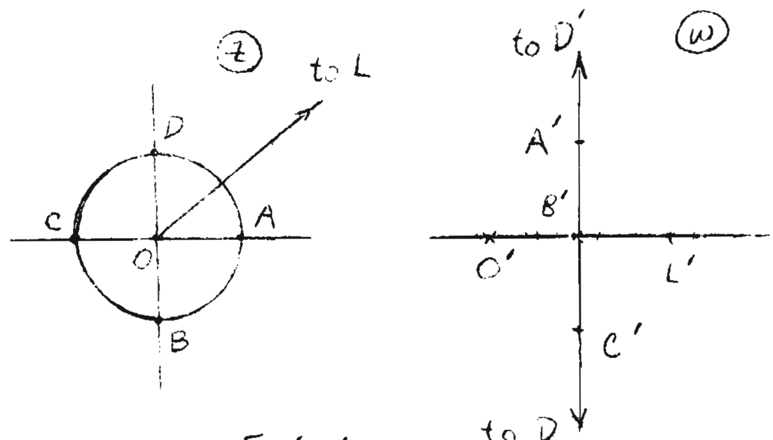


Fig. 1

Since the point at infinity in the z-plane corresponds to the point L' in the right-hand half of the w-plane, the outside of the circle corresponds to that half, i.e., to $Rw > 0$. This we may also see by noting as

No. 16 continued.

we go round the circle in the direction ABCD, the outside of the circle is on our left hand, and as we go along the straight line in the corresponding direction A'B'C'D' the right-hand side of the plane, $Rw > 0$, is also on our left hand.

Second Part: First Method: Analytically

To find analytically the equation of the curve in the w-plane (with $w = u + iv$) that corresponds to a given curve in the z-plane under a bilinear transformation, it is best to express the equation of the curve in the z-plane in terms of z and $\bar{z}$, where

$$x^2 + y^2 = z\bar{z}, \quad x = \tfrac{1}{2}(z + \bar{z}), \quad y = \frac{1}{2i}(z - \bar{z}),$$

and to transform to an equation in w and $\bar{w}$, etc.

When either arg z = const. or $|z|$ = constant, the equations in terms of z and $\bar{z}$ may be immediately written down. Since $z = re^{i\theta}$, if arg z = λ (= constant),

$$\frac{1}{2i}\left(e^{-i\lambda} z - e^{i\lambda}\bar{z}\right) = 0$$

Then $z = i\dfrac{w+1}{w-1}$, so $\bar{z} = -i\dfrac{\bar{w}+1}{\bar{w}-1}$,

and the transformed equation is:

$$\frac{1}{2}\left[e^{-i\lambda}\frac{w+1}{w-1} + e^{i\lambda}\frac{\bar{w}+1}{\bar{w}-1}\right] = 0$$

i.e., $$\frac{e^{-i\lambda}}{2}(w+1)(\bar{w}-1) + \frac{e^{i\lambda}}{2}(\bar{w}+1)(w-1) = 0$$

i.e. $$\frac{e^{-i\lambda}}{2}(w\bar{w} - 1 + \bar{w} - w) + \frac{e^{i\lambda}}{2}(w\bar{w} - 1 - \bar{w} + w) = 0$$

i.e. $$\tfrac{1}{2}(e^{i\lambda} + e^{-i\lambda})(w\bar{w} - 1) + \tfrac{1}{2}(e^{i\lambda} - e^{-i\lambda})(w - \bar{w}) = 0$$

i.e. $$(w\bar{w} - 1)\cos\lambda + i(w - \bar{w})\sin\lambda = 0$$

i.e. $$u^2 + v^2 - 1 - 2v\tan\lambda = 0$$

i.e. $$u^2 + (v - \tan\lambda)^2 = 1 + \tan^2\lambda = \sec^2\lambda \qquad (1)$$

No. 16 continued.

This is the equation of a system of circles, with centers at $(0, \tan\lambda)$ and radii $|\sec\lambda|$. When $v = 0$, $u = \pm 1$, and these circles all pass through O' and L'.

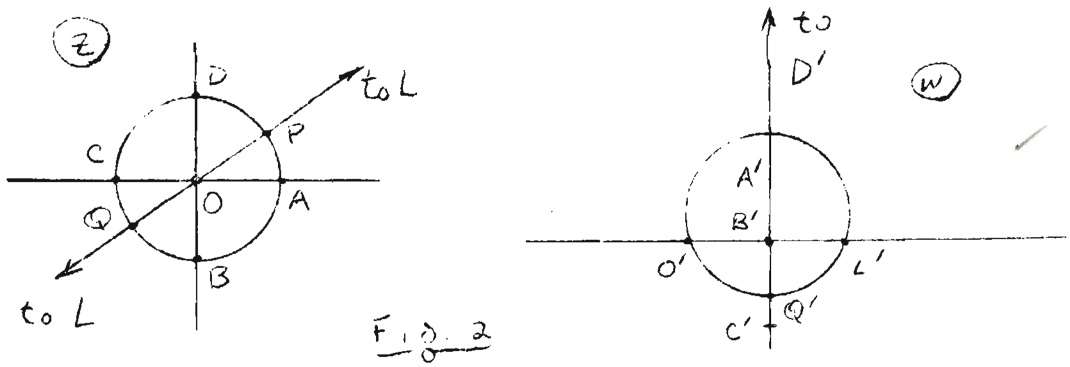


Fig. 2

The points O', L' divide each circle into two arcs, Each straight line arg z = constant goes from O to L once, and corresponds to a curve which goes from O' to L' once. Each of the semi-infinite lines OPL, OQL (for which agr $z = -(\pi - \lambda)$ and arg $z = \lambda$) correspond to one of the arcs of the circle joining O' to L'. The points P and Q at which the line cuts the circle in the z-plane correspond to the points P', Q' at which the circle cuts the imaginary axis ($u = 0$) in the w-plane. In Fig. 2, P lies between D and A, and Q between B and C, so P' lies between B' and A' (i.e., on the upper half of the imaginary axis beyond A') and Q' lies between B' and C'. The positions of P' and Q' are similarly determined in other cases. Then OPL corresponds to the arc O'P'L', and the part PL of that line (arg $z = \lambda$, $|z| \geq 1$) corresponds to that P'L' of that arc.

Consider next $|z| = r$. The curves in the z-plane are circles with center at O (and they cut the lines arg z = constant orthogonally). They are closed curves, and the corresponding curves in the w-plane will be closed curves. Their equations corresponding to

$$|z| = r, \quad \text{i.e.} \quad z\bar{z} = r^2 \qquad \text{will be}$$

$$-i^2 \, \frac{w+1}{w-1} \, \frac{\bar{w}+1}{\bar{w}-1} = r^2$$

i.e. $w\bar{w} + 1 + w + \bar{w} = r^2\left[w\bar{w} + 1 - w - \bar{w}\right]$

i.e. $(r^2-1)(w\bar{w}+1) - (r^2+1)(w+\bar{w}) = 0$; $(r^2-1)(u^2+v^2+1) - 2(r^2+1)u = 0$

when $r = 1$, this gives $u = 0$ (as before).

when $r \neq 1$, let $\sigma = \dfrac{r^2+1}{r^2-1}$. The eq'n is

$u^2 + v^2 + 1 - 2\sigma u = 0$; i.e. $(u-\sigma)^2 + v^2 = \sigma^2 - 1 = \dfrac{4r^2}{(r^2-1)^2}$

No. 16 Continued.

Any one of these circles never meets the imaginary axis ($u = 0$), and each lies entirely either to the right or to the left of that axis. The centers are at $(\sigma, 0)$, and the radii are $R = 2r/(r^2-1)$. For $r > 1$, σ is positive, and the circles lie entirely in the right hand side of the w-plane.

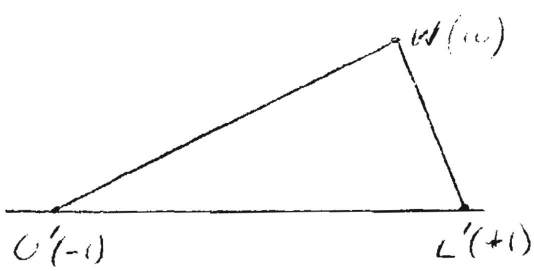
Note that when we change r into $\frac{1}{r}$, and are considering the circle $|z| = \frac{1}{r}$ which is the locus of the inverse points with respect to O of the points on $|z| = r$ (with unit radius of inversion), σ changes sign and R is unaltered. The circles $|z| = \frac{1}{r}$ $(r > 1)$ are the minor images in the imaginary axis of the circles corresponding to $|z| = r$. The circles in the w-plane are coaxal circles, and O', L' (corresponding to $\frac{1}{r} = 0$, $r = \infty$) are the limiting points $(R = 0)$. Since in the z-plane, $|z| =$ const. and arg z $=$ const. give curves which cut orthogonally, and the transformation is orthogonal, the corresponding curves in the w-plane must cut orthogonally - i.e., the system of coaxal circles with O',L' as linking points must cut orthogonally the system of circles through O' and L'. This is a known theorem of geometry.

Second Part: Second Method: Geometrically.

If $\arg z = \lambda$, $\arg i + \arg \frac{w+1}{w-1} = \lambda$: $\arg \frac{w+1}{w-1} = \lambda - \frac{\pi}{2}$

and $\arg \frac{w-1}{w+1} = \frac{\pi}{2} - \lambda$

(If $-\pi < \frac{\pi}{2} - \lambda \leq \pi$, this is the principal value of the argument. Otherwise, we may subtract 2π to bring $\frac{1}{2}\pi - \lambda$ into the required range. The results will be found to be the same in all cases.)



Now, if W is the part that represents w,

$\arg \frac{w-1}{w+1} = \measuredangle$ O'W L'

so $\measuredangle$ O'WL' $= \frac{1}{2}\pi - \lambda$

This is constant, and the locus of W is an arc of a circle through O' and L'. If $-\pi/2 < \lambda < \frac{\pi}{2}$, then $0 < \frac{\pi}{2} - \lambda < \pi$, and the arc is above O'L';

No. 16 Continued.

if $-\pi < \lambda < -\frac{\pi}{2}$, or $\frac{\pi}{2} < \lambda < \pi$, then the arc is below O'L'. [When $\lambda = -\frac{1}{2}\pi$, $\angle O'WL' = \pi$, & W is on the line O'L' between O' and L'; when $\lambda = \pi/2$, $\angle O'WL' = 0$, and W is on the line O'L', to the right of L' or the left of O'.] [Note also that in Fig. 2, $\angle B'P'L' = \frac{1}{2}\angle O'P'L' = \frac{1}{4}\pi - \frac{1}{2}\lambda$, and this checks, since from the analysis, putting u = 0 and taking v positive, B'P' =

$$v = \tan\lambda + \sec\lambda, \text{ from 1. So } \tan \angle B'P'L' = \frac{B'L'}{B'P'} =$$

$$\frac{1}{\tan\lambda + \sec\lambda} = \frac{\cos\lambda}{1+\sin\lambda}$$

$$= \frac{\sin\left(\frac{\pi}{2} - \lambda\right)}{1 + \cos\left(\frac{\pi}{2} - \lambda\right)} = \frac{2\sin\left(\frac{\pi}{4} - \frac{\lambda}{2}\right)\cos\left(\frac{\pi}{4} - \frac{\lambda}{2}\right)}{2\cos^2\left(\frac{\pi}{4} - \frac{\lambda}{2}\right)}$$

$$= \tan\left(\frac{\pi}{4} - \frac{\lambda}{2}\right)$$

Also if $|z| = r$, then $\left|\frac{w+1}{w-1}\right| = r$, i.e. $\frac{O'W}{L'W} = r$, $O'W = r\,L'W$. When this condition is satisfied, with r constant, W is known to lie on a circle, and for various values of r we obtain a system of coaxal circles with O' and L' as limiting points. When $r > 1$, the circle is entirely to the right of the line bisecting O'L' at right angles (the radical one's).

17. If $w = \dfrac{1 + iz}{i + z}$, $z = \dfrac{1 - iw}{w - i}$, $\bar{z} = \dfrac{1 + i\bar{w}}{\bar{w} + i}$

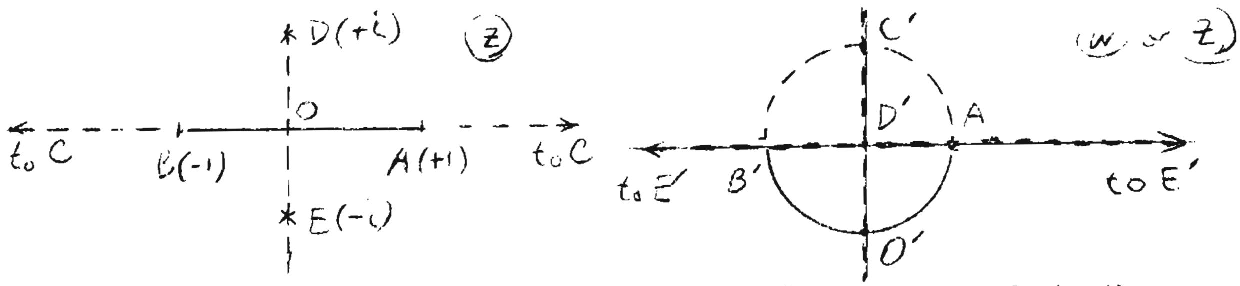
On the real axis $z - \bar{z} = 0$, so the corresponding curve is

$$\frac{1 - iw}{w - i} - \frac{1 + i\bar{w}}{\bar{w} + i} = 0$$

i.e. $\bar{w} + i - iw\bar{w} + w - \left[w - i + iw\bar{w} + \bar{w} \right] = 0$

$$i\,(w\bar{w} - 1) = 0, \qquad w\bar{w} = u^2 + v^2 = 1$$

The real axis on the z-plane is mapped onto the unit circle $|w| = 1$ in the w-plane. Since $z = -1, 0, 1$ correspond to $w = -1, -i, +1$ respectively, the points B', O', A' in the figure corresponds to B, O, A, respectively,



and the segment BOA of the real axis in the z-plane corresponds to the semicircle B'O'A' in the lower half of the w-plane.

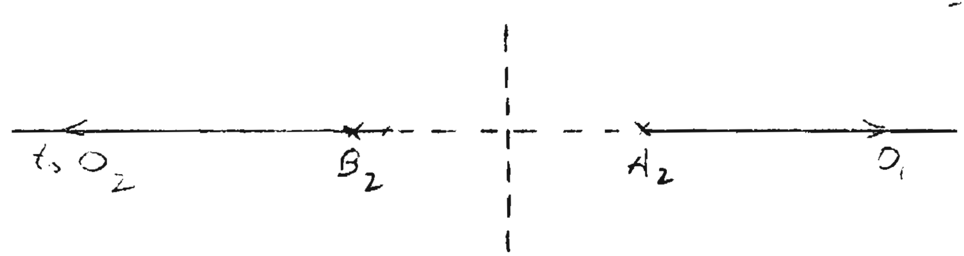
(Since the bilinear transformation is shown to transform a segment of a straight line into a segment of a straight line or an arc of a circle, and BOA corresponds to a curve through B'O'A', this is sufficient to prove that the curve in the w-plane through B'O'A' must be a semi-circle, etc.)

Note also that the points $z = \infty, i, -i$, correspond to $w = i, 0, \infty$ respectively, so the points C', D', E' correspond to C, D, E respectively.

If we now denote w by z_1 and write

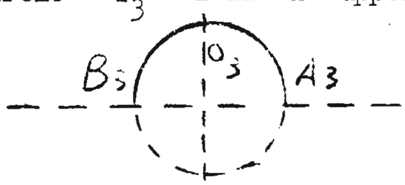
$$z_2 = \frac{1 + iz_1}{i + z_1}$$

and transform B'O'A' to the z_2 plane, B'O'A'($z_1 = -1, -i, +1$) become $z_2 = -1, \infty$, 1 respectively, and BOA in the z-plane or B'O'A' in the z_1-plane corresponds to the real axis outside $-1 < \mathcal{R}z_2 < 1$ in the z_2-plane.



Next write $z_3 = \dfrac{1 + iz_2}{i + z_2}$

Then the points $z_2 = -1, \infty, +1$ correspond to $z_3 = -1, +i, +1$ respectively, and the corresponding figure (to $A\ B_2\ O_2\ A_2$ in the z_2- plane) is the semi-circle $z_3 = 1$ in the upper half of the z_3 - plane.



Finally write $z_4 = \dfrac{1 + iz_3}{i + z_3}$; then

$z_3 = -1, i, 1$ corresponding to $z_4 = -1, 0, +1$ respectively, and the figure in the z_4 - plane corresponds to semicircle $B_3 O_3 A_3$ is the real axis from -1 to + 1, i.e. it is the same as the original figure in the z-plane.

No. 21.

Since $\frac{1}{2\pi i}\int_C \frac{f(z)}{z-\alpha}\,dz = f(\alpha)$; $\frac{1}{2\pi i}\int_C \frac{f(z)}{z-\beta}\,dz = f(\beta)$

and $\frac{1}{(z-\alpha)(z-\beta)} = \frac{1}{\alpha-\beta}\,\frac{1}{z-\alpha} + \frac{1}{\beta-\alpha}\,\frac{1}{z-\beta}$

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-\alpha)(z-\beta)}\,dz = \frac{1}{2\pi i}\left[\frac{1}{\alpha-\beta}\int_C \frac{f(z)}{z-\alpha}\,dz + \frac{1}{\beta-\alpha}\int_C \frac{f(z)}{z-\beta}\,dz\right]$$
$$= f(\alpha)/(\alpha-\beta) + f(\beta)/(\beta-\alpha)$$

Now let f be an integral function, founded in the entire z-plane. Let C be the circle $|z| = R$. On C, $|f(z)| \le M$, where M is a constant independent of R (since f is bounded in the whole plane). Also f is regular in and on C for all R (since f is an integral function).

$|z - \alpha|$ is the distance from the point α to the point z. Where z is on C, this distance is shortest when z is on the radius of the circle through α, and its value is then $R - |\alpha|$. The smallest value of $|z - \alpha|$ is therefore $R - |\alpha|$ and similarly the smallest value of $|z - \beta|$ is $R - |\beta|$. Hence,

$$\left|\frac{f(z)}{(z-\alpha)(z-\beta)}\right| \le \frac{M}{\{R-|\alpha|\}\{R-|\beta|\}}$$

on C, the length of C is $2\pi R$. Hence

$$\left|\frac{f(\alpha)}{\alpha-\beta} + \frac{f(\beta)}{\beta-\alpha}\right| = \left|\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-\alpha)(z-\beta)}\,dz\right| \le \frac{1}{2\pi}\,\frac{M\,[2\pi R]}{\{R-|\alpha|\}\{R-|\beta|\}}$$

No. 21 Continued.

Let $R \to \infty$. Then $\dfrac{R^2}{\{R-|\alpha|\}\{R-|\beta|\}} \to 1$, and the right-hand of the above inequality $\to 0$. The left side is independent of R. Hence

$$\frac{f(\alpha)}{\alpha-\beta} + \frac{f(\beta)}{\beta-\alpha} = 0 \; ; \qquad f(\alpha) = f(\beta)$$

This is true for any two points α and β, so f(z) has a constant value.

22. $\cosh\left(z + \frac{1}{z}\right) = \cosh w$, where $w = z + \frac{1}{z}$. Now $\cosh w$ is an integral function of w, regular in any finite region of the w- plane; w is a regular function of z everywhere except at the origin. So $\cosh(z + 1/z)$ is a regular function of z everywhere except at the origin, and by Laurent's theorem

$$\cosh(z + 1/z) = \sum_{-\infty}^{\infty} a_n z^n$$

everywhere in the annulus $R_2 < |z| < R_1$, no matter how small R_2 may be, or how large R_1 may be. Also by Laurent's theorem (Second remark)

$$a_n = \frac{1}{2\pi i} \int_{\gamma} \frac{\cosh(z + 1/z)}{z^{n+1}}\, dz$$

where γ is any circle $|z| = r$ (since we may always choose R_1, R_2 so that $R_2 < r < R_1$). Let P be the circle $|z| = 1$. On γ we may write,

$$z = e^{i\theta},\ 1/z = e^{-i\theta},\ z + 1/z = 2\cos\theta$$

$$\frac{1}{z^{n+1}} = e^{-(n+1)i\theta},\quad dz = ie^{i\theta}d\theta,\quad \frac{dz}{z^{n+1}} = ie^{-ni\theta}d\theta$$
$$= i\left[\cos n\theta - i\sin n\theta\right] d\theta$$

Hence

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cosh(2\cos\theta)\cos n\theta\, d\theta - \frac{i}{2\pi}\int_{-\pi}^{\pi}\cosh(2\cos\theta)\sin n\theta\, d\theta.$$

In the second integral, the integrand is an odd function of θ, so the second integral vanishes. In the first integral the integral is an even function of θ, so

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cosh(2\cos\theta)\cos n\theta\, d\theta = \frac{1}{\pi}\int_{0}^{\pi}\cosh(2\cosh\theta)\cos n\theta\, d\theta.$$

Since $\cos(-n\theta) = \cos n\theta$, $a_{-n} = a_n$, and

$$\cosh(z + 1/z) = a_o + \sum_{n=1}^{\infty} a_n\left(z^n + \frac{1}{zn}\right)$$

15. Let $\Omega = f'(z)$ and $\zeta = \log \Omega = \log|\Omega| + i \arg \Omega$

If we make any cut in the Ω - plane from the origin to infinity, ζ is a regular function of Ω in the cut plane. If we exclude the origin in the Ω - plane, i.e. of $f'(z) \neq 0$, any point considered is in a domain in which ζ is a regular function of Ω, and Ω is a regular function of z, so ζ is a regular function of z. Thus $\log|\Omega| = \log|f'(z)|$ is the real part of a regular function of z.

Hence $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \log|f'(z)| = 0.$

Let $|\Omega| = |f'(z)| = X(x) \cdot Y(y)$,
where X is a function of x alone, and Y a function of y alone. Then

$$\log|\Omega| = \log X + \log Y$$

$$0 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \log|\Omega| = \frac{d^2}{dx^2} \log x + \frac{d^2}{dy^2} \log Y.$$

The first term $\left(\frac{d^2}{dx^2} \log X\right)$ is a function of x alone, and the second term a function of y alone. Since their sum is identically zero, each must be equal to a constant, and the sum of the two constants must be zero. The expressions are real, so the constants are real. Denote the constants by $\pm$ 2a, where a is a real constant. Then

$$\frac{d^2}{dx^2} \log X = 2a, \qquad \frac{d^2}{dy} \log y = -2a.$$

Hence $\log X = ax^2 + b_1 x + c_1$, $\log Y = -ay^2 - b_2 y + c_2$

where b_1, c_1, b_2, c_2 are also real constants. Hence, if

$$\log \Omega = \zeta = \xi + i\eta$$

$$\xi = \log|\Omega| = \log X + \log Y = a(x^2 - y^2) + b_1 x - b_2 y + c_3$$

where c_3 ($= c_1 + c_2$) is a real constant.

Then $\frac{d\zeta}{dz} = \frac{\partial \xi}{\partial x} + i \frac{\partial \eta}{\partial x} = \frac{\partial \xi}{\partial x} - i \frac{\partial \xi}{\partial y} = 2ax + b_1 + i(2ay + b_2)$

$$= 2az + \beta$$

where β (= $b_1 + ib_2$) is a complex constant.

Hence $\zeta = \log \Omega = \log[f'(z)] = az^2 + \beta z + \gamma$

where γ is a complex constant (whose real part is equal to c_3). Thus

$$f'(z) = \exp(az^2 + \beta z + \gamma)$$

No. 23 $\dfrac{z^2-1}{(z+2)(z+3)} = \dfrac{z^2-1}{z^2+5z+6} = 1 - \dfrac{5z+7}{(z+2)(z+3)} = 1 + \dfrac{3}{z+2} - \dfrac{8}{z+3}$

Now $\dfrac{1}{1+\rho} = (1+\rho)^{-1} = 1-\rho+\rho^2-\rho^3+\dots+(-1)^n\rho^n+\dots = \sum_{n=0}^{\infty}(-1)^n\rho^n$ for $|\rho|<1$

Hence in $|z|<2$

$$\frac{1}{z+2} = \frac{1}{2(1+\frac{z}{2})} = \sum_{n=0}^{\infty}(-1)^n\frac{z^n}{2^{n+1}}$$

In $|z|>2$

$$\frac{1}{z+2} = \frac{1}{z(1+\frac{2}{z})} = \sum_{n=0}^{\infty}(-1)^n\frac{2^n}{z^{n+1}}$$

In $|z|<3$

$$\frac{1}{z+3} = \sum_{n=0}^{\infty}(-1)^n\frac{z^n}{3^{n+1}}$$

In $|z|>3$

$$\frac{1}{z+3} = \sum_{n=0}^{\infty}(-1)^n\left(z^{-n-1}\right)3^n$$

Therefore:

(1) In $|z|<2$,

$$\frac{z^2-1}{(z+2)(z+3)} = 1+3\sum_{n=0}^{\infty}(-1)^n\frac{z^n}{2^{n+1}} - 8\sum_{n=0}^{\infty}(-1)^n\frac{z^n}{3^{n+1}}$$

$$= -\frac{1}{6} + \sum_{n=1}^{\infty}(-1)^n\left[\frac{3}{2^{n+1}} - \frac{8}{3^{n+1}}\right]z^n$$

$$= -\frac{1}{6} + \sum_{n=1}^{\infty}(-1)^n\left[\frac{3^{n+2}-2^{n+4}}{6^{n+1}}\right]z^n$$

(11) In $2<|z|<3$,

$$\frac{z^2-1}{(z+2)(z+3)} = 1+3\sum_{n=0}^{\infty}(-1)^n\frac{2^n}{z^{n+1}} - 8\sum_{n=0}^{\infty}(-1)^n\frac{z^n}{3^{n+1}}$$

$$= -5/3 - 8\sum_{n=1}^{\infty}(-1)^n\frac{z^n}{3^{n+1}} + 3\sum_{n=0}^{\infty}(-1)^n\frac{2^n}{z^{n+1}}$$

$$= -5/3 + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{8}{3^{n+1}}z^n + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{3\cdot 2^{n-1}}{z^n}$$

No. 23 continued.

(111) In $|z|>3$,

$$\frac{z^2-1}{(z+2)(z+3)} = 1+3\sum_{n=0}^{\infty}(-1)^n\frac{2^n}{z^{n+1}} - 8\sum_{n=0}^{\infty}(-1)^n\frac{3^n}{z^{n+1}}$$

$$= 1+\sum_{n=0}^{\infty}(-1)^n\left[3\cdot 2^n-8\cdot 3^n\right]\Big/z^{n+1}$$

$$= 1+\sum_{n=0}^{\infty}(-1)^{n+1}\left[8\cdot 3^n-3\cdot 2^n\right]\Big/z^{n+1}$$

$$= 1+\sum_{n=0}^{\infty}\frac{(-1)^n\left[8\cdot 3^{n-1}-3\cdot 2^{n-1}\right]}{z^n}$$

Solution No. 25.

The points $z=0$, $z=\alpha$ are within C, and since $|R^2/\bar\alpha| = |R^2/\alpha| > R$, the point $z = R^2/\bar\alpha$ is outside C. Hence

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{z}\,dz = f(0);\ \frac{1}{2\pi i}\int_C \frac{f(z)}{z-\alpha}\,dz = f(\alpha);\ \frac{1}{2\pi i}\int_C \frac{f(z)}{z-R^2/\bar\alpha}\,dz =$$

Then

$$\frac{R^2\alpha - z^2\bar\alpha}{z(z-\alpha)(R^2-z\bar\alpha)} + \frac{1}{z} = \frac{R^2\alpha - z^2\bar\alpha + (z-\alpha)(R^2-z\bar\alpha)}{z(z-\alpha)(R^2-z\bar\alpha)}$$

$$= \frac{R^2 - 2z\bar\alpha + \alpha\bar\alpha}{(z-\alpha)(R^2-z\bar\alpha)} = \frac{1}{z-\alpha} - \frac{\bar\alpha}{R^2-z\bar\alpha}$$

$$= \frac{1}{z-\alpha} - \frac{1}{z-R^2/\bar\alpha}$$

and

$$\frac{1}{2\pi i}\int_C \left\{\frac{R^2\alpha - z^2\bar\alpha}{z(z-\alpha)(R^2-z\bar\alpha)} + \frac{1}{z}\right\} f(z)\,dz = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-\alpha}\,dz + \frac{1}{2\pi i}\int_C \frac{f(z)}{z-R^2/\bar\alpha}\,d$$

$$= f(\alpha) \ \ldots\ldots\ldots\ldots\ (1)$$

[Note also that

$$\frac{1}{2\pi i}\int_C \frac{R^2\alpha - z^2\bar\alpha}{z(z-\alpha)(R^2-z\bar\alpha)} f(z)\,dz = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-\alpha}\,dz + \frac{1}{2\pi i}\int_C \frac{f(z)}{z-R^2/\bar\alpha}\,d$$

$$-\frac{1}{2\pi i}\int_C \frac{f(z)}{z}\,dz$$

$$= f(\alpha) - f(0) \ \ldots\ldots\ldots\ldots\ (2)$$

Let $\alpha = re^{i\theta}$ on C, let $z = Re^{i\phi}$. Then on C, $\frac{dz}{z} = i\,d\phi$, an

$$\frac{R^2\alpha - z^2\bar\alpha}{(z-\alpha)(R^2-z\bar\alpha)} = \frac{R^2re^{i\theta} - R^2re^{i(2\phi-\theta)}}{(Re^{i\phi}-re^{i\theta})(R^2-Rre^{i(\phi+\theta)})}$$

Solution No. 25 continued.

$$= \frac{R^2 r e^{i\phi}\{e^{i(\theta-\phi)} - e^{-i(\theta-\phi)}\}}{Re^{i\phi}\{Re^{i\phi}-re^{i\theta}\}\{Re^{-i\phi}-re^{-i\theta}\}}$$

$$= \frac{2i\,Rr\sin(\theta-\phi)}{R^2-2Rr\cos(\theta-\phi)+r^2}$$

[Hence, from (2)

$$\frac{i}{\pi}\int_{-\pi}^{\pi}\frac{Rr\sin(\theta-\phi)}{R^2-2Rr\cos(\theta-\phi)+r^2}f(Re^{i\phi})\,d\phi = f(re^{i\theta}) - f(0)\ldots.(3)$$

and from (1)

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left\{\frac{2iRr\sin(\theta-\phi)}{R^2-2Rr\cos(\theta-\phi)+r^2}+1\right\}f(Re^{i\phi})\,d\phi = f(re^{i\theta})\ldots(4)$$

$$\text{i.e. } \frac{i}{\pi}\int_{-\pi}^{\pi}\left\{\frac{Rr\sin(\theta-\phi)}{R^2-2Rr\cos(\theta-\phi)+r^2}-\frac{i}{2}\right\}f(Re^{i\phi})\,d\phi = f(re^{i\theta})\ldots..(5)$$

If, as in Ex. 19, we put

$$f(Re^{i\phi}) = u(R,\phi) + i\,v(R,\phi)$$
$$f(re^{i\theta}) = u(r,\theta) + i\,v(r,\theta)$$

(u, v real), and now take the imaginary parts of eqns. 3 and 5, we find that

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{Rr\sin(\theta-\phi)}{R^2-2Rr\cos(\theta-\phi)+r^2}\,u(R,\phi)\,d\phi = v(r,\theta) - v(0)\ldots..(6)$$

and

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\left\{\frac{Rr\sin(\theta-\phi)}{R^2-2Rr\cos(\theta-\phi)+r^2}\,u(R,\phi)+\frac{1}{2}v(R,\phi)\right\}d\phi = v(r,\theta)\ldots.(7)$$

If u is the real part, and v the imaginary part, of a regular function of z, we saw in Ex. 19 that we could express u at any point inside the circle in terms of its values on the circle. We now see that we can express, not $v(r,\theta)$, but $v(r,\theta) - r(0)$, in terms of the values of u on the circle. In fact, if u is given on the circle, v is indeterminate to the extent of an additive constant. (This remains true if u is given everywhere - See Exs. 6 and 8).

Solution No. 25 continued.

The result of Ex. 19, and the formula 7 above, may be combined into a simple formula.

$$f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left\{\frac{[R^2 - r^2 + 2iRr\sin(\theta-\phi)]\,u(R,\phi)}{R^2 - 2Rr\cos(\theta-\phi) + r^2} + iv(R,\phi)\right\}d\phi$$

If $z = Re^{i\phi}$, $\alpha = re^{i\theta}$

$$\frac{z+\alpha}{z-\alpha} = \frac{(z+\alpha)(\bar z - \bar\alpha)}{(z-\alpha)(\bar z - \bar\alpha)} = \frac{z\bar z - \alpha\bar\alpha + \bar z\alpha - z\bar\alpha}{z\bar z + \alpha\bar\alpha - \bar z\alpha - z\bar\alpha}$$

$$= \frac{R^2 - r^2 + Rr\{e^{i(\theta-\phi)} - e^{-i(\theta-\phi)}\}}{R^2 + r^2 - Rr\{e^{i(\theta-\phi)} + e^{-i(\theta-\phi)}\}} = \frac{R^2 - r^2 + 2iRr\sin(\theta-\phi)}{R^2 - 2Rr\cos(\theta-\phi) + r^2}$$

Hence

$$f(\alpha) = u(r,\theta) + iv(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left\{\frac{z+\alpha}{z-\alpha}\,u(R,\phi) + iv(R,\phi)\right\}d\phi$$

Where

$$z = Re^{i\phi}, \quad \alpha = re^{i\theta}$$

Solution No. 26.

If $\zeta = w - \frac{1}{w}$, $\exp \frac{1}{2}z\zeta$ is an integral function of ζ, and ζ is a regular function of w everywhere except at the origin. So $\exp \frac{1}{2}z(w - \frac{1}{w})$ is a regular function of w everywhere except at the origin, and has a Laurent expansion. The coefficients are functions of z, so we may write

$$\exp \tfrac{1}{2}z\ (w - \tfrac{1}{w}) = \sum_{n=-\infty}^{\infty} w^n\ Jn\ (z) \qquad (1)$$

everywhere in any annulus $R_2 < |z| < R_1$, no matter how small R_2 maybe, or how large R_1 may be.

(1) In (1), replace w by $-\frac{1}{t}$. Then

$$\exp \tfrac{1}{2}z\ (t - \tfrac{1}{t}) = \sum_{n=-\infty}^{\infty} (-\tfrac{1}{t})^n\ Jn\ (z) = \sum_{n=-\infty}^{\infty} (-1)^n\ t^{-n}\ Jn(z)$$

In the summation, let $n = -m$, $(-1)^{-m} = (-1)^m$, so

$$\exp \tfrac{1}{2}z\ (t - \tfrac{1}{t}) = \sum_{n=-\infty}^{\infty} (-1)^m\ t^m\ J_{-m}\ (z).$$

But by direct substitution of t for w in (1),

$$\exp \tfrac{1}{2}z(t - \tfrac{1}{t}) = \sum_{n=-\infty}^{\infty} t^m\ J_m\ (z).$$

But a Laurent expansion of a function is unique. Hence $(-1)^m\ J_{-m}\ (z) = J_m\ (z)$ for all m; i.e.

$$J_{-n}\ (z) = (-)^n\ Jn\ (z)$$

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for all (integral) n,

(ii) By Laurent's theorem, the coefficient Jn (z) is given by

$$Jn\ (z) = \frac{1}{2\pi i}\int_{\Gamma} \left\{ \frac{\exp \frac{1}{2}\ z\ (w - 1/w)}{w^{n+1}} \right\} dw,$$

and for Γ we may take the circle $|w| = 1$, on which $w = e^{i\theta}$, $1/w = e^{-i\theta}$,
$w - 1/w = 2i \sin\theta$, $\frac{dw}{w^{n+1}} = \frac{ie^{i\theta}d\theta}{e^{(n+1)i\theta}} = i\,e^{-ni\theta}d\theta.$ so

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Solution No. 26 continued.

$$Jn\ (z) = \frac{1}{2\pi i} \int_{-\pi}^{\pi} \exp\ (iz \sin\theta)\ i\ e^{-ni\theta}\ d\theta$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp\left\{ -i(n\theta - z \sin\theta) \right\} d\theta$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos\ (n\theta - z \sin\theta)\ d\theta - \frac{i}{2\pi} \int_{-\pi}^{\pi} \sin(n\theta - z \sin\theta)\ d\theta .$$

The first intergrand is an even function of θ, and the second intergrand an odd function of . Hence the second integral vanishes, and the first is twice the integral from 0 to π.

$$Jn\ (z) = \frac{1}{\pi} \int_{0}^{\pi} \cos\ (n\theta - z \sin\theta)\ d\theta .$$

(iii) exp $\frac{1}{2}z(w - 1/w)$ = $e^{\frac{1}{2}zw}$ $e^{-z/2w}$, and $e^{\frac{1}{2}zw}$ can be expanded in an absolutely convergent series of ascending powers of w, and $e^{-z/2w}$ can be expanded in an absolutely convergent series of descending powers of w for all values of w except zero. The series may be multiplied together, and the product arranged according to powers of w.

$$\exp\left\{\tfrac{1}{2}z(w - 1/w)\right\} = \sum_{r=0}^{\infty} \frac{(\frac{1}{2}z)^r w^r}{r!} \cdot \sum_{m=0}^{\infty} \frac{(-\frac{1}{2}z)^m w^{-m}}{m!}$$

When the product is arranged in powers of w, since a Laurent series is unique, the result must be

$$\sum_{n=-\infty}^{\infty} w^n J_n(z).$$

To obtain a term in w^n in the product, where n is a positive integer or zero, we must multiply each term in w^{-m} in the second series by the term in w^{n+m} in the first series, and then sum for all m. The term from w^{-m} in the second series is

$$\frac{(\frac{1}{2}z)^{n+m} w^{n+m}}{(n+m)!} \cdot \frac{(-\frac{z}{2})^m w^{-m}}{m!} = \frac{(-1)^m (\frac{1}{2}z)^{n+2m}}{m!(n+m)!} w^n$$

To find the coefficient of w^n in the product, we must sum for all values of m. Hence

cont. on next page

Solution No. 26 continued.

$$J_n(z) = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{1}{2}z\right)^{n+2m}}{m!\,(n+m)!}$$

$$= \frac{z^n}{2^n\, n!}\left\{1 - \frac{z^2}{2^2\cdot 1\cdot(n+1)} + \frac{z^4}{2^4\cdot 1\cdot 2\cdot(n+1)(n+2)} + \dots\right\}$$

when n is a positive interger or zero.

In particular

$$Jo(z) = 1 - \frac{z^2}{2^2} + \frac{z^4}{2^2\cdot 4^2\cdot \cancel{6^2}} - \frac{z^6}{2^2\cdot 4^2\cdot 6^2} \dots\dots$$

Jn(z) is Bessel's function of order n; it may be shown to be a solution for y of the differential equation

$$z^2 \frac{d^2y}{dz^2} + z\frac{dy}{dz} + (z^2 - n^2)y = 0.$$

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Solution No. 27.

Let $\zeta = \frac{1}{z}$. If $\zeta = \xi + i\eta$, $0 = e^{\zeta} - c = e^{\xi+i\eta} - Ke^{i\gamma}$; $(K \neq 0)$

$0 = e^{\xi+i\eta} - Ke^{i(\gamma-2\pi n)}$; $n = 0, \pm 1, \ldots$

When $e^{\xi} = K$, $\eta = \gamma - 2\pi n$

Let $k = \log K$. Then $\xi = k$, $\eta = \gamma - 2\pi n$

When $z = \frac{1}{\zeta}$ the line $\xi = k$ maps into a circle, symmetrical about the real axis. The point in the z-plane corresponding to a point P^1 in the ζ - plane is found by inverting with respect to the origin, and taking the mirror image in the real axis. The inverse of $\zeta = \infty$ is $z = 0$, and of $\zeta = k$ is $z = 1/k$. Hence the map of the line $\xi = k$ is the circle with center at $z = \frac{1}{2k}$ of radius $= 1/2k$, i.e.,

$$\left| z - \frac{1}{2k} \right| = \frac{1}{2k}$$

This we check as follows.

If $z = \frac{1}{\zeta} = \frac{1}{\xi + i\eta} = \frac{1}{k + i\eta}$, when $\xi = \frac{1}{2}$

$$z - \frac{1}{2k} = \frac{1}{k+i\eta} - \frac{1}{2k} = \frac{1}{2k}\,\frac{k - i\eta}{k + i\eta}, \text{ and}$$

$$\left| z - \frac{1}{2k} \right| = \frac{1}{2k}$$

When $\eta = \gamma - 2\pi n$, $z - \frac{1}{2k} = \frac{1}{2k}\,\frac{k + i(2\pi n - \gamma)}{k - i(2\pi n - \gamma)}$

and $\arg(z - \frac{1}{2k}) = 2 \arctan \frac{2\pi n - \gamma}{k}$

This gives the position of the zeros of $e^{1/z} - c$ on the circle $\left| z - \frac{1}{2k} \right| = \frac{1}{2k}$.

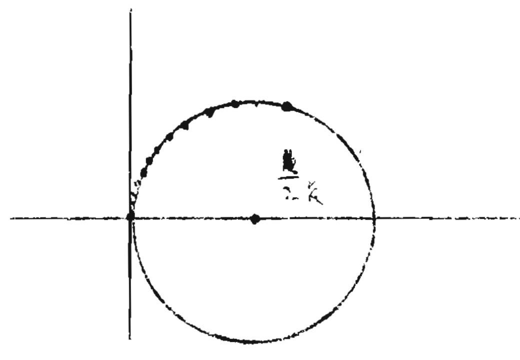
When n changes sign, $\arg(z - \frac{1}{2k})$ changes sign: the two zeros corresponding to $\pm n (n \neq 0)$ are mirror images of one another in the real axis.

When n is large and positive, write

$$(1) \ldots\ldots \quad \arg\left(z - \frac{1}{2k}\right) = 2\left(\frac{\pi}{2} - \arctan \frac{k}{2\pi n - \gamma}\right) = \pi - 2 \arctan \frac{k}{2\pi n - \gamma}$$

Solution No. 27 continued.

The point $\left|z - \frac{1}{2k}\right| = \frac{1}{2k}$; arg $\left(z - \frac{1}{2k}\right) = \pi$ is $z = 0$. From (1) we see that there is an infinite sequence of zeros on the circle $\left|z - \frac{1}{2k}\right| = \frac{1}{2k}$ with the origin as a limit point, and there are an infinite number of such zeros inside any circle $|z| = r$, however small r may be.



when n is large and negative, write $\arg\left(z - \frac{1}{2k}\right) = -\pi + 2 \arctan \frac{k}{2\pi|n| + k}$, ect

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Solution No. 28.

If we write
$$f_1(z) = \frac{\cos \pi z}{(z-1)^2 \sin \pi z}$$

$$f_2(z) = \frac{z}{\sin z}$$

$$f_3(z) = \frac{z^4}{(c^2+z^2)^4}$$

$$f_4(z) = \frac{1}{z(e^z-1)}$$

Then in each case the numerator and denominator are integral functions, so the only singularities of the functions may be at the zeros of the denominators. The zeros are isolated, and the singularities are poles.

(i) The zeros of $\sin \pi z$ are at $z = n$ $(n = 0, \pm 1, \pm 2, \ldots\ldots)$ and are of order one. $(z-1)^2$ has a zero of order 2 at $z = 1$. The zeros of the denominator are not zeros of the numerator. Hence $f_1(z)$ has a pole of order 3 at $z = 1$ (a triple pole), and simple poles at $z = n(n = 0, -1, \pm 2, \pm 3, \pm 4, \ldots)$.

(ii) $\sin z$ has simple zeros at $z = n\pi$ $(n = 0, \pm 1, \pm 2, \pm 3, \ldots)$. $z = 0$ is also a simple zero of the numerator. Hence $f_2(z)$ has simple poles at $z = n\pi(n = \pm 1, \pm 2, \pm 3, \ldots\ldots)$.

(iii) $(c^2 + z^2)^4$ has zeros of the fourth order at the zeros of $c^2 + z^2$, and these are not zeros of the numerator. So $f_3(z)$ has poles of the fourth order at these points. There are two distinct zeros of $c^2 + z^2$. They occur when
$$z^2 = -c^2 = c^2\, e^{(2n+1)\pi i} \qquad (n = 0, \pm 1, \pm 2, \ldots)$$
$$z^2 = c\, e^{(n+\frac{1}{2})\pi i}$$

The values n 0, -1, corresponding to $z = c\, e^{\frac{1}{2}\pi i}$, $z = c\, e^{-\frac{1}{2}\pi i}$, i.e., to $z = \pm ic$ give the two distinct zeros, and f_3 has poles of order 4 at these points.

(iv) $e^z - 1$ has zeros when $e^z = 1 = e^{2n\pi i}$ $(n = 0, \pm 1, \pm 2, \ldots\ldots)$ i.e., at $z = 2n\pi i$. Hence the denominator has a double zero at $z = 0$, and simple zeros at $z = 2n\pi i$ $(n = \pm 1, \pm 2, \ldots)$. $f_4(z)$ has a double pole at $z = 0$ and simple poles at $z = 2n\pi i(n = \pm 1, \pm 2, \ldots\ldots)$.

Solution No. 29.

(i) $f_1(z)$ has simple poles at $z = n$ ($n = 0, -1, \pm 2, \pm 3, \pm 4, \ldots$) and a triple pole at $z = 1$. To find the residues at the simple poles, let $z - n = \rho$ (where n has the values above).

Then $f_1(z) = \dfrac{\cot(\pi\rho + n\pi)}{(\rho + n - 1)^2} = \dfrac{\cot \pi\rho}{(n-1+\rho)^2} = \dfrac{\cos \pi\rho}{(n-1+\rho)^2 \sin \pi\rho}$

$$= \frac{\left(1 - \frac{1}{2}\pi^2\rho^2 + \ldots\right)}{(n-1)^2\left[1 + \frac{2\rho}{n-1} + \frac{1}{(n-1)^2}\right]\left[\pi\rho - \frac{\pi^3\rho^3}{6} + \ldots\right]} \qquad (n \neq +1)$$

for sufficiently small values of $|\rho|$,

$$f_1(z) = \frac{1}{\pi(n-1)^2}\,\frac{1}{\rho} + \ldots\ldots\ldots = \frac{1}{\pi(n-1)^2}\,\frac{1}{z-n} + \ldots$$

and the residue at $z = n$ ($n \neq +1$) is $\boxed{\dfrac{1}{\pi(n-1)^2}}$.

When $n = 1$, $z - 1 = \rho$,

$$f_1(z) = \frac{\cos \pi\rho}{\rho^2 \sin \pi\rho} = \frac{1}{\rho^2}\,\frac{1 - \frac{1}{2}\pi^2\rho^2 + \ldots}{\pi\rho\left[1 - \frac{1}{6}\pi^2\rho^2 + \ldots\right]} = \frac{1}{\pi\rho^3}\left[1 - \frac{1}{2}\pi^2\rho^2 + \ldots\right] \Big/ \left[1 - \frac{1}{6}\pi^2\rho^2 + \ldots\right]$$

$$= \frac{1}{\pi\rho^3}\left[1 - \frac{1}{2}\pi^2\rho^2 + \ldots\right]\left[1 + \frac{\pi^2\rho^2}{6} + \ldots\right]$$ for sufficiently small $|\rho|$

$$= \frac{1}{\pi\rho^3}\left[1 - \frac{\pi^2\rho^2}{3} + \ldots\right] = \frac{1}{\pi\rho^3} - \frac{\pi}{3}\,\frac{1}{\rho} + \ldots$$

and the residue at $z = 1$ is $\boxed{-\dfrac{\pi}{3}}$

(ii) $f_2(z)$ has simple poles at $z = n\pi$ ($n = \pm 1, \pm 2, \pm 3, \ldots$).

Let $\rho = z - n\pi$; then

$$f_2(z) = \frac{z}{\sin z} = \frac{\rho + n\pi}{\sin(\rho + n\pi)} = (-1)^n \frac{n\pi + \rho}{\sin \rho} = (-1)^n \frac{n\pi + \rho}{\rho\left(1 - \frac{\rho^2}{6} + \ldots\right)}$$

$$= (-1)^n \frac{n\pi + \rho}{\rho}\left[1 - \frac{\rho^2}{6} + \ldots\right]^{-1} = \frac{(-1)^n n\pi}{\rho} + \ldots$$ (for sufficient small $|\rho|$)

and the residue at $z = n$ is $\boxed{(-1)^n\ n\pi}$

Cp Ex. 9b (i).

Solution No. 29 continued.

(iii) $f_3(z)$ has poles of order 4 at $z = c_n = c\,e^{(n+\frac{1}{2})\pi i}$, $n = 0$ or -1.
then $c_n^2 = -c^2$, $c_n^4 = c^4$.

Let $z - c_n = \zeta$. Then

$$f_3(z) = \frac{(c_n+\zeta)^4}{[c^2+(c_n+\zeta)^2]^4} = \frac{c_n^4\left(1+\frac{\zeta}{c_n}\right)^4}{(2c_n\zeta+\zeta^2)^4} = \frac{1}{16\zeta^4}\left(1+\frac{\zeta}{c_n}\right)^4\left(1+\frac{\zeta}{2c_n}\right)^{-4}$$

Hence the residue at $z = c_n$ is $\frac{1}{16}$ (coefficient of ζ^3 in the expansion of $\left[1+\frac{\zeta}{c_n}\right]^4\left[1+\frac{\zeta}{2c_n}\right]^{-4}$). Let $\zeta/c_n = \alpha$. Then

$$(1+\alpha)^4\left(1+\frac{\alpha}{2}\right)^{-4} = (1+4\alpha+6\alpha^2+4\alpha^3+\alpha^4)\left(1-4\frac{\alpha}{2}+10\frac{\alpha^2}{2^2}-20\frac{\alpha^3}{2^3}+\dots\right)$$

$$= (1+4\alpha+6\alpha^2+4\alpha^3+\alpha^4)\left(1-2\alpha+\frac{5\alpha^2}{2}-\frac{5\alpha^3}{2}+\dots\right)$$

, for sufficient small $|\alpha|$.

The coefficient of α^3 is $-\frac{5}{2}+10-12+4 = -\frac{1}{2}$

The residue at $z = c_n = \frac{1}{16}\left(-\frac{1}{2c_n^3}\right) = -\frac{1}{32c^3}e^{-(3n+\frac{3}{2})\pi i} = \frac{1}{32c^3}e^{-(3n+\frac{1}{2})\pi i}$

Take $n = 0$ and -1 in turn.

$n = 0$: The residue at $z = ic$ is $\frac{1}{32c^3}e^{-\frac{1}{2}\pi i} = \boxed{-\frac{i}{32c^3}}$

$n = -1$: The residue at $z = -ic$ is $\frac{1}{32c^3}e^{\frac{5\pi i}{2}} = \boxed{+\frac{i}{32c^3}}$

(iv) f_4 has simple poles at $z = 2n\pi i$ ($n = \pm 1, \pm 2, \dots$) and a double pole at $z = 0$. With the above values of n, let $\zeta = z - 2n\pi i$. Then

$$f_4 = \frac{1}{(\zeta+2n\pi i)(e^{\zeta+2n\pi i}-1)} = \frac{1}{2n\pi i}\left[1+\frac{\zeta}{2n\pi i}\right]^{-1}\left(\frac{1}{e^{\zeta}-1}\right)$$

$$= \frac{1}{2n\pi i}\left[1+\frac{\zeta}{2n\pi i}\right]^{-1}\frac{1}{\zeta\left[1+\frac{\zeta}{2}+\dots\right]} = \frac{1}{2n\pi i}\,\frac{1}{\zeta}\left[1+\frac{\zeta}{2n\pi i}\right]^{-1}\left[1+\frac{\zeta}{2}+\dots\right]^{-1}$$

$$= \frac{1}{2n\pi i}\cdot\frac{1}{\zeta}+\dots$$

for sufficient small $|\zeta|$;

and the residue at $z = 2n\pi i$ is $= \boxed{-\frac{i}{2n\pi}}$

Also
$$f_4 = \frac{1}{z(e^z-1)} = \frac{1}{z^2\left(1+\frac{z}{2}+\dots\right)} = \frac{1}{z^2}\left[1+\frac{z}{2}+\dots\right]^{-1}$$

$$= \frac{1}{z^2}\left[1-\frac{z}{2}+\dots\right]$$ for sufficiently small $|z|$

$$= \frac{1}{z^2}-\frac{1}{2z}+\dots$$

and the residue at $z = 0$ is $\boxed{-\frac{1}{2}}$

Solution No. 30.

(i) $f_1\left(\frac{1}{z}\right) = \dfrac{\cos(\pi/z)}{(1/z-1)^2 \sin(\pi/z)}$ and has poles at $z = \pm \frac{1}{n}$,

Where n is a positive integer. Hence the origin is a limit point of poles and is a non-isolated essential singularity. Hence the point at infinity is a non-isolated essential singularity of $f_1(z)$.

(ii) $f_2\left(\frac{1}{z}\right) = \dfrac{1/z}{\sin(1/z)}$ has poles at $z = \pm \frac{1}{2n\pi}$, where n is a positive integer, and the point at infinity is a non-isolated essential singularity of $f_2(z)$.

(iii) The point at infinity is easily seen (by considering the nature of the origin for $f_3\left(\frac{1}{z}\right)$ to be a point at which $f_3(z)$ has no singularity.

(iv) $e^{1/z} - 1$ has zeros at $z = \pm \dfrac{1}{2n\pi i}$ (n a positive integer) and $f_4\left(\frac{1}{z}\right)$ has poles at these points, for which the origin is a limit point. So $f_4(z)$ has a non-isolated essential singularity at infinity.

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Solution No. 31.

(i) cot z = $\frac{\cos z}{\sin z}$, and sin z has a simple zero at z = 0, which is not a zero of cos z. So cot z has a simple pole at z = 0, and since $\lim_{z\to 0}$ [z cot z] = $\lim_{z\to 0}$ (z [sin z]$^{-1}$ cos z) = 1, the residue is 1.

(ii) $$\operatorname{cosec}^2 z \log(1-z) = \frac{\log(1-z)}{\sin^2 z} = \frac{-z-z^2/2-\dots}{(z-z^3/6+\dots)^2} \quad \text{for } |z|<1$$

$$= -\frac{z(1+\frac{z}{2}+)}{z^2}\left[1-\frac{z^2}{6}+\dots\right]^{-2} = -\frac{1}{z}\left(1+\frac{z}{2}+\dots\right)\left(1+\frac{z^2}{3}+\dots\right)$$

$$= -\frac{1}{z}+\dots \quad \text{for z sufficient small}$$

Hence $\operatorname{cosec}^2 z \log(1-z)$ has a simple pole at z = 0, with residue -1.

(iii) $$\frac{z}{\sin z-\tan z} = -\frac{z\cos z}{\sin z(1-\cos z)} = \frac{-z(1-\frac{z^2}{2}+\dots)}{(z-\frac{z^3}{6}+\dots)(\frac{z^2}{2}-\frac{z^4}{24}+\dots)}$$

$$= -\frac{2}{z^2}\left(1-\frac{z^2}{2}+\dots\right)\left(1-\frac{z^2}{6}+\dots\right)^{-1}\left(1-\frac{z^2}{12}+\dots\right)^{-1}$$

$$= -\frac{2}{z^2}\left(1-\frac{z^2}{2}+\dots\right)\left(1+\frac{z^2}{6}+\right)\left(1+\frac{z^2}{12}+\dots\right) \quad \text{for sufficient small z}$$

$$= -\frac{2}{z^2}\left(1-\frac{z^2}{4}+\dots\right) = -2/z^2+1/2+\dots$$

Hence $\frac{z}{\sin z-\tan z}$ has a double pole at z = 0 with residue <u>zero</u>.

Solution No. 32.

The principal part at $z = -1$ is $\frac{1}{z+1}$, and the principal part at $z = 2$ is $\frac{2}{z-2} + \frac{b}{(z-2)^2}$, where b is an unknown function.

Since f(z) has no essential singularities in the whole z - plane, it is a rational function; in fact

$$f(z) - \frac{1}{z+1} - \frac{2}{z-2} - \frac{b}{(z-2)^2}$$

is regular everywhere in the ~~center~~ entire plane, and so is founded for all z, and by Liouville's theorem is a constant A (say). Hence

$$f(z) = \frac{1}{z+1} + \frac{2}{z-2} + \frac{b}{(z-2)^2} + A$$

Since $f(0) = 7/4$, $f(1) = 5/2$

$$7/4 = 1 - 1 + \frac{b}{4} + A, \quad \text{i.e.} \quad b + 4A = 7$$

$$5/2 = \frac{1}{2} - 2 + b + A, \quad \text{i.e.} \quad b + A = 4$$

Hence $b = 3$, $A = 1$, and

$$f(z) = \frac{1}{z+1} + \frac{2}{z-2} + \frac{3}{(z-2)^2} + 1$$

For $|z| > 1$, $\frac{1}{z+1} = \frac{1}{z}\left(1 + \frac{1}{z}\right)^{-1} = \frac{1}{z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \dots + (-1)^{n-1}\frac{1}{z^{n-1}} + \dots\right)$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{z^n}$$

For $|z| < 2$, $\frac{1}{z-2} = -\frac{1}{2}\left(1 - \frac{z}{2}\right)^{-1} = -\frac{1}{2} \sum_{n=0}^{\infty} \frac{z^n}{2^n}$

$$\frac{1}{(z-2)^2} = \frac{1}{4}\left(1 - \frac{z}{2}\right)^{-2} = \frac{1}{4}\left(1 + 2\cdot\frac{z}{2} + 3\frac{z^2}{2^2} + \dots + (n+1)\frac{z^n}{2^n} + \dots\right)$$

Hence $f(z) = 1 - \sum_{n=0}^{\infty} \frac{z^n}{2^n} + \frac{3}{4}\sum_{n=0}^{\infty} \frac{(n+1)z^n}{2^n} + \sum_{n=1}^{\infty} (-1)^{n-1}/z^n$

$$= 3/4 + \sum_{n=1}^{\infty}\left(\frac{3n}{4} - \frac{1}{4}\right)\frac{z^n}{2^n} + \sum_{n=1}^{\infty}(-1)^{n-1}/z^n$$

$$= \frac{3}{4} + \sum_{n=1}^{\infty}\left(\frac{3n-1}{2^{n+2}}\right)z^n + \sum_{n=1}^{\infty}(-1)^{n-1}/z^n, \quad 1 < |z| < 2$$

Solution No. 35.

If $\quad w = \cosh^{-1} z,$

$$\cosh w = z$$
$$\sinh w = (z^2 - 1)^{\frac{1}{2}}$$
$$e^w = \cosh w + \text{sin-hw} = z + (z^2 - 1)^{\frac{1}{2}}$$
and $$w = \operatorname{Log} \left\{ z + (z^2 - 1)^{\frac{1}{2}} \right\}.$$

If w_1 is any one branch, and w_2 any other value, then

$$z = \cosh w_1 = \cosh w_2$$

Hence $$\cosh w_2 - \cosh w_1 = 0.$$

Now $\cos iw = \cosh w$, and $\sin iw = i \cdot \sinh w$, so
$$0 = \cosh w_2 - \cosh w_1 = \cos iw_2 - \cos iw_1 = 2 \sin \frac{i(w_1 + w_2)}{2} \cdot \sin \frac{i(w_1 - w_2)}{2}$$

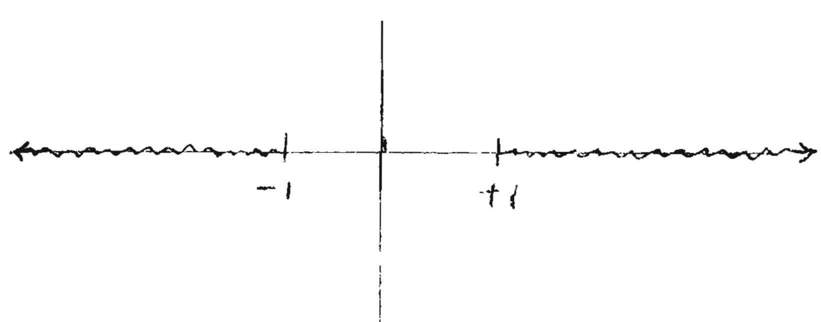
$$= 2i^2 \sinh \frac{w_1 + w_2}{2} \cdot \sinh \frac{w_1 - w_2}{2} = 2 \sinh \frac{w_2 + w_1}{2} \cdot \left(\sinh \frac{w_2 - w_1}{2} \right).$$

The zeros of $\sinh w$ are at $w = n\pi i$, where n is a positive or negative integer, or zero. Hence

$$\frac{w_2 + w_1}{2} \quad n\pi i \quad \text{or} \quad \frac{w_2 - w_1}{2} = n\pi i, \text{ i.e., } w_2 = 2n\pi i \pm w_1.$$

If w_1 is one branch, the other values are given by $2n\pi i \pm w_1$.

If $\zeta = z + (z^2 - 1)^{\frac{1}{2}}$, $w = \operatorname{Log} \zeta$, then ζ is a doubley-valued function of z with branch points at $z = \pm 1$; w is an infinitely many-valued function of ζ, with branch points at $\zeta = 0$ and $\zeta = \infty$. But ζ has no zero (since $\zeta = 0$ would imply $-z = (z^2 - 1)^{\frac{1}{2}}$, $z^2 = z^2 - 1$, which is impossible). $\zeta = \infty$ when $z = \infty$. Hence $w = \operatorname{Log} \left\{ z + (z^2 - 1)^{\frac{1}{2}} \right\}$ has branch points at $z = \pm 1$, $z = \infty$, and suitable cuts in the z-plane go from $+1$ to ∞, and from -1 to ∞.



Solution No. 35, continued.

When the cuts are determined (i.e., the routes taken from +1 and from −1 to ∞), the branches are each completely defined, and each branch is a regular function in any region of the z-plane which does not contain $z = 1$, $z = -1$, $z = \infty$ or any point of the cuts. (For each branch is single-valued in every such region and is bounded in the neighbourhood of every point of such a region, and so is regular at every such point.)

For any branch, $\frac{dz}{dw} = \sinh w$, $\quad \frac{dw}{dz} = \frac{1}{\sinh w} = \frac{1}{(z^2-1)^{\frac{1}{2}}} = \frac{i}{(1-z^2)^{\frac{1}{2}}}$

i.e., $\frac{dw}{dz} = \pm i\left[1 + \frac{z^2}{2} + \frac{1.3}{2.4} z^4 + \ldots\right]$ for $|z| < 1$.

and $\quad w = \text{constant} \pm i\left[z + \frac{1}{2}\frac{z^3}{3} + \frac{1.3}{2.4}\frac{z^5}{5} + \frac{1.3.5}{2.4.6}\frac{z^7}{7} + \ldots\right]$ for $|z| < 1$.

Since $\cosh \frac{1}{2}\pi i = \cos\frac{1}{2}\pi = 0$, for one branch $w = \frac{1}{2}\pi i$ when $z = 0$. For this branch $\sinh w = \sinh\frac{1}{2}\pi i = i \sin\frac{1}{2}\pi = i$ at $z = 0$, so

$\frac{dw}{dz} = \frac{1}{i} = -i$ at $z = 0$, and

$$\frac{dw}{dz} = -i\left[1 + \frac{z^2}{2} + \ldots\right], \qquad (|z| < 1)$$

so for this branch

$$w = \frac{1}{2}\pi i - i\left[z + \frac{1}{2}\frac{z^3}{3} + \frac{1.3}{2.4}\frac{z^5}{5} + \frac{1.3.5}{2.4.6}\frac{z^7}{7} + \ldots\right] \quad (|z| < 1).$$

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Solution No. 33.

The poles and zeros of $f(z)$ are isolated.

In the neighborhood of any point other than a pole, $f(z)$ is regular, so $f'(z)$ is regular (by Cauchy's formula for the derivatives of a regular function). Hence the only singularities of $zf'(z)/f(z)$ are at the poles and zeros of $f(z)$. Since they are isolated, we may draw small circles Γ_k with each pole b_k as center, and small circles C_p with each zero a_p as center, such that none of these circles intersect and they all lie entirely within the closed contour C. Then on C and on every Γ_k and C_k, and at all points between them, $zf'(z)/f(z)$ is regular, so

$$\frac{1}{2\pi i}\int_C \frac{zf'(z)}{f(z)}\,dz = \sum_{p=1}^{m} \frac{1}{2\pi i}\int_{C_p} \frac{zf'(z)}{f(z)}\,dz + \sum_{k=1}^{n} \frac{1}{2\pi i}\int_{\Gamma_k} \frac{zf'(z)\,dz}{f(z)} \quad \ldots\ldots (1)$$

At a zero a_p of order S_p,

$$f(z) = (z - a_p)^{S_p}\,\phi(z),$$

where $\phi(z)$ is regular and non-zero in a neighborhood of a_p; and we may take C_p to lie within this neighborhood. Then

$$z\frac{f'(z)}{f(z)} = \frac{S_p z}{z-a_p} + \frac{z\phi'(z)}{\phi(z)} = S_p + \frac{S_p a_p}{z-a_p} + \frac{z\phi'(z)}{\phi(z)},$$

and $z\phi'(z)/\phi(z)$ is regular in and on C_p. Hence

$$\int_{C_p} \frac{z\phi'(z)}{\phi(z)}\,dz = 0$$

Also

$$\int_{C_p} dz = 0, \quad \frac{1}{2\pi i}\int_{C_p} \frac{dz}{z-a_p} = 1$$

Hence

$$\frac{1}{2\pi i}\int_{C_p} z\frac{f'(z)}{f(z)}\,dz = S_p a_p$$

In the neighborhood of a pole b_k of order r_k,

$$(2) \qquad f(z) = \frac{A}{(z-b_k)^{r_k}} + \ldots \qquad (A \neq 0)$$

so there is a neighborhood of b_k in which $(z-b_k)^{r_k} f(z)$ is regular and non-zero,

i.e., $$f(z)=(z-b_k)^{-r_k}\psi(z)$$

where $\psi(z)$ is regular and non-zero in a neighborhood of b_k in which we may take Γ_k to lie.

Then $$\frac{zf'(z)}{f(z)} = z\frac{\psi'(z)}{\psi(z)} - \frac{r_k z}{z-b_k} = \frac{z\psi'(z)}{\psi(z)} - r_k - \frac{r_k b_k}{z-b_k}$$

and $z\psi'/\psi(z)$ is regular in and on Γ_k. Hence (as before)

$$\frac{1}{2\pi i}\int_{\Gamma_k}\frac{zf'(z)}{f(z)}\,dz = -r_k b_k$$

From (1), (2), and (3), $\frac{1}{2\pi i}\int_C \frac{zf'(z)}{f(z)}\,dz = \sum_{\rho=1}^{m} s_\rho a_\rho - \sum_{k=1}^{n} r_k b_k$

Solution No. 34.

Let $f(z) = z^4 + z^3 + 4z^2 + 2z + 3.$

When z is real and positive, $f(z)$ is real and greater than 3, and $f(z) = 0$ has no roots.

When z is real and negative, set $z = -x$ $(x > 0)$, so

$$f(z) = x^4 - x^3 + 4x^2 - 2x + 3.$$

$f(0) = 3$, $f(-1) = 5$, and

$$f(z) = x^2(x^2 - x + 4) + 3 - 2x = x^2\left\{(x - \tfrac{1}{2})^2 + \tfrac{15}{16}\right\} + 3 - 2x$$

so $f(z) = 0$ has no root for which $0 \leq x \leq 1$. If we write

$$f = x^3(x - 1) + 4\left[x^2 - \frac{x}{2} + \frac{3}{4}\right] = x^3(x - 1) + 4\left\{(x - \tfrac{1}{4})^2 + \frac{11}{16}\right\}$$

we see that $f = 0$ has root for $x > 1$. So $f = 0$ has no real root.

When $z = iy$, $f = y^4 - 4y^2 + 3 \; - i\left\{y^3 - 2y\right\}$,

and the real and imaginary parts do not vanish together. [The imaginary part vanishes only when $y = 0$ or $y^2 = 2$, and the real part does not then vanish.]

Now consider the change in the argument of $f(z)$, written $\Delta \arg f(z)$, taken round the part of the first quadrant bounded by $|z| = R$, where R is large. On the arc of the circle, $z = Re^{i\theta}$, and

$$\Delta \arg f(z) = \Delta \arg (R^4 e^{4i\theta}) + \Delta \arg\left(1 + \frac{1}{z} + \frac{4}{z^2} + \frac{2}{z^3} + \frac{3}{z^4}\right)$$
$$= 4 \times \tfrac{1}{2}\pi + \eta = 2\pi + \eta$$

where $\eta \to 0$ as $R \to \infty$.

On the imaginary axis $(z = iy)$, we may take

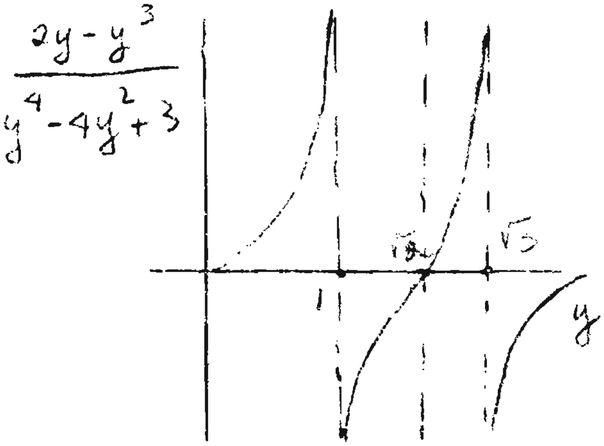
$$\arg f = \arctan \frac{2y - y^3}{y^4 - 4y^2 + 3}$$

$2y - y^3$ vanishes on the positive imaginary axis at $y = \sqrt{2}$, is negative for $y > \sqrt{2}$, and positive for $0 < y < \sqrt{2}$. $y^4 - 4y^2 + 3 = (y^2 - 3)(y^2 - 1)$ and vanishes on the positive imaginary axis when $y = \sqrt{3}$ and when $y = 1$. It is positive for $y > \sqrt{3}$, negative for $\sqrt{3} > y > 1$, and positive for $0 < y < 1$.

Solution No. 34, continued.

$\frac{2y-y^3}{y^4-4y^2+3} \to 0$ as $y \to \infty$; is negative for $y > \sqrt{3}$; tends to $\pm\infty$ as $y \to \sqrt{3}$; positive for $\sqrt{3} > y > \sqrt{2}$; vanishes when $y = \sqrt{2}$; is negative for $\sqrt{2} > y > 1$; tends to $\pm\infty$ as $y \to 1$; is positive for $1 > y > 0$, and zero at $y = 0$. (see figure).

Hence as y decreases from ∞, arg f decreases by $\frac{1}{2}\pi$ as y decreases to $\sqrt{3}$; decreases by a further π as y decreases from $\sqrt{3}$ to $\sqrt{2}$, and by further $\frac{1}{2}\pi$ as y decreases from $\sqrt{2}$ to 0, i.e., arg f decreases by 2π in all, Δ arg f $= -2\pi$ on the positive imaginary axis.



arg f does not change on the positive real axis. Hence for the whole boundary of the first quadrant

$$\Delta \text{ arg } f = 0.$$

The equation has no roots in the first quadrant. Since the roots occur in conjugate pairs (the coefficients being real), there are no roots in the fourth quadrant; and for the same reason, and since there are four roots altogether, there must be

Two roots in the second quadrant, and two roots in the third quadrant

- be -

Solution No. 36.

$$\text{Let } f_1(z) = z - \frac{z^2}{2} + \frac{z^3}{3} + \ldots$$

$$\text{and } f_2(z) = \frac{1}{2}\log(1+b^2) + i \arctan b + \mathcal{G}, \text{ where}$$

$$\mathcal{G} = \frac{z-ib}{1+ib} - \frac{1}{2}\left(\frac{z-ib}{1+ib}\right)^2 + \ldots$$

Note first that

(i) $\mathrm{Log}\,(1+ib) = \log|1+ib| + i \arg(1+ib) = \frac{1}{2}\log(1+b^2) + i \arctan b$, in general (with neither arg $(1+ib)$ nor arc tan b necessarily having their principal values).

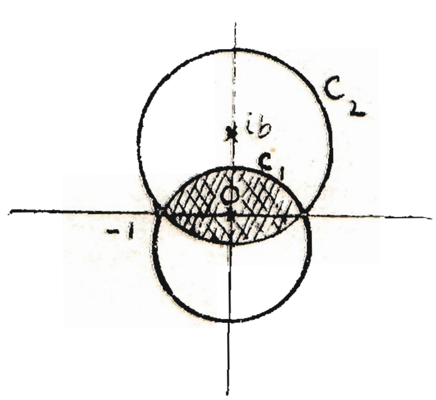
(ii) The power series for $f_1(z)$ converges for $|z| < 1$ and inside the circle C_1 of convergence $f_1(z)$ is equal to that branch of $\mathrm{Log}\,(1+z)$ which is zero at $z=0$. (This defines the branch inside the circle of convergence, the branch point being at $z = -1$.)

(iii) The power series for $\mathcal{G}$ converges for $\left|\frac{z-ib}{1+ib}\right| < 1$, i.e., inside the circle C_2 with center at ib and radius $(1+b^2)^{\frac{1}{2}}$. Inside the circle C_2, $\mathcal{G}$ is equal to that branch of $\mathrm{Log}\left(1 + \frac{z-ib}{1+ib}\right)$ $= \mathrm{Log}([1+z]/[1+ib])$ which is zero at $z = ib$.

(iv) Inside C_2, $f_2(z)$ is equal to some branch of $\mathrm{Log}([1+z]/[1+ib]) + \mathrm{Log}\,(1+ib)$, i.e., to some branch of $\mathrm{Log}\,(1+z)$.

(v) Since any branch of $\mathrm{Log}\,(1+z)$ may be obtained by analytic continuation from any other branch, $f_2(z)$ is an analytic continuation of $f_1(z)$, or a continuation of a continuation, or is obtained by continuation in some finite number of steps.

(vi) The regions of convergence of the power series for $f_1(z)$ and $\mathcal{G}$ overlap, and we may show that if we choose arc tan b to lie between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$, i.e., if $\arg(1+ib) = \phi$, where $\tan\phi = b$, $-\frac{1}{2}\pi < \phi < \frac{1}{2}\pi$ (b being either positive or negative), then $f_2(z)$ is the direct continuation of $f_1(z)$ -- i.e., $f_1(z)$ and $f_2(z)$ have the same values at all points of the region common to the interiors of C_1 and C_2.



Since each of $f_1(z)$ and $f_2(z)$ is equal to a branch of $\mathrm{Log}(1+z)$, it is necessary to show this only for one point of the common region, say at $z=0$.

Solution No. 36, continued.

Now $f_1(z) = 0$ at $z = 0$.

$\mathcal{J}$ is equal to the branch of $\mathrm{Log}\,\frac{1+z}{1+ib}$ which is zero at $z = ib$; therefore $\mathcal{J}$ $\mathrm{Log}(1+z) - \log|1+ib| - i\arg(1+ib) = \mathrm{Log}(1+z) - \frac{1}{2}\log(1+b^2) - i\phi$

if we take that branch of $\mathrm{Log}(1+z)$ inside C_2 which is equal to $\frac{1}{2}\log(1+b^2) + i\phi$ at $z = ib$. Hence if we write $\mathrm{Log}(1+z) = \log|1+z|$ $i\arg(1+z)$, we must take $\arg(1+z) = \phi$ at $z = ib$. If z remains within C_2, and moves along any contour inside C_2 from ib to 0, $\arg(1+z)$ changes from ϕ at $z = ib$ to zero at $z = 0$. Also $\log|1+z| = 0$ at zero. So $\mathcal{J} = -\frac{1}{2}\log(1+b^2) - i\phi$ at $z = 0$ and $f_2(z) = 0 = f_1(z)$ at $z = 0$, and therefore throughout the region common to the interiors of C_1 and C_2.

Note that if we cut the z-plane along the negative real axis from -1 to $-\infty$, the branch of $\mathrm{Log}(1+z)$ which is zero at $z = 0$ is one-valued and regular in the cut plane; for $-\pi < \arg(1+z) < \pi$ and $|1+z| > 0$ its value is defined by

$$f_3(z) = \log(1+z) = \log|1+z| + i\arg(1+z).$$

It follows from the arguments above that $f_3(z) = f_1(z)$ at all points within C_1, and that $f_3(z) = f_2(z)$ at all points within C_2.

If preferred, the reasoning may be altered to show first that $f_3(z) = f_1(z)$ at all points within C_1, and that $f_3(z) = f_2(z)$ at all points within C_2, and it then follows that $f_1(z) = f_2(z)$ at all points of the region common to the interiors of C_1 and C_2.

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Solution No. 37.

Write $f_1(z) = z + \frac{1}{2} z^2 + \frac{1}{3} z^3 + \ldots$

and $f_2(z) = i\pi - \varphi$

where $\varphi = (z-2) - \frac{1}{2}(z-2)^2 + \frac{1}{3}(z-2)^2 - \ldots$

(1) The series for $f_1(z)$ converges for $z < 1$, and inside the circle C_1 of convergence $f_1(z)$ is equal to the branch of $-\operatorname{Log}(1-z)$ which is zero at $z = 0$.

(11) The series for φ converges for $|z-2| < 1$, i.e., inside a circle C_2 with center at $z = 2$ and radius 1. Inside the circle C_2, φ is equal to the branch of $\operatorname{Log}\{1 + (z-2)\} = \operatorname{Log}(z-1)$ which is zero at $z = 2$.

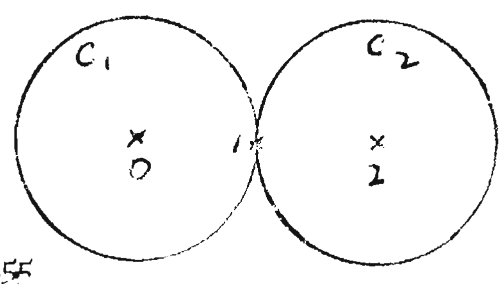
(111) One value of $\operatorname{Log}(-1)$ is $-i\pi$. If we take this value of $\operatorname{Log}(-1)$,

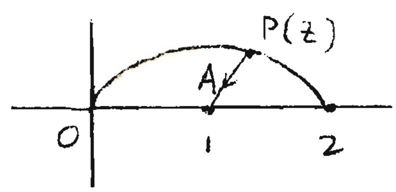
$$f_2(z) = -\operatorname{Log}(-1) - \operatorname{Log}(z-1) = -\operatorname{Log}(1-z).$$

(iv) Since any branch of $\operatorname{Log}(1-z)$ may be obtained by analytical continuation from any other branch, $f_2(z)$ may be obtained from $f_1(z)$, and $f_1(z)$ from $f_2(z)$, by continuation in a finite number of steps.

(v) To show that $f_1(z)$ and $f_2(z)$ are continuations of the same function, we need to show that only two steps are necessary in (iv).

Since the regions of convergence have no common point, we need to pay attention to the route by which the continuation from $z = 0$ to $z = 2$ is effected. If we write $f_1(z) = -\log|1-z| - i\arg(1-z)$, $(|z| < 1)$ ..... (1) we must take $\arg(1-z) = 0$ at $z = 0$. At $z = 2$, $f_2(z) = i\pi$, and if we continue $f_1(z)$ by the formula (1) (without the restriction $|z| < 1$) to $z = 2$, we would have $f_1(z) = f_2(z)$ at $z = 2$ if $\arg(1-z) = -\pi$ at $z = 2$. If A represents the point 1 in the Argand diagram, and P the point z, and $\arg(1-z) = 0$ at $z = 0$, then $\arg(1-z)$ is the angle which the vector PA makes with the positive direction of the real axis. If P describes a path from 0 to 2 lying in the first





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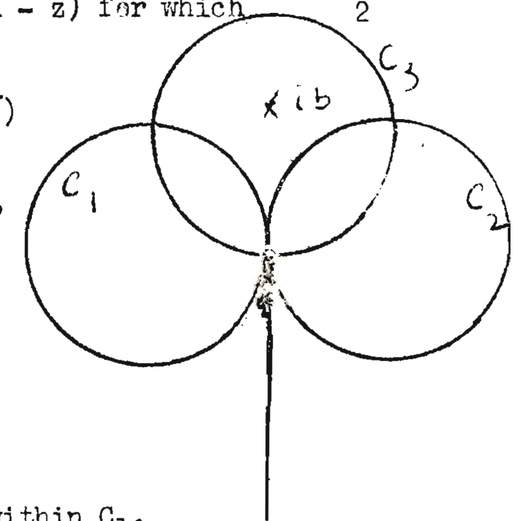
quadrant, the vector PA rotates in the negative sense, and $\arg(1+z)$ decreases from 0 at $z=0$ to $-\pi$ at $z=2$. Hence we expect that the route by which we continue $f_1(z)$ to obtain $f_2(z)$ must lie in the first quadrant. We can now carry out the construction of a function, defined by a power-series, of which $f_1(z)$ and $f_2(z)$ are direct continuations.

(vi) Let $\arg(1-z)$ lie in the range $\frac{1}{2}\pi > \arg(1-z) > -\frac{3}{2}\pi$, and define $f_3(z)$ as that branch of $-\mathrm{Log}(1-z)$ for which

$$f_3(z) = \log|1-z| - i\arg(1-z)$$
$$(|1-z|>0,\ \tfrac{1}{2}\pi > \arg(1-z) > -\tfrac{3}{2}\pi)$$



If we cut the z-plane by a line parallel to the negative imaginary axis from $z=1$ to $z=1-i\infty$, then $f_3(z)$ is regular and single-valued in the cut plane.

At $z=0$, $|1-z|=1$, $\arg(1-z)=0$, so $f_3(z)=0=f_1(z)$.

Since $f_3(z)$ and $f_1(z)$ are both branches of $-\mathrm{Log}(1-z)$, $f_3(z)=f_1(z)$ at all points within C_1.

At $z=2$, $|1-z|=1$, $\arg(1-z)=-\pi$, so $f_3(z)= i\pi = f_2(z)$. Since $f_3(z)$ and $f_2(z)$ are both branches of $-\mathrm{Log}(1-z)$, $f_3(z)= f_2(z)$ at all points within C_2.

At $z=1+bi$, $(b>0)$ $|1-z|=b$, $\arg(1-z)=\arg(-bi)=-\frac{1}{2}\pi$,
$f_3(z) = -\log b + \frac{1}{2}\pi i$.

Let $f_4(z) = -\log b + \frac{1}{2}\pi i + w$,

where
$$w = \frac{i(z-1-ib)}{b} + \frac{1}{2}\left\{\frac{i(z-1-ib)}{b}\right\}^2 + \frac{1}{3}\left\{\frac{i(z-1-ib)}{b}\right\}^3 + \dots .$$

The series for w converges for $\left|\frac{z-1-ib}{b}\right| < 1$, i.e., inside a circle C_3 with center at $1+ib$ and readius b. Inside the circle C_3, w is equal to that branch of

$$-\mathrm{Log}\left\{1-\frac{i(z-1-ib)}{b}\right\} = -\mathrm{Log}\,\frac{b+i(1-z)-b}{b} = -\mathrm{Log}\,\frac{i(1-z)}{b}$$

$= -\mathrm{Log}(1-z) + \log b - \frac{1}{2}\pi i$, which is equal to zero when $z=1+bi$,

Solution No. 37, continued.

i.e., for which $-\operatorname{Log}(1-z) = -\log b + \frac{1}{2}\pi i$ when $z = 1 + bi$. This branch of $-\operatorname{Log}(1-z)$ is $f_3(z)$. Hence inside C_3, $w = \log - \frac{1}{2}\pi i + f_3(z)$, so $f_4(z) = f_3(z)$.

If D_1, D_2, D_3 are the regions of convergence of the power series for $f_1(z)$, φ, w, respectively, D_3 overlaps both D_1 and D_2; in region common to D_3 and D_1, $f_1(z) = f_3(z) = f_4(z)$; in the region common to D_3 and D_2, $f_2(z) = f_3(z) = f_4(z)$. Hence both $f_1(z)$ and $f_2(z)$ are continuations of $f_4(z)$.

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Solution No. 38.

Note first that any function $f(\sin\theta, \cos\theta)$ of $\sin\theta$ and $\cos\theta$ is periodic in θ, with period 2π, and that if $F(\theta)$ is periodic in π, with period 2π, so that $F(\theta + 2\pi) = F(\theta)$, then for any constant α

$$\int_{-\pi+\alpha}^{\pi+\alpha} F(\theta)\,d\theta = \int_{-\pi}^{\pi} F(\theta)\,d\theta$$

In particular
$$\int_{-\pi}^{\pi} F(\theta)\,d\theta = \int_0^{2\pi} F(\theta)\,d\theta$$

If $F(\theta)$ is an even function of θ, with period 2π,

$$\int_0^{2\pi} F(\theta)\,d\theta = \int_{-\pi}^{\pi} F(\theta)\,d\theta = 2\int_0^{\pi} F(\theta)\,d\theta$$

If $F(\theta)$ is an odd function of θ, with period 2π,

$$\int_0^{2\pi} F(\theta)\,d\theta = \int_{-\pi}^{\pi} F(\theta)\,d\theta = 0$$

Let $I = \int_0^{\pi} \frac{\cos^2 3\theta}{1-2p\cos 2\theta + p^2}\,d\theta \qquad (0 < p < 1).$

A direct attack, writing $\quad I = \frac{1}{2}\int_{-\pi}^{\pi} \frac{\cos^2 3\theta}{1-2p\cos 2\theta + p^2}\,d\theta$

and, with $z = e^{i\theta}$, $\cos 3\theta = \frac{1}{2}(z^3 + \frac{1}{z^3})$, $\cos 2\theta = \frac{1}{2}(z^2 + \frac{1}{z^2})$,
$$d\theta = \frac{dz}{iz},$$

and then evaluating the resultant contour integral round the unit circle, is possible. But the evaluation of the residues is rather laborious if this direct approach is used. (Try it.) It is easier to proceed as follows:

First, since $2\cos^2\theta = 1 + \cos 2\theta$, and $2\cos^2 3\theta = 1 + \cos 6\theta$

$$I = \frac{1}{2}\int_0^{\pi} \frac{1+\cos 6\theta}{1-2p\cos 2\theta + p^2}\,d\theta = \frac{1}{4}\int_0^{2\pi} \frac{1+\cos 3\phi}{1-2p\cos\phi + p^2}\,d\phi$$

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Solution No. 38, continued.

the last equality arising from the substitution $\phi = 2\theta$ in the integral. Again, with $z = e^{i\phi}$, we may write $\cos 3\phi = \frac{1}{2}(z^3 + \frac{1}{z^3})$, $\cos\phi = \frac{1}{2}(z + \frac{1}{z})$, and the work is now easier than it would have been before. But the evaluation is still fairly laborious. (Try it.) It is clear that labor can be saved by writing

$$I = RJ, \text{ where } J = \frac{1}{4}\int_0^{2\pi} \frac{1 + 3\phi + i \sin 3\phi}{1 - 2\rho\cos\phi + \rho^2}\, d\phi$$

$$= \frac{1}{4}\int_0^{2\pi} \frac{1 + e^{3i\phi}}{1 - 2\rho\cos\phi + \rho^2}\, d\phi$$

[In fact $\operatorname{Im} J = \frac{1}{4}\int_0^{2\pi} \frac{\sin 3\phi}{1 - 2p\cos\phi + \rho^2}\, d\phi = \frac{1}{4}\int_{-\pi}^{\pi} \frac{\sin 3\phi}{1 - 2\rho\cos\phi + \rho^2}\, d\phi$

and since the integrand is an odd function of ϕ, this is zero, so $I = J$]. Now write $z = e^{i\phi}$, and let C be the unit circle $|z| = 1$. Then

$$J = \frac{1}{4}\oint_C \frac{1 + z^3}{1 - \rho(z + \frac{1}{z}) + \rho^2}\left[\frac{dz}{iz}\right] = \frac{1}{4i}\oint_C \frac{1 + z^3}{(1+\rho^2)z - \rho z^2 - \rho}\, dz$$

$$= \frac{1}{4i}\oint_C \frac{1 + z^3}{(z - p)(1 - \rho z)}\, dz$$

The integrand has simple poles at $z = p$ and $z = \frac{1}{p}$; since $0 < p < 1$, only the pole at $z = p$ is inside the circle. The residue there is

$$\left[\frac{1 + z^3}{1 - \rho z}\right]_{z=\rho} = \frac{1 + \rho^3}{1 - \rho^2} = \frac{(1+\rho)(1 - \rho + \rho^2)}{(1+\rho)(1-\rho)} = \frac{1 - \rho + \rho^2}{1 - \rho}$$

and $I = J = \frac{1}{4i} \times 2\pi i \times \frac{1 - \rho + \rho^2}{1 - \rho} = \frac{\pi}{2}\,\frac{1 - \rho + \rho^2}{1 - \rho}$

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Solution No. 39.

It is immediately clear that $\int_0^{2\pi} \cos^n\theta\, d\theta = 0$ if n is odd. For $\cos^n\theta$ is an even function of θ with period 2π, so

$$\int_0^{2\pi} \cos^n\theta\, d\theta = \int_{-\pi}^{\pi} \cos^n\theta\, d\theta = 2\int_0^{\pi} \cos n\theta\, d\theta$$

$$= 2\left[\int_0^{\frac{\pi}{2}} \cos^n\theta\, d\theta + \int_{\frac{\pi}{2}}^{\pi} \cos^n\theta\, d\theta\right]$$

$\cos\theta$ is odd about $\theta = \frac{1}{2}\pi$ i.e., $\cos(\pi - \theta) = -\cos\theta]$, so if n is odd, $\cos^n\theta$ is odd about $\theta = \frac{1}{2}\pi$, and $\int_0^{\pi}\cos^n\theta\, d\theta$ must be zero. Formally, substitute $\phi = \pi - \theta$ in the last integral on the right above, which becomes

$$\int_{\frac{\pi}{2}}^{0} \cos^n(\pi-\phi)(-d\phi) = \int_0^{\frac{\pi}{2}} \cos^n(\pi - \phi)\, d\phi$$

$$= (-1)^n \int_0^{\frac{\pi}{2}} \cos^n\phi\, d\phi = -\int_0^{\frac{\pi}{2}} \cos^n\phi\, d\phi$$

(since n is odd).

Let $I = \int_0^{2\pi} \cos^n\theta\, d\theta$;

$$z = e^{i\theta}, \quad \cos\theta = \tfrac{1}{2}\left(z + \tfrac{1}{z}\right), \quad d\theta = \frac{dz}{iz}$$

Then $I = \frac{1}{2^n i} \oint_C \left(z - \frac{1}{z}\right)^n \frac{dz}{z}$, where C is the unit circle $|z| = 1$.

$$I = \frac{1}{2^n i} \oint_C \frac{(1+z^2)^n}{z^{n+1}}\, dz$$

The integrand has a pole of order $n + 1$ at $z = 0$; the residue is the coefficient of $\frac{1}{z}$ in the expansion of $\frac{(1+z^2)^n}{z^{n+1}}$ in powers of z, and therefore is the coefficient of z^n in the expansion of $(1+z^2)^n$. Now

$$(1+z^2)^n = 1 + nz^2 + \frac{n(n-1)}{2!} z^4 + \dots + \frac{n(n-1)\dots(n-s+1)}{s!} z^{2s}$$
$$+ \dots,$$

so if n is odd the residue is zero, and if n is even, it is found by putting $s = \frac{1}{2}n$ in the general term above, and is

$$\frac{n(n-1)\cdots(\frac{n}{2}+1)}{(\frac{n}{2})!}$$

When n is even, write n 2m. Then

$$I = \frac{1}{2^{2m}i} \times 2\pi i \times \frac{2m(2m-1)\cdots(m+1)}{m!} = 2\pi\,\frac{2m(2m-1)\cdots(m+1)}{2^{2m}\cdot m!}$$

$$= 2\pi\,\frac{(2m)!}{2^{2m}m!\,m!} \quad \left(\text{since } \frac{(2m)!}{m!} = 2m(2m-1)\cdots(m+1)\right)$$

$$= 2\pi\,\frac{1\cdot3\cdot5\cdot\ldots\cdot(2m-1)\cdot2\cdot4\cdot6\cdot\ldots\cdot2m}{2^{2m}\cdot m!\,m!}$$

$$= 2\pi\,\frac{1\cdot3\cdot5\cdot\ldots(2m-1)\cdot(m!)}{2^{m}\cdot m!\,m!} \quad (\text{since } 2^{m}m! = 2\cdot4\cdot6\ldots2m$$

$$= 2\pi\,\frac{1\cdot3\cdot5\cdot\ldots\cdot(2m-1)}{2^{m}\cdot m!} = 2\pi\,\frac{1\cdot3\cdot5\cdot\ldots\cdot(2m-1)}{2\cdot4\cdot6\cdot\ldots\cdot2m}$$

$$= 2\pi\,\frac{1\cdot3\cdot5\cdot\ldots\cdot(n-1)}{2\cdot4\cdot6\cdot\ldots\cdot n}$$

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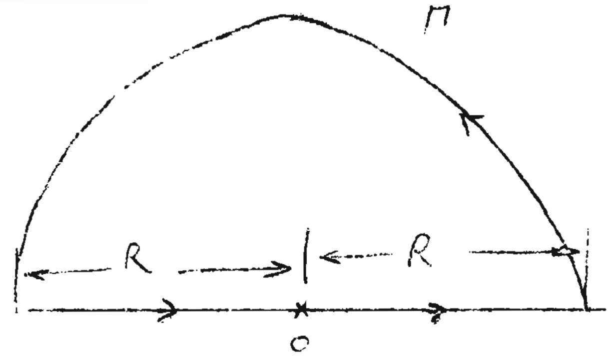
Solution No. 40.

Let $\quad I = \displaystyle\int_0^\infty \frac{x^6\,dx}{(a^4+x^4)^2}$



The integral is convergent.

Consider $\displaystyle\int_C \frac{z^6}{(a^4+z^4)^2}\,dz$

round a contour C consisting of the real axis from $-R$ to $+R$, and the semi-circle, Γ, on which $|z| = R$, $0 \leq \arg z \leq \pi$. In the sequel, let $R \to \infty$. $z = \pm a e^{\frac{3}{4}\pi i}$

The integrand has double poles at $z^4 = -a^4$, i.e., at $z = \pm a e^{\frac{1}{4}\pi i}$. The poles at $z = a e^{\frac{1}{4}\pi i}$ and at $z = a e^{\frac{3}{4}\pi i}$ are within C; the other two are outside C.

The residue at $z = a e^{\frac{1}{4}\pi i}$ is the coefficient of $\left\{z - a e^{\frac{1}{4}\pi i}\right\}^{-1}$ in the expansion of $f(z) = \dfrac{z^6}{(a^4+z^4)^2}$ in powers of $(z - a e^{\frac{1}{4}\pi i})$.

To find the residue at $z = \beta$, where β may be either $a e^{\frac{1}{4}\pi i}$ or $a e^{\frac{3}{4}\pi i}$ (so that $\beta^4 + a^4 = 0$), put $\quad \zeta = z - \beta$

and expand $f(z)$ in powers of ζ. The residue is the coefficient of ζ^{-1}.

$$a^4 + z^4 = a^4 + (\beta+\zeta)^4 = a^4 + \beta^4 + 4\beta^3\zeta + 6\beta^2\zeta^2 + 4\beta\zeta^3 + \zeta^4$$

$$= 4\beta^3\zeta\left[1 + \frac{3}{2}\frac{\zeta}{\beta} + \dots\right]$$

$$\frac{1}{(a^4+z^4)^2} = \frac{1}{16\beta^6\zeta^2}\left[1 + \frac{3}{2}\frac{\zeta}{\beta} + \dots\right]^{-2} = \frac{1}{16\beta^6\zeta^2}\left[1 - 3\frac{\zeta}{\beta} + \dots\right]$$

$$z^6 = (\beta+\zeta)^6 = \beta^6\left(1 + \frac{\zeta}{\beta}\right)^6 = \beta^6\left(1 + 6\frac{\zeta}{\beta} + \dots\right)$$

$$f(z) = \frac{1}{16\zeta^2}\left(1 + 6\frac{\zeta}{\beta} + \dots\right)\left(1 - 3\frac{\zeta}{\beta} + \dots\right) = \frac{1}{16\zeta^2}\left(1 + \frac{3\zeta}{\beta} + \dots\right)$$

$$= \frac{1}{16\zeta^2} + \frac{3}{16\beta}\frac{1}{\zeta} + \dots$$

The residue at $z = \beta$ is $3/16\beta$. The sum of the residues at $z = a e^{\frac{1}{4}\pi i}$ and $z = a e^{\frac{3}{4}\pi i} = -a e^{-\frac{1}{4}\pi i}$ is

$$\frac{3}{16a}\left(e^{-\frac{1}{4}\pi i} - e^{\frac{1}{4}\pi i}\right) = -\frac{3}{16a}\left(2i\sin\frac{\pi}{4}\right) = \frac{-3i}{8\sqrt{2}\,a} = -\frac{3\sqrt{2}\,i}{16a}$$

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Solution No. 40, continued.

Hence

$$\int_{-R}^{R} \frac{x^6\,dx}{(a^4+x^4)^2} + \int_{\Gamma} \frac{z^6}{(a^4+z^4)^2}\,dz = 2\pi i\left(-\frac{3\sqrt{2}\,i}{16a}\right) = \frac{3\sqrt{2}\,\pi}{8a}$$

If w_1 and w_2 are any too complex numbers, and $|w_2| > |w_1|$, then $|w_1 + w_2| \geq |w_2| - |w_1|$.

On Γ, $|z| = R$, so $|a^4 + z^4| \geq R^4 - a^4$, $\left|\frac{1}{(a^4+z^4)^2}\right| \leq \frac{1}{(R^4-a^4)^2}$, and

$$|f(z)| \leq \frac{R^6}{(R^4-a^4)^2}$$

Hence

$$\left|\int_{\Gamma} f(z)\,dz\right| \leq \frac{R^6}{(R^4-a^4)^2}\cdot \pi R = \frac{\pi R^7}{(R^4-a^4)^2} \to 0 \text{ as } R \to \infty$$

Also

$$\int_{-R}^{R} \frac{x^6\,dx}{(a^4+x^4)^2} = 2\int_{0}^{R} \frac{x^6\,dx}{(a^4+x^4)^2}$$

Since the integrand is an even function of x.

Hence $$\int_{0}^{\infty} \frac{x^6\,dx}{(a^4+x^4)^2} = \frac{3\sqrt{2}\,\pi}{16a}$$

The method used above is not very laborious, but the work can be shortened. The alternative method works very well for

$$\int_{0}^{\infty} \frac{x^4\,dx}{(a^4+x^4)^2} \quad \text{or} \quad \int_{0}^{\infty} \frac{x^2\,dx}{(a^4+x^4)^2}$$

it is a little more troublesome for $\int_{0}^{\infty} \frac{x^6\,dx}{(a^4+x^4)^2}$ as will be explained.

Note that $a^4 + x^4 = (x^2+ia^2)(x^2-ia^2)$

and $$\frac{x^{2n}}{(x^2-ia^2)^2} = \frac{x^{2n}(x^2+ia^2)^2}{(a^4+x^4)^2} = \frac{x^{2n}(x^4-a^4)}{(a^4+x^4)^2} + \frac{2ia^2x^{2n+2}}{(a^4+x^4)^2}$$

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Solution No. 40, continued.

Hence by considering $\displaystyle\int_C \frac{z^{2n}\,dz}{(z^2-ia^2)^2}$

and then by taking real and imaginary parts, we may obtain the values of

$$\int_0^\infty \frac{x^{2n}(x^4-a^4)\,dx}{(a^4+x^4)^2} \quad \text{and} \quad \int_0^\infty \frac{x^{2n+2}\,dx}{(x^4+a^4)^2} \quad \ldots\ldots \text{(1)}$$

If these integrals are convergent, and if

$$\left|\int_\Gamma \frac{z^{2n}}{(z-ia^2)^2}\,dz\right| \to 0 \text{ as } R \to \infty \quad \ldots\ldots\ldots \text{(2)}$$

these conditions hold for $n = 0$ and $n = 1$. The case we want is, however, $n = 2$; in this case

$\dfrac{z^4}{(z^2-ia^2)^2} \longrightarrow 1$ as $|z| \to \infty$, so (2) is not satisfied (and the first of the integrals in (1) is not convergent).

We can overcome this difficulty by considering

$$\int_C \left[1 - \frac{z^4}{(z^2-ia^2)^2}\right] dz$$

Let

$$f(z) = 1 - \frac{z^4}{(z^2-ia^2)^2} = \frac{-2ia^2z^2 - a^4}{(z^2-ia^2)^2}$$

Then on Γ,

$$|f(z)| \leq \frac{2a^2R^2 + a^4}{(R^2-a^2)^2}$$

and

$$\left|\int_\Gamma f(z)\,dz\right| \to 0, \text{ as } R \to \infty$$

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Solution No. 40, continued.

The only singularity within C is a double pole at $z = \beta = a e^{\frac{1}{4}\pi i}$; $(\beta^2 = i a^2)$

If $\zeta = z - \beta$,

$$f(z) = 1 - \frac{(\beta+\zeta)^4}{(\beta^2 - i a^2 + 2\beta\zeta + \zeta^2)^2} = 1 - \frac{\beta^4 (1 + \zeta/\beta)^4}{\left[2\beta\zeta(1 + \frac{\zeta}{2\beta})\right]^2}$$

$$= 1 - \frac{\beta^2}{4\zeta^2}\left(1 + \frac{4\zeta}{\beta} + \ldots\right)\left(1 - \frac{\zeta}{\beta} + \ldots\right) = 1 - \frac{\beta^2}{4\zeta^2}\left(1 + \frac{3\zeta}{\beta} + \ldots\right)$$

$$= 1 - \frac{\beta^2}{4\zeta^2} - \frac{3\beta}{4}\frac{1}{\zeta} + \ldots$$

The residue is $-\frac{3}{4}\beta = -\frac{3}{4} a e^{\frac{1}{4}\pi i}$. Hence

$$2\int_0^\infty \left[1 - \frac{x^4}{(x^2 - i a^2)^2}\right] dx = 2\int_0^\infty \left[1 - \frac{x^4(x^4 - a^4)}{(a^4 + x^4)^2}\right] dx - 4 i a^2 \int \frac{x^6 dx}{(a^4 + x^4)^2}$$

$$= 2\pi i \left[-\frac{3}{4} a e^{\frac{1}{4}\pi i}\right] = -\frac{3\pi a}{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i\right) = \frac{3\sqrt{2}\,\pi a}{4}(1 - i)$$

Equate imaginary parts.

$$\int_0^\infty \frac{x^6 dx}{(a^4 + x^4)^2} = \frac{3\sqrt{2}\,\pi}{16 a}$$

(Note that we have also obtained the result

$$\int_0^\infty \left[1 - \frac{x^4(x^4 - a^4)}{(a^4 + x^4)^2}\right] dx = a^4 \int_0^\infty \frac{a^4 + 3x^4}{(a^4 + x^4)^2} dx = \frac{3\sqrt{2}\,\pi a}{8} \quad)$$

-be- 12.15.55

Solution No. 41.

Let
$$I = \int_0^\infty \frac{x^3 \sin mx}{x^4 + a^4}\,dx \qquad [m>0,\ a>0]$$

Since $\int^x \sin mx\, dx$ is bounded, and $\frac{x^3}{x^4+a^4}$ is a decreasing function of x for sufficiently large x and $\to 0$ as $x \to \infty$, the integral converges by Dirichlet's test.

It is quite easy, but unnecessary, to work with m and a separately throughout. The integral is clearly a function of ma only, for if we make the substitution $\xi = mx$, we find that

$$I = \int_0^\infty \frac{\xi^3 \sin \xi}{\xi^4 + m^4 a^4}\,d\xi = \int_0^\infty \frac{x^3 \sin x}{x^4 + b^4}\,dx$$

where $b = ma$.

First, if $F(\theta) = \tan\theta - \theta$; $F(0) = 0$ and for $0 < \theta < \frac{\pi}{2}$

$$F'(\theta) = \sec^2\theta - 1 > 0, \quad \text{so}$$

$$F(\theta) > 0, \quad \theta < \tan\theta$$

Then
$$\frac{d}{d\theta}\left(\frac{\sin\theta}{\theta}\right) = \frac{\theta\cos\theta - \sin\theta}{\theta^2} = \frac{\cos\theta}{\theta^2}(\theta - \tan\theta) < 0 \quad \boxed{0 < \theta \leq \frac{\pi}{2}}$$

$\frac{\sin\theta}{\theta}$ is a decreasing function of θ for $0 \leq \theta \leq \frac{\pi}{2}$, and

$$1 \geq \frac{\sin\theta}{\theta} \geq \frac{2}{\pi};\ \sin\theta > \frac{2\theta}{\pi};\ e^{-R\sin\theta} < e^{-2R\theta/\pi}, \text{ for } 0 < \theta \leq \frac{\pi}{2}$$

Now let $f(z) = \frac{z^3 e^{iz}}{z^4 + b^4}$ $(b > 0)$

and consider $\int_C f(z)\,dz$, where C is the same contour as in Ex. 40, and Γ is the same semicircle.

On Γ,
$$|f(z)| = \frac{R^3 \left|e^{iR(\cos\theta + i\sin\theta)}\right|}{|z^4 + b^4|} = \frac{R^3 e^{-R\sin\theta}}{|z^4 + b^4|}$$
$$\leq \frac{R^3 e^{-R\sin\theta}}{R^4 - b^4},$$

so
$$\left|\int_\Gamma f(z)\,dz\right| = \left|\int_0^\pi f(Re^{i\theta})\, iRe^{i\theta}\,d\theta\right| \leq \int_0^\pi |f(Re^{i\theta})| R\,d\theta \leq \frac{R^4}{R^4 - a^4}\int_0^\pi e^{-R\sin\theta}\,d\theta$$

Since $\sin\theta$ is symmetrical about $\theta = \frac{\pi}{2}$, i.e., $\sin(\pi - \theta) = \sin\theta$,

$$\int_0^{\pi} e^{-R\sin\theta}\, d\theta = 2\int_0^{\frac{\pi}{2}} e^{-R\sin\theta}\, d\theta \le 2\int_0^{\frac{\pi}{2}} e^{-2R\frac{\theta}{\pi}}\, d\theta$$

$$= 2\left[-\frac{\pi}{2R}\right]\left[e^{-2R\frac{\theta}{\pi}}\right]_0^{\frac{\pi}{2}} = \frac{\pi}{R}\left[1 - e^{-R}\right] < \frac{\pi}{R}$$

and $$\left|\int_\Gamma f(z)\,dz\right| \le \frac{\pi R^3}{R^4 - b^4} \to 0 \text{ as } R \to \infty$$

(Application of Jordan's lemma.)

The singularities of $f(z)$ within C are simple poles at $z = be^{\frac{1}{4}\pi i}$, $z = be^{\frac{3}{4}\pi i}$. To find the residue at $z = \gamma$, where γ is either $be^{\frac{1}{4}\pi i}$, or $be^{3/4\pi i}$ ($\gamma^4 + b^4 = 0$ in either case), we may put $\rho = z - \gamma$ and expand $f(z)$ in powers of ρ. The residue is the coefficient of ρ^{-1}

$$f(z) = \frac{(\gamma+\rho)^3 e^{i(\gamma+\rho)}}{\gamma^4 + a^4 + 4\gamma^3\rho + 6\gamma^2\rho^2 + \dots} = \frac{\gamma^3 e^{i\gamma}}{4\gamma^3\rho}\left(1+\frac{\rho}{\gamma}\right)^3 e^{i\rho}\left(1 + \frac{3}{2}\frac{\rho}{\gamma} + \dots\right)^{-1}$$

$$= \frac{e^{i\gamma}}{4}\,\frac{1}{\rho}\left(1 + \frac{3\rho}{\gamma} + \dots\right)\left(1 + i\rho + \dots\right)\left(1 - \frac{3}{2}\frac{\rho}{\gamma} + \dots\right)$$

The residue is $\frac{1}{4}e^{i\gamma}$. Or, since the singularity at $z = \gamma$ is a simple pole, we may find the residue as

$$\lim_{z\to\gamma}(z-\gamma)f(z) = \lim_{z\to\gamma}\frac{(z-\gamma)z^3 e^{iz}}{z^4 - \gamma^4} = \lim_{z\to\gamma}\frac{z^3 e^{iz}}{(z+\gamma)(z^2+\gamma^2)} = \frac{1}{4}e^{i\gamma}$$

The sum of the residues at $be^{\frac{1}{4}\pi i} = \frac{b}{\sqrt{2}}(1+i)$ and at $be^{\frac{3}{4}\pi i} = -be^{-\frac{1}{4}\pi i} = -\frac{b}{\sqrt{2}}(1-i)$ is $= \frac{1}{4}\left\{e^{i\frac{b}{\sqrt{2}}(1+i)} + e^{-i\frac{b}{\sqrt{2}}(1-i)}\right\}$

$$= \frac{1}{4}e^{-\frac{b}{\sqrt{2}}}\left\{e^{i\frac{b}{\sqrt{2}}} + e^{-ib/\sqrt{2}}\right\} = \frac{1}{2}e^{-b/\sqrt{2}}\cos\frac{b}{\sqrt{2}}$$

Now

$$\int_{-R}^{R}\frac{x^3 e^{ix}}{x^4+b^4}\,dx = \int_{R}^{R}\frac{x^3\cos x}{x^4+b^4}\,dx + i\int_{-R}^{R}\frac{x^3\sin x}{x^4+b^4}\,dx$$

-be-

Solution No. 41, continued.

The first integral vanishes, since the integrand is odd. The second term is

$$2i\int_0^R \frac{x^3 \sin x}{x^4+b^4}\,dx$$

since the integrand is even. Let $R \to \infty$. The integral $\int_\Gamma f(z)\,dz$, so we obtain

$$2i\int_0^\infty \frac{x^3 \sin x\,dx}{x^4+b^4} = (2\pi i)\left(\frac{1}{2}e^{-b/\sqrt{2}}\cos\frac{b}{\sqrt{2}}\right)$$

i.e.,

$$\int_0^\infty \frac{x^3 \sin x}{x^4+b^4}\,dx = \frac{\pi}{2}e^{-b/\sqrt{2}}\cos\frac{b}{\sqrt{2}}$$

and

$$\int_0^\infty \frac{x^3 \sin mx}{x^4+a^4}\,dx = \frac{1}{2}\pi e^{-\frac{ma}{\sqrt{2}}}\cos\frac{ma}{\sqrt{2}}$$

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Solution No. 42.

Let $I_1 = \int_0^\infty \frac{(1+x^2)\cos ax}{1+x^2+x^4}\,dx \qquad (a>0)$

$$I_2 = \int_0^\infty \frac{x \sin ax}{1+x^2+x^4}\,dx \qquad (a>0)$$

Now $1+x^2+x^4 = (1+x^2)^2 - x^2 = (1+x+x^2)(1-x+x^2)$

and $\frac{1+x^2}{1+x^2+x^4} = \frac{1}{2}\left\{\frac{1}{1-x+x^2} + \frac{1}{1+x+x^2}\right\}$

$$\frac{x}{1+x^2+x^4} = \frac{1}{2}\left\{\frac{1}{1-x+x^2} - \frac{1}{1+x+x^2}\right\}$$

If $g(x) = \frac{1}{1-x+x^2}$

$I_1 = \frac{1}{2}\int_0^\infty [g(x)+g(-x)]\cos ax\,dx \;;\; I_2 = \frac{1}{2}\int_0^\infty [g(x)-g(-x)]\sin ax\,dx$

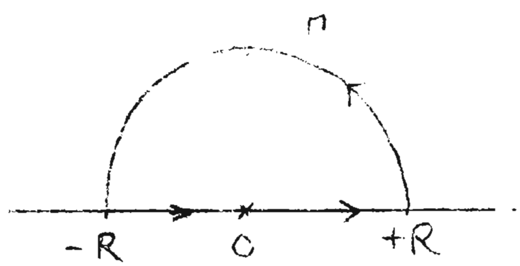
In $\int_0^\infty g(-x)\cos ax\,dx$ and $\int_0^\infty g(-x)\sin ax\,dx$, substitute $\xi = -x$, and thereafter write x for ξ again as the (dummy) variable of integration. In this way we find that

$$I_1 = \frac{1}{2}\int_{-\infty}^{\infty} g(x)\cos ax\,dx \;;\; I_2 = \frac{1}{2}\int_{-\infty}^{\infty} g(x)\sin ax\,dx,\; I_1 + iI_2 = \frac{1}{2}\int_{-\infty}^{\infty} g(x)e^{iax}\,dx$$

All the infinite integrals written above are convergent. Consider

$$\oint_C f(z)\,dz, \text{ where } f(z) = e^{iaz}g(z) = \frac{e^{iaz}}{1-z+z^2},$$

and C is the closed contour consisting of the real axis from $-R$ to $+R$ and the semicircle Γ, $|z| = R$, in the upper half-plane, $0 \le \arg z \le \pi$.



In the sequel let $R \to \infty$.

The integrand has simple poles at the zeros of $1 - z + z^2$, i.e., at $z = \alpha = \frac{1}{2} + \frac{\sqrt{3}}{2}i$, and at $z = \beta = \frac{1}{2} - \sqrt{3}\,i/2$. The pole at $z = \alpha$ is within C; the pole at $z = \beta$ is in the lower half-plane, outside C. If $\zeta = z - \alpha$,

$$f(z) = \frac{e^{ia(\zeta+\alpha)}}{1-\alpha+\alpha^2-\zeta+2\alpha\zeta+\zeta^2} = \frac{e^{ia\alpha}}{(2\alpha-1)}\,\frac{1}{\zeta}\,e^{ia\zeta}\left[1+\frac{\zeta}{2\alpha-1}\right]^{-1}$$

Solution No. 42, continued.

$$= \frac{e^{ia\alpha}}{\sqrt{3}\, i} \frac{1}{\rho}\left[1 + ia\rho + \ldots\right]\left[1 - \frac{\rho}{\sqrt{3}\, i} + \ldots\right]$$

since $1 - \alpha + \alpha^2 = 0$; $2\alpha - 1 = \sqrt{3}\, i$

The residue at $z = \alpha$ is

$$\frac{e^{ia\alpha}}{\sqrt{3}\, i} = \frac{e^{-\sqrt{3}\,a/2}\, e^{\frac{1}{2}ai}}{\sqrt{3}\, i};$$

On Γ, $|e^{iaz}| = |e^{ia(x+iy)}| = e^{-ay} \leq 1$, in the upper half-plane, $y \geq 0$, since $a > 0$.

Hence $|f(z)| = \dfrac{|e^{iaz}|}{|z-\alpha||z-\beta|} \leq \dfrac{1}{\{R - |\alpha|\}\{R - |\beta|\}}$, and $\left|\int_\Gamma f(z)\,dz\right| \leq \dfrac{\pi R}{\{R - |\alpha|\}\{R - |\beta|\}} \to 0$, as $R \to \infty$

Hence

$$\frac{1}{2}\int_{-\infty}^{\infty} g(x)\, e^{iax}\,dx = I_1 + i I_2 = \frac{1}{2}\frac{2\pi i}{\sqrt{3}\, i}\, e^{-\sqrt{3}\,a/2}\, e^{\frac{1}{2}ai}$$

$$= \frac{\pi}{\sqrt{3}}\, e^{-\sqrt{3}\,a/2}\left(\cos\frac{a}{2} + i \sin\frac{a}{2}\right),$$

$$I_1 = \frac{\pi}{\sqrt{3}}\, e^{-\sqrt{3}\,a/2} \cos a/2, \quad I_2 = \frac{\pi}{\sqrt{3}}\, e^{-\frac{\sqrt{3}\,a}{2}} \sin\frac{a}{2}$$

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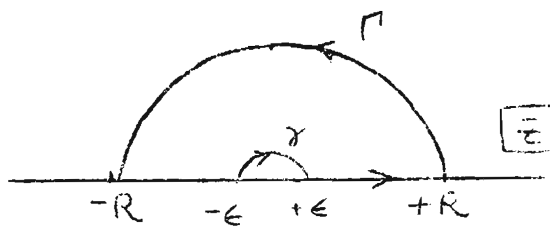
Solution No. 43.

Consider $\oint_C f(z)dz$, where C is the closed contour consisting of the real axis from $-R$ to $-\epsilon$, and from $+\epsilon$ to $+R$, the semicircle γ, $|z| = \epsilon$, in the upper half-plane, $0 \le \arg z \le \pi$, and the semicircle Γ, $|z| = R$ in the upper half-plane; and $f(z) = \frac{z^{a-1}}{1+z+z^2}$, $0 < \arg z < \pi$, for z within C. On the positive real axis, $\arg z = 0$; if $|z| = r$, then $z = r$, $z^{a-1}\,dz$ $r^{a-1}\,dr$, r goes from ϵ to R. On the negative real axis $\arg z = \pi$, $z = re^{i\pi}$, $z^{a-1} = r^{a-1}e^{i\pi(a-1)} = -r^{a-1}e^{i\pi a}$



and $dz = e^{i\pi}dr = -dr$, $z^{a-1}dz = r^{a-1}e^{i\pi a}dr$, $1+z+z^2 = 1-r+r^2$; and r goes from $+R$ to $+\epsilon$. (r is $|z|$ and is positive; it is <u>not</u> $R(z)$).

In the sequel, we let $\epsilon \to 0$ and $R \to \infty$.

The singularities of the integrand are the branch point at $z = 0$ and simple poles at $z = \alpha = -\frac{1}{2} + \sqrt{3}i/2$, $z = \beta = -\frac{1}{2} - \sqrt{3}i/2$ (α and β being the zeros of $1 + z + z^2$). The only singularity within C is at $z = \alpha$, and $\lim_{z\to\alpha} (z - \alpha) f(z) = \lim_{z\to\alpha} \frac{z^{a-1}}{z-\beta} = \frac{(\alpha)^{a-1}}{\alpha-\beta} = \frac{(\alpha)^{a-1}}{\sqrt{3}\,i}$.

Now $|\alpha| = \sqrt{\{\frac{1}{4}+\frac{3}{4}\}} = 1$, $\cos(\arg\alpha) = -\frac{1}{2}$, $\sin\arg\alpha = \sqrt{3}/2$, $0 < \arg\alpha < \pi$, so $\arg\alpha = \frac{2\pi}{3}$, $\alpha = e^{\frac{2\pi i}{3}}$

(This also follows from $z^3 - 1 = (z-1)(z^2+z+1) = (z-1)(z-\alpha)(z-\beta)$).

Hence $(\alpha)^{a-1} = e^{(2\pi/3)(a-1)i}$, and the residue at $z = \alpha$ is

$$\frac{1}{\sqrt{3}\,i}e^{(2\pi/3)(a-1)i}$$

Hence
$$\int_{+R}^{+\epsilon} e^{i\pi a}\frac{r^{a-1}dr}{1-r+r^2} + \int_{\epsilon}^{R}\frac{r^{a-1}dr}{1+r+r^2} + \int_\gamma f(z)dz + \int_\Gamma f(z)dz$$
$$= \frac{2\pi}{\sqrt{3}}e^{\frac{2\pi}{3}(a-1)i}$$

On Γ, $|f(z)| \le \frac{R^{a-1}}{\{R-|\alpha|\}\{R-|\beta|\}}$, $\left|\int_\Gamma f(z)dz\right| \le \frac{\pi R^a}{\{R-|\alpha|\}\{R-|\beta|\}} \to 0$ as $R \to \infty$, since $a < 2$

On γ, $|f(z)| \le \frac{\epsilon^{a-1}}{\{|\alpha|-\epsilon\}\{|\beta|-\epsilon\}}$, $\left|\int_\gamma f(z)dz\right| \le \frac{\pi\epsilon^a}{\{|\alpha|-\epsilon\}\{|\beta|-\epsilon\}}$ $\to 0$ as $\epsilon \to 0$, since $a > 0$.

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Solution No. 43, continued.

Hence

$$-e^{i\pi a}\int_0^\infty \frac{r^{a-1}\,dr}{1-r+r^2}+\int_0^\infty \frac{r^{a-1}\,dr}{1+r+r^2}=\frac{2\pi}{\sqrt{3}}e^{\frac{2\pi}{3}(a-1)i} \quad\text{.........(1)}$$

since $e^{\pi i}=1$, in (1)

We may now write, if we wish, x in place of r for the dummy variable of integration. We may write $\frac{2\pi}{\sqrt{3}}e^{\left(\frac{2\pi a}{3}-\frac{2\pi}{3}\right)i}=-\frac{2\pi}{\sqrt{3}}e^{\left(\frac{2\pi a}{3}+\frac{\pi}{3}\right)i}$

Then equate imaginary parts. We find that

$$\int_0^\infty \frac{x^{a-1}}{1-x+x^2}\,dx=\frac{2\pi}{\sqrt{3}}\sin\left(\frac{2\pi a}{3}+\frac{\pi}{3}\right)\Big/(\sin\pi a)$$

Divide (1) by $e^{i\pi a}$ $\longrightarrow$ and write $e^{-\frac{2\pi i}{3}}=e^{-\frac{1}{2}\pi i}e^{-\frac{\pi i}{6}}$ $=-i\,e^{-\frac{\pi i}{6}}$

Then $$-\int_0^\infty \frac{x^{a-1}\,dx}{1-x+x^2}+e^{-i\pi a}\int_0^\infty \frac{x^{a-1}\,dx}{1+x+x^2}=\frac{2\pi}{\sqrt{3}}e^{-\left(\frac{\pi a}{3}+\frac{2\pi}{3}\right)i}=-i\frac{2\pi}{\sqrt{3}}e^{-\left(\frac{\pi a}{3}+\frac{\pi}{6}\right)i}$$

Equate imaginary parts. We find that

$$\int_0^\infty \frac{x^{a-1}\,dx}{1+x+x^2}=\frac{2\pi}{\sqrt{3}}\,\frac{\cos\left(\frac{\pi a}{3}+\frac{\pi}{6}\right)}{\sin\pi a}$$

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Solution No. 44.

Let $I = P\int_0^\infty \frac{x^4\,dx}{x^6-1}$

Now $x^6-1=(x^3-1)(x^3+1)$ and $\frac{x^4}{x^6-1} = \frac{1}{2}\left[\frac{x}{x^3+1} + \frac{x}{x^3-1}\right]$

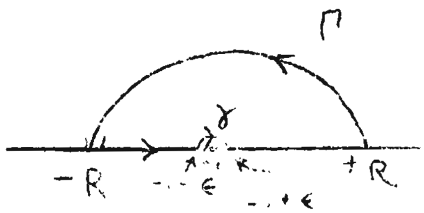
$$I = \frac{1}{2}P\int_0^\infty \left[\frac{x}{x^3+1}+\frac{x}{x^3-1}\right]dx = \frac{1}{2}P\int_0^\infty [f(x)+f(-x)]\,dx,$$

where $f(x) = \frac{x}{x^3+1}$.

If $\xi = -x$ in the second integral, we find that

$$I = +\frac{1}{2}\int_0^\infty f(x)\,dx + \frac{1}{2}P\int_{-\infty}^0 f(\xi)\,d\xi$$

i.e., $$I = \frac{1}{2}P\int_{-\infty}^\infty f(x)\,dx$$



Now consider $\oint_C f(z)\,dz$, where $f(z) = \frac{z}{z^3+1}$. C is the closed contour consisting of the real axis from $-R$ to $+R$, and the semicircle Γ, $|z| = R$, in the upper half-plane, $0 \le \arg z \le \pi$. Since $f(z)$ has a pole at $z = -1$, we indent the contour there by a small semicircle γ, $|z+1| = \epsilon$, in the upper half-plane. Along γ, $\arg(z+1)$ decreases from π to 0. In the sequel let $\epsilon \to 0$, $R \to \infty$.

$$\int_{-R}^{-1-\epsilon} f(x)\,dx + \int_{-1+\epsilon}^{R} f(x)\,dx + \int_\gamma f(z)\,dz + \int_\Gamma f(z)\,dz = 2\pi S i$$

where S is the sum of the residues of $f(z)$ at its poles within C. $f(z)$ has poles at the zeros of z^3+1. There are 3 zeros, at $z = -1$, $z = \alpha = e^{\pi i/3}$, $z = \beta = e^{-\pi i/3}$. Only the pole of $f(z)$ at $z = \alpha$ is within C, and the residue there $=$

$$\lim_{z\to\alpha} \frac{z(z-\alpha)}{z^3+1} = \alpha \lim_{z\to\alpha} \frac{z-\alpha}{z^3+1} = \alpha \lim_{z\to\alpha} \frac{1}{3z^2} = \frac{1}{3\alpha} = \frac{e^{-\frac{1}{3}\pi i}}{3}$$

$$= \frac{1}{3}\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right) = \frac{1}{3}\left(\frac{1}{2} - \frac{\sqrt{3}\,i}{2}\right) = \frac{1}{6} - \frac{\sqrt{3}\,i}{6}$$

Solution No. 44, continued.

The residue at $z = -1$ is $\lim_{z \to -1} \frac{(1+z)z}{1+z^3} = (-1) \lim_{z \to -1} \frac{1+z}{1+z^3} = (-1) \lim_{z \to -1} \frac{1}{3z^2}$

$$= -\frac{1}{3}$$

Hence, $\lim_{\epsilon \to 0} \int_\gamma f(z)\,dz = -\frac{1}{3}(-\pi i) = \frac{1}{3}\pi i$

Also on Γ,

$$|z f(z)| = \left|\frac{z^2}{z^3+1}\right| \leq \frac{|z|^2}{|z|^3-1} = \frac{R^2}{R^3-1} \to 0, \text{ as } R \to \infty$$

So

$$\int_\Gamma f(z)\,dz \to 0, \text{ as } R \to \infty$$

Hence $\mathcal{P}\int_{-\infty}^{\infty} f(x)\,dx + \frac{1}{3}\pi i = 2\pi\left(\frac{1}{6} - \frac{\sqrt{3}\,i}{6}\right) = \frac{1}{3}\pi i + \frac{\sqrt{3}\pi}{3}$

and $I = \frac{1}{2}\mathcal{P}\int_{-\infty}^{\infty} f(x)\,dx = \frac{\sqrt{3}\pi}{6}$

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Solution No. 45.

Let $$I = \int_{-\infty}^{\infty} \frac{\sinh ax}{\sinh \pi x}\, dx$$

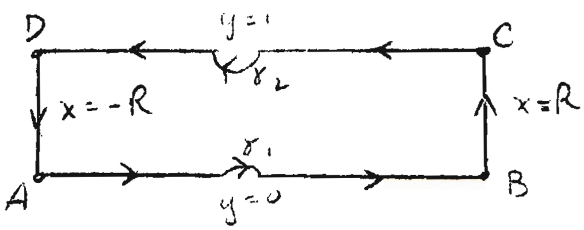
Since $-\pi < a < \pi$; $\int^{\infty} \frac{\sinh ax}{\sinh \pi x}\, dx$ and $\int_{-\infty} \frac{\sinh ax}{\sinh \pi x}\, dx$

converge , as we see by writing

$$\sinh ax = \frac{1}{2}(e^{ax} - e^{-ax}); \quad \sinh \pi x = \frac{1}{2}(e^{\pi x} - e^{-\pi x})$$

The integrand is an even function of x, and $I = 2\int_0^{\infty} \frac{\sinh ax}{\sinh \pi x}\, dx$

Consider $\oint_C f(z)\,dz$, where $f(z) = e^{az}/\sinh \pi z$, and C is the rectangle of sides y = 0, y = 1, x = ± R, indented at z = 0 and z = i by the semicircles γ_1 and γ_2, of radius ϵ, and with centers at z = 0, z = i respectively. In the sequel, let $\epsilon \to 0$ and $R \to \infty$. f(z) has simple poles at the zeros of $\sinh \pi z$, i.e., at $z = ni$ ($n = 0, \pm 1, \pm 2, \ldots$). There are no singularities of f(z) within C. The residue at z = 0 is



$$\lim_{z \to 0} \frac{z e^{az}}{\sinh \pi z} = \frac{1}{\pi}$$; On γ_1, arg z

decreases by π. Hence $\int_{\gamma_1} f(z)\,dz \to -\pi i \left(\frac{1}{\pi}\right) = -i$, as $\epsilon \to 0$

The residue at $z = i$ is $$\lim_{z \to i} \frac{(z-i)e^{az}}{\sinh \pi z} = e^{ai} \lim_{z \to i} \frac{z-i}{\sinh \pi z}$$

$$= e^{ai} \lim_{z \to i} \frac{1}{\pi \cosh \pi z}$$

On γ_2, $\arg(z-i)$ decreases by π. Hence $\int_{\gamma_2} f(z)\,dz \to -\pi i\left(-\frac{e^{ai}}{\pi}\right)$

$$= i e^{ai} \quad \text{as } \epsilon \to \infty$$

On BC ($x = R$), $z = R + iy$, and

$$|f(z)| = \left| \frac{2e^{aR + ayi}}{e^{\pi(R+iy)} - e^{-\pi(R+iy)}} \right|$$

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Solution No. 45, continued.

Since, if $|w_1| > |w_2|$, $|w_1 - w_2| \geq |w_1| - |w_2|$; $\left|e^{\pi(R+iy)} - e^{-\pi(R+iy)}\right| \geq e^{\pi R} - e^{-\pi R}$

$$|f(z)| \leq \frac{2e^{aR}}{e^{\pi R} - e^{-\pi R}} \to 0, \text{ as } R \to \infty, \text{ since } a < \pi$$

In exactly the same way, on DA

$$|f(z)| \leq \frac{2e^{-aR}}{e^{\pi R} - e^{-\pi R}} \to 0, \text{ as } R \to \infty, \text{ since } a > -\pi;\ -a < \pi.$$

BC and DA are of length 1, and therefore

$$\int_{BC} f(z)\,dz;\ \int_{DA} f(z)\,dz \text{ both} \to 0, \text{ as } R \to \infty$$

After we have let $\epsilon \to 0$, $R \to \infty$

$$\int_{AB} f(z)\,dz = P\int_{-\infty}^{\infty} \frac{e^{ax}}{\sinh \pi x}\,dx$$

$$\int_{CD} f(z)\,dz = P\int_{+\infty}^{-\infty} \frac{e^{a(x+i)}\,dx}{\sinh \pi(x+i)} = e^{ai}\, P\int_{-\infty}^{\infty} \frac{e^{ax}}{\sinh \pi x}\,dx$$

$$\text{since } \sinh \pi(x+i) = -\sinh x$$

$$\left[\int^{\infty} \frac{e^{ax}}{\sinh \pi x}\,dx,\ \int_{-\infty} \frac{e^{ax}}{\sinh \pi x}\,dx \text{ converge for } -\pi < a < \pi\right]$$

Hence, after we have let $\epsilon \to 0$, $R \to \infty$, we obtain

$$(1+e^{ia})\,P\int_{-\infty}^{\infty} \frac{e^{ax}}{\sinh \pi x}\,dx - i(1-e^{ia}) = 0$$

$$P\int_{-\infty}^{\infty} \frac{e^{ax}\,dx}{\sinh \pi x} = -i\,\frac{e^{ia}-1}{e^{ia}+1} = -i\,\frac{e^{\frac{a}{2}i} - e^{-\frac{a}{2}i}}{e^{\frac{ai}{2}} + e^{-\frac{ai}{2}}}$$

$$= -i\,\frac{2i \sin\frac{a}{2}}{2\cos\frac{a}{2}} = \tan\frac{a}{2}.$$

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Solution No. 45, continued.

But $e^{ax} = \cosh ax + \sinh ax$

$$P\int_{-\infty}^{\infty} \frac{\cosh ax}{\sinh \pi x}\,dx + \int_{-\infty}^{\infty} \frac{\sinh ax}{\sinh \pi x}\,dx = \tan \tfrac{1}{2} a$$

(No "P" is required in front of the second integral, since $\frac{\sinh ax}{\sinh \pi x}$ has no discontinuity at x = 0.)

$\frac{\cosh ax}{\sinh \pi x}$ is odd in x, so $\int_{-R}^{-\epsilon} + \int_{\epsilon}^{R} \frac{\cosh ax}{\sinh \pi x}\,dx = 0$

and $P\int_{-\infty}^{\infty} \frac{\cosh ax}{\sinh \pi x}\,dx = 0$

Hence $\int_{-\infty}^{\infty} \frac{\sinh ax}{\sinh \pi x}\,dx = 2\int_{0}^{\infty} \frac{\sinh ax\,dx}{\sinh \pi x} = \tan \tfrac{1}{2} a$

[Or change a into -a:

$$P\int_{-\infty}^{\infty} \frac{e^{ax}}{\sinh \pi x}\,dx = \tan \tfrac{1}{2} a \quad (-\pi < a < \pi)$$

$$P\int_{-\infty}^{\infty} \frac{e^{-ax}}{\sinh \pi x}\,dx = -\tan \tfrac{1}{2} a \quad (-\pi < a < \pi)$$

and $\int_{-\infty}^{\infty} \frac{\sinh ax}{\sinh \pi x}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{ax} - e^{-ax}}{\sinh \pi x}\,dx$

$$= \frac{1}{2} P\int_{-\infty}^{\infty} \frac{e^{ax}}{\sinh \pi x}\,dx - \frac{1}{2} P\int_{-\infty}^{\infty} \frac{e^{-ax}}{\sinh \pi x}\,dx$$

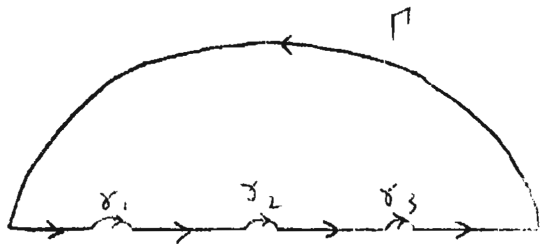
$$= \tan \tfrac{1}{2} a$$

-be- 12.15.55

Solution No. 46.

Consider $\oint_C f(z)\,dz$, where $f(z) = \dfrac{z^b}{z^2-1}$, and C is the contour consisting of the real axis from $-R$ to $+R$, and the semicircle Γ, $|z| = R$, in the upper half plane, $0 \le \arg z \le \pi$, indented at $z = -1, 0, +1$, by small semicircles γ_1, γ_2, γ_3 in the upper half-plane, with centers at $z = -1, 0, +1$, and of radii, ϵ_1, ϵ_2, ϵ_3. In the sequel, let $R \to \infty$, $\epsilon_1, \epsilon_2, \epsilon_3$ all $\to 0$.

f(z) has no singularities within C. The residues of f(z) at $z = \pm 1$ are



$\lim_{z\to\pm1} \dfrac{z^b(z\mp1)}{z^2-1}$, respectively,

i.e., $\lim_{z\to\pm1} \dfrac{z}{z\pm1}$ respectively,

i.e., $\frac{1}{2}, -\frac{1}{2}e^{\pi b i}$ respectively,

On γ_1 and γ_2, arg (z + 1) and arg (z − 1), respectively, decrease by π. So

$$\lim_{\epsilon_1\to0}\int_{\gamma_1} f(z)\,dz = -\pi i\left(\tfrac{1}{2}\right) = -\tfrac{1}{2}\pi i, \quad \lim_{\epsilon_3\to0}\int_{\gamma_3} f(z)\,dz = -\pi i\left(-\tfrac{1}{2}e^{\pi b i}\right) = \tfrac{1}{2}\pi i\, e^{\pi b i}$$

$$\text{also } |z f(z)| = \left|\frac{z^{b+1}}{z^2-1}\right| \le \frac{r^{b+1}}{1-r^2} \text{ for } r<1, \text{ when } |z| = r$$

So $|z f(z)| \to 0$, as $|z| \to 0$, for $b > -1$, and $\lim_{\epsilon_2\to0}\int_{\gamma_2} f(z)\,dz = 0$

On Γ, with $r > 1$, $|z f(z)| \le \dfrac{r^{b+1}}{r^2-1} \to 0$ for $b+1<2$; $b<1$

Hence $\int_\Gamma f(z)\,dz \to 0$, as $R \to \infty$

On the positive real axis, $z = r$. On the negative real axis, $z = re^{i\pi}$.

Hence $$P\int_0^\infty \frac{r^b}{r^2-1}\,dr + P\int_\infty^0 \frac{(re^{i\pi})^b e^{i\pi}}{r^2-1}\,dr - \frac{1}{2}\pi i\left(1-e^{\pi b i}\right) = 0$$

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Solution No. 46, continued.

i.e., $$(1+e^{\pi bi})\, P\int_0^\infty \frac{r^b}{r^2-1}\,dr = \tfrac{1}{2}\pi i\,(1-e^{\pi bi})$$

$$P\int_0^\infty \frac{r^b}{r^2-1}\,dr = -\tfrac{1}{2}\pi i\,\frac{e^{\pi bi}-1}{e^{\pi bi}+1} = \tfrac{1}{2}\pi i\,(1-e^{\pi bi})$$

$$P\int_0^\infty \frac{r^b}{r^2-1}\,dr = -\tfrac{1}{2}\pi i\,\frac{e^{\pi bi}-1}{e^{\pi bi}+1} = -\tfrac{1}{2}\pi i\,\frac{e^{\frac{1}{2}\pi bi}-e^{-\frac{1}{2}\pi bi}}{e^{\frac{1}{2}\pi bi}+e^{-\frac{1}{2}\pi bi}}$$

$$= -\tfrac{1}{2}\pi i\,\frac{2i\sin\frac{1}{2}\pi b}{2\cos\frac{1}{2}\pi b} = \tfrac{1}{2}\pi\tan\tfrac{1}{2}\pi b \quad (-1<b<1)$$

Since $-1 < -b < 1$, if $-1 < b < 1$, by changing the sign of b we have

$$P\int_0^\infty \frac{r^{-b}}{r^2-1}\,dr = -\tfrac{1}{2}\pi\tan\tfrac{1}{2}\pi b$$

Subtract the results, [No "P" is required now in front of the integral, since the integral is not infinite at $r = 1$.]. Change the variable of integration into x.

$$\frac{1}{\pi}\int_0^\infty \frac{x^b-x^{-b}}{x^2-1}\,dx = \tan\tfrac{1}{2}\pi b \qquad (-1<b<1)$$

: Put $x = e^{\pi\xi}$, $dx = \pi e^{\pi\xi}\,d\xi$

Also, let $\pi b = a$

$$\frac{1}{\pi}\int_0^\infty \frac{x^b-x^{-b}}{x^2-1}\,dx = \frac{1}{\pi}\int_{-\infty}^\infty \frac{e^{a\xi}-e^{-a\xi}}{e^{2\pi\xi}-1}\,\pi e^{\pi\xi}\,d\xi$$

$$= \int_{-\infty}^\infty \frac{e^{a\xi}-e^{-a\xi}}{e^{\pi\xi}-e^{-\pi\xi}}\,d\xi = \int_{-\infty}^\infty \frac{\sinh a\xi}{\sinh\pi\xi}\,d\xi$$

$$= \tan\tfrac{1}{2}a.$$

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Solution No. 47.

Since the integrand is an even function of x, $\int_0^{\pi} \frac{x \sin x}{1-2a\cos x+a^2}\,dx = \frac{1}{2}\int_{-\pi}^{\pi} \frac{x \sin x}{1-2a\cos x+a^2}\,dx$

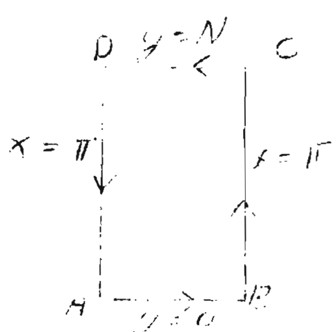
$1-2a\cos x+a^2 = (a-\cos x)^2+\sin^2 x = (a-\cos x+i\sin x)(a-\cos x-i\sin x)$,

so $\frac{x}{a-\cos x+i\sin x} = \frac{x(a-\cos x)-ix\sin x}{1-2a\cos x+a^2}$

and $$\int_{-\pi}^{\pi} \frac{x\sin x}{1-2a\cos x+a^2}\,dx = -\mathcal{I}\int_{-\pi}^{\pi}\frac{x}{a-\cos x+i\sin x}\,dx$$

$$= -\mathcal{I}\int_{-\pi}^{\pi}\frac{x}{a-e^{-ix}}\,dx.$$

Consider $\oint_C f(z)\,dz$, where $f(z)=\frac{z}{a-e^{-iz}}$ and C is the rectangle of sides $y=0$, $y=N$, $x=\pm\pi$.



f(z) has poles at the zeros of $a-e^{-iz}$.

$e^{-iz} = a = e^{\log a - 2n\pi i}$

$(n = 0, \pm 1, \pm 2, \ldots.)$

at $-iz = \log a - 2n\pi i$

$z = i\log a + 2n\pi$

With $a > 1$, $\log a > 0$, and there is a simple pole at $z = i\log a$ within C.

All the other singularities are outside C.

The residue at $z = i\log a$ is $\lim_{z\to i\log a}\frac{(z-i\log a)z}{a-e^{-iz}}$

$$= i\log a \lim_{z\to i\log a}\frac{z-i\log a}{a-e^{-iz}} = i\log a \lim_{z\to i\log a}\frac{1}{ie^{-iz}}$$

$$= \frac{i\log a}{ie^{-\log a}} = \frac{\log a}{a}.$$

Solution No. 47, continued.

$$\int_{BC} f(z)\,dz + \int_{DE} f(z)\,dz = \int_0^N \frac{\pi + iy}{a - e^{-i(\pi+iy)}}\, i\,dy$$

$$+ \int_N^0 \frac{-\pi + iy}{a - e^{-i(-\pi+iy)}}\, i\,dy = \int_0^N \frac{\pi + iy}{a + e^y}\, i\,dy$$

$$- \int_0^N \frac{-\pi + iy}{a + e^y}\, i\,dy = \int_0^N \frac{2\pi i\,dy}{a + e^y} = 2\pi i \int_0^N \frac{dy}{a + e^y}$$

On $y = N$, $f(z) = \dfrac{x + iN}{a - e^{-i(iN+x)}} = \dfrac{x + iN}{a - e^{N - ix}}$

$$|f(z)| \le \frac{(N^2 + x^2)^{\frac{1}{2}}}{e^N - a} \longrightarrow 0 \text{ as } N \longrightarrow \infty.$$

The length of CD is 2π, so $\displaystyle\int_{CD} f(z)\,dz \to 0$ as $N \to \infty$.

Let $N \to \infty$. We obtain

$$\int_{-\pi}^{\pi} \frac{x}{a - e^{-ix}}\,dx + 2\pi i \int_0^{\infty} \frac{dy}{a + e^y} = 2\pi i\, \frac{\log a}{a}.$$

Now, if we substitute $e^y = t$, we have $\displaystyle\int_0^{\infty} \frac{dy}{a + e^y} = \int_1^{\infty} \frac{dt}{t(a+t)}$

$\left(\text{since } \dfrac{dt}{t} = dy\right)$

$$= \frac{1}{a}\int_1^{\infty} \left[\frac{1}{t} - \frac{1}{a+t}\right] dt = \frac{1}{a}\left[\log \frac{t}{a+t}\right]_{t=1}^{t=\infty} = -\frac{1}{a}\log \frac{1}{1+a} = \frac{1}{a}\log(1+a).$$

Hence $\displaystyle\int_{-\pi}^{\pi} \frac{x}{a - e^{-ix}} = \frac{2\pi i}{a}\left[\log a - \log(1+a)\right]$

$$= -\frac{2\pi i}{a} \log \frac{1+a}{a}.$$

i.e., $$\int_{-\pi}^{\pi} \frac{x(a - \cos x) - i x \sin x}{1 - 2a\cos x + a^2}\,dx = \int_{-\pi}^{\pi} \frac{x(a - \cos x)}{1 - 2a\cos x + a^2}\,dx$$

$$- i \int_{-\pi}^{\pi} \frac{x \sin x}{a^2 - 2a\cos x + a^2}\,dx = -\frac{2\pi i}{a} \log \frac{1+a}{a}.$$

Solution No. 47, continued.

Take imaginary parts.

$$\int_{-\pi}^{\pi} \frac{x \sin x}{1 - 2a \cos x + a^2}\, dx = \frac{2\pi}{a} \log \frac{1+a}{a}.$$

The integrand is an even function of x. Hence

$$\int_{0}^{\pi} \frac{x \sin x}{1 - 2a \cos x + a^2}\, dx = \frac{\pi}{a} \log \frac{1+a}{a}.$$

[The main reason for considering the integral from $-\pi$ to π, instead of from 0 to π, is that $a - e^{-iz}$ has a period 2π, so that the denominator in $f(z)$ has the same value $(a + e^2)$ on BC and AD. The pole is on $x = 0$, but we could always indent there, and avoid that difficulty. But because the numerator is $a - e^y$ on $x = 0$, the calculation becomes much harder.]

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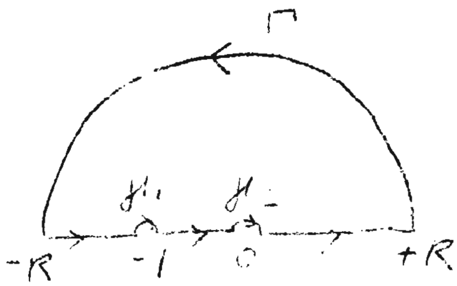
Solution No. 48.

Consider $\oint_C f(z)\,dz$, where $f(z) = \frac{\log z}{1+z^5}$, and C is the contour consisting of the real axis from $-R$ to $+R$, and the semicircle Γ, $|z| = R$, in the upper half-plane $0 \le \arg z \le \pi$, indented at $z = -1$ and $z = 0$ by small semicircles, γ_1 and γ_2, in the upper half-plane, with centers at $z = -1$ and $z = 0$, and with radii ϵ_1 and ϵ_2, respectively. Let $z = re^{i\theta}$, $\log z = \log r + i\theta$, in C, where $0 < \theta < \pi$.
On the positive real axis, $\log z = \log r$; on the negative real axis, $\log z = \log r + \pi i$. Since $|z f(z)| = \left|\frac{z \log z}{1+z^5}\right| \to 0$ uniformly in arg z both as $r \to \infty$ and as $r \to 0$,



$\int_{\gamma_2} f(z)\,dz \to 0$ as $\epsilon_2 \to 0$,
$\int_{\Gamma} f(z)\,dz \to 0$ as $R \to \infty$.
$f(z)$ has poles when $z^5 + 1 = 0$, i.e., at $z = -1$, $z = \alpha = e^{\frac{1}{5}\pi i}$, $z = \beta$ i, $z = e^{-\frac{1}{5}\pi i}$, $z = e^{-\frac{3}{5}\pi i}$.

The poles at $z = \alpha$, $z = \beta$ are within C; the others are outside C. The residue at $z = -1$ is $\lim_{z \to e^{\pi i}} \frac{(z+1)\log z}{1+z^5} = \log(e^{\pi i}) \lim_{z \to -1} \frac{z+1}{1+z^5}$

$$= \pi i \lim_{z \to -1} \frac{1}{5z^4} = \frac{1}{5}\pi i$$

Hence, as $\epsilon_1 \to 0$, $\int_{\gamma_1} f(z)\,dz \to -\pi i\left(\frac{1}{5}\pi i\right) = \frac{\pi^2}{5}$

The residue at $z = \mu$, where $\mu^5 = -1$ is $\lim_{z \to \mu} \frac{(z-\mu)\log z}{z^5+1} = \log \mu \lim_{z \to \mu} \frac{1}{5z^4}$

$$= \frac{1}{5\mu^4}\log \mu = \frac{\mu}{5\mu^5}\log \mu = -\frac{1}{5}\mu \log \mu$$

The sum of the residues at α and β is $-\frac{1}{5}(\alpha \log \alpha + \beta \log \beta) =$

$$= -\frac{1}{5}\left(\frac{1}{5}\pi i e^{\frac{1}{5}\pi i} + \frac{3}{5}\pi i e^{\frac{3}{5}\pi i}\right) = -\frac{\pi i}{25}\left(e^{\frac{1}{5}\pi i} + 3e^{\frac{3}{5}\pi i}\right)$$

Hence $\int_{\epsilon_2}^{R} \frac{\log r}{1+r^5}\,dr + P\int_{+R}^{\epsilon_2} \frac{\log r + i\pi}{1+r^5} e^{i\pi}\,dr + \frac{\pi^2}{5} = -\frac{2\pi^2 i^2}{25}\left(e^{\frac{1}{5}\pi i} + 3e^{\frac{3}{5}\pi i}\right)$

Solution No. 48, continued.

We have taken the limit as $\varepsilon_1 \to 0$. Let $R \to \infty$, $\varepsilon_2 \to 0$.

$$P\int_0^\infty \left\{ \log r \left[\frac{1}{1+r^5} + \frac{1}{1-r^5} \right] + \frac{i\pi}{1-r^5} \right\} dr + \frac{\pi^2}{5} = \frac{2\pi^2}{25}\left[e^{\frac{1}{5}\pi i} + 3e^{\frac{3}{5}\pi i} \right]$$

Since $\frac{1}{1+r^5} + \frac{1}{1-r^5} = \frac{2}{1-r^{10}}$, and $e^{\frac{3}{5}\pi i} = -e^{-\frac{2}{5}\pi i}$,

$$2P\int_0^\infty \frac{\log r}{1-r^{10}}\,dr + i\pi P\int_0^\infty \frac{dr}{1-r^5} = -\frac{\pi^2}{5} + \frac{2\pi^2}{25}\left[e^{\frac{1}{5}\pi i} - 3e^{-\frac{2}{5}\pi i} \right]$$

Take real and imaginary parts. Write x in place of r for the variable of integration.
Change the sign in the real part.

$$P\int_0^\infty \frac{\log x}{x^{10}-1}\,dx = \frac{\pi^2}{25}\left\{ \frac{5}{2} - \cos\frac{\pi}{5} + 3\cos\frac{2\pi}{5} \right\}$$

$$P\int_0^\infty \frac{dx}{1-x^5} = \frac{2\pi}{25}\left[\sin\frac{\pi}{5} + 3\sin\frac{2\pi}{5} \right]$$

Solution No. 49.

Let

$$I = \int_0^{\frac{1}{2}\pi} \frac{a \sin 2\theta}{1 - 2a\cos 2\theta + a^2}\,\theta\, d\theta \qquad \text{(A real, } a \neq 1\text{)}$$

Put $x = \tan\theta$. Then $\sin 2\theta = \frac{2x}{1+x^2}$, $\cos 2\theta = \frac{1-x^2}{1+x^2}$, $d\theta = \frac{dx}{\sec^2\theta} = \frac{dx}{1+\tan^2\theta} = \frac{dx}{1+x^2}$

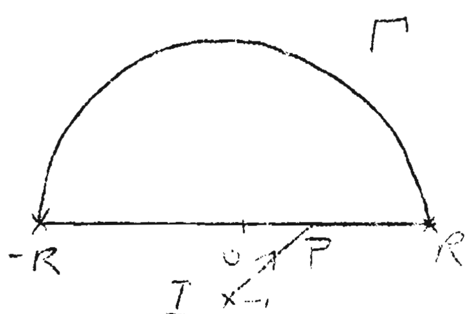
$$I = \int_0^{\infty} \frac{2a\,x\tan^{-1}x}{\{1+x^2\}\{(1-a)^2+(1+a)^2x^2\}}\,dx$$

The integrand is even in x, so

$$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{2a\,x\tan^{-1}x}{\{1+x^2\}\{(1-a)^2+(1+a)^2x^2\}}\,dx \qquad \text{where } -\tfrac{1}{2}\pi \leq \tan^{-1}x \leq \tfrac{1}{2}\pi$$

Consider $\oint_C f(z)\,dz$, where $f(z) = \frac{2a\,z\log(1-iz)}{\{1+z^2\}\{A^2+B^2z^2\}}$

with $A = |1-a|$, $B = |1+a|$;



and C is the contour consisting of the real axis from $-R$ to R and the semicircle Γ, $|z| = R$, in the upper half-plane, $0 \leq \arg z \leq \pi$.

$\log(1-iz)$ has a branch point at $z = -i$, which is outside C. Define the value of $\log(1-iz)$ within and on C as follows; $\log(1-iz) = \log(-i) + \log(z+i) = -\frac{1}{2}\pi i + \log(z+i)$.

If I represents $-i$ and P represents z in the Argand diagram, $z+i$ may be represented by the vector IP. As z goes from $+\infty$ to $-\infty$ on the real axis, the vector rotates through the angle π in the positive direction - i.e., $\arg(z+i)$ increases by π. When z is at $+\infty$, $\overrightarrow{IP}$ is parallel to, and in the same sense as, the positive real axis, so we may take $\arg(z+i) = 0$.

Hence on the whole real axis $0 \leq \arg(z+i) \leq \pi$. For any point within C, $0 < \arg(z+i) < \pi$ for any R. Then $\log(1-iz) = \log|z+i| + \arg(z+i) - \frac{1}{2}\pi\, i$, i.e., $-\frac{1}{2}\pi < \operatorname{Im}\log(1-iz) < \frac{1}{2}\pi$. On the real axis, therefore,

$$\log(1-iz) = \log(1-ix) = \log(1+x^2)^{\frac{1}{2}} - i\tan^{-1}x, \quad -\tfrac{1}{2}\pi \leq \tan^{-1}x \leq \tfrac{1}{2}\pi.$$

-be-

Solution No. 49, continued.

The poles of $f(z)$ within C are at $z = i$, $z = \frac{iA}{B}$. The residue at $z = i$ is

$$\lim_{z\to i} \frac{2a(z-i)z\log(1-iz)}{(1+z^2)(A^2+B^2z^2)} = \frac{2ai\log 2}{A^2-B^2}\lim_{z\to i}\frac{z-i}{(z-i)(z+i)} = \frac{2ai\log 2}{2i(A^2-B^2)} = \frac{a\log 2}{A^2-B^2}$$

But $A^2 - B^2 = -4a$. So the residue at $z = i$ is $-\frac{1}{4}\log 2$.

The residue at $z = \frac{iA}{B}$ is similarly

$$\frac{2a\left(i\frac{A}{B}\right)\log\left(1+\frac{A}{B}\right)}{\left(1-\frac{A^2}{B^2}\right)B^2}\lim_{z\to i\frac{A}{B}}\frac{z-i\frac{A}{B}}{z^2+\frac{A^2}{B^2}} = \frac{2i\frac{A}{B}a\log\left(1+\frac{A}{B}\right)}{B^2-A^2}\,\frac{1}{2i\frac{A}{B}} =$$

$$= \frac{a\log\left(1+\frac{A}{B}\right)}{B^2-A^2} = a\log\left(1+\frac{A}{B}\right)/4a = \frac{1}{4}\log\left(1+\frac{A}{B}\right)$$

As $|z| \to \infty$, $z f(z) = 2a\frac{z^2}{(1+z^2)(A^2+B^2z^2)}\log(1-iz) \to 0$ uniformly in $\arg z$ for $0 \le \arg z \le \pi$. Hence $\int_\Gamma f(z)\,dz \to 0$ as $R \to \infty$.

Let $R \to \infty$. Then

$$\int_{-\infty}^{\infty}\frac{2ax\log(1-ix)}{(1+x^2)(A^2+B^2x^2)}\,dx = (2\pi i)\frac{1}{4}\left[\log\left(1+\frac{A}{B}\right) - \log 2\right]$$

Take imaginary parts, and divide by -2.

$$I = \frac{1}{2}\int_{-\infty}^{\infty}\frac{2ax\tan^{-1}x}{(1+x^2)(A^2+B^2x^2)}\,dx = -\frac{\pi}{4}\log\frac{A+B}{2B}$$

When $-1 < a < 1$, $A = 1-a$, $B = 1+a$, $A+B = 2$,

$$I = -\frac{\pi}{4}\log\frac{1}{1+a} = \frac{\pi}{4}\log(1+a)$$

When $a > 1$, $A = a-1$, $B = a+1$, $A+B = 2a$, $\frac{A+B}{2B} = \frac{a}{a+1}$.

When $a < -1$, $A = 1-a$, $B = -(a+1)$, $A+B = -2a$, $\frac{A+B}{2B} = \frac{a}{a+1}$.

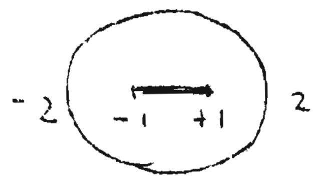
Hence when $a > 1$ or $a < -1$,

$$I = -\frac{\pi}{4}\log\frac{a}{a+1} = \frac{\pi}{4}\log\left(\frac{a+1}{a}\right) = \frac{\pi}{4}\log\left(1+\frac{1}{a}\right)$$

be

Solution No. 50.

If the z-plane be cut from -1 to +1, $\log \frac{z+1}{z-1}$ is single-valued in the cut plane, since $\arg(z+1)$ and $\arg(z-1)$ both increase by 2π when we go round a contour enclosing the cut, and $\arg(z+1)$ and $\arg(z-1)$ both return to their original values as we go round a contour enclosing no point of the cut. The cut lies entirely within C.



Since $\oint_C z^2 = 0$, and since any two branches of $\log \frac{z+1}{z-1}$ differ by a constant when z is on C,

$\oint_C z^2 \log\frac{z+1}{z-1}\, dz$ has the same value whatever branch of the logarithm be chosen.

z^3 and $\log \frac{z+1}{z-1}$ are regular in a domain in the cut z-plane which includes C, so we may integrate by parts round C.

$$\oint_C z^2 \log\frac{z+1}{z-1}\, dz = \frac{1}{3}\oint_C \left(\frac{d}{dz} z^3\right) \log\frac{z+1}{z-1}\, dz =$$

$$= \left[\frac{z^3}{3}\log\frac{z+1}{z-1}\right] - \oint_C \frac{z^3}{3}\left(\frac{1}{z+1} - \frac{1}{z-1}\right) dz$$

$\log \frac{z+1}{z-1}$ returns to its original value after one complete circuit of C. (It is single-valued in the cut plane.) z^3, of course, also returns to its original value. Hence

$$\left[\frac{z^3}{3}\log\frac{z+1}{z-1}\right] = 0$$

and $\oint_C z^2 \log\frac{z+1}{z-1}\, dz = -\frac{1}{3}\oint_C z^3\left(\frac{1}{z+1} - \frac{1}{z-1}\right) dz$

There are poles of the integrand on the right at $z = -1$ and at $z = 1$. The residue at $z = -1$ is $(-1)^3$; the residue at $z = 1$ is $-(+1)^3$. The sum of the residues is -2, and the value of the integral on the right is $-4\pi i$.

Hence $\oint_C z^2 \log\frac{z+1}{z-1}\, dz = \frac{4}{3}\pi i$

-be

Solution No. 51.

Since $f(z) = \dfrac{1}{1+z^4}$, the integrand has poles at $z^4 = -1$, i.e., at $z = e^{\frac14 \pi i}$, $e^{-\frac14 \pi i}$, $e^{-\frac34 \pi i}$, $e^{3/4 \pi i}$

i.e., at $z = \frac{1}{\sqrt2}(1+i)$, $\frac{1}{\sqrt2}(1-i)$, $\frac{1}{\sqrt2}(-1+i)$, $\frac{1}{\sqrt2}(-1-i)$.

The (x,y) coordinates of these poles are $(\frac{1}{\sqrt2}, \frac{1}{\sqrt2})$, $(\frac{1}{\sqrt2}, -\frac{1}{\sqrt2})$, $(-\frac{1}{\sqrt2}, \frac{1}{\sqrt2})$, $(-\frac{1}{\sqrt2}, -\frac{1}{\sqrt2})$.

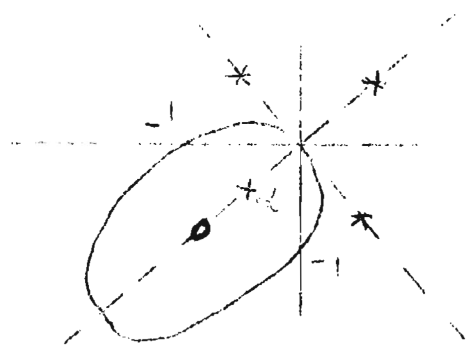
Let $F(x,y) = x^2 - xy + y^2 + x + y$. Then C is the curve $F(x,y) = 0$.

Since $F(\infty, 0) > 0$, points for which $F(x,y) > 0$ are outside C, and points for which $F(x,y) < 0$ are within C.

$F(\frac{1}{\sqrt2}, \frac{1}{\sqrt2}) = \frac12 - \frac12 + \frac12 + \frac{1}{\sqrt2} + \frac{1}{\sqrt2} > 0.$
$F(1/\sqrt2, -1/\sqrt2) = 1/2 + 1/2 + 1/2 + 1/\sqrt2 - 1/\sqrt2 > 0.$
$F(-1/\sqrt2, 1/\sqrt2) = 1/2 + 1/2 + 1/2 - 1/\sqrt2 + 1/\sqrt2 > 0.$
$F(-1/\sqrt2, -1/\sqrt2) = 1/2 - 1/2 + 1/2 - 1/\sqrt2 - 1/\sqrt2 = 1/2 - \sqrt2 < 0.$

Only the pole at $z = \alpha$ (say) $= e^{-\frac34 \pi i}$ is within C.

C is an ellipse; the equation is $\frac32\left(\frac{x+y}{\sqrt2}\right)^2 + \frac12\left(\frac{x-y}{\sqrt2}\right)^2 + \sqrt2\left(\frac{x+y}{\sqrt2}\right) = 0$, and by rotating the axes through 45°, C is easily seen to be the ellipse with center at $(-1, -1)$ with semi-axes of lengths $\sqrt2$, $\sqrt{2/3}$, and principal axes along $x = y$, and parallel to $x = -y$. Also $x = -y$ is tangent to the ellipse.



x - poles
o - center of ellipse

The residue at $z = \alpha$ is $\lim\limits_{z \to \alpha} \dfrac{z-\alpha}{1+z^4} = \lim\limits_{z \to \alpha} \dfrac{1}{4z^3} = \dfrac{1}{4\alpha^3} = \dfrac{\alpha}{4\alpha^4} = -\dfrac{\alpha}{4}$

$$= -\frac14 e^{-3/4 \pi i} = \frac14 e^{1/4 \pi i}$$

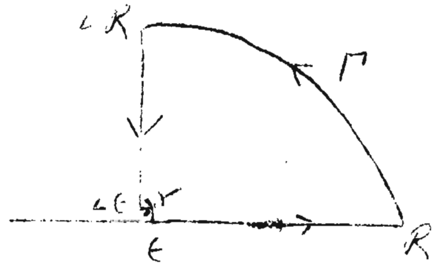
Hence

$$\oint_C f(z)\,dz = \frac{2\pi i}{4} e^{\frac14 \pi i} = \frac{2\pi i}{4\sqrt2}(1+i) = \frac{\pi\sqrt2}{4}(-1+i)$$

-be-

Solution No. 52.

Consider $\oint_C f(z)\,dz$, where $f(z) = z^{-\frac{1}{2}}e^{iz}$, and C is the contour consisting of the positive real axis from ϵ to R, the quadrant Γ of $|z| = R$ for which $0 \le \arg z \le \frac{1}{2}\pi$, the positive imaginary axis from iR to $i\epsilon$, and the quadrant γ of $|z| = \epsilon$ for which $\frac{1}{2}\pi \ge \arg z \ge 0$. Within C, let $z = re^{i\theta}$ $(0 < \theta < \frac{1}{2}\pi)$, $z^{\frac{1}{2}} = r^{\frac{1}{2}} e^{\frac{1}{2}i\theta}$. On the positive real axis, $z = x$, $z^{\frac{1}{2}} = x^{\frac{1}{2}}$. On the positive imaginary axis, $z = e^{\frac{1}{2}\pi i}y$, $z^{\frac{1}{2}} = e^{\frac{1}{4}\pi i}y^{\frac{1}{2}}$.



On $|z| = r$, $|f(z)| = \dfrac{e^{-r\sin\theta}}{r^{\frac{1}{2}}} \le \dfrac{1}{r^{\frac{1}{2}}}$ for $\sin\theta > 0$.

$$\left|\int_\gamma f(z)\,dz\right| \le \left|\int_{\pi}^{0} \frac{1}{\epsilon^{\frac{1}{2}}}\,\epsilon\, d\theta\right| = \frac{\pi}{2}\epsilon^{\frac{1}{2}} \to 0 \text{ as } \epsilon \to 0.$$

$$\left|\int_\Gamma f(z)\,dz\right| \le \int_0^{\frac{1}{2}\pi} \frac{e^{-R\sin\theta}}{R^{\frac{1}{2}}}\, R\,d\theta \le R^{\frac{1}{2}}\int_0^{\frac{1}{2}\pi} e^{-\frac{2R\theta}{\pi}}\, d\theta$$

(since $\frac{\sin\theta}{\theta} \ge \frac{2}{\pi}$ for $0 \le \theta \le \frac{1}{2}\pi$) $= -R^{\frac{1}{2}}\dfrac{\pi}{2R}\left[e^{-2R\theta/\pi}\right]_0^{\frac{1}{2}\pi}$

$= \dfrac{\pi}{2R^{\frac{1}{2}}}\left[1 - e^{-R}\right] \le \dfrac{\pi}{2R^{\frac{1}{2}}} \to 0$ as $R \to \infty$.

The integrand has no singularities within C. Let $\epsilon = 0$, $R \to \infty$. Then

$$\int_0^\infty \frac{e^{ix}}{x^{\frac{1}{2}}}\,dx + \int_\infty^0 \frac{e^{-y}}{e^{\frac{1}{4}\pi i}y^{\frac{1}{2}}}\, e^{\frac{1}{2}\pi i}\,dy = 0$$

i.e.,

$$\int_0^\infty \frac{e^{ix}}{x^{\frac{1}{2}}}\,dx = e^{\frac{1}{4}\pi i}\int_0^\infty \frac{e^{-y}}{y^{\frac{1}{2}}}\,dy$$

Let $y = \eta^2$, $\displaystyle\int_0^\infty e^{-y}y^{-\frac{1}{2}}\,dy = 2\int_0^\infty e^{-\eta^2}\,d\eta = \sqrt{\pi}$.

Hence

$$\int_0^\infty \frac{e^{ix}}{x^{\frac{1}{2}}}\,dx = \sqrt{\pi}\, e^{\frac{1}{4}\pi i} = \sqrt{\frac{\pi}{2}}\,(1+i)$$

-be-

Solution No. 52, continued.

Take real and imaginary parts.

$$\int_0^\infty \frac{\cos x}{x^{1/2}}\,dx = \int_0^\infty \frac{\sin x}{x^{1/2}}\,dx = \sqrt{\frac{\pi}{2}}$$

-be-

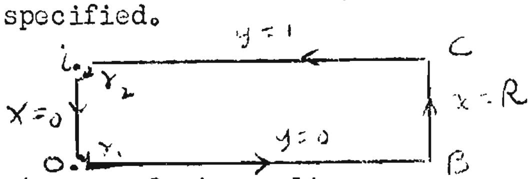
Solution No. 53.

To investigate the value of $\int_0^\infty \frac{\sin ax}{e^{2\pi x}-1}\,dx$, we begin by thinking of the imaginary part of integral of

$$f(z) = \frac{e^{iaz}}{e^{2\pi z}-1}$$

round a contour which includes the positive real axis. Now $f(z)$ has simple poles at the zeros of $e^{2\pi z}-1$, i.e., at $z = ni$ $(n = 0, \pm 1, \pm 2, \ldots)$. There are thus an infinite number of poles of $f(z)$, at equal distances apart, on the imaginary axis; at these poles $f(z)$ becomes infinite. Any idea of including a large semicircle, or large quadrant, of radius R, and letting $R \to \infty$, is therefore bound to fail; $f(z)$ will not $\to 0$ as $|z| \to \infty$ if $\arg z = \pm \frac{1}{2}\pi$.

We therefore make use of the special circumstance that the denominator in $f(z)$ is periodic, with period in $[e^{2\pi(z+ni)} = e^{2\pi z}]$, and include the positive real axis, $(y = 0,\ x > 0)$ and also $y = 1,\ x > 0$, in the contour. Now along $x = R$, $0 \le y \le 1$, it is easy to see that $\int f(z)dz$ will $\to 0$ as $R \to \infty$, but to close the contour we have to include also $x = 0$, $0 \le y \le 1$, and hope that the integral along this side of the rectangle can be otherwise evaluated. This we soon persuade ourselves is possible for the imaginary part, which is all that we require. Since the integrand $f(z)$ has poles at $z = 0$ and $z = i$ we must indent the contour there by small quadrants, γ_1 and γ_2, with centers at those points and of radii ϵ, and let $\epsilon \to 0$. We are thus led to the contour specified.



Then $f(z)$ has no singularities within the contour. The residue at $z = 0$ is

at z 0 is lim

$$\lim_{z\to 0} \frac{z e^{iaz}}{e^{2\pi z}-1} = \lim_{z\to 0} \frac{z}{e^{2\pi z}-1} = \lim_{z\to 0} \frac{1}{2\pi e^{2\pi z}} = \frac{1}{2\pi}$$

On γ_1, arg z decreases by $\frac{\pi}{2}$. Hence $\int_{\gamma_1} f(z)\,dz \to \left(-\frac{1}{2}\pi i\right)\left(\frac{1}{2\pi}\right) = -\frac{i}{4}$, as $\epsilon \to 0$

The residue at $z = i$ is $\lim_{z\to i} \frac{(z-i)e^{iaz}}{e^{2\pi z}-1} = e^{-a} \lim_{z\to i} \frac{z-i}{e^{2\pi z}-1}$

$$= e^{-a} \lim_{z\to i} \frac{1}{2\pi e^{2\pi z}}$$

On γ_2, arg $(z - i)$ decreases by $\frac{1}{2}\pi$. Hence

$$\int_{\gamma_2} f(z)\,dz \to \left(-\frac{1}{2}\pi i\right)\left(\frac{1}{2\pi} e^{-a}\right) = -\frac{1}{4} e^{-a} i, \text{ as } \epsilon \to 0$$

Solution No. 53, continued.

$$\frac{e^{iaz}}{e^{2\pi z}-1}$$

Also, on BC,

$$|f(z)| = \left|\frac{e^{ia(R+iy)}}{e^{2\pi(R+iy)}-1}\right| = \frac{e^{-ay}}{|e^{2\pi(R+iy)}-1|} \le \frac{e^{-ay}}{e^{2\pi R}-1} \le \frac{1}{e^{2\pi R}-1}$$

so $$\left|\int_{BC} f(z)\,dz\right| \le \frac{1}{e^{2\pi R}-1}\cdot 1 \to 0 \text{ as } R\to\infty$$

Hence, if we let $\epsilon \to 0$ and $R \to \infty$, we find that

$$\lim_{\epsilon\to 0}\int_\epsilon^\infty \frac{e^{iax}}{e^{2\pi x}-1}\,dx + \lim_{\epsilon\to 0}\int_\infty^\epsilon \frac{e^{ia(x+i)}}{e^{2\pi(x+i)}-1}\,dx + \lim_{\epsilon\to 0}\int_{1-\epsilon}^{\epsilon} \frac{e^{-ay}\,i\,dy}{e^{2\pi i y}-1}$$

$$-\frac{1}{4}i - \frac{1}{4}e^{-a}i = 0$$

The second term is $-e^{-a}\lim\limits_{\epsilon\to 0}\int_\epsilon^\infty \frac{e^{iax}\,dx}{e^{2\pi x}-1}$, so

$$(1-e^{-a})\lim_{\epsilon\to 0}\int_\epsilon^\infty \frac{e^{iax}}{e^{2\pi x}-1}\,dx = \frac{1}{4}i(1+e^{-a}) + i\lim_{\epsilon\to 0}\int_\epsilon^{1-\epsilon}\frac{e^{-ay}\,dy}{\cos 2\pi y - 1 + i\sin 2\pi y}$$

Now: $(\cos 2\pi y - 1)^2 + (\sin 2\pi y)^2 = 2(1-\cos 2\pi y)$, so

$$\frac{e^{-ay}}{\cos 2\pi y - 1 + i\sin 2\pi y} = \frac{1}{2}e^{-ay}\frac{\cos 2\pi y - 1 - i\sin 2\pi y}{1-\cos 2\pi y} = -\frac{1}{2}e^{-ay} - \frac{i}{2}e^{-ay}\frac{\sin 2\pi y}{1-\cos 2\pi y}$$

$$= -\frac{1}{2}e^{-ay} - \frac{i}{2}e^{-ay}\cot \pi y, \text{ and}$$

$$(1-e^{-a})\lim_{\epsilon\to 0}\int_\epsilon^\infty \frac{\cos ax + i\sin ax}{e^{2\pi x}-1}\,dx = \frac{i}{4}(1+e^{-a}) - \frac{i}{2}\lim_{\epsilon\to 0}\int_\epsilon^{1-\epsilon} e^{-ay}\,dy$$

Take the imaginary part.

$$(1-e^{-a})\lim_{\epsilon\to 0}\int_\epsilon^\infty \frac{\sin ax}{e^{2\pi x}-1}\,dx = \frac{1}{4}(1+e^{-a}) - \frac{1}{2}\lim_{\epsilon\to 0}\int_\epsilon^{1-\epsilon} e^{-ay}\,dy + \frac{1}{2}\lim_{\epsilon\to 0}\int_\epsilon^{1-\epsilon} e^{-ay}\cot \pi y\,dy$$

$\frac{\sin ax}{e^{2\pi x}-1}$ is finite at $x = 0$, so the integral on the right converges with zero as the lower limit of integration. Also e^{-ay} is finite at $y = 0$, $y = 1$.

(continued on next page)

Solution No. 53, continued.

Hence

$$(1-e^{-a})\int_0^\infty \frac{\sin ax}{e^{2\pi x}-1}\,dx = \frac{1}{4}(1+e^{-a}) \qquad \frac{1}{2}\int_0^1 e^{-ay}\,dy$$

$$\int_0^1 e^{-ay}\,dy = \frac{1}{a}\Big[-e^{-ay}\Big]_0^1 = \frac{1}{a}(1-e^{-a})$$

Also

$$\frac{1+e^{-a}}{1-e^{-a}} = \frac{e^{\frac{1}{2}a}+e^{-\frac{1}{2}a}}{e^{\frac{1}{2}a}-e^{-\frac{1}{2}a}} = \coth \tfrac{1}{2}a$$

Hence

$$\int_0^\infty \frac{\sin ax}{e^{2\pi x}-1}\,dx = \frac{1}{4}\coth \tfrac{1}{2}a - \frac{1}{2a}$$

Solution No. 54.

(i) $$\frac{z}{e^z-1}+\frac{z}{2}=\frac{z}{2}\left\{1+\frac{2}{e^z-1}\right\}=\frac{z}{2}\,\frac{e^z+1}{e^z-1}=\frac{z}{2}\,\frac{e^{\frac{z}{2}}+e^{-\frac{z}{2}}}{e^{\frac{z}{2}}-e^{-\frac{z}{2}}}$$

$$=\frac{z}{2}\,\frac{\cosh\frac{z}{2}}{\sinh\frac{z}{2}}=\frac{z}{2}\coth\frac{z}{2}$$

Since $\cosh\frac{1}{2}z$ is an even function of z, and z and $\sinh\frac{1}{2}z$ are odd functions, $\frac{1}{2}z\cot\frac{1}{2}z$, and therefore $\frac{z}{e^z-1}+\frac{1}{2}z$, is an even function.

(ii) $z/(e^z-1)$ can have singularities only at the zeros of e^z-1. Now e^z 1 at $z = 2n\pi i (n=0,\pm 1,\pm 2,\dots)$. When $|z|$ is sufficiently small,

$$\frac{z}{e^z-1}=\frac{z}{z+\frac{z^2}{2!}+\frac{z^3}{3!}+\dots}=\frac{1}{1+\frac{z}{2!}+\frac{z^2}{3!}+\dots}$$

So there is no singularity at $z=0$. The nearest singularities to the origin are at $z=\pm 2\pi i$. Hence the function can be expanded in powers of z for $|z|<2\pi$. Now $z/(e^z-1)+\frac{1}{2}z$ is an even function of z; it can also be expanded in powers of z for $|z|<2\pi$, and the expansion will contain even powers of z only; when $z=0$, it is equal to 1, so the constant term is 1. Hence we may write

$$\frac{z}{e^z-1}=1-\frac{z}{2}+\sum_{n=1}^{\infty}(-1)^{n-1}B_n\frac{z^{2n}}{(2n)!}\quad\dots\dots\dots\quad (1)$$

This equation may serve as the definition of the B_n, which are called Bernoullian numbers.

For sufficiently small $|z|$,

$$\frac{z}{e^z-1}=\left(1+\frac{z}{2!}+\frac{z^2}{3!}+\dots\right)^{-1}=1-\left(\frac{z}{2}+\frac{z^2}{6}+\dots\right)+\left(\frac{z}{2}+\frac{z^2}{6}+\dots\right)^2+\dots$$

$$=1-\frac{1}{2}z+z^2\left(\frac{1}{4}-\frac{1}{6}\right)+\dots$$

so $\frac{B_1}{2!}=\frac{1}{4}-\frac{1}{6}$; $B_1=\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$

(iii) Multiply the equation

$$\cot z-\frac{1}{z}=2z\sum_{n=1}^{\infty}\frac{1}{z^2-n^2\pi^2}$$

by z, and write $\frac{1}{2}i\,y$ for z. Since $\cos\frac{iy}{2}=\cosh\frac{y}{2}$;

$\sin\frac{iy}{2}=i\sinh\frac{y}{2}$,

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Solution No. 54, continued.

we find that

$$\frac{\rho}{2}\coth\frac{\rho}{2}-1=2i^2\frac{\rho^2}{4}\sum_{p=1}^{\infty}\frac{1}{\frac{1}{4}i^2\rho^2-p^2\pi^2}=2\rho^2\sum_{p=1}^{\infty}\frac{1}{\rho^2+4p^2\pi^2}$$

Hence

$$\frac{z}{e^z-1}=\frac{z}{2}\coth\frac{z}{2}-\frac{z}{2}=1-\frac{z}{2}+2z^2\sum_{p=1}^{\infty}\frac{1}{z^2+4p^2\pi^2} \quad \ldots\ldots\ldots\ldots (2)$$

(iv) Divide the identity

$$(1+x)(x-x^2+x^3-\ldots\ (-1)^{m-1}x^m)+(-1)^m x^{m+1}=x.$$

by $(1+x)$. Then

$$\frac{x}{1+x}=x-x^2+x^3+\ldots\ldots(-1)^{m-1}x^m+\frac{(-1)^m x^{m+1}}{1+x}.$$

In this identity, substitute $z^2/(4p^2\pi^2)$ for x. Then

$$\frac{x}{1+x}=\frac{z^2}{z^2+4p^2\pi^2}=\left(\frac{z}{2p\pi}\right)^2-\left(\frac{z}{2p\pi}\right)^4+\ldots+(-1)^{m-1}\left(\frac{z}{2p\pi}\right)^{2m}+(-1)^m\frac{z^{2m+2}}{(2p\pi)^{2m}}\frac{1}{4p^2\pi^2+z^2}$$

Substitute into each term of the series in (2). Write

$$s_k=\sum_{p=1}^{\infty}\frac{1}{p^k}\qquad (k>1)$$

(The series for s_k converges for $k>1$.) Thus

$$\frac{z}{e^z-1}=1-\frac{z}{2}+2\left\{\frac{z^2}{(2\pi)^2}s_2-\frac{z^4}{(2\pi)^4}s_4+\ldots+(-1)^{m-1}\frac{z^{2m}}{(2\pi)^{2m}}s_{2m}\right\}$$
$$+(-1)^m R_m,$$

where $R_m = 2\dfrac{z^{2m+2}}{(2\pi)^{2m}}\displaystyle\sum_{p=1}^{\infty}\frac{1}{p^{2m}}\frac{1}{4p^2\pi^2+z^2}$

-be-

Solution No. 54, continued.

In general $$|R_m| \le \frac{2|z|^{2m+2}}{(2\pi)^{2m}} \sum_{p=1}^{\infty} \frac{1}{p^{2m}} \frac{1}{4p^2\pi^2 - |z|^2},$$

but for our purposes it suffices to take z real and equal to x, so that $4p^2\pi^2 + x^2 \geq 4p^2\pi^2$ and

$$|R_m| \le \frac{2x^{2m+2}}{(2\pi)^{2m}} \sum_{p=1}^{\infty} \frac{1}{p^{2m}} \frac{1}{4p^2\pi^2} = 2\left(\frac{x}{2\pi}\right)^{2m+2} S_{2m+2}$$

As $m \to \infty$, $S_{2m+2} \to 1$, so for $|x| < 2\pi$, $R_m \to 0$ as $m \to \infty$.

For z real,

$$\frac{z}{e^z - 1} = 1 - \frac{z}{2} + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2}{(2\pi)^{2n}} S_{2n} z^{2n}$$

and this identity persists inside the circle of convergence. By comparison with (1), since the coefficients in a Taylor series about the origin are uniquely determined, it follows that

$$\frac{B_n}{(2n)!} = \frac{2}{(2\pi)^{2n}} S_{2n}.$$

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12.16.55

Solution No. 55.

From the symbolic equation

$(\ \ 1)^3 - A^3 = 0,\quad 3A^2 + 3A + 1 = 0$

we have $3A_2 + 3A_1 + 1 = 0$,

i.e. $3B_1 - \frac{3}{2} + 1 = 0,\quad B_1 = \frac{1}{6}$.

Since, symbolically, $(A+1)^n - A^n = 0,\ n > 1$,

$A^n(A-1)^{n+1} = (A+1)^n\ A^{n+1}\quad (n > 1)$,

when each side is written as a polynomial in A and A_m substituted for A^m.

Now $A^{n+1}(A+1)^n = A_{2n+1} + nA_{2n} + \frac{n(n-1)}{2!}A_{2n-1} + \cdots \frac{n(n-1)\cdots(n-r+1)}{r!}A_{2n+1-r} + \cdots + (-1)^{n+1}A_n + \cdots A_{n-1}$

$A^n(A-1)^{n+1} = A_{2n+1} - (n+1)A_{2n} + \frac{(n+1)n}{2!}A_{2n-1} + \cdots (-)^r\frac{(n+1)n\cdots(n-r+2)}{r!}A_{2n+1-r} +$

Now $A_3 = A_5 = \ldots A_{2n+1} = 0$. Thus $A_{2n+1-r} = 0$ if r is even. If r is odd, write $r = 2m+1$ in the coefficient of $A_{2n+1-r} =$

$= A_{2n-2m}$ in $A^{n+1}(A+1)^n - A^n(A-1)^{n+1}$. This coefficient is

$$\frac{n(n-1)\cdots(n-2m)}{(2m+1)!} - (-1)^{2m+1}\frac{(n+1)n\cdots(n-2m+1)}{(2m+1)!}$$

$$= \frac{n(n-1)\cdots(n-2m+1)}{(2m+1)!}(n-2m+n+1) = \frac{2n-2m+1}{(2m+1)!}n(n-1)\cdots(n-2m+1)$$

Hence

$$0 = (2n+1)A_{2n} + \frac{2n-1}{3!}n(n-1)A_{2n-2} + \frac{2n-2m+1}{(2m+1)!}n(n-1)\cdots$$
$$\cdots(n-2m+1)A_{2n-2m} + \cdots\qquad (n>)$$

If n is odd, $A_n = 0$, and the last term is a multiple of A_{n+1}, obtained by putting $m = \frac{1}{2}(n-1)$ in the general term; it is $\frac{(n+2)}{n!}n(n-1)\ldots 2.$ $A_{n+1} = (n+2)A_{n+1}$.

If n is even, the last term is a multiple of A_n. Since A_n does not occur in $A^{n+1}(A+1)^n$, this term comes from $-A^n(A-1)^{n+1}$ and is $(-1)^n A_n = A_n$. It is easy to verify that this term is correctly given by substituting $m = \frac{1}{2}n$ in the general term.

Since $A_{2m} = (-1)^{m-1}B_m$,

$(2n+1)B_n - \frac{2n-1}{3!}n(n-1)B_{n-1} + \ldots \frac{(-1)^m(2n-2m+1)}{(2m+1)!}n(n-1)\ldots$
$(n-2m+1)B_{n-m} + \ldots = 0\quad (n>1)$,

(cont. on next page)

Solution No. 55, continued.

the last term being $(-1)^{\frac{1}{2}(n-1)}(n+2)B_{\frac{1}{2}(n+1)}$ if n is odd, and $(-1)^{\frac{1}{2}n} B_{\frac{1}{2}n}$ if n is even.

The advantage of this recurrance relation is that it contains only the Bernoullian numbers between B_n and $B_{\frac{1}{2}(n+1)}$ or $B_{\frac{1}{2}n}$.

The computation is as follows:

n

2: $5B_2 - B_1 = 0$ $\qquad 5B_2 = B_1 = \frac{1}{6}$, $B_2 = \frac{1}{30}$

3: $7B_3 - 5B_2 = 0$ $\qquad 7B_3 = 5B_2 = \frac{1}{6}$, $B_3 = \frac{1}{42}$

4: $9B_4 - \frac{7}{3!}4.3B_3 + B_2 = 0$ $\qquad 9B_4 = 14B_3 - B_2 = \frac{1}{3} - \frac{1}{30} = \frac{9}{30}$

5: $11B_5 - \frac{9}{3!}5.4.B_4 + 7B_3 = 0$ $\qquad B_4 = \frac{1}{30}$

Since $\qquad 11B_5 = 30B_4 - 7B_3 = 1 - \frac{1}{6} = \frac{5}{6}$ $\qquad B_5 = \frac{5}{66}$

$$\frac{B_n}{(2n)!} = \frac{2}{(2\pi)^{2n}} \sum_{p=1}^{\infty} \frac{1}{p^{2n}}$$

$$\sum_{p=1}^{\infty} \frac{1}{p^2} = \frac{1}{2}\frac{(2\pi)^2}{2!}B_1 = \pi^2 B_1 = \frac{\pi^2}{6}$$

$$\sum_{p=1}^{\infty} \frac{1}{p^4} = \frac{1}{2}\frac{(2\pi)^4}{4!}B_2 = \frac{\pi^4}{90}$$

$$\sum_{p=1}^{\infty} \frac{1}{p^6} = \frac{1}{2}\frac{(2\pi)^6}{6!}B_3 = \frac{1}{2}\frac{(2\pi)^4}{4!}\frac{(2\pi)^2}{5.6}B_3 = \frac{1}{3}\cdot\frac{4\pi^6}{5.6}B_3 = \frac{2}{45}\pi^6 B_3 = \frac{\pi^6}{945}$$

$$\sum_{p=1}^{\infty} \frac{1}{p^8} = \frac{1}{2}\frac{(2\pi)^8}{8!}B_4 = \frac{2}{45.7.8}(2\pi)^2\pi^6 B_4 = \frac{2}{45}\cdot\frac{4}{56}\pi^8 B_4 = \frac{\pi^8}{9450}$$

etc.

Solution No. 56.

Since $\frac{z}{e^{z}-1} = \frac{1}{2}z\coth\frac{1}{2}z - \frac{1}{2}z = 1 - \frac{1}{2}z + \sum_{n=1}^{\infty}(-1)^{n-1}B_n\frac{z^{2n}}{(2n)!}\quad |z|<2\pi$

$$\tfrac{1}{2}z\coth\tfrac{1}{2}z = 1 + \sum_{n=1}^{\infty}(-1)^{n-1}B_n\frac{z^{2n}}{(2n)!},\qquad |z|<2\pi$$

Substitute iz for z.

$$\tfrac{1}{2}z\cot\tfrac{1}{2}z = 1 - \sum_{n=1}^{\infty}B_n\frac{z^{2n}}{(2n)!}\qquad |z|<2\pi$$

Let z $2x$

$$x\cot x = 1 - \sum_{n=1}^{\infty}\frac{2^{2n}B_n}{(2n)!}x^{2n}\qquad |x|<\pi$$

Let c stand for $\cot x$. Since

$\sin 2x = 2\sin x\cos x,\quad \cos 2x = \cos^2 x - \sin^2 x,\quad 1 = \cos^2 x + \sin^2 x,$

$$\cot 2x = \frac{\cos^2 x - \sin^2 x}{2\sin x\cos x} = \frac{c^2-1}{2c}$$

$$\operatorname{cosec} 2x = \frac{\cos^2 x + \sin^2 x}{2\sin x\cos x} = \frac{c^2+1}{2c}$$

Hence

$$2\cot 2x = \frac{c^2-1}{c} = \cot x - \tan x,$$

i.e., $\tan x = \cot x - 2\cot 2x.$

Also

$$\operatorname{cosec} 2x = \frac{c^2+1}{2c} = \frac{2c^2-(c^2-1)}{2c} = \cot x - \cot 2x$$

i.e, $\operatorname{cosec} x = \cot\frac{1}{2}x - \cot x$

APPLIED MATHEMATICS

Solution No. 56, continued.

Then tan x = cot x−2 cot 2x $= \frac{1}{x}\left\{1 - \sum_{n=1}^{\infty} \frac{2^{2n} B_n}{(2n)!} x^{2n} - 1 + \sum_{n=1}^{\infty} \frac{2^{2n} \cdot 2^{2n} B_n}{(2n)!} x^{2n}\right\} = \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1) B_n}{(2n)!} x^{2n-1} \quad (|x| < \tfrac{1}{2}\pi)$

$$x \operatorname{cosec} x = x \cot \tfrac{1}{2}x - x \cot x = 2 - \sum_{n=1}^{\infty} 2B_n \frac{x^{2n}}{(2n)!} - 1 + \sum_{n=1}^{\infty} \frac{2^{2n} B_n}{(2n)!} x^{2n}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(2^{2n}-2) B_n}{(2n)!} x^{2n} \qquad (|x| < \pi)$$

Inside the circle of convergence, $|z| < \pi$,

$$\cot z - \frac{1}{z} = -\sum_{n=1}^{\infty} \frac{2^{2n} B_n}{(2n)!} z^{2n-1}$$

We may integrate term-by-term within the circle of convergence. Integrate along the real axis from 0 to x.

$$\log \frac{\sin x}{x} = \int_0^x \left(\cot z - \frac{1}{z}\right) dz = -\sum_{n=1}^{\infty} \frac{2^{2n-1} B_n}{n(2n)!} x^{2n}$$

$$|x| < \pi$$

By analytic continuation, the results all hold for x complex, (if the branch of log $\frac{\sin x}{x}$ is taken which is zero at the origin, the origin being the only branch point in $|x| < \pi$).

Solution No. 57.

Consider $\oint_C f(z)dz$, where $f(z) = \pi\ z^{-2m} \cot \pi z$, and C is the rectangle of sides $x = \pm(s+\frac{1}{2})\pi$, $y = \pm(s+\frac{1}{2})\pi$.

On C, $|\cot \pi z|$ is bounded, and is $\leq M$(say) where M is independent of s.

$|z| \geq (s+\frac{1}{2})\pi$, so $|f(z)| \leq \dfrac{\pi M}{(s+\frac{1}{2})^{2m}\pi^{2m}}$, and for $m \geq 1$,

$(4s+2)|f(z)| \to 0$ as $s \to \infty$

$\oint_C f(z)\,dz \to 0$ as $s \to \infty$

$f(z)$ has simple poles at $z = n(n = \pm 1, \pm 2, \ldots)$, with residue equal to

$$\lim_{z \to n} \frac{\pi(z-n)\cos \pi z}{z^{2m}\sin \pi z} = \frac{\pi \cos \pi n}{n^{2m}} \lim_{z \to n} \frac{z-n}{\sin \pi z} = \frac{\pi \cos \pi n}{n^{2m}} \lim_{z \to n} \frac{1}{\pi \cos \pi z} = \frac{1}{n^{2m}}$$

The only other singularity of $f(z)$ is at $z = 0$, where there is a pole of order $2m+1$.

Now $z \cot z = 1 - \sum_{p=1}^{\infty} \dfrac{2^{2p} B_p z^{2p}}{(2p)!}$

Hence $f(z) = \dfrac{\pi \cot \pi z}{z^{2m}} = \dfrac{\pi z \cot \pi z}{z^{2m+1}} =$

$$= \frac{1}{z^{2m+1}} - \sum_{p=1}^{\infty} \frac{(2\pi)^{2p} B_p z^{2p}}{(2p)!\, z^{2m+1}}$$

The residue of $f(z)$ at $z = 0$ is the coefficient of z^{-1}. For this we take $p = m$, and the coefficient is $-\dfrac{(2\pi)^{2m} B_m}{(2m)!}$

Now let $s \to \infty$, so that $\oint_C f(z)\,dz \to 0$ 0. The sum of the residues $\to$ 0.

$$\sum_{n=-\infty}^{+\infty}{}' \frac{1}{n^{2m}} - \frac{(2\pi)^{2m} B_m}{(2m)!} = 0$$

(The prime on $\sum$ denotes that the term with $n = 0$. is omitted.).

For $m \geq$ $\sum_1^{\infty} \frac{1}{n^{2m}}$, $\sum_{-\infty}^{-1} \frac{1}{n^{2m}}$ converge. Take the terms with the same $|n|$ from n positive and n negative together.

cont. on next page.

Solution No. 57, continued.

$$\therefore \sum_{n=1}^{\infty} \frac{1}{n^{2m}} = \frac{(2\pi)^{2m} B_m}{(2m)!}$$

i.e., $$\sum_{n=1}^{\infty} \frac{1}{n^{2m}} = \frac{(2\pi)^{2m} B_m}{2\,(2m)!}$$

Solution No. 58. (a)

Let $\zeta = \overset{u}{\xi} + i \overset{v}{\eta} = z^{\frac{1}{2}}$, where $z = x + iy = re^{i\theta}$, $0 \le \theta \le 2\pi$

Then $x = 0$, $y \ge 0$ is $\theta = 0$ and $\theta = 2\pi$.

On the positive real axis, $\zeta = x^{\frac{1}{2}}$, so $\phi = \beta \xi$

If $w = \phi + i\psi$ is a function of ζ, then $\nabla^2 \phi = \dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} = 0$

Hence if $w = \phi + i\psi = \beta \zeta = \beta(\overset{u}{\xi} + i \overset{v}{\eta})$

$\nabla^2 \phi = 0$ and $\phi = \beta x^{\frac{1}{2}}$ on the positive real axis (both sides).

Also $\dfrac{dw}{dz} = \dfrac{dw}{d\zeta}\dfrac{d\zeta}{dz} = \dfrac{\beta}{2z^{\frac{1}{2}}}$, $\left|\dfrac{dw}{dz}\right| = |\mathrm{grad}\,\phi| = \dfrac{\beta}{2r^{\frac{1}{2}}}$

and $\to 0$, as $r \to \infty$

w is a regular function of ζ and ζ is a regular function of z in the cut z-plane, so w is a regular function of z in the cut plane.

Hence $\phi = \beta \xi$ is a solution.

Since $\xi + i\eta = r^{\frac{1}{2}} e^{i\frac{\theta}{2}}$, $\xi = r^{\frac{1}{2}} \cos\frac{\theta}{2}$; $\xi^2 = r\cos^2\frac{\theta}{2}$

$$= \frac{r}{2}(1 + \cos\theta) = \frac{r + x}{2}$$

In terms of x and y,

$$\phi = \beta\sqrt{\frac{r + x}{2}}, \text{ where } r = \sqrt{x^2 + y^2}$$

when ξ is positive(in the upper half-plane), and $\phi = -\beta\sqrt{\dfrac{r+x}{2}}$ when ξ is negative (in the lower half-plane). (There is a singularity at $z = 0$)

(b) $\psi = Uy = 2U\xi\eta$ on $\eta = \eta_0$, where $\zeta = \xi + i\eta = z^{\frac{1}{2}}$

i.e. $\psi = 2U\xi\eta_0$ on $\eta = \eta_0$

Hence if $w = \phi + i\psi = 2iU\eta_0(\xi + i\eta)$, $\psi = 2U\eta_0\xi$

Solution No. 58, continued.

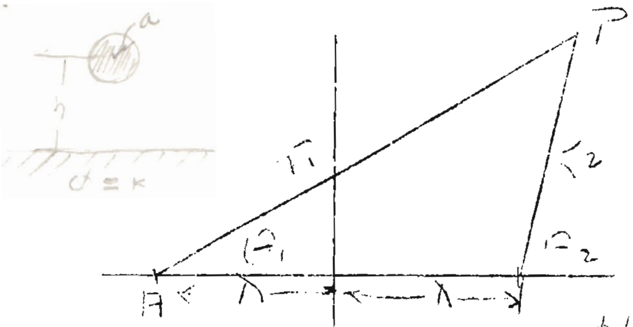
ψ satisfies $\dfrac{\partial^2\psi}{\partial x^2} + \dfrac{\partial^2\psi}{\partial y^2} = 0$ and $\psi = Uy$ on $\eta = \eta_0$

Also $$\frac{dw}{dz} = \frac{dw}{dy}\frac{dy}{dz} = \frac{2iU\eta_0}{2z^{1/2}} = \frac{iU\eta_0}{z^{1/2}}$$

$$\left|\frac{dw}{dz}\right| = \frac{U\eta_0}{r^{1/2}} \to 0 \text{ as } r \to \infty$$

Hence $\psi = 2U\eta_0\xi$ is a solution.

Solution No. 60.

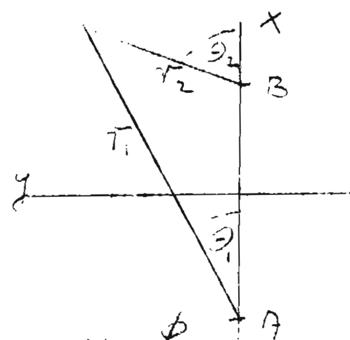


Consider $w = \phi + i\psi = 2q \log \frac{z+\lambda}{z-\lambda}$

If A, B, P are the points $-\lambda, \lambda, z$, AP = r_1, BP = r_2, and AP, BP make angles θ_1, θ_2 with the real axis, as shown, $\phi = 2q \log \frac{r_1}{r_2}$, and we may take

$$\psi = 2q(\theta_1-\theta_2) = -2q(\theta_2-\theta_1) = -2q\angle APB$$

The equipotentials are $r_1/r_2 = e^{\phi/2q}$ = constant = k(say). These are coaxial circles, with A and B as limiting points. When $\phi = 0$, $k = 1$, $r_1 = r_2$, and the equipotential $\phi = 0$ is $x = 0$, the radical axis.



Analytically, $r_1/r_2 = k$, $r_1^2 = k^2 r_2^2$

$$(x+\lambda)^2 + y^2 = k^2\{(x-\lambda)^2 + y^2\}$$

$$x^2 + y^2 + \lambda^2 - 2\lambda \frac{k^2+1}{k^2-1} x = 0$$

But $$\frac{k^2+1}{k^2-1} = \frac{k+\frac{1}{k}}{k-\frac{1}{k}} = \frac{e^{\phi/2q} + e^{-\phi/2q}}{e^{\phi/2q} - e^{-\phi/2q}} = \coth\frac{\phi}{2q}$$

Hence $r_1/r_2 = k$ is $x^2 + y^2 + \lambda^2 - 2\lambda x \coth\frac{\phi}{2q} = 0$

i.e., since $\coth^2\frac{\phi}{2q} - 1 = \operatorname{cosech}^2\frac{\phi}{2q}$,

$$\left(x - \lambda\coth\frac{\phi}{2q}\right)^2 + y^2 = \lambda^2 \operatorname{cosech}^2\frac{\phi}{2q}$$

The equipotential ϕ is therefore a circle of radius $a = \lambda\left|\operatorname{cosech}\frac{\phi}{2q}\right|$ with its center at (h,0), where $h = \lambda\coth\frac{\phi}{2q}$. Hence, (for ϕ positive),

$$\frac{h}{a} = \cosh\frac{\phi}{2q}, \quad \phi = 2q\cosh^{-1}\frac{h}{a} \qquad \text{Also } \frac{\lambda}{a} = \sinh\frac{\phi}{2q}$$

This applies to the circular cable above the ground, with the ground at x = 0. The only singularities of ϕ are at A and B. A is not in the space considered, above the ground; as we shall see, B is inside the circle. The potential difference is $\phi_1 = 2q\cosh^{-1}\frac{h}{a}$; h is positive and equal to $\lambda\coth\frac{\phi_1}{2q}$;

Solution No. 60, continued.

$$a = \lambda \operatorname{csch} \frac{\phi_1}{2q}$$

$$x = \lambda\left(\coth\frac{\phi_1}{2q} \pm \operatorname{csch}\frac{\phi_1}{2q}\right) = \frac{\lambda}{\sinh\frac{\phi_1}{2q}}\left(\cosh\frac{\phi_1}{2q} \pm 1\right) = \lambda\coth\frac{\phi_1}{4q} \text{ and } \lambda\tanh\frac{\phi_1}{4q}$$

Since (with ϕ_1 positive), $0 < \tanh\frac{\phi_1}{4q} < 1$ and $\coth\frac{\phi_1}{4q} > 1$ B inside the circle and A is outside it. Hence if $[\psi]$ is the charge in ψ as P goes round the circle once in the positive sense,

$$-[\psi] = 2q[\theta_2 - \theta_1] = 2q[\theta_2] - 2q[\theta_1] = 4\pi q$$

Since $[\theta_2] = 2\pi$, $[\theta_1] = 0$. Hence q is the charge per unit length on the cable. The capacity is

$$-\frac{1}{4\pi}\frac{[\psi]}{\phi_1} = \frac{q}{\phi_1} = \frac{1}{2\cosh^{-1}\frac{h}{a}}$$

If $\frac{a}{h}$ is small, we write

$$\cosh^{-1}\frac{h}{a} = \log\left\{\frac{h}{a} + \sqrt{\frac{h^2}{a^2} - 1}\right\} = \log\left\{\frac{h}{a} + \frac{h}{a}\left(1 - \frac{a^2}{h^2}\right)^{\frac{1}{2}}\right\}$$
$$= \log\left\{\frac{h}{a} + \frac{h}{a}\left(1 - \frac{a^2}{2h^2} + O\left(\frac{a^4}{h^4}\right)\right)\right\}$$
$$= \log\left\{\frac{2h}{a} - \frac{a}{2h} + O\left(\frac{a^3}{h^3}\right)\right\}$$
$$= \log\left\{\frac{2h}{a}\left(1 - \frac{a^2}{4h^2} + O\left(\frac{a^4}{h^4}\right)\right)\right\}$$
$$= \log\frac{2h}{a} + \log\left(1 - \frac{a^2}{4h^2} + O\left(\frac{a^4}{h^4}\right)\right)$$
$$= \log\frac{2h}{a} - \frac{a^2}{4h^2} + O\left(\frac{a^4}{h^4}\right)$$

and the capacity is $\dfrac{1}{2\left\{\log\frac{2h}{a} - \frac{a^2}{4h^2}\right\}}$ approximately.

<u>Note 1.</u>
Note that if we took the "image" of the cable in the ground as if it were a line charge, we should find $\phi = 2q\log\frac{r_1}{r_2}$, where r_2 and r_1 are now the distances from the center of the circle and its image, instead of from the limiting points. r_2 would be constant over the circle, but r_1 would not be quite constant. The approximation consists of neglecting the variation of r_1 over the circle. We should then find for the capacity $\left\{2\log\frac{2h}{a}\right\}^{-1}$. If a/h is small, the error is quite small.

<u>Note 2.</u>
We obtain the form for w as follows. We know that if λ is correctly chosen, the bilinear transformation $\zeta = \frac{z-\lambda}{z+\lambda}$ transforms the half of the z-plane for which $R(z) > 0$ and which is outside the given

Solution No. 60, continued.

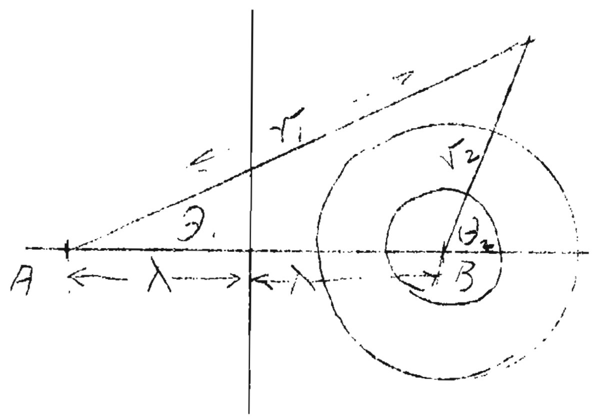
circle into the region between two concentric circles, and that $\omega = -i a \log \zeta$ solves the problem in the ζ - plane for the two concentric circles. It is, in fact, easier to work with Φ, directly, rather than with the radii of the circles in the ζ - plane as an intermediate step.

Solution No. 61.

Let $w = 2q \log \frac{z+\lambda}{z-\lambda} = \phi + i\psi$

Use the notation of Ex. 60. Then, as we have seen, the equipotentials $\phi=\phi_0$, $\phi=\phi_1$ are circles with centers at $(h_0,0)$ and $(h_1,0)$ and radii a and b, respectively, where

$$\phi_0 = 2q \cosh^{-1}\frac{h_0}{a}, \quad \phi_1 = 2q \cosh^{-1}\frac{h_1}{b}$$



The singularities are at A and B, and there are no singularities between the circles. Also

$$\lambda^2 = a^2 \sinh^2\frac{\phi_0}{2q} = a^2\left(\cosh^2\frac{\phi_0}{2q} - 1\right) = h_0^2 - a^2$$

Similarly $\lambda^2 = h_1^2 - b^2$

so $h_1^2 - h_0^2 = b^2 - a^2$

But the distance between the centers is $d = h_1 - h_0$.

Hence $h_1 + h_0 = \frac{b^2-a^2}{d}$ and $h_0 = \frac{b^2-a^2-d^2}{2d}$, $h_1 = \frac{b^2-a^2+d^2}{2d}$

The charge on the outer surface of the inner cylinder is $-\frac{1}{4\pi}[\psi]$. B is inside the circle and A is outside it, so $[\theta_2] = 2\pi$, $[\theta_1] = 0$, and

$$-\frac{1}{4\pi}[\psi] = \frac{q}{2\pi}[\theta_2 - \theta_1] = \frac{q}{2\pi}[\theta_2] - \frac{q}{2\pi}[\theta_1] = q$$

The capacity is therefore $-\frac{1}{4\pi}\frac{[\psi]}{\phi_1-\phi_0} = \frac{1}{2\left\{\cosh^{-1}\frac{h_0}{a} - \cosh^{-1}\frac{h_1}{b}\right\}}$

We must now eliminate h_0 and h_1.

If $\mu_0 = \cosh^{-1}\frac{h_0}{a}$ and $\mu_1 = \cosh^{-1}\frac{h_1}{b}$, then

$$\cosh\mu_0 = \frac{h_0}{a}, \quad \cosh\mu_1 = \frac{h_1}{b}, \quad \sinh\mu_0 = \frac{\sqrt{h_0^2-a^2}}{a} = \frac{\lambda}{a}; \quad \sinh\mu_1 = \frac{\lambda}{b}$$

so

$$\cosh(\mu_0-\mu_1) = \cosh\mu_0\cosh\mu_1 - \sinh\mu_0\sinh\mu_1 =$$

$$= \frac{h_0h_1 - \lambda^2}{ab} = \frac{h_0h_1 - h_0^2 + a^2}{ab} = \frac{h_0(h_1-h_0)+a^2}{ab}$$

$$= \frac{h_0 d + a^2}{ab} = \frac{\frac{1}{2}(b^2-a^2-d^2)+a^2}{ab} = \frac{a^2+b^2-d^2}{2ab}$$

Solution No. 61, continued.

and $\cosh^{-1} - \cosh^{-1}\frac{h_1}{b} = \mu_0 - \mu_1 = \cosh^{-1}\frac{a^2+b^2-d^2}{2ab}$

The capacity is therefore $\left\{ 2\cosh^{-1}\frac{a^2+b^2-d^2}{2ab} \right\}^{-1}$

Note 1. The equations above show how to find λ, h_0, h_1 given a, b, and d.

Note 2. The bilinear transformation $\zeta = \frac{z+\lambda}{z-\lambda}$ transforms the region between the circles into the region between two concentric circles in the ζ - plane, and $w = \phi + i\psi = -2q \log \zeta$ solves the problem, as we know, for the two concentric circles in the ζ plane. It turns out to be a little easier to work with ϕ_0 and ϕ_1 directly, rather than with radii of the circles in the ζ plane as an intermediate step.

Solution No. 59

$z = e^{\zeta}$, $\zeta = \operatorname{Log} z$.

Then $re^{i\theta} = e^{\xi + i\eta}$, $r = e^{\xi}$, $\theta = \eta$

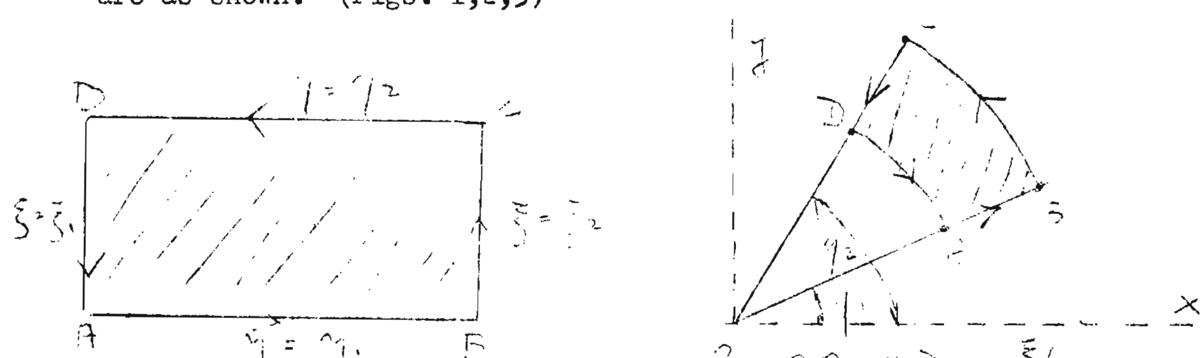
To specify the branch of Log z we may put in a cut in the z-plane along $\theta = \alpha$ and $\theta = \alpha + 2\pi$, and in the cut z-plane and on the cut $\alpha \leq \theta \leq \alpha + 2\pi$, so in the ζ -plane $\alpha \leq \eta \leq \alpha + 2\pi$.

We know that if ζ_1, is any branch of Log z, any other value is given by $\zeta_1 + 2n\pi i$ $(n = \pm 1, \pm 2 \ldots)$

$\xi = -\infty$ corresponds with $r = 0$, $\xi = +\infty$ with $r = \infty$. The infinite straight line $-\infty \leq \xi \leq \infty$, $\eta = \eta_1$, corresponds with the semi-infinite straight line, $\theta = \eta_1$, $0 \leq r \leq \infty$.

Also ξ = constant = ξ_1, $\eta_1 \leq \eta \leq \eta_2$, corresponds with the circular arc $r = e^{\xi_1}$, $\eta_1 \leq \theta \leq \eta_2$ (where we take $\eta_2 - \eta_1$, less than 2π as we must to keep to one branch of Log z - to avoid over lapping in the z-plane.)

Hence the correspondence of the rectangles in the ζ plane bounded by (i) $\eta = \eta_1$, $\eta = \eta_2$, $\xi = \xi_1$, $\xi = \xi_2$ (ii) $\eta = 0$, $\eta = \pi$, $\xi = \xi_1$, $\xi = \xi_2$ are as shown. (Figs. 1,2,3)



$OA = OD = e^{\xi_1}$, $OB = OC = e^{\xi_2}$

Fig 1 $(\eta_2 - \eta_1 < \pi)$

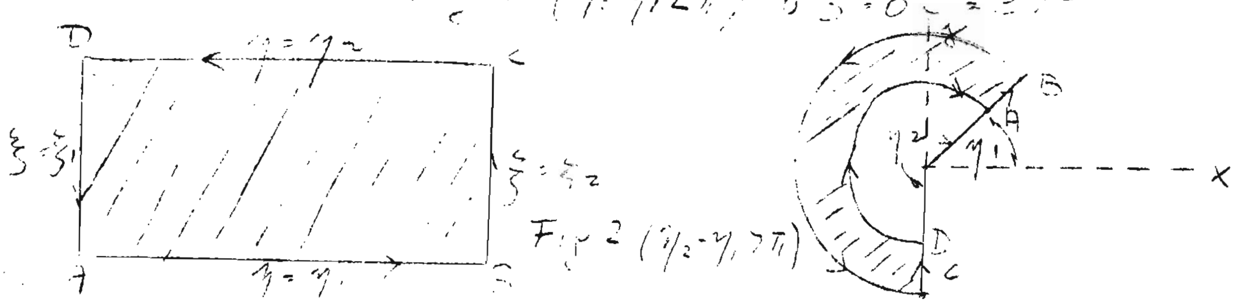


Fig 2 $(\eta_2 - \eta_1 > \pi)$

Solution No. 59, continued.

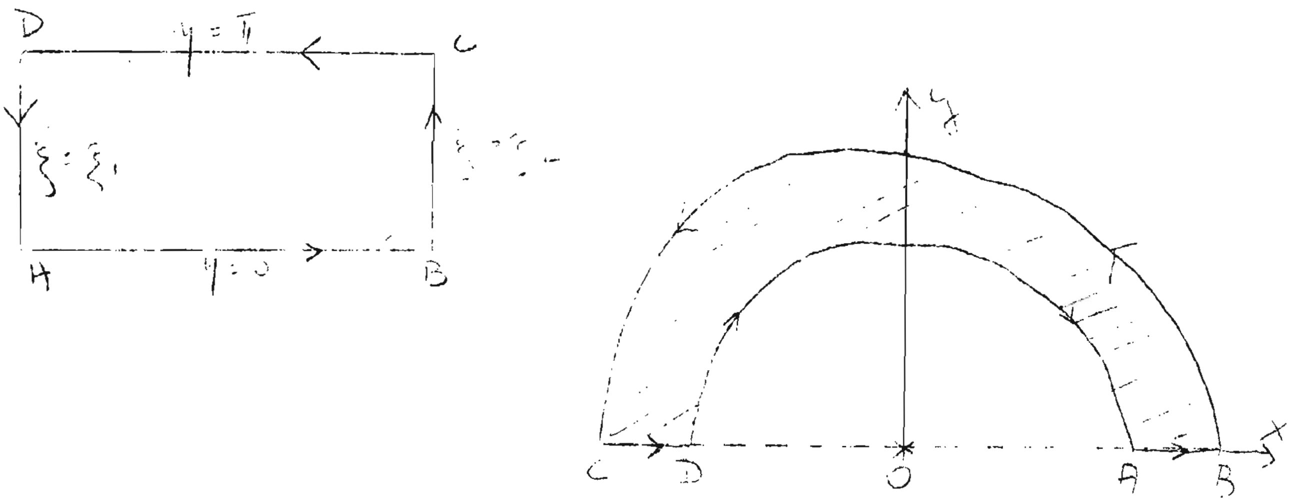


Fig 3

The shaded regions correspond because they are on the left as the boundaries are traversed in corresponding directions. If (in Fig. 3) we let $\xi_2 \to \infty$, we see that the semi-infinite strip (open rectangle) $\xi_1 \leq \xi \leq \infty$, $\eta = 0$, $\eta = \pi$ corresponds with the upper half-plane above the semi-circle $r = e^{\xi_1}$, and its boundary. (Fig. 4)

If now we let $\xi_1 \to \infty$, the infinite strip, $-\infty \leq \xi \leq \infty$; $\eta = 0$; $\eta = \pi$ is seen to correspond with the upper half of the z-plane. (Fig. 5)

If every case, we suppose the cut chosen so that α is outside the range $\eta_1 \leq \theta \leq \eta_2$ (In Figs. 3,4,5, $\eta_1 = 0$; $\eta_2 = \pi$.)

Solution No. 59, continued.

Similarly, if $0 \le \theta \le 2\pi$, $r = r_1$ corresponds with $\xi = \log r_1$, $0 \le \eta \le 2\pi$, $\eta = 2\pi$, $0 \le \xi \le \infty$ with the lower side of the cut from A to ∞ along the positive real axis, $\eta = 0$, $0 \le \xi \le \infty$ with the upper side of the cut, and $\xi = \infty$ with a circle of infinite radius. (Fig. 7) The inside of the semi-infinite strip (open rectangle) $\eta = 0$, $\eta = 2\pi$,

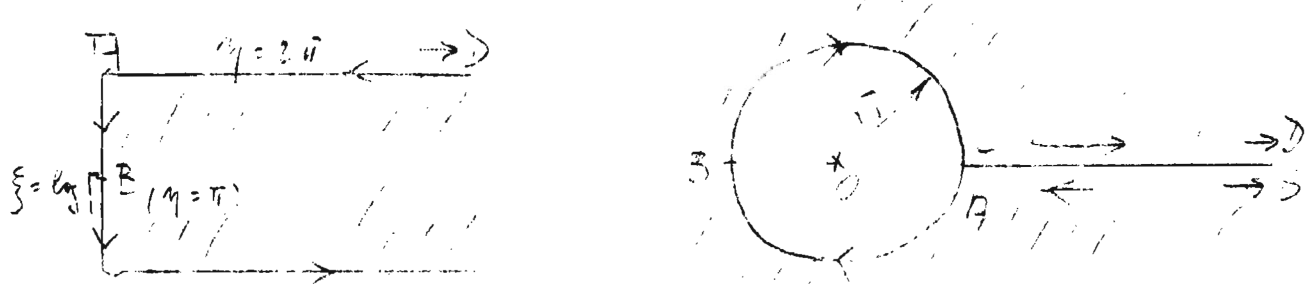


Fig 7.

$\log r_1 \le \xi \le \infty$ corresponds with the outside of the circle $r = r_1$, with a barriar along the positive real axis from C(or A) to ∞.

In Fig. 4, if we put $\xi_1 = 0$, the semi-infinite strip $\eta = 0$, $\eta = \pi$, $0 \le \xi \le \infty$ corresponds to the upper half of the z-plane above the semicircle of unit radius ($r_1 = 1$ in Fig. 4.). This is with the transformation $z = e^{\zeta}$. But with the transformation $z_1 = \cosh \zeta_1$, the same semi-infinite strip corresponds with the upper half of the z_1-plane and the real axis. Hence the two regions in the z- and z_1-planes must correspond under the transformation obtained from $z = e^{\zeta}$, $z_1 = \cosh \zeta = \frac{1}{2}(e^{\zeta} + e^{-\zeta})$ i.e. $z_1 = \frac{1}{2}(z + \frac{1}{z})$. This easily verified if we take the correct branch of z as a function of z_1 by writing $z = re^{i\theta}$, $z_1 = \frac{1}{2}(re^{i\theta} + \frac{1}{r}e^{-i\theta})$ and $z_1 = \frac{1}{2}(x + \frac{1}{x})$ when z is on the real axis of the z-plane, etc. ($= \cos\theta$ on $r = 1$)

Solution No. 59, continued.

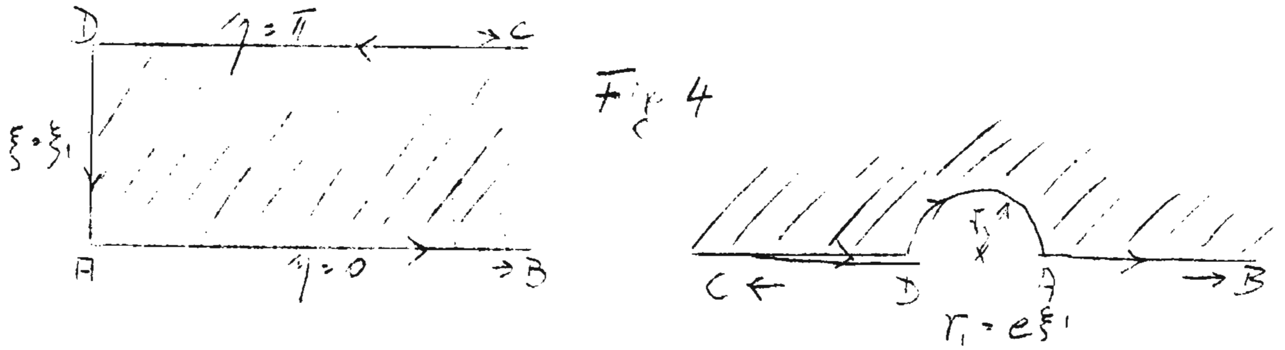


Fig 4

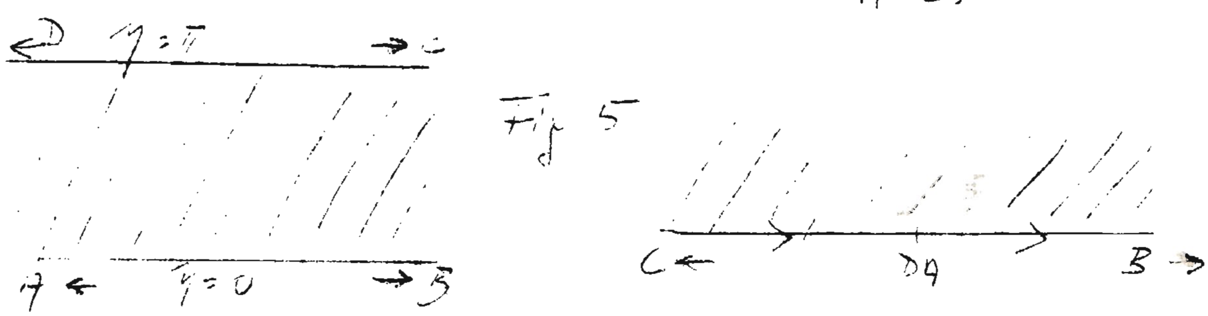


Fig 5

Now take the branch for which Log z $=$ $\log r + i\theta$, $-\pi \leq \theta \leq \pi$, and put in the negative real axis in the z-plane as an impassible barrier. Then (Fig. 6) the circle $r = r_1$ corresponds with $\xi = \xi_1 = \log r_1$, $-\pi < \theta \leq \pi$, $\eta = \pi$, $-\infty \leq \xi \leq \xi_1$, with the upper side of the cut CO, and $\eta = -\pi$, $-\infty \leq \xi \leq \xi_1$, with the lower side of the cut, and $\xi = -\infty$, $-\pi \leq \eta \leq \pi$ with a point circle at the origin. The correspondence is as shown; the inside of the semi-infinite strip (open rectangle) corresponds with the inside of the circle with an impassible barrier from O to A or C.

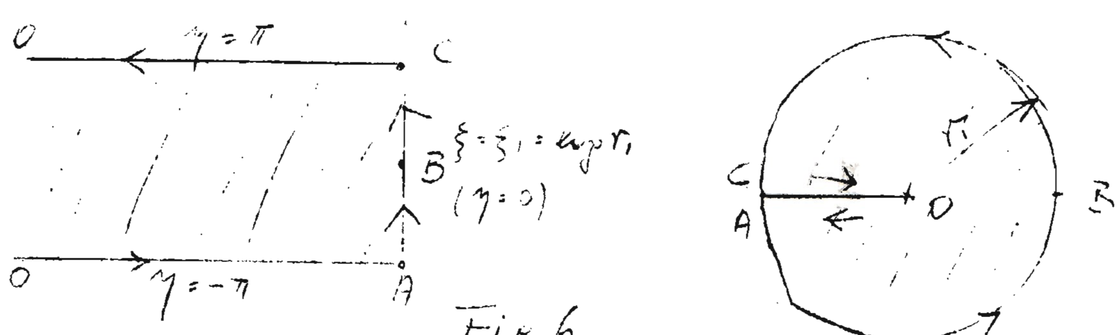


Fig 6

Solution No. 62.

For the undisturbed potential (since $E_x = -\frac{\partial \phi}{\partial x}$, $E_y = -\frac{\partial \phi}{\partial y}$), we may take

$$\phi_0 = -Ax, \quad w_0 = -Az.$$

Let $\phi = -Ax + \phi' = -Ar\cos\theta + \phi'$. Then we require $\nabla^2\phi' = 0$, $\phi' = Aa\cos\theta$ when r = a, $\phi = O(\frac{1}{r})$ at ∞, ϕ shall have no singularities outside the circle. Since $\frac{1}{z} = \frac{e^{-i\theta}}{r}$ has a real part equal to $\frac{\cos\theta}{r}$, $\frac{\cos\theta}{r}$ satisfies $\nabla^2\phi' = 0$. The only singularity is inside the circle at z = 0; there are no singularities outside the circle. $\phi' = O(\frac{1}{r})$ at ∞ and $\phi' = \frac{\cos\theta}{a}$ on the circle. Hence $\phi' = Aa^2\frac{\cos\theta}{r}$, $w = -Az + \frac{Aa^2}{z}$ satisfies all the requirements.

$$w = A\left(\frac{a^2}{z} - z\right)$$

$$\phi = A\left(\frac{a^2}{r} - r\right)\cos\theta$$

$$\chi = -A\left(\frac{a^2}{r} + r\right)\sin\theta$$

Solution No. 53.

For the undisturbed potential we may take

$$\phi_0 = -Ax \;, \quad \omega_0 = -Az = -Ac \cosh \zeta .$$

Then the undisturbed potential $\phi_0 = -Ac \cosh \xi \cos \eta$,and on $\xi = \xi_0$,

$$\phi_0 = -Aa \cos \eta .$$

Let $\phi = \phi_0 + \phi'$. Then we require $\nabla^2 \phi' = 0$, $\phi' = O(\frac{1}{r})$ at ∞, and $\phi' = Aa \cos \eta$ on $\xi = \xi_0$, ϕ' shall have no singularities outside the ellipse. We look for a solution of $\nabla^2 \phi' = 0$ which is proportial to $\cos \eta$ when ξ is constant, and which vanishes at ∞. Since $e^{-\zeta} = e^{-\xi}(\cos \eta + i \sin \eta)$, $e^{-\xi} \cos \eta$ is such a solution. Also we know that $e^{+\xi} \sim \frac{2r}{c}$ at ∞, so $e^{-\xi} \sim \frac{c}{2r}$ and if $\phi' = Be^{-\xi} \cos \eta$, where B is a real constant, it will satisfy $\nabla^2 \phi' = 0$, $\phi' = O(\frac{1}{r})$ at ∞, and also $\phi' = Aa \cos \eta$ on $\xi = \xi_0$ if $B = Aae^{\xi_0}$. Also $e^{-\zeta}$ has singularities at any finite point in the ζ- plane as a function of ζ, so the only singularities as a function of z are the singularities of $\cosh^{-1} z/c$, which are inside the ellipse.

Since $a = c \cosh \xi_0$, $b = c \sinh \xi_0$, $ce^{\xi_0} = a + b$, and $c = \sqrt{a^2 - b^2}$. So $e^{\xi_0} = \sqrt{\frac{a+b}{a-b}}$, $B = Aa \left(\frac{a+b}{a-b}\right)^{1/2}$.

Hence

$$\phi = -Ax + Aa \left(\frac{a+b}{a-b}\right)^{\frac{1}{2}} e^{-\xi} \cos \eta ,$$

$$\omega = -Az + Aa \left(\frac{a+b}{a-b}\right)^{\frac{1}{2}} e^{-\zeta} ,$$

$$\chi = -Ay - Aa \left(\frac{a+b}{a-b}\right)^{\frac{1}{2}} e^{-\xi} \sin \eta ,$$

where $z = c \cosh \zeta$, $c = (a^2 - b^2)^{1/2}$, and $x = c \cosh \xi \cos \eta$,

$y = c \sinh \xi \sin \eta$.

Problem # 64

For flow past a circular cylinder of radius R, if the section of the cylinder is in the Z - plane,

$$w = -U\left(Ze^{i\alpha} + \frac{R^2}{Z}e^{i\alpha}\right) - \frac{ck}{2\pi}\log Z.$$

Under the transformation

$$z = Z + \frac{a^2}{Z} \quad (a < R)$$

the circle $|Z| = R$ transforms into the ellipse $\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$

where $A = R + \frac{a^2}{R}$, $B = R - \frac{a^2}{R}$, and the outside of the circle transforms into the outside of the ellipse, that branch of Z, as a function of z, being chosen which make $z = \infty$ and $Z = \infty$ correspond with one another, the z - plane being cut along the real axis from -2a to 2a. Also, as $z \to \infty$, $Z = z + O\left(\frac{1}{z}\right)$, so under the transformation w becomes

$$w = -Uze^{i\alpha} - \frac{ck}{2\pi}\log z + O(1/z),$$

and the undisturbed flow at infinity, and the circulation, are unaltered by the transformation.

Also $R = \frac{A+B}{2}$, $\frac{a^2}{R} = \frac{A-B}{2}$, so $a^2 = \frac{A^2-B^2}{4}$, $a = \frac{\sqrt{A^2-B^2}}{2}$

Hence if $z = Z + \frac{A^2-B^2}{4Z}$, $Z = \frac{1}{2}\left\{z + \left(z^2 - (A^2-B^2)\right)^{\frac{1}{2}}\right\}$ and the branch of Z is defined by cutting the z-plane from $-\sqrt{A^2-B^2}$ to $+\sqrt{A^2-B^2}$ and choosing the branch of $\left(z^2-(A^2-B^2)\right)^{\frac{1}{2}}$ which is equal to $z\left[1 - \frac{A^2-B^2}{2z^2} + \cdots\right]$ for $|z| > \sqrt{A^2-B^2}$, then the flow in the z-plane has a complex potential

$$w = -U\left[Ze^{i\alpha} + \frac{(A^2-B^2)}{4Z}e^{-i\alpha}\right] - \frac{ck}{2\pi}\log Z \quad (1)$$

By elliptic coordinates we found for the complex potential in the z-plane,

$$w = -Uce^{i\alpha}\cosh\zeta - U\sqrt{\frac{A+B}{A-B}}\,e^{-\zeta}(B\cos\alpha - iA\sin\alpha) - \frac{ck}{2\pi}\zeta + \text{const} \quad (2)$$

where $z = c\cosh\zeta$, $c = \sqrt{A^2-B^2}$

If now we let $Z = \frac{1}{2}ce^{\zeta}$ then $z = Z + \frac{A^2-B^2}{4Z} = Z + \frac{c^2}{4Z}$ becomes

$z = \frac{1}{2}c\left(e^{\zeta} + e^{-\zeta}\right) = c\cosh\zeta$ so

$Z = \frac{1}{2}ce^{\zeta}$ gives the relation between

Z and ζ. In terms of Z, (2) becomes

$$w = -Ue^{i\alpha}\left(Z + \frac{A^2-B^2}{4Z}\right) - U\sqrt{\frac{A+B}{A-B}}\,\frac{\sqrt{A^2-B^2}}{2Z}(B\cos\alpha - iA\sin\alpha) - \frac{ck}{2\pi}\log\frac{2Z}{c} + \text{const.}$$

$$= -U\left[Ze^{i\alpha} + \frac{1}{Z}\left\{\frac{A^2-B^2}{4}e^{i\alpha} + \frac{A+B}{2}(B\cos\alpha - iA\sin\alpha)\right\}\right] - \frac{ck}{2\pi}\log Z + \text{const}$$

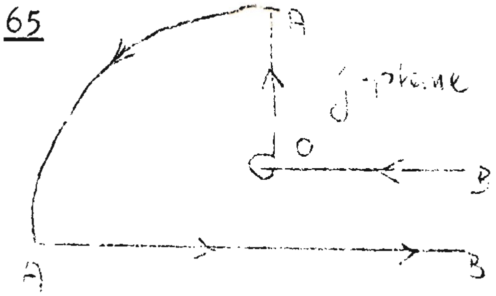
PROBLEM 64 continued

The constant is irrelevant, since it does not affect the velocity; in any case, it may be chosen to be zero so (2) becomes

$$w = -U\left[Ze^{i\alpha} + \frac{1}{4Z}(a+b)^2(\cos\alpha - i\sin\alpha)\right] - \frac{iK}{2\pi}\log_U Z$$

$$= U\left[Ze^{i\alpha} + \frac{(a+b)^2}{4Z}e^{-i\alpha}\right] - \frac{iK}{2\pi}\log_U Z$$

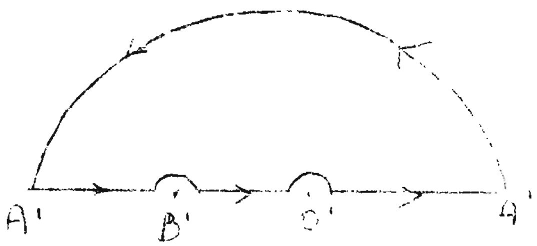
which is the same as (1).



Since B and B' corr espond to $\zeta = 0$, and A A' to $\zeta = \infty$, we change the notation, and denote both B and B' simply by B, and A and A' simply by A. We then denote by ', B', o' the points in the ζ -plane corresponding to A, B , o, respectively.

In the notation used in lectures, we take α_1, α_2, α_3 at B', o', A' respectively, so

$$\alpha_1 = 0, \quad \alpha_2 = 1, \quad \alpha_3 = \infty$$



Also $\mu_1 = 1$, $\mu_2 = -\frac{1}{2}$, $\mu_3 = \frac{3}{2}$

Sin ce $\alpha_3 = \infty$, we then take $F'(\zeta) = \frac{K}{\zeta}(\zeta - 1)^{\frac{1}{2}}$

where K is a constant, and for any real α, we define $\arg(\zeta - \alpha)$ to be such that

$$0 < \arg(\zeta - \alpha) < \pi$$

in the upper half-plane, with $\arg(\zeta - \alpha) = 0$ on the real axis for $\zeta > \alpha$, and $\arg(\zeta - \alpha) = \pi$ on the real axis for $\zeta < \alpha$.

Also $F'(\zeta) = \frac{iK}{\zeta} + \frac{K}{\zeta}\{(\zeta - 1)^{1/2} - i\}$

and n ear B', $z = F(\zeta) = iK \log \zeta + \phi(\zeta)$

where ϕ is regular near $\zeta = 0$. At B, z increases by ib; at B', $\log \zeta$ decreases by $i\pi$, so $F(\zeta)$ increases by $-i^2 K\pi = K\pi$. H ence we must have $K\pi = ib$, $K = ib/\pi$,

and $F'(\zeta) = \frac{ib}{\pi}\frac{(\zeta - 1)^{1/2}}{\zeta} = -\frac{b}{\pi\zeta} + \frac{ib}{\pi}\frac{(\zeta-1)^{1/2} - i}{\zeta}$ (1)

We integrate (1) $F(\zeta) = -\frac{b}{\pi}\log \zeta + \frac{ib}{\pi}\int \frac{(\zeta - 1)^{1/2} - i}{\zeta}\, d\zeta$

To evaluate the indefinite integral, let $\zeta - 1 = w^2$,

$$\frac{(\zeta-1)^{1/2} - i}{\zeta}\, d\zeta = \frac{w - i}{w^2 + 1}\, 2w\, dw = \frac{2w}{w - i}\, dw = 2\left\{1 - \frac{i}{w + i}\right\} dw$$

$$z = F(\zeta) = -\frac{b}{\pi}\log \zeta + \frac{2ib}{\pi}\{w - i\log(w + i)\} + C$$

where C is a constant. Since at O, z = o, $\zeta = 1$, w = o, $C = \frac{2ib}{\pi}(i \log i)$ and

$$F(\zeta) = -\frac{b}{\pi}\log \zeta + \frac{2ib}{\pi}\left\{w - i\log\frac{w + i}{i}\right\}$$

$$= -\frac{b}{\pi}\log \zeta + \frac{2ib}{\pi}\{w - i\log(1 - iw)\} \qquad (2)$$

Problem # 65 continued -

where $\quad w = (\zeta - 1)^{1/2}$ (3)

and we have chosen a branch of $\log(1-iw)$, as a function of w, which is zero at w = 0 (We return to this point later)

If we integrate the first form of $F'(\zeta)$ in (1) we have

$$F(\zeta) = \frac{ib}{\pi}\int \frac{(\zeta-1)^{1/2}}{\zeta}\,d\zeta = \frac{2ib}{\pi}\int \frac{w^2}{w^2+1}\,dw = \frac{2ib}{\pi}\int\left(1 - \frac{1}{w^2+1}\right)dw$$

$$= \frac{2ib}{\pi}\left\{ w - \arctan w \right\} \qquad (4)$$

the additio constant being zero if we take a branch for which $\arctan w = 0$ when $w = 0$, since $z = F(\zeta) = 0$ when $w = 0$

It is easily seen directly that (2) and (4) are equivalent if the correct branches are chosen, since (Ex. 10)

$$\arctan w = \frac{1}{2i}\operatorname{Log}\frac{1+iw}{1-iw} \qquad \text{so}$$

$$\arctan w = -\frac{i}{2}\operatorname{Log}\frac{1+w^2}{(1-iw)^2} = -\frac{i}{2}\operatorname{Log}\frac{\zeta}{(1-iw)^2} = i\left\{\operatorname{Log}(1-iw) - \frac{1}{2}\operatorname{Log}\zeta\right\} \qquad (5)$$

*

This is all that is re uired to answer the question as set. But is is instructive to check carefully that our formulae are correct. We check (i) that the formula for $F'(\zeta)$ in (1) satisfies all the requirements on $F'(\zeta)$ (ii) that the branches of $\log(1-iw)$ in (2) and of $\arctan w$ in (4) can be fully specified (iii) that the boundary values of F (ζ) are such that the transformation of the boundary is easily checked.

The requirements on F' (ζ) are as follows:

(a) At O, arg z decreases by $\frac{3}{2}\pi$ when arg $(\zeta-1)$ decreases by π , so near $\zeta = 1$

z = F (ζ) = $(\zeta-1)^{3/2}\,\psi(\zeta)$, where ψ is regular near 1 and ψ (1) is not zero.
Hence F' (ζ) = $(\zeta-1)^{1/2}$ X (ζ), where X is regular near 1 and X (1) isnot zero.

(b) At A, arg z increases by $\frac{1}{2}\pi$ when arg ζ increases by π , so near $\zeta = \infty$

$z = F(\zeta) = \zeta^{1/2}\,\psi_1(\zeta)$ where ψ_1 , is regular and non-zero at ∞

Hence F' (ζ) = $\zeta^{-\frac{1}{2}}\,X_1(\zeta)$ where X, is regular and non-zero at ∞

(c) At B, z increases by ib when arg ζ increases by π so

$$F(\zeta) = -\frac{b}{\pi}\log\zeta + \psi_2(\zeta)$$ where ψ_2 is regular at $\zeta = 0$

$$F'(\zeta) = -\frac{b}{\pi\zeta} + X_2(\zeta)$$ where X_2 is regular at $\zeta = 0$

(d) F' is regular in the upper half-plane and on the real axis except at B' and O'

(e) the real axis of ζ

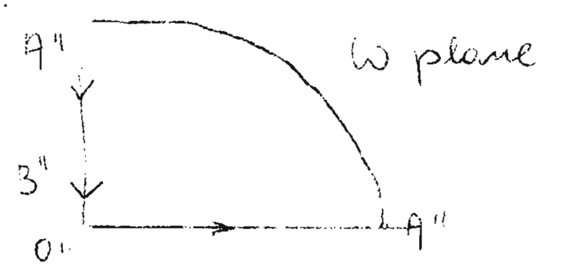
$$\arg F'(\zeta) = \arg\frac{dz}{d\zeta} = \arg dz = \begin{cases} 0 & \text{on } A'B' \\ +\pi & \text{on } B'O' \\ \frac{1}{2}\pi & \text{on } O'A' \end{cases}$$

Problem #65 continued -

and these are satisfied since $\arg F'(\zeta) = \frac{1}{2}\pi + \frac{1}{2}\arg(\zeta-1) - \arg\zeta$

$$\left.\begin{aligned}\arg(\zeta-1) &= 0 \quad \text{for } \zeta > 1\\ &= \pi \quad\ \ \zeta < 1\end{aligned}\right\} \qquad \begin{aligned}\arg\zeta &= 0 \quad \text{for } \zeta > 0\\ &= \pi \quad\ \ \zeta < 0\end{aligned}$$

All the requirements on $F'(\zeta)$ are therefore satisfied.



The upper half of the ζ -plane corresponds to the first quadrant of the w-plane, as shown, where A'', B'', O'' correspond to A', B', 08, respectively, and B'' is at w = i. Then $\arg(1-iw) = \arg(-i) + \arg(w+i) = \arg(w+i) - \frac{1}{2}\pi$
At O'', $\arg(1-iw)$ is to be zero, since $\log(1-iw)$ is to be zero at w = o; hence $\arg(w+i)$ is to be $\frac{1}{2}\pi$ at O''. We therefore take $\arg(w+i)$ to be zero at A'' on the real axis at ∞; as w moves along the real axis to O'', $\arg(w+i)$ increases to $\frac{1}{2}\pi$; on O'' B'' A'', $\arg(w+i) = \frac{1}{2}\pi$. Hence $\arg(1-iw) = -\frac{1}{2}\pi$ at ∞ on the real axis; $-\frac{1}{2}\pi < \arg(1-iw) < 0$ on A'' O''; $\arg(1-iw) = 0$ on O'' B'' A''. And in the first quadrant of w, on the upper half -plane of ζ, $-\frac{1}{2}\pi < \arg(1-iw) < 0$.

This defines $\log(1-iw)$ in (2) completely.

We have already defined $\log\zeta$ by $0 < \arg\zeta < \pi$ in the upper half -plane of ζ. We use the same value of $\log\zeta$, and the value of $\log(1-iw)$ above, to define arctan w is (5), and this is the definition that must be used in (4). Now consider the boundary values of $F(\zeta)$. Write

$$u = (\xi-1)^{\frac{1}{2}} \text{ when } \xi \geq 1 \qquad v = (1-\xi)^{1/2} \text{ when } \xi \leq 1$$

Note that $\tanh^{-1} v = \frac{1}{2}\log\frac{1+v}{1-v} = \frac{1}{2}\log\frac{(1+v)^2}{1-v^2} = \frac{1}{2}\log\frac{(1+v)^2}{\xi} =$

$$= \log(1+v) - \frac{1}{2}\log\xi \qquad \text{for } \xi > 0$$

$$\coth^{-1} v = \frac{1}{2}\log\frac{v+1}{v-1} = \frac{1}{2}\log\frac{(1+v)^2}{v^2-1} = \frac{1}{2}\log\frac{(1+v)^2}{|\xi|} = \log(1+v) - \frac{1}{2}\log\xi$$

and $\arctan iv = i\tanh^{-1} v = \frac{1}{2}\pi + i\coth^{-1} v \qquad$ for $\xi < 0$

Then on O' A', where ζ is real and equal to ξ, $\xi \geq 1$

$w = u$, $\log\zeta = \log\xi$, and $\log(1-iw) = \frac{1}{2}\log\xi - i\phi$

where $\tan\phi = u$, $\frac{1}{2}\pi \geq \phi \geq 0$

so $\arctan w = \phi = \arctan u$.

on $O'B'A'$ $\quad w = iv$, $\log(1-iw) = \log(1+v)$

on $O'B'$ $\quad \log\zeta = \log\xi$, so $\arctan w = i\{\log(1+v) - \frac{1}{2}\log\xi\} = i\tanh^{-1} v$

on $B'A'$ $\quad \log\zeta = i\pi + \log|\xi|$

Problem #65 continued -

so $\arctan w = \frac{1}{2}\pi + i\left\{\log(1+v) - \frac{1}{2}\log|\xi|\right\}$
$= \frac{1}{2}\pi + i\coth^{-1} v$

(Since $\arctan iv = i\tanh^{-1} v = \frac{1}{2}\pi + i\coth^{-1} v$, these were anticipated, but we have checked them carefully.) Hence the boundary values of F (ζ) are as follows, and from them the transformation of the boundary is checked.

on $A'B'$ $\quad F(\zeta) = \frac{2b}{\pi}\left\{\coth^{-1} v - v\right\} - ib \quad (\xi < 0)$

on $B'O'$ $\quad F(\zeta) = \frac{2b}{\pi}\left\{\tanh^{-1} v - v\right\} \quad (0 < \xi \le 1)$

on $O'A'$ $\quad F(\zeta) = \frac{2bi}{\pi}\left\{u - \tan^{-1} u\right\} \quad \xi \ge 1$

Problem # 66

If $\pi\mu_s$ are the external angles of a polygon in the z -plane, z = F(ζ) maps the outside of the polygon on the upper half -plane of ζ if

$$\frac{dF}{d\zeta} = A\,\frac{\prod(\zeta-\alpha_s)^{\mu_s}}{(\zeta-\beta)^2(\zeta-\bar\beta)^2} \tag{1}$$

where A is a constant, the α_s are real and correspond to the vertices of the polygon, R (β) > 0 , and $\zeta = \beta$ corresponds to $z = \infty$ and

$$\sum \frac{\mu_s}{\beta-\alpha_s} = \frac{2}{\beta-\bar\beta} \tag{2}$$

in order that the mapping should be conformal at $\zeta = \beta$

$$\text{Also}\quad \omega = -i\,\frac{\zeta-\bar\beta}{\zeta-\beta} \qquad \zeta = \frac{i\omega\beta-\bar\beta}{i\omega-1} \tag{3}$$

maps the upper half -plane of ζ on the exterior of unit circle $|\omega| = 1$, with $\zeta = \beta$ corresponding to $\omega = \infty$. So z = $G(\omega)$ maps the exterior of the polygon on the exterior of the unit circle, with the infinite points corresponding, if

$$\frac{dG}{d\omega} = \frac{dF}{d\zeta}\,\frac{d\zeta}{d\omega} = A\,\frac{\prod(\zeta-\alpha_s)^{\mu_s}}{(\zeta-\beta)^2(\zeta-\bar\beta)^2}\,\frac{d\zeta}{d\omega} \tag{4}$$

where ζ is given by (3). Let $\omega = \mu_s$ be the point corresponding to $\zeta = \alpha_s$ under the transformation (3), ie

$$\alpha_s = \frac{i\mu_s\beta-\bar\beta}{i\mu_s-1} \tag{5}$$

Then by direct substitution

$$\zeta-\alpha_s = \frac{i(\bar\beta-\beta)}{(i\mu_s-1)}\,\frac{\omega-\mu_s}{i\omega-1}$$

$$\zeta-\beta = \frac{\beta-\bar\beta}{i\omega-1}, \qquad \zeta-\bar\beta = \frac{i\omega(\beta-\bar\beta)}{i\omega-1}, \qquad \frac{d\zeta}{d\omega} = \frac{i(\bar\beta-\beta)}{(i\omega-1)^2}$$

#65 continued -

Also $\sum_1 \mu_s = 2$

Hence
$$\frac{dG}{dw} = -\frac{A(\beta-\bar\beta)^2}{\prod(i\nu_s-1)^{\mu_s}}\,\frac{\prod(w-\nu_s)^{\mu_s}}{(iw-1)^2}\,\frac{(iw-1)^4}{(iw)^2(\beta-\bar\beta)^4}\,\frac{i(\beta-\bar\beta)}{(iw-1)^2}$$
$$= A'\,\frac{\prod(w-\nu_s)^{\mu_s}}{w^2} \qquad (6)$$

where A' is a constant.

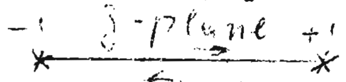
Near ∞ $\frac{dG}{dw} = A'\prod\left(1-\frac{\nu_s}{w}\right)^{\mu_s} = A'\left[1-\frac{\sum_1 \mu_s\nu_s}{w}+\dots\right]$

and in order that the transformation should be conformal at ∞, G contains no term in log w, and we require $\sum_1 \mu_s\nu_s = 0$

Since $\nu_s = -i\,\frac{\alpha_s-\bar\beta}{\alpha_s-\beta}$ this is the same as

$$0 = \sum_1 \mu_s\left(\frac{\alpha_s-\bar\beta}{\alpha_s-\beta}\right) = \sum \mu_s\left\{1+\frac{\beta-\bar\beta}{\alpha_s-\beta}\right\} = \sum_1 \mu_s + (\beta-\bar\beta)\sum_1\frac{\mu_s}{\alpha_s-\beta}$$

and since $\sum_1 \mu_s = 2$, this is the same as (2)



To map the z-plane cut from −1 to +1 on the upper half-plane of ζ, let $\zeta = i$ correspond to $z = \infty$, $\zeta = -1$ to $z = 1$, $\zeta = +1$ to $z = -1$. Then $\alpha_1 = -1$, $\alpha_2 = +1$, $\mu_1 = \mu_2 = 1$

Since $\beta = i$, $\bar\beta = -i$, it is easily checked that the condition (2) is satisfied.

$$F'(\zeta) = A\,\frac{\zeta^2-1}{(\zeta^2+1)^2} = A\left\{\frac{2\zeta^2-(\zeta^2+1)}{(\zeta^2+1)^2}\right\}$$
$$z = F(\zeta) = -\frac{A\zeta}{\zeta^2+1} + B$$

when B is a constant. Since $\zeta = \pm 1$ correspond to $z = \mp 1$

$$\pm 1 = \mp\frac{A}{2} + B \qquad B = 0,\ A = -2$$

$$z = \frac{2\zeta}{\zeta^2+1} \qquad (7) \qquad (7)$$

This is easily checks. If we transform the upper -plane of ζ into the exterior of unit circle, $|w| = 1$ by
$$w = -i\,\frac{\zeta+i}{\zeta-i}$$

we have
$$\zeta = i\,\frac{iw+1}{iw-1},\quad \zeta^2+1 = -\frac{4iw}{(iw-1)^2},$$
$$z = 2i\,\frac{iw+1}{iw-1}\,\frac{(iw-1)^2}{(-4iw)} = \frac{1}{2}\left(w+\frac{1}{w}\right) \longrightarrow (8)$$

Problem #66 continued -

which is a known transformation of the cut z -plane into the exterior of the circle in the w -plane.

The transformation (8) may also be obtained directly from (6) by taking $\alpha_1 = -1$, $\alpha_2 = 1$ with $\mu_1 = \mu_2 = 1$ as before. (So $\sum \mu_s \alpha_s = 0$ is satisfied.) Then

$$\frac{dG}{dw} = A' \frac{w^2-1}{w^2} = A'\left(1 - \frac{1}{w^2}\right)$$

$$z = G = A'\left(w + \frac{1}{w}\right) + B'$$

If we take w = ± 1 to correspond with z = ± 1 , respectively, we find $B' = 0$, $A' = \frac{1}{2}$, and recover (8)

Assignment #1
Problem Set #1
Problem 2

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DEAP-AP G1
Applied Math 201
9-26-60

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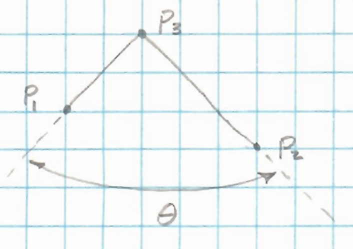
Excercise 2: P_1, P_2, P_3 correspond to an Argand diagram of z_1, z_2, z_3 with one possible case shown below:



$\angle P_1 P_3 P_2 \equiv \theta$ and is positive

Show: $\theta = \arg \dfrac{z_3 - z_2}{z_3 - z_1}$

The procedure will be as follows:

1. The points will be translated without deformation such that P_3 forms the origin of a new complex plane.
2. The group of points will now be rotated without deformation such that the line segment $\overrightarrow{P_3 P_1}$ lies on the real axis of a second new complex plane with P_3 as the origin.
3. θ is now the argument of the complex number in the second plane corresponding to z_2 in the original plane.

The transformation which effects procedure 1 is:

(1) $w = z - z_3$

which maps all points in the z plane into the w plane with a translation z_3.

Procedure 2 is wrought by:

(2) $s = e^{i\varphi} w$

which maps all points in the w plane into the s plane with a rotation φ. Since it is desired to bring the line segment $\overrightarrow{P_3 P_1}$ into coincidence with the real axis of the s plane, it is plane that

(3) $\varphi = 2\pi - \arg w_1$ (principle values of $\arg w_1$ only)

(4) $\therefore e^{i\varphi} = \dfrac{1}{w_1}$

Thus,

$$(5) \quad s = \frac{w}{w_1} = \frac{z - z_3}{z_1 - z_3}$$

The point s_2 corresponding to the point z_2 in the z plane is

$$(6) \quad s_2 = \frac{z_2 - z_3}{z_1 - z_3} = \frac{z_3 - z_2}{z_3 - z_1}$$

It is now plain that the angle θ is merely the argument of s_2, viz,

$$(7) \quad \theta = \arg s_2 = \arg \frac{z_3 - z_2}{z_3 - z_1}$$

The result is perfectly general for all arrangements of P_1, P_2, P_3 as no assumptions as to juxtaposition were made in the proof: QED.

Assignment #2
Problem Set #1
Prob. 1, 3, 4

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1\. a. (1) Show $|z_1 - z_2|^2 + |z_1 + z_2|^2 = 2|z_1|^2 + 2|z_2|^2$

(2) $z_1 \equiv x_1 + i y_1, \quad |z_1|^2 = x_1^2 + y_1^2$

$z_2 \equiv x_2 + i y_2, \quad |z_2|^2 = x_2^2 + y_2^2$

(3) $z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2), \quad |z_1 - z_2|^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

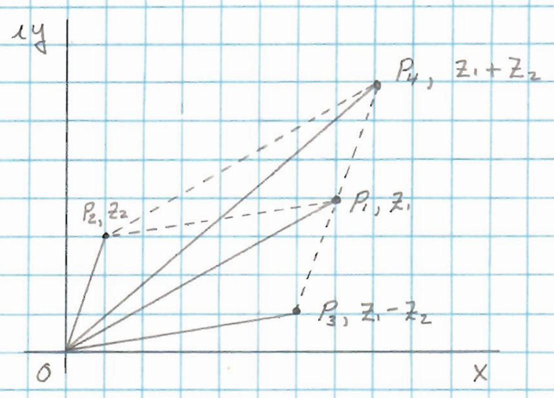
$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2), \quad |z_1 + z_2|^2 = (x_1 + x_2)^2 + (y_1 + y_2)^2$

(4) (3) into (1):

$x_1^2 - 2x_1x_2 + x_2^2 + x_1^2 + 2x_1x_2 + x_2^2 + y_1^2 - 2y_1y_2 + y_2^2 + y_1^2 + 2y_1y_2 + y_2^2$

$= 2(x_1^2 + y_1^2) + 2(x_2^2 + y_2^2) = 2|z_1|^2 + 2|z_2|^2$, QED

b. Consider the following Argand Diagram:



(1) $|z_1 + z_2|$ is length of line segment $\overrightarrow{OP_4}$

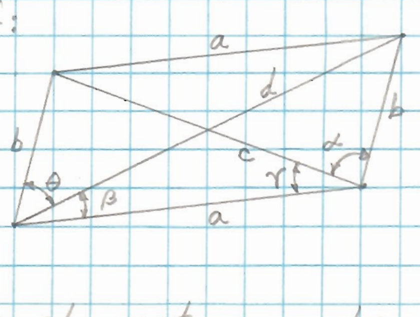
$|z_1 - z_2|$ " " " " " $\overrightarrow{OP_3}$

$|z_1|$ " " " " " $\overrightarrow{OP_1}$

$|z_2|$ " " " " " $\overrightarrow{OP_2}$

(2) It follows that equation a(1) describes the following property of parallelograms: The sum of the squares of the diagonals of a parallelogram is equal to twice the sum of the squares of its two unequal sides. This can be seen by considering the parallelogram $OP_2P_4P_1$ of the Argand diagram.

(3) Proof:



$a^2 = b^2 + d^2 - 2bd\cos\theta$

$a^2 = b^2 + c^2 - 2bc\cos\alpha$

$\frac{a}{\sin\theta} = \frac{b}{\sin\beta} = \frac{d}{\sin(\alpha+\gamma)}$

$\frac{a}{\sin\alpha} = \frac{b}{\sin\gamma} = \frac{c}{\sin(\theta+\beta)}$

$\alpha + \beta + \gamma + \theta = 180°$

(4) Upon eliminating angles from Eqs. (3): $c^2 + d^2 = 2(a^2 + b^2)$

1 c. (1) $|z|^2 = |z|\,|z| = |z \cdot z| = |z^2|$

(2) $\therefore\ |(z_1 - z_2)^2| + |(z_1 + z_2)^2| = 2\,|z_1^2| + 2\,|z_2^2|$

$|(z_1^2 - 2z_1z_2 + z_2^2)| + |(z_1^2 + 2z_1z_2 + z_2^2| = 2\,|z_1^2| + 2\,|z_2^2|$

(3) Let $z_1^2 = \dfrac{\alpha - \beta}{2}$; $z_2^2 = \dfrac{\alpha + \beta}{2}$

(4) $z_1^2 - 2z_1z_2 + z_2^2 = \dfrac{\alpha-\beta}{2} - \sqrt{\alpha^2-\beta^2} + \dfrac{\alpha+\beta}{2} = \alpha - \sqrt{\alpha^2-\beta^2}$

(5) $z_1^2 + 2z_1z_2 + z_2^2 = \alpha + \sqrt{\alpha^2-\beta^2}$

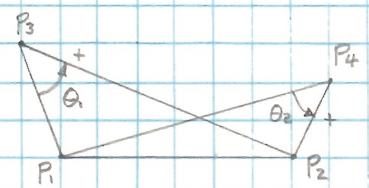
(6) Substitute in (2): $|\alpha - \sqrt{\alpha^2-\beta^2}| + |\alpha + \sqrt{\alpha^2-\beta^2}| = |\alpha-\beta| + |\alpha+\beta|$

---

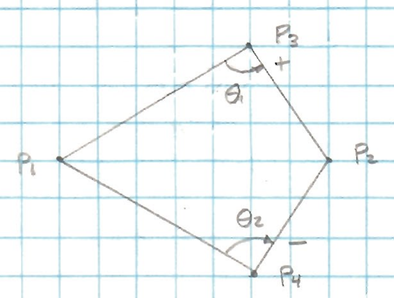
3. (1) Let us assign the following geometrical meanings to the points $P_1, \ldots P_4$ on the Argand plane:

$z_3 - z_1 = \overrightarrow{P_3P_1}$
$z_3 - z_2 = \overrightarrow{P_3P_2}$
$z_4 - z_1 = \overrightarrow{P_4P_1}$
$z_4 - z_2 = \overrightarrow{P_4P_2}$

(2) There are two general possible cases:



I



II

(3) Clearly case II is excluded by the results of Problem 2 and the given equation, viz,

$$\theta_1 = \arg \frac{z_3 - z_2}{z_3 - z_1} = \arg \frac{z_4 - z_2}{z_4 - z_1} = \theta_2$$

Therefore, if $\theta_1 = \theta_2$, case I results, and P_3 and P_4 must lie on the same side of the line $\overrightarrow{P_1P_2}$. Case II would give equal angles but opposite in direction thus not fitting the given equation and its sign convention as stated in Problem 2.

Assignment #2
Problem Set #1
Prob. 1, 3, 4

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3. continued:

(4) A geometric proof that equal angles which intercept the same arc are inscribed angles can be found in #277, Plane Geometry, by Welchons and Krickenberger.

One takes the $\Delta P_1P_4P_2$, circumscribes a circle about it, and is then led by the above theorem that the point P_3 lies on the circumscribed circle.

(5) The converse may also be proven geometrically, but let us try an analytic approach this time:

If the points $z_1, \dots z_4$ are concyclic, they must each satisfy the equation of a circle in the complex plane, viz,

$$|z-a| = r^2$$

where r is a real constant and $a = g+ih$ is a complex constant

(6) Expanding (5) and substituting the above points:

$$(x_1-g)^2 + (y_1-h)^2 = r^2$$
$$(x_2-g)^2 + (y_2-h)^2 = r^2$$
$$(x_3-g)^2 + (y_3-h)^2 = r^2$$
$$(x_4-g)^2 + (y_4-h)^2 = r^2$$

(7) As we have 4 equations in 3 unknowns, we may eliminate completely g, h, and r and obtain the following relation between $x_1,\dots x_4$, $y_1,\dots y_4$:

$$\frac{(x_3-x_1)(y_3-y_2)-(x_3-x_2)(y_3-y_1)}{(x_3-x_2)(x_3-x_1)+(y_3-y_2)(y_3-y_1)}$$

$$= \frac{(x_4-x_1)(y_4-y_2)-(x_4-x_2)(y_4-y_1)}{(x_4-x_2)(x_4-x_1)+(y_4-y_2)(y_4-y_1)}$$

after considerable manipulation.

(8) If $\arg \frac{z_3 - z_2}{z_3 - z_1} = \arg w_3 = \arg \frac{z_4 - z_2}{z_4 - z_1} = \arg w_4$

then, $\frac{\operatorname{Im} w_4}{\operatorname{Re} w_4} = \frac{\operatorname{Im} w_3}{\operatorname{Re} w_3}$, if P_3, P_4 are on the same side of $\overrightarrow{P_1P_2}$

(9) Hence: $\frac{z_3 - z_2}{z_3 - z_1} = \frac{[(x_3-x_2) + i(y_3-y_2)][(x_3-x_1) - i(y_3-y_1)]}{(x_3-x_1)^2 + (y_3-y_1)^2}$

$$= \frac{[(x_3-x_2)(x_3-x_1) + (y_3-y_2)(y_3-y_1)]}{(x_3-x_1)^2 + (y_3-y_1)^2} + i \frac{[(x_3-x_1)(y_3-y_2) - (x_3-x_2)(y_3-y_1)]}{(x_3-x_1)^2 + (y_3-y_1)^2}$$

with a similar result for w_4 when the 3 subscript is replaced by 4 in the above. It is now plain that (7) will result when the operation (8) is made on (9).

---

4. (1) $\left(\frac{z+1}{z}\right)^5 = 32$, Let $w = \frac{z+1}{z} = u + iv$

(2) $w = 32^{1/5} = 2\left(\cos \frac{2\pi k}{5} + i \sin \frac{2\pi k}{5}\right)$, $k = 0, 1, 2, 3, 4$

thus $u = 2\cos\theta$, $v = 2\sin\theta$; $\theta = \frac{2\pi k}{5}$

this, of course, the equation of a circle in the w plane, considering k a dummy parameter.

(3) $z + 1 - wz = 0$; $z = \frac{1}{w-1}$

(4) $x + iy = \frac{1}{(u-1) + iv} = \frac{(u-1) - iv}{(u-1)^2 + v^2}$

(5) $x = \frac{u-1}{(u-1)^2 + v^2}$; $y = \frac{-v}{(u-1)^2 + v^2}$

(6) Now: $x^2 + y^2 = \frac{1}{(u-1)^2 + v^2}$

(7) The points in the w plane which lie on a circle have the cartesian equation:

$u^2 + v^2 = 4$

Assignment # 2
Problem Set # 1
Prob. 1, 3, 4

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4. Continued

(8) $$u + iv = \frac{x + iy + 1}{x + iy} = \frac{x^2 + x + y^2}{x^2 + y^2} + i \frac{-y}{x^2 + y^2}$$

(9) Substituting in (7):

$$\frac{(x^2 + x + y^2)^2 + y^2}{(x^2 + y^2)^2} = 4$$

expanding: $3x^4 - 2x^3 - x^2 + 6x^2y^2 - 2xy^2 - y^2 + 3y^4 = 0$

(10) This may be factored:

$$(3x^2 - 2x + 3y^2 - 1)(x^2 + y^2) = 0$$

The second factor is trivial as it represents mapping into the origin.

(11) Completing the square on the first factor:

$$(x - 1/3)^2 + y^2 = 4/9$$

Thus the 5 points on the circle $u^2 + v^2 = 4$ in the w plane will map into 5 points on the circle with radius 2/3 and center at $x = 1/3$ in the z plane. The location of the points can be found from eq. (5) and equation (2) for $k = 0, 1, 2, 3, 4$. This is a special case of the bilinear transformation.

Prove: $$\frac{d}{dt} e^{(a+ib)t} = (a+ib)\, e^{(a+ib)t}$$

(1) $e^{(a+ib)t} = e^{at} e^{ibt} = s$

(2) $$\frac{ds}{dt} = a e^{at} e^{ibt} + ib e^{at} e^{ibt} = (a+ib)\, e^{(a+ib)t}$$

Assignment # 3
Problem Set # 1
Prob. 5, 6, 7

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5. (1) $z = x + iy, \quad \bar{z} = x - iy, \quad x = \frac{1}{2}(z + \bar{z})$
$y = -\frac{i}{2}(z - \bar{z})$

(2) $\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$

(3) $\frac{\partial}{\partial z} = \frac{\partial}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial}{\partial y}\frac{\partial y}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$

(4) Now: $u_{xx} + u_{yy} = \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) u$

$= \left(2\frac{\partial}{\partial \bar{z}}\right)\left(2\frac{\partial}{\partial z}\right) u = 4\frac{\partial^2 u}{\partial \bar{z}\partial z} = 4\frac{\partial^2 V}{\partial \bar{z}\partial z}$

(5) Thus Laplace's equation may be written:

$u_{xx} + u_{yy} = 4V_{\bar{z}z} = 4V_{z\bar{z}} = 0$

or $V_{\bar{z}z} = V_{z\bar{z}} = 0$

10

6. (1) Given: $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

(2) $u_x = 3x^2 - 3y^2 + 6x \quad ; \quad u_{xx} = 6x + 6$

$u_y = -6xy - 6y \quad ; \quad u_{yy} = -6x - 6$

(3) $\therefore \quad u_{xx} + u_{yy} = 0$

(4) Now $f'(z) = u_x - i u_y = 3x^2 - 3y^2 + 6x + i\,6xy + i\,6y$

(5) Set $x = z, \ y = 0$: $f'(z) = 3z^2 + 6z$

(6) $f(z) = z^3 + 3z^2 + 1 + iB$

(7) Check: $f(z) = u(x,y) + i v(x,y) = (x+iy)^3 + 3(x+iy)^2 + 1 + iB$

$= x^3 - 3xy^2 + 2i x^2 y + i x^2 y - i y^3 + 3x^2 - 3y^2 + 6ixy + 1 + iB$

(8) $\left.\begin{aligned} u(x,y) &= x^3 - 3xy^2 + 3x^2 - 3y^2 + 1 \\ v(x,y) &= 3x^2 y - y^3 + 6xy + B \end{aligned}\right\}$ Checks

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7. (1) we have: $\dfrac{\partial^2 u}{\partial \bar z\, \partial z} = \dfrac{\partial}{\partial \bar z}\left(\dfrac{\partial}{\partial z}\right) = 0$

with $$\left.\begin{aligned} x &= \frac{z+\bar z}{2} = r\cos\theta \\ y &= \frac{z-\bar z}{2i} = r\sin\theta \end{aligned}\right\} \quad z = re^{i\theta}, \quad \bar z = re^{-i\theta}$$

(2) $\theta = \tan^{-1}\dfrac{z-\bar z}{i(z+\bar z)}$; $r = \dfrac{1}{2}\left[(z+\bar z)^2 - (z-\bar z)^2\right]^{1/2}$

(3) $\dfrac{\partial}{\partial z} = \dfrac{\partial}{\partial r}\dfrac{\partial r}{\partial z} + \dfrac{\partial}{\partial \theta}\dfrac{\partial \theta}{\partial z}$; $\dfrac{\partial}{\partial \bar z} = \dfrac{\partial}{\partial r}\dfrac{\partial r}{\partial \bar z} + \dfrac{\partial}{\partial \theta}\dfrac{\partial \theta}{\partial \bar z}$

(4) $\dfrac{\partial r}{\partial z} = \dfrac{\bar z}{2}\left[(z+\bar z)^2 - (z-\bar z)^2\right]^{-1/2} = \dfrac{e^{-i\theta}}{2}$

(5) $\dfrac{\partial r}{\partial \bar z} = \dfrac{e^{i\theta}}{2}$

(6) $$\frac{\partial \theta}{\partial z} = \frac{-i}{1-\left(\frac{z-\bar z}{z+\bar z}\right)^2}\cdot\frac{(z+\bar z)-(z-\bar z)}{(z+\bar z)^2} = \frac{(z+\bar z)-(z-\bar z)}{(z+\bar z)^2-(z-\bar z)^2}\cdot -i$$

$$= \frac{2\bar z \cdot -i}{4z\bar z} = \frac{-i}{2z} = \frac{-i}{2r}e^{-i\theta}$$

(7) $$\frac{\partial \theta}{\partial \bar z} = \frac{(z+\bar z)^2\cdot -i}{(z+\bar z)^2-(z-\bar z)^2}\cdot\frac{-(z+\bar z)-(z-\bar z)}{(z+\bar z)^2} = \frac{+2zi}{4z\bar z} = \frac{+i}{2r}e^{i\theta}$$

(8) Now: $$\frac{\partial}{\partial \bar z}\left(\frac{\partial u}{\partial z}\right) = \frac{\partial r}{\partial z}\cdot\frac{\partial}{\partial \bar z}\left(\frac{\partial u}{\partial r}\right) + \frac{\partial u}{\partial r}\cdot\frac{\partial}{\partial \bar z}\left(\frac{\partial r}{\partial z}\right) + \frac{\partial u}{\partial \theta}\cdot\frac{\partial}{\partial \bar z}\left(\frac{\partial \theta}{\partial z}\right) + \frac{\partial \theta}{\partial z}\cdot\frac{\partial}{\partial \bar z}\left(\frac{\partial u}{\partial \theta}\right)$$

(9) $$\frac{\partial}{\partial \bar z}\left(\frac{\partial u}{\partial r}\right) = \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\right)\cdot\frac{\partial r}{\partial \bar z} + \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial r}\right)\cdot\frac{\partial \theta}{\partial \bar z} = \frac{e^{i\theta}}{2}\frac{\partial^2 u}{\partial r^2} + \frac{i}{2r}e^{i\theta}\frac{\partial^2 u}{\partial \theta\, \partial r}$$

(10) $$\frac{\partial}{\partial \bar z}\left(\frac{\partial r}{\partial z}\right) = \frac{\partial}{\partial r}\left(\frac{\partial r}{\partial z}\right)\cdot\frac{\partial r}{\partial \bar z} + \frac{\partial}{\partial \theta}\left(\frac{\partial r}{\partial z}\right)\frac{\partial \theta}{\partial \bar z} = 0 - \frac{ie^{-i\theta}}{2}\cdot\frac{ie^{i\theta}}{2r} = \frac{1}{4r}$$

(11) $$\frac{\partial}{\partial \bar z}\left(\frac{\partial \theta}{\partial z}\right) = \frac{\partial}{\partial r}\left(\frac{\partial \theta}{\partial z}\right)\cdot\frac{\partial r}{\partial \bar z} + \frac{\partial}{\partial \theta}\left(\frac{\partial \theta}{\partial z}\right)\cdot\frac{\partial \theta}{\partial \bar z} = \frac{i}{2r^2}e^{-i\theta}\cdot\frac{e^{i\theta}}{2} - \frac{e^{-i\theta}}{2r}\cdot\frac{ie^{i\theta}}{2r} = 0$$

(12) $$\frac{\partial}{\partial \bar z}\left(\frac{\partial u}{\partial \theta}\right) = \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial \theta}\right)\cdot\frac{\partial r}{\partial \bar z} + \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial \theta}\right)\cdot\frac{\partial \theta}{\partial \bar z} = \frac{e^{i\theta}}{2}\frac{\partial^2 u}{\partial r\, \partial \theta} + \frac{ie^{i\theta}}{2r}\frac{\partial^2 u}{\partial \theta^2}$$

(13) $$\frac{\partial}{\partial \bar z}\left(\frac{\partial u}{\partial z}\right) = V_{\bar z z} = \frac{1}{4}\frac{\partial^2 u}{\partial r^2} + \frac{i}{4r}\frac{\partial^2 u}{\partial \theta\, \partial r} + \frac{1}{4r}\frac{\partial u}{\partial r} - \frac{i}{4r}\frac{\partial^2 u}{\partial \theta\, \partial r} + \frac{1}{4r^2}\frac{\partial^2 u}{\partial \theta^2}$$

(14) From Prob. 5; $u_{xx} + u_{yy} = 4V_{\bar z z} = \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2}$

or, for Laplace's Equation: $r^2 u_{rr} + r u_r + u_{\theta\theta} = 0$

10

Assignment #4
Problems 8, 9, 10

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8. (1) $u = \sin x \cosh y + 2\cos x \sinh y + x^2 - y^2 + 4xy$

10

$u_x = \cos x \cosh y - 2\sin x \sinh y + 2x + 4y$

$u_{xx} = -\sin x \cosh y - 2\cos x \sinh y + 2$

$u_y = \sin x \sinh y + 2\cos x \cosh y - 2y + 4x$

$u_{yy} = \sin x \cosh y + 2\cos x \sinh y - 2$

(2) $u_{xx} + u_{yy} = -\sin x \cosh y - 2\cos x \sinh y + 2 + \sin x \cosh y + 2\cos x \sinh y - 2 = 0$

(3) $f'(z) = u_x - i u_y = u_x(z,0) - i u_y(z,0)$

Put $x = z$, $y = 0$:

(4) $f'(z) = \cos z + 2z - i2\cos z - i4z$

$= (1-2i)\cos z + 2(1-2i)z = (1-2i)[\cos z + 2z]$

(5) $f(z) = (1-2i)[\sin z + z^2] + A + iB$

(6) At $y=0$: $f(x) = (1-2i)[\sin x + x^2] + A + iB = u(x,0) + iv(x,0)$

$= \sin x + x^2 + iv(x,0)$, $\therefore$ $A = 0$

(7) Finally, $f(z) = (1-2i)[\sin z + z^2] + iB$

Check:

(8) $f(z) = (1-2i)[\sin(x+iy) + (x^2-y^2) + 2ixy] + iB$

$= \sin x \cosh y + i\cos x \sinh y + x^2 - y^2 + 2ixy - 2i\sin x \cosh y + 2\cos x \sinh y - 2ix^2 + 2iy^2 + 4xy + iB$

$= \sin x \cosh y + 2\cos x \sinh y + x^2 - y^2 + 4xy + i(\cos x \sinh y - 2\sin x \cosh y + 2xy - 2x^2 + 2y^2 + B) = u(x,y) + iv(x,y)$

(9) $\therefore$ $u(x,y) = \sin x \cosh y + 2\cos x \sinh y + x^2 - y^2 + 4xy$,

which equals equation (1), thus completing the check.

9\. a. i. (1) $\cos z = \cos(x+iy) = \underbrace{\cos x \cosh y}_{} - i \underbrace{\sin x \sinh y}_{}$

$\cosh y \neq 0$; $\cos x = 0$; $\therefore x = \frac{n\pi}{2}$, $n = \pm 1, \pm 3, \pm 5$

$x = \frac{n\pi}{2}$, n odd, $\sin x \neq 0$; $\therefore \sinh y = 0$, $y = 0$

(2) Zeroes at $y = 0$; $x = (2n+1)\frac{\pi}{2}$, $n = 0, \pm 1, \pm 2, \ldots$

ii. (1) $\sinh z = \sinh(x+iy) = \underbrace{\sinh x \cos y}_{} + i \underbrace{\cosh x \sin y}_{}$

$\cos y \neq 0$; $\sinh x = 0$; $x = 0$

$\cosh x \neq 0$; $\sin y = 0$; $y = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$

(2) $\therefore$ Zeroes at $x = 0$; $y = n\pi$, $n = 0, \pm 1, \pm 2, \pm 3, \ldots$

iii. (1) $\cosh z = \cosh(x+iy) = \underbrace{\cosh x \cos y}_{} + i \underbrace{\sinh x \sin y}_{}$

$\cosh x \neq 0$; $\cos y = 0$; $\therefore y = (2n+1)\frac{\pi}{2}$, $n = 0, \pm 1, \pm 2, \ldots$

$\sin y \neq 0$; $\sinh x = 0$; $x = 0$

(2) $\therefore$ Zeroes at $x = 0$; $y = (2n+1)\pi/2$, $n = 0, \pm 1, \pm 2, \ldots$

b. i. (1) $\lim_{z \to n\pi} (z - n\pi) \operatorname{cosec} z = \lim_{z \to n\pi} \frac{z - n\pi}{\sin z}$

(2) Apply L'Hôpital's Rule, since both $z - n\pi$ and $\sin z$ are analytic at the point approached. explain more fully

(3) $\therefore \lim_{z \to n\pi} \frac{1}{\cos z} = \frac{1}{\cos n\pi} = (-1)^n$, $n = 0, \pm 1, \pm 2, \ldots$

ii. (1) $\lim_{z \to n\pi} (z - n\pi) \cot z = \lim_{z \to n\pi} \frac{z - n\pi}{\tan z}$

(2) Apply L'Hôpital's Rule: $\lim_{z \to n\pi} \frac{1}{\sec^2 z} = \lim_{z \to n\pi} \cos^2 z = 1$

iii. (1) $\lim_{z \to (n+\frac{1}{2})\pi} \{z - (n + 1/2)\pi\} \sec z = \lim_{z \to (n+\frac{1}{2})\pi} \frac{z - (n+1/2)\pi}{\cos z}$

(2) L'Hôpital's Rule can be used for the same reason as before:

$$\lim_{z \to (n+\frac{1}{2})\pi} \frac{1}{-\sin z} = \frac{1}{-\sin(n+\frac{1}{2})\pi} = (-1)^{n+1}, \quad n = 0, \pm 1, \pm 2, \ldots$$

Assignment #4
Continued

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9. b. iv. (1) $\displaystyle \lim_{z \to (n+\frac{1}{2})\pi} \left\{ z - (n+\tfrac{1}{2})\pi \right\} \tan z = \lim_{z \to (n+\frac{1}{2})\pi} \frac{z - (n+\frac{1}{2})\pi}{\cot z}$

(2) Apply L'Hôpital's Rule:

$$\lim_{z \to (n+\frac{1}{2})\pi} \frac{1}{-\csc^2 z} = -\sin^2 (n+\tfrac{1}{2})\pi = -1$$

c. As all the hyperbolic functions of $n\pi$ and $(n+\frac{1}{2})\pi$ are finite for finite n, it is clearly seen that when the trigonometric functions in (b) i, ii, iii, iv are replaced by their respective hyperbolic functions, the resulting limits will tend to zero.

By "corresponding" limits is meant limits of the type $\frac{0}{0}$

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10. a. (1) $\displaystyle z = \tanh w = \frac{\sinh w}{\cosh w} = \frac{e^{w} - e^{-w}}{e^{w} + e^{-w}} = \frac{e^{2w} - 1}{e^{2w} + 1}$

(2) $\displaystyle z e^{2w} + z - e^{2w} + 1 = 0 \quad , \quad e^{2w} = \frac{-z-1}{z-1} = \frac{1+z}{1-z}$

(3) $\displaystyle 2w = \mathrm{Log} \frac{1+z}{1-z} \quad , \quad w = \frac{1}{2} \mathrm{Log} \frac{1+z}{1-z}$

b. (1) $\displaystyle z = \tan w = \frac{\sin w}{\cos w} = \frac{1}{i} \frac{e^{iw} - e^{-iw}}{e^{iw} + e^{-iw}} = \frac{1}{i} \frac{e^{i2w} - 1}{e^{i2w} + 1}$

(2) $\displaystyle iz e^{i2w} + iz - e^{i2w} + 1 = 0 \, , \quad e^{i2w} = \frac{-iz-1}{iz-1} = \frac{1+iz}{1-iz}$

(3) $\displaystyle i2w = \mathrm{Log} \frac{1+iz}{1-iz} \; ; \quad w = \frac{1}{2i} \mathrm{Log} \frac{1+iz}{1-iz}$

Assignment #5
Problems 11, 12

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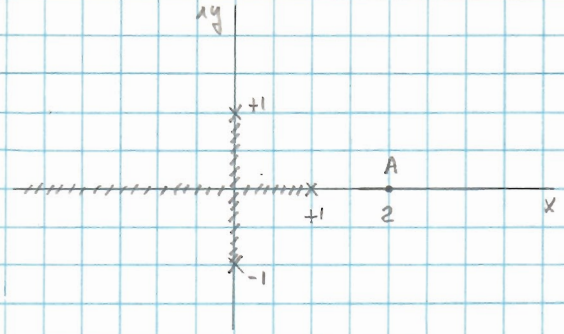
11. (1) $w^2 = (1-z)(1+z^2) = (1-z)(z+i)(z-i)$



(2) when $z = 2$,

$w = \pm\sqrt{5}\,i$

(3) We must choose from these two values of:

$w = -\sqrt{5}\,i$

$w = \sqrt{5}\,i$

(4) At the origin $w=1$ and the path of z is restricted to the first quadrant. The solutions of (1) are of the form:

$$i\,w_1 = (r_1 r_2 r_3)^{1/2} e^{i\,1/2\,(\theta_1+\theta_2+\theta_3)}$$

$$i\,w_2 = -(r_1 r_2 r_3)^{1/2} e^{i\,1/2\,(\theta_1+\theta_2+\theta_3)}$$

all square roots are implied positive.

We investigate each individually at the origin for the proper one to be used elsewhere in the first quadrant.

$$i\,w_1 = (1)^{1/2} e^{i\,1/2\,(\pi/2 - \pi/2 - \pi)} = i \;;\quad w_1 = 1$$

$$i\,w_2 = -(1)^{1/2} e^{i\,1/2\,(\pi/2 - \pi/2 - \pi)} = -i \;;\quad w_2 = -1$$

Thus the w_1 form is the proper one.

(5) At $z = 2$:

$$i\,w_1 = \left[\sqrt{5}\sqrt{5}\sqrt{1}\right]^{1/2} e^{i\,1/2\,(\theta_1-\theta_1+0)} = -i\sqrt{5}$$

$\therefore\; w_1 = -\sqrt{5}\,i$ Q.E.D.

12. a. (1) Use D'Alembert's Test: $\lim_{n\to\infty} \left| \frac{\alpha_{n+1}}{\alpha_n} \right| < 1$

(2) $$\lim_{n\to\infty} \left| \frac{\alpha(\alpha-1)\cdots(\alpha-n)}{(n+1)!} \cdot \frac{n!}{\alpha(\alpha-1)\cdots(\alpha-n+1)} \right| |z|$$

$$= \left| \frac{\alpha - n}{n+1} \right| |z| = |z| < 1$$

(3) If $f(z)$ is a point inside the circle of convergence $|z| < 1$ then $f(z)$ is a regular point, also, it may be differentiated term by term inside the circle of convergence.

b. (1) $$f'(z) = \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{(n-1)!} z^{n-1}$$

(2) $$(1+z) f'(z) = \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{(n-1)!} z^{n-1} + \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{(n-1)!} z^{n}$$

$$= \alpha + \alpha z + \frac{\alpha(\alpha-1)}{1!} z + \frac{\alpha(\alpha-1)}{1!} z^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{2!} z^2 + \cdots$$

$$= \alpha \left\{ 1 + z + \alpha z - z + \alpha z^2 - z^2 + \frac{\alpha^2 z^2}{2!} - \frac{3\alpha z^2}{2!} + 2\frac{z^2}{2!} + \cdots \right.$$

$$= \alpha \left\{ 1 + \alpha z + \frac{\alpha(\alpha-1)}{2} z^2 + \cdots \right\}$$

$$= \alpha f(z)$$

(3) $$\therefore \quad f'(z) = \frac{\alpha f(z)}{1+z}$$

c. (1) $$\frac{d}{dz}(1+z)^{-\alpha} f(z) = -\alpha(1+z)^{-\alpha-1} f(z) + (1+z)^{-\alpha} f'(z)$$

$$= -\alpha(1+z)^{-\alpha-1} f(z) + (1+z)^{-\alpha} \left\{ \alpha(1+z)^{-1} f(z) \right\} = 0$$

(2) $\frac{d}{dz}(1+z)^{-\alpha} f(z) = 0 \;;\; \therefore \quad f(z) = c\,(1+z)^{\alpha}$

(3) $f(0) = 1 = c\,(1)^{\alpha} \;;\quad c = 1$

(4) $\therefore \quad f(z) = (1+z)^{\alpha}$ and may be represented by the series when $|z| < 1$.

Problems 13, 14, 15

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13. (1) Given that $z = x + iy$, $\zeta = \xi + i\eta = f(z)$, where $f(z)$ is a regular function in Domain D with $f'(z) \neq 0$, we have:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = |f'(z)|^2 \left(\frac{\partial^2 V}{\partial \xi^2} + \frac{\partial^2 V}{\partial \eta^2} \right)$$

(2) We choose $f(z) = z = r\cos\theta + i r \sin\theta = \xi + i\eta$ which is analytic everywhere with $f'(z) = 1 \neq 0$ so that equation (1) may be applied.

(3)
$$\frac{\partial V}{\partial \xi} = \frac{\partial V}{\partial r}\frac{\partial r}{\partial \xi} + \frac{\partial V}{\partial \theta}\frac{\partial \theta}{\partial \xi} \; ; \quad \frac{\partial V}{\partial \eta} = \frac{\partial V}{\partial r}\frac{\partial r}{\partial \eta} + \frac{\partial V}{\partial \theta}\frac{\partial \theta}{\partial \eta}$$

(4)
$$r = \sqrt{\xi^2 + \eta^2} \qquad \theta = \tan^{-1}\frac{\eta}{\xi}$$

$$\frac{\partial r}{\partial \xi} = [\xi^2 + \eta^2]^{-1/2}\xi = \cos\theta \qquad \frac{\partial r}{\partial \eta} = [\xi^2 + \eta^2]^{-1/2}\eta = \sin\theta$$

$$\frac{\partial \theta}{\partial \xi} = \frac{-\eta/\xi^2}{1 + \eta^2/\xi^2} = -\frac{\sin\theta}{r} \qquad \frac{\partial \theta}{\partial \eta} = \frac{1/\xi}{1 + \eta^2/\xi^2} = \frac{\cos\theta}{r}$$

(5)
$$\frac{\partial V}{\partial \xi} = \cos\theta\frac{\partial V}{\partial r} - \frac{\sin\theta}{r}\frac{\partial V}{\partial \theta} \; ; \quad \frac{\partial V}{\partial \eta} = \sin\theta\frac{\partial V}{\partial r} + \frac{\cos\theta}{r}\frac{\partial V}{\partial \theta}$$

(6)
$$\frac{\partial^2 V}{\partial \xi^2} + \frac{\partial^2 V}{\partial \eta^2} = \left(\frac{\partial}{\partial \xi} + i\frac{\partial}{\partial \eta}\right)\left(\frac{\partial V}{\partial \xi} - i\frac{\partial V}{\partial \eta}\right)$$

$$= \frac{\partial}{\partial \xi}\left(\frac{\partial V}{\partial \xi}\right) + \frac{\partial}{\partial \eta}\left(\frac{\partial V}{\partial \eta}\right)$$

(7)
$$\frac{\partial}{\partial \xi}\left(\frac{\partial V}{\partial \xi}\right) = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\cos\theta\frac{\partial V}{\partial r} - \frac{\sin\theta}{r}\frac{\partial V}{\partial \theta}\right)$$

$$= \cos^2\theta\frac{\partial^2 V}{\partial r^2} + \frac{\sin\theta\cos\theta}{r^2}\frac{\partial V}{\partial \theta} - \frac{\sin\theta\cos\theta}{r}\frac{\partial^2 V}{\partial r\partial\theta}$$

$$+ \frac{\sin^2\theta}{r}\frac{\partial V}{\partial r} - \frac{\sin\theta\cos\theta}{r}\frac{\partial^2 V}{\partial r\partial\theta} + \frac{\sin\theta\cos\theta}{r^2}\frac{\partial V}{\partial \theta}$$

$$+ \frac{\sin^2\theta}{r^2}\frac{\partial^2 V}{\partial \theta^2}$$

(8) $\frac{\partial}{\partial \eta}\left(\frac{\partial V}{\partial \eta}\right) = \left(\sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\sin\theta \frac{\partial V}{\partial r} + \frac{\cos\theta}{r}\frac{\partial V}{\partial \theta}\right)$

$= \sin^2\theta \frac{\partial^2 V}{\partial r^2} - \frac{\sin\theta\cos\theta}{r^2}\frac{\partial V}{\partial \theta} + \frac{\sin\theta\cos\theta}{r}\frac{\partial^2 V}{\partial r\partial\theta}$

$+ \frac{\cos^2\theta}{r}\frac{\partial V}{\partial r} + \frac{\cos\theta \sin\theta}{r}\frac{\partial^2 V}{\partial\theta\partial r} - \frac{\sin\theta\cos\theta}{r^2}\frac{\partial V}{\partial\theta} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 V}{\partial\theta^2}$

(9) $\frac{\partial}{\partial \xi}\left(\frac{\partial V}{\partial \xi}\right) + \frac{\partial}{\partial \eta}\left(\frac{\partial V}{\partial \eta}\right) = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial\theta^2}$

(10) Since $|f'(z)|^2 = 1$,

$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial\theta^2}$

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14. (1) $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\cdot |f(z)|^2$

(2) $\frac{\partial}{\partial x}|f(z)|^2 = \frac{\partial}{\partial x}(u+iv)(u-iv) = (u_x + iv_x)(u-iv)$
$+ (u+iv)(u_x - iv_x)$

$\frac{\partial}{\partial y}|f(z)|^2 = \frac{\partial}{\partial y}(u+iv)(u-iv) = (u_y + iv_y)(u-iv)$
$+ (u+iv)(u_y - iv_y)$

(3) $\frac{\partial^2}{\partial x^2}|f(z)|^2 = (u_{xx} + iv_{xx})(u-iv) + (u+iv)(u_{xx} - iv_{xx})$
$+ 2(u_x + iv_x)(u_x - iv_x)$

$\frac{\partial^2}{\partial y^2}|f(z)|^2 = (u_{yy} + iv_{yy})(u-iv) + (u+iv)(u_{yy} - iv_{yy})$
$+ 2(u_y + iv_y)(u_y - iv_y)$

Now: $u_{xx} + u_{yy} = v_{xx} + v_{yy} = 0$, $u_x = v_y$, $u_y = -v_x$

(4) $\therefore \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\cdot|f(z)|^2 = 2(u_x + iv_x)(u_x - iv_x) + 2(-v_x + iu_x)(-v_x - iu_x)$

$= 2|f'(z)|^2 + (-2)(-iv_x - u_x)(-iv_x + u_x)$

$= 2|f'(z)|^2 + (+2)(u_x + iv_x)(u_x - iv_x) = 4|f'(z)|^2$

since $|f'(z)|^2 = (u_x + iv_x)(u_x - iv_x)$

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Problems 13, 14, 15
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15. a. (1) $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \cdot \ln|f'(z)|$

(2) If $f(z)$ is analytic, it has been shown that $f'(z)$ is analytic and therefore obeys the Cauchy - Riemann Relations. It will also be necessary in what follows that $f'(z) \neq 0$. Now:

$$f'(z) = u_x + i v_x = v_y - i u_y$$

or $f'(z) \equiv \mathfrak{f} = \xi + i\eta$, $\xi = u_x = v_y$, $\eta = v_x = -u_y$

and since $\mathfrak{f}$ is analytic: $\begin{matrix} \xi_x = \eta_y \\ \xi_y = -\eta_x \end{matrix} \Bigg\} \begin{matrix} \xi_{xx} + \xi_{yy} = 0 \\ \eta_{xx} + \eta_{yy} = 0 \end{matrix}$

(3) $\ln|f'(z)| = \frac{1}{2}\ln(\xi^2 + \eta^2)$

(4) $\frac{\partial}{\partial x}\ln|f'(z)| = \frac{\xi\xi_x + \eta\eta_x}{\xi^2+\eta^2}$

$\frac{\partial}{\partial y}\ln|f'(z)| = \frac{\xi\xi_y + \eta\eta_y}{\xi^2+\eta^2}$

(5) $\frac{\partial^2}{\partial x^2}\ln|f'(z)| = \frac{(\xi^2+\eta^2)(\xi_x^2 + \xi\xi_{xx} + \eta\eta_{xx} + \eta_x^2) - 2(\xi\xi_x + \eta\eta_x)^2}{(\xi^2+\eta^2)^2}$

$\frac{\partial^2}{\partial y^2}\ln|f'(z)| = \frac{(\xi^2+\eta^2)(\xi_y^2 + \xi\xi_{yy} + \eta\eta_{yy} + \eta_y^2) - 2(\xi\xi_y + \eta\eta_y)^2}{(\xi^2+\eta^2)^2}$

$= \frac{(\xi^2+\eta^2)(\xi_x^2 - \xi\xi_{xx} - \eta\eta_{xx} + \eta_x^2) - 2(\xi_x\eta - \xi\eta_x)^2}{(\xi^2+\eta^2)^2}$

(6) It is only necessary to consider the numerator since $f'(z) \neq 0$

$\xi^2\xi_x^2 + \xi^2\eta_x^2 + \eta^2\xi_x^2 + \eta^2\eta_x^2 - 2\xi^2\xi_x^2 - 4\xi\xi_x\eta\eta_x - 2\eta^2\eta_x^2$

$- 2\xi_x^2\eta^2 + 4\xi_x\eta\xi\eta_x - 2\xi^2\eta_x^2 = 0$

(7) $\therefore \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\ln|f'(z)| = 0$

b. (1) $|f'(z)| = \psi(x)\varphi(y)$; $\ln|f'(z)| = \ln\psi(x) + \ln\varphi(y)$

(2) Substituting in (a 1):

$$\frac{d}{dx}\left(\frac{d}{dx}\ln\psi\right) + \frac{d}{dy}\left(\frac{d}{dy}\ln\varphi\right) = 0$$

Making the time-honored arguments about separation of variables, we postulate a separation constant $2a$ which must be real.

$$\frac{d}{dx}\left(\frac{d}{dx}\ln\psi\right) = 2a \quad ; \quad \ln\psi = ax^2 + Cx + C'$$

$$\frac{d}{dy}\left(\frac{d}{dy}\ln\varphi\right) = -2a \quad ; \quad \ln\varphi = -ay^2 + Dy + D'$$

$$\ln|f'(z)| = a(x^2 - y^2) + Cx + Dy + E$$

(3) Let us examine the function $\log f'(z) = \ln|f'(z)| + i\theta$. Since $f'(z)$ is analytic and $\neq 0$, $\log f'(z)$ is analytic (p. 41, Churchill). Let us say:

$$\log f'(z) = g(z) = \sigma + i\omega \; ; \quad \sigma = \ln|f'(z)|, \; \omega = \theta$$

with $\sigma_x = \omega_y$, $\sigma_y = -\omega_x$

(4) $g'(z) = \sigma_x - i\sigma_y = 2ax + c + i2ay - iD$, (put $x = z$, $y = 0$)

$$= 2az + (C - iD) = 2az + \beta$$

$$g(z) = az^2 + \beta z + F + iG = az^2 + \beta z + \gamma = \log f'(z)$$

(5) $\therefore \; f'(z) = \exp(az^2 + \beta z + \gamma)$, where a is real constant, β and γ are complex.

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Problems 18, 24

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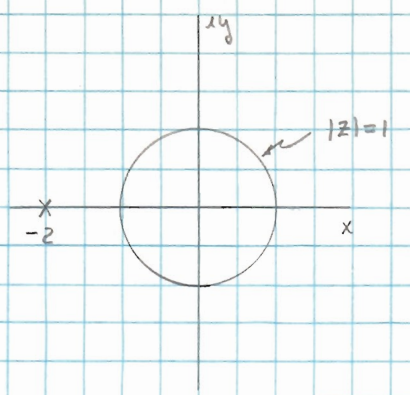
18.



(1) Consider the integral $\int_C \frac{dz}{z+2}$

where C is the path $|z|=1$.
$f(z) = \frac{1}{z+2}$ has a singularity at $x=-2$

but this is outside the domain enclosed by C in which $f(z)$ is analytic within and on C.

(2) $\therefore$ by Cauchy's Theorem:

$$\int_{|z|=1} \frac{dz}{z+2} = 0$$

(3) Letting the path of integration be $z = e^{i\theta}$, with $dz = ie^{i\theta}\,d\theta$, we have:

$$\frac{dz}{z+2} = \frac{(i\cos\theta - \sin\theta)\,d\theta}{\cos\theta + i\sin\theta + 2}$$

$$= \frac{(i\cos\theta - \sin\theta)\left[(\cos\theta + 2) - i\sin\theta\right]d\theta}{(\cos\theta+2)^2 + \sin^2\theta}$$

$$= \frac{(i\cos^2\theta + 2i\cos\theta + \cos\theta\sin\theta - \sin\theta\cos\theta - 2\sin\theta + i\sin^2\theta)\,d\theta}{\cos^2\theta + 4\cos\theta + 4 + \sin^2\theta}$$

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$$= \frac{i(1+2\cos\theta)\,d\theta}{5+4\cos\theta} - \frac{2\sin\theta\,d\theta}{5+4\cos\theta}$$

(4) Therefore we have:

$$\int_{|z|=1} \frac{dz}{z+2} = i\int_{-\pi}^{\pi} \frac{(1+2\cos\theta)\,d\theta}{5+4\cos\theta} - \int_{-\pi}^{\pi} \frac{2\sin\theta\,d\theta}{5+4\cos\theta} = 0$$

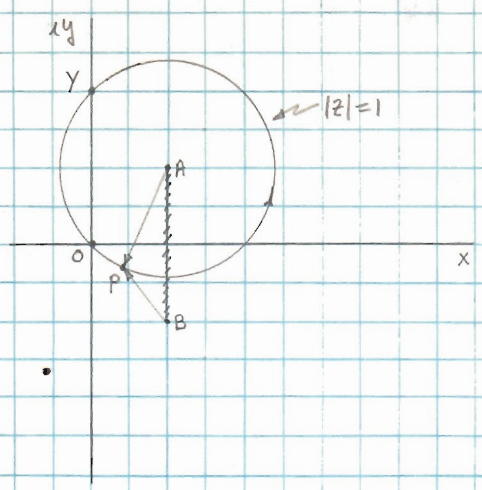
(5) Equating real and imaginary parts:

$$\int_{-\pi}^{\pi} \frac{(1+2\cos\theta)\,d\theta}{5+4\cos\theta} = 0$$

(6) Since the integrand is an even function,

$$\int_{0}^{\pi} \frac{(1+2\cos\theta)\,d\theta}{5+4\cos\theta} = 0$$

24. a.



(1) Given: $w^2 = z^2 - 2z + 2$

$$= (z - \{1+i\})(z - \{1-i\})$$

$$= (z-A)(z-B)$$

where P moves on the cartesian circle:

$$(x-1)^2 + (y-1)^2 = 2$$

and starts from $z=0$ in a ccw (+) direction where $w = +\sqrt{2}$

(2)
$$\left.\begin{aligned} z - A &= r_1 e^{i\theta_1} \\ z - B &= r_2 e^{i\theta_2} \end{aligned}\right\} \quad \begin{aligned} w_1 &= [r_1 r_2]^{1/2} e^{i(\theta_1+\theta_2)} \\ w_2 &= -[r_1 r_2]^{1/2} e^{i(\theta_1+\theta_2)} \end{aligned}$$

(3) when P is at $z=0$, $\theta_1+\theta_2=0$, $w=+\sqrt{2}$ so we are considering the first branch w_1. However, as P moves ccw, it passes thru the cut and $w_1 \to w_2$ so when P arrives at $Y \neq 0$ we are considering the second branch w_2.

(4) At Y: $1 + (y-1)^2 = 2$; $y-1 = 1$, $y = 2$

$$r_1 = |z-A| = |i2 - 1 - i| = [2]^{1/2} = \sqrt{2}$$
$$r_2 = |z-B| = |i2 - 1 + i| = [1+9]^{1/2} = \sqrt{10}$$

$$\theta_1 = \arg(z-A) = \tan^{-1}(-1) = 3\pi/4$$
$$\theta_2 = \arg(z-B) = \tan^{-1}(-3) = \underline{\pi - \tan^{-1}(3)}$$ ✗ ✓

(5) $w = -[2\sqrt{5}]^{1/2} \exp \frac{i}{2}\{\tan^{-1}(-1) + \tan^{-1}(-3)\}$

$$= -[2\sqrt{5}]^{1/2} \exp \frac{i}{2}\{7\pi/4 - \tan^{-1}(3)\}$$

10

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(6) At O: $w = -\sqrt{2}$, because $r_1 = r_2 = \sqrt{2}$

and $\theta_1 = -3\pi/4$, $\theta_2 = +3\pi/4$

Problems 19, 20

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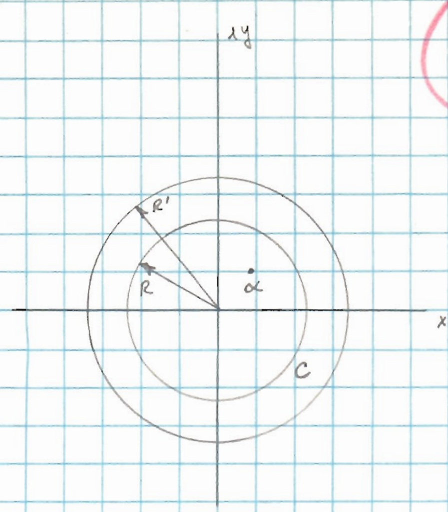
19.



(1) $f(\alpha) = \dfrac{1}{2\pi i} \displaystyle\int_C \dfrac{f(z)\,dz}{z-\alpha}$

C is circle $|z| = R$

(2) Consider: $\dfrac{1}{2\pi i} \displaystyle\int_C \dfrac{R^2 - |\alpha|^2}{(z-\alpha)(R^2-\bar{\alpha} z)} f(z)\,dz$

(3) $$\frac{1}{(z-\alpha)(R^2-\bar{\alpha}z)} = \frac{-1/\bar{\alpha}}{(z-\alpha)\left(z-\frac{R^2}{\bar{\alpha}}\right)} = \frac{-\frac{\alpha}{|\alpha|^2}}{(z-\alpha)\left(z-\left\{\frac{R^2}{|\alpha|^2}\right\}\alpha\right)} = \frac{K_1}{z-\alpha}$$

$$+ \frac{K_2}{z-\left(\frac{R^2}{|\alpha|^2}\right)\alpha} = \frac{1/R^2-|\alpha|^2}{z-\alpha} - \frac{\alpha/R^2(1-\alpha)}{z-\left(\frac{R^2}{|\alpha|^2}\right)\alpha}$$

(4) $$\frac{1}{2\pi i}\int_C \frac{R^2-|\alpha|^2}{(z-\alpha)(R^2-\bar{\alpha}z)} f(z)\,dz = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z-\alpha}$$

$$-\frac{1}{2\pi i}\cdot\frac{\alpha(R^2-|\alpha|^2)}{R^2(1-\alpha)}\int_C \frac{f(z)\,dz}{z-\left(\frac{R^2}{|\alpha|^2}\right)\alpha}$$

(5) Consider the point $\left(\frac{R^2}{|\alpha|^2}\right)\alpha$: its distance from the origin is $\frac{R^2}{|\alpha|}$ which is $> R$ for $|\alpha| < R$ as stated. Therefore the integrand of the second integral is regular within the contour of integration and the integral vanishes by Cauchy's Theorem.

(6) $$\therefore \frac{1}{2\pi i}\int_C \frac{R^2-|\alpha|^2}{(z-\alpha)(R^2-\bar{\alpha}z)} f(z)\,dz = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z-\alpha} = f(\alpha)$$ by Cauchy's Integral Formula.

(7) For $0 < r < R$; $C: |z| = R$, Let $\alpha = re^{i\theta}$, $z = Re^{i\varphi}$, $dz = iRe^{i\varphi}d\varphi$: Then;

$$f(re^{i\theta}) = \frac{1}{2\pi i}\int_{-\pi}^{\pi} \frac{(R^2-r^2)\,f(Re^{i\varphi})\cdot iRe^{i\varphi}d\varphi}{(Re^{i\varphi}-re^{i\theta})(R^2-rRe^{i(\varphi-\theta)})}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(R^2-r^2)\,f(Re^{i\varphi})\,d\varphi}{(R-re^{i(\theta-\varphi)})(R-re^{-i(\theta-\varphi)})} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(R^2-r^2)\,f(Re^{i\varphi})\,d\varphi}{R^2-2rR\cos(\theta-\varphi)+r^2}$$

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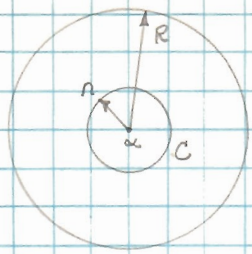
20.

(1) $f(z)$ is regular in $|z-\alpha| < R$



(2) $z = \alpha + re^{i\theta}$; $f(z) = f(\alpha + re^{i\theta})$;

$dz = ire^{i\theta}d\theta$; $z - \alpha = re^{i\theta}$

(3) $$f'(\alpha) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-\alpha)^2}\,dz$$

$$= \frac{1}{2\pi i}\int_{-\pi}^{\pi} \frac{f(\alpha + re^{i\theta})}{r^2 e^{i2\theta}} \cdot ire^{i\theta}\,d\theta$$

$$= \frac{1}{2\pi r}\int_{-\pi}^{\pi} f(\alpha + re^{i\theta})\,e^{-i\theta}\,d\theta = \frac{1}{2\pi r}\left[\int_{-\pi}^{\pi} P(\theta)\,e^{-i\theta}\,d\theta + i\int_{-\pi}^{\pi} Q(\theta)\,e^{-i\theta}\,d\theta\right]$$

where $P(\theta) = \operatorname{Re} f(\alpha + re^{i\theta})$; $Q(\theta) = \operatorname{Im} f(\alpha + re^{i\theta})$

(4) Now: $\int_C f(z)\,dz = 0$; $\int_{-\pi}^{\pi} f(\alpha + re^{i\theta}) \cdot ire^{i\theta}\,d\theta = 0$

then $\int_{-\pi}^{\pi} P(\theta)\,e^{i\theta}\,d\theta + i\int_{-\pi}^{\pi} Q(\theta)\,e^{i\theta}\,d\theta = 0$;

(5) $$\int_{-\pi}^{\pi} P(\theta)\cos\theta\,d\theta + i\int_{-\pi}^{\pi} P(\theta)\sin\theta\,d\theta + i\int_{-\pi}^{\pi} Q(\theta)\cos\theta\,d\theta - \int_{-\pi}^{\pi} Q(\theta)\sin\theta\,d\theta = 0$$

(6) $$\int_{-\pi}^{\pi} P(\theta)\,e^{-i\theta}\,d\theta + i\int_{-\pi}^{\pi} Q(\theta)\,e^{-i\theta}\,d\theta = \int_{-\pi}^{\pi} P(\theta)\cos\theta\,d\theta - i\int_{-\pi}^{\pi} P(\theta)\sin\theta\,d\theta$$

$$+ i\int_{-\pi}^{\pi} Q(\theta)\cos\theta\,d\theta + \int_{-\pi}^{\pi} Q(\theta)\sin\theta\,d\theta = \int_{-\pi}^{\pi} P(\theta)\cos\theta\,d\theta - i\int_{-\pi}^{\pi} P(\theta)\sin\theta\,d\theta$$

$$- i\int_{-\pi}^{\pi} P(\theta)\sin\theta\,d\theta + \int_{-\pi}^{\pi} P(\theta)\cos\theta\,d\theta = 2\int_{-\pi}^{\pi} P(\theta)\,e^{-i\theta}\,d\theta$$

(7) $$\therefore\quad f'(\alpha) = \frac{1}{2\pi r} \cdot 2\int_{-\pi}^{\pi} P(\theta)\,e^{-i\theta}\,d\theta = \frac{1}{\pi r}\int_{-\pi}^{\pi} P(\theta)\,e^{-i\theta}\,d\theta$$

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Problems
16, 17, 21

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16. 10

(1) The required transformation is of the bilinear type, viz.,

$$w = \frac{az+b}{cz+d}, \quad ad \neq bc$$

(2) $$cwz + dw - az - b = 0$$

or, if $c \neq 0$, $\quad wz + Aw + Bz + C = 0$; $A = \frac{d}{c}$, $B = \frac{-a}{c}$, $C = -\frac{b}{c}$

(3) If three required points of the transformation are known; A, B, C may be found.

$$\begin{vmatrix} w_1 & z_1 & 1 \\ w_2 & z_2 & 1 \\ w_3 & z_3 & 1 \end{vmatrix} \begin{vmatrix} A \\ B \\ C \end{vmatrix} = \begin{vmatrix} -w_1 z_1 \\ -w_2 z_2 \\ -w_3 z_3 \end{vmatrix}$$

(4) In this problem: $w_1 = i$, $z_1 = 1$
$w_2 = 0$, $z_2 = -i$
$w_3 = -i$, $z_3 = -1$

$$\begin{vmatrix} i & 1 & 1 \\ 0 & -i & 1 \\ -i & -1 & 1 \end{vmatrix} \begin{vmatrix} A \\ B \\ C \end{vmatrix} = \begin{vmatrix} -i \\ 0 \\ -i \end{vmatrix}$$

(5) $$\Delta = i \begin{vmatrix} -i & 1 \\ -1 & 1 \end{vmatrix} - i \begin{vmatrix} 1 & 1 \\ -i & 1 \end{vmatrix} = 1 + i - i + 1 = 2$$

$$\Delta_A = -2i$$

$$\Delta_B = i \begin{vmatrix} 0 & 1 \\ -i & 1 \end{vmatrix} - i \begin{vmatrix} i & 1 \\ 0 & 1 \end{vmatrix} = -1 - 1 = -2$$

$$\Delta_C = i \begin{vmatrix} -i & 0 \\ -1 & -i \end{vmatrix} - i \begin{vmatrix} 1 & -i \\ -i & 0 \end{vmatrix} = -i - i = -2i$$

$$A = -i, \quad B = -1, \quad C = -i$$

(6) $$wz - iw - z - i = 0$$

$$w(z - i) = z + i$$

(7) $$w = \frac{z+i}{z-i}$$

1. (8) Consider now the lines $\arg z = \theta = \text{constant}$ in the z plane. We may write this in the cartesian form:

$$y = mx \quad , \text{where } m = \tan\theta$$

(9) $$w = \frac{z+i}{z-i} \;; \quad z = i\left(\frac{w+1}{w-1}\right) \;, \quad x+iy = i\left(\frac{u+iv+1}{u+iv-1}\right)$$

$$= \frac{2v}{(u-1)^2+v^2} + i\,\frac{(v^2+u^2-1)}{(u-1)^2+v^2}$$

(10) $$\therefore \quad \frac{(v^2+u^2-1)}{(u-1)^2+v^2} = \frac{2mv}{(u-1)^2+v^2}$$

(11) Finally we have the circle in the w plane:

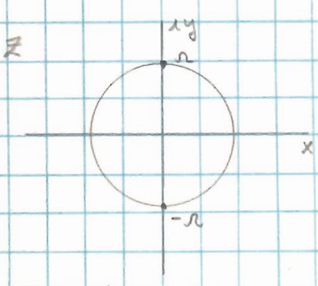
$$u^2 + (v-m)^2 = m^2+1$$

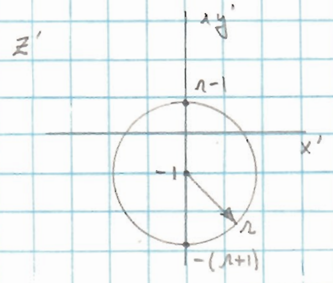
as $\theta \to \pm \pi/2$, the circle degenerates to the line $v = 1$

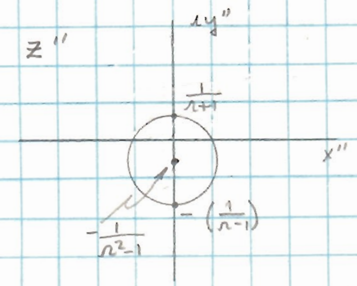
II. (12) Let us now consider the following sets of transformations:

$w = \dfrac{az+b}{cz+d}$	$w = \dfrac{z+i}{z-i}$
$z' = cz+d$	$z' = z-i$
$z'' = \dfrac{1}{z'}$	$z'' = \dfrac{1}{z'} \;; \quad r'' = \dfrac{1}{r'} \,, \quad \theta'' = -\theta'$
$z''' = \left(\dfrac{bc-ad}{c}\right) z''$	$z''' = \left(\dfrac{i+i}{1}\right) z'' = 2i\, z''$
$w = \left(\dfrac{a}{c}\right) + z'''$	$w = 1 + z'''$

How are the circles $|z| = r$ transformed to the w plane?







$r' = r-1 \qquad \theta' = \pi/2$

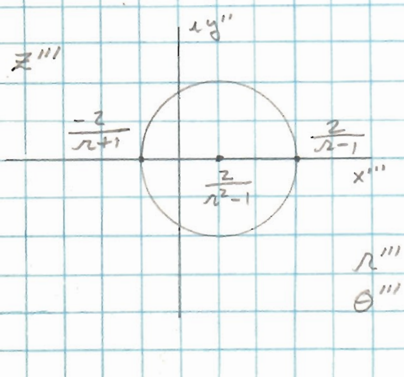
$r'' = \dfrac{1}{r-1} \qquad \theta'' = -\pi/2$

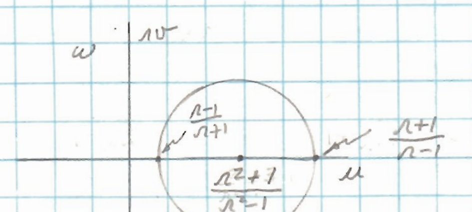
$r' = r+1 \qquad \theta' = -\pi/2$

$r'' = \dfrac{1}{r+1} \qquad \theta'' = \pi/2$

Center: $\frac{1}{2}\left(\frac{1}{r+1} - \frac{1}{r-1}\right)$

$= \dfrac{-1}{r^2-1}$





$\therefore$ The mapping is into the circles

$$r''' = 2r''$$
$$\theta''' = \theta'' + \pi/2$$

$$\left(u - \frac{r^2+1}{r^2-1}\right)^2 + v^2 = \frac{4r^2}{(r^2-1)^2}$$

provided $r > 1$

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Problems 16, 17, 21 Continued

17. (1) Consider the bilinear transformation: $w = \dfrac{iz+1}{z+i}$

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(2) Solving for z: $z = \dfrac{-iw+1}{w-i}$

or $x + iy = \dfrac{(v+1) - iu}{u + i(v-1)} = \dfrac{2u}{u^2+(v-1)^2} + i\,\dfrac{1-u^2-v^2}{u^2+(v-1)^2}$

(3) Now consider the whole x axis in the z plane whose cartesian equation is $y=0$. We easily see that the x axis maps into the following circle in the w plane:

$$u^2+v^2=1$$

(4) We now consider the segment $y=0,\ -1 \le x \le 1$, along with the following transformations:

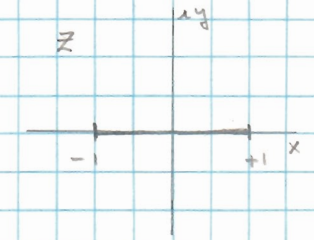
$w = \dfrac{iz+1}{z+i}$

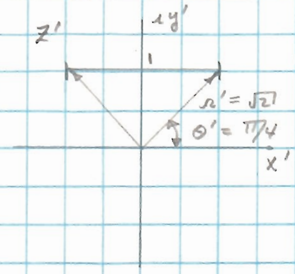
$z' = z + i$

$z'' = \dfrac{1}{z'}$

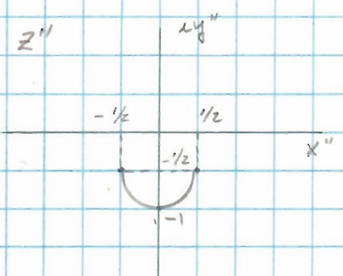
$z''' = 2z''$

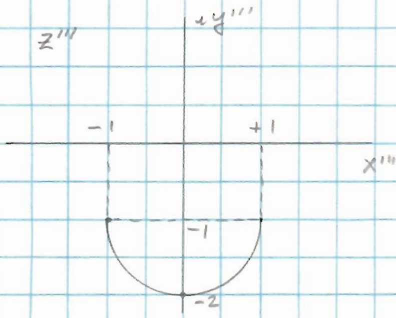
$w = z''' + i$

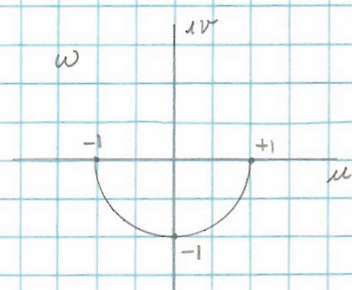




$r' = \sqrt{2}$: $r'' = \dfrac{1}{\sqrt{2}}$
$\theta' = \pi/4$: $\theta'' = -\pi/4$
$\theta' = 3\pi/4$: $\theta'' = -3\pi/4$
$r' = 1$: $r'' = 1$
$\theta' = \pi/2$: $\theta'' = -\pi/2$





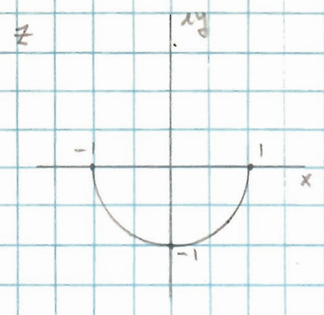


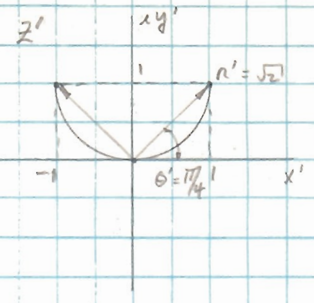
Thus we see that the line segment in the z plane is mapped into the w plane as a semi-circle of radius 1 in the lower half-plane. Since the whole x axis maps into a circle, we must conclude that the part of the x axis, $-\infty < x < -1$, $1 < x < \infty$, maps into the upper semi-circle by this transformation.

Let us call the above transformation T. Then successive transformations will be:

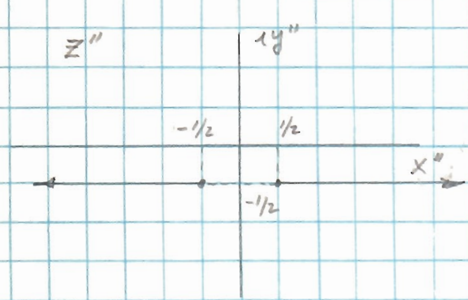
(5) $w = T, T^2, T^3, \ldots, T^n$, and we shall see how many independent applications there are.

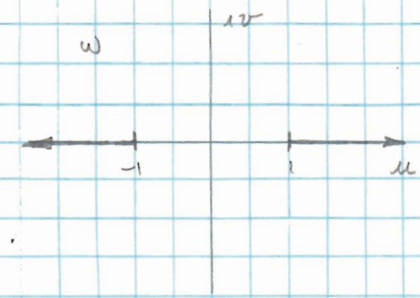
(6) For T^2, we apply T on the result of T which we now place in a "new" z plane and operate on it with T.



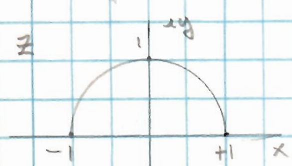


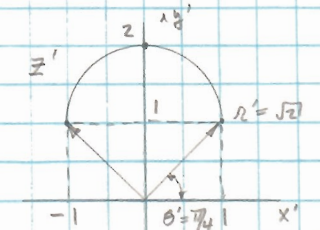
$r' = 0 : r'' = \infty,$
$\theta'' = 0, \pi$
$r' = \sqrt{2} : r'' = \frac{1}{\sqrt{2}},$
$\theta'' = -\pi/4, -3\pi/4$
z'' does not exist for $\pi/4 < \theta' < 3\pi/4$



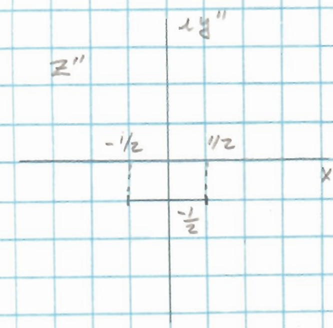


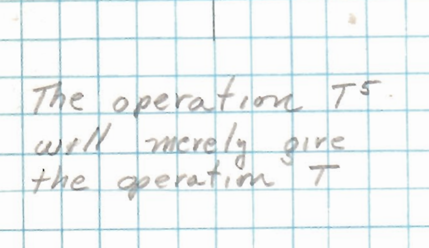
(7) For T^3, it is clear from before that the line segment $-\infty < x < -1$, $1 < x < \infty$ will map into a positive semi-circle of radius 1. We then proceed to T^4.



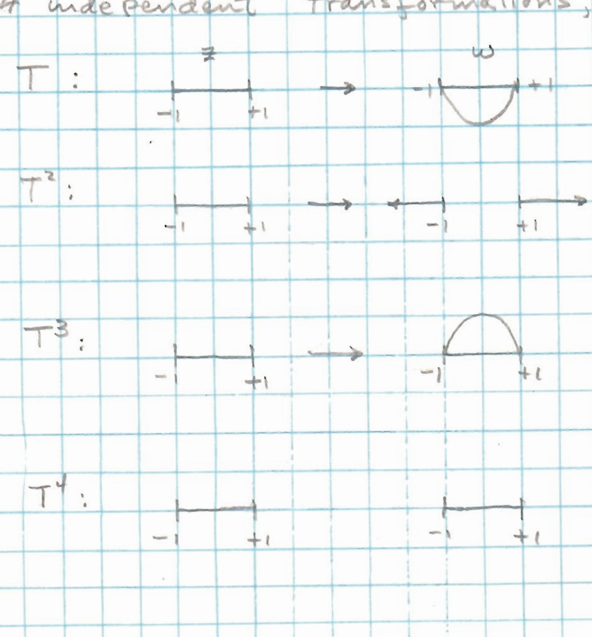


$r' = \sqrt{2} : r'' = \frac{1}{\sqrt{2}}$
$\theta' = \pi/4 : \theta'' = -\pi/4$
$\theta' = 3\pi/4 : \theta'' = -3\pi/4$
$r' = 2 : r'' = 1/2$
$\theta' = \pi/2 : \theta'' = -\pi/2$
z'' does not exist for $\theta' < \pi/4$, $\theta' > 3\pi/4$





Thus we see that there are only 4 independent transformations, T, T^2, T^3, T^4.



The operation T^5 will merely give the operation T

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21. (1) Given:

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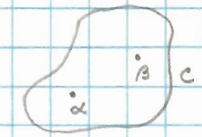

$f(z)$ regular within and on C:

Evaluate: $\dfrac{1}{2\pi i}\displaystyle\int_C \dfrac{f(z)\,dz}{(z-\alpha)(z-\beta)}$

(2) $$\frac{1}{(z-\alpha)(z-\beta)} = \frac{K_1}{z-\alpha} + \frac{K_2}{z-\beta}\ ; \quad K_1 = \frac{1}{\alpha-\beta}, \quad K_2 = \frac{1}{\beta-\alpha}$$

$$= \frac{(\alpha-\beta)^{-1}}{z-\alpha} + \frac{(\beta-\alpha)^{-1}}{z-\beta}$$

(3) $$\frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-\alpha)(z-\beta)} = \underbrace{\frac{(\alpha-\beta)^{-1}}{2\pi i}\int_C \frac{f(z)\,dz}{z-\alpha}}_{=f(\alpha)} + \underbrace{\frac{(\beta-\alpha)^{-1}}{2\pi i}\int_C \frac{f(z)\,dz}{z-\beta}}_{=f(\beta)}$$

$$= \frac{f(\alpha)}{\alpha-\beta} + \frac{f(\beta)}{\beta-\alpha}, \text{ QED.}$$

by Cauchy's Integral Formula

(4) It follows from the Theorem that the modulus of the sum is less than the sum of the module that:

$$\left|\frac{f(\alpha)-f(\beta)}{\alpha-\beta}\right| \le \frac{1}{2\pi}\int_C \frac{|f(z)|\,|dz|}{|z-\alpha|\,|z-\beta|}$$

Let the circle C be taken so large that $|z-\alpha| \doteqdot |z-\beta| \cong R$. Then:

$|z-\alpha| \ge R-|\alpha| \qquad |z-\beta| \ge R-|\beta|$

(5) $$\left|\frac{f(\alpha)-f(\beta)}{\alpha-\beta}\right| \le \frac{1}{2\pi R^2} ML = \frac{M\cdot 2\pi R}{2\pi R^2} = \frac{M}{R}$$

Now let $f(z)$ be an integral function and everywhere bounded. As R increases without limit, $\frac{M}{R}$ will vanish because M is finite everywhere

(6) $$\therefore \left|\frac{f(\alpha)-f(\beta)}{\alpha-\beta}\right| = 0$$

But this is true everywhere since α and β are arbitrary.

(7) Thus $f(\alpha) = f(\beta)$ for arbitrary α, β with the conclusion that $f(z)$ must be a constant everywhere.

A more not necessarily rigourous way to show this would be to take an elliptical path with α and β as its focii.

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22. (1) $\cosh\left(z + \frac{1}{z}\right) = f(z)$ has a singularity at $z = 0$ and at $z = \infty$.

We shall expand it in the Laurent series in the region $0 < |z| < \infty$.

(2) $f(z) = \sum_0^\infty a_n z^n + \sum_1^\infty b_n \frac{1}{z^n} = a_0 + \sum_1^\infty \left[a_n z^n + b_n \frac{1}{z^n}\right]$.

$$a_n = \frac{1}{2\pi i}\int_{C_1} \frac{f(z)\,dz}{z^{n+1}}, \quad b_n = \frac{1}{2\pi i}\int_{C_2} z^{n-1} f(z)\,dz$$

(3) Since $f(z)$ is regular everywhere except at $z=0$, we may take C_1 and C_2 to be the same path Γ.

Let Γ be a circle $|z| = r$, $r > 0$. In particular, let $r = 1$ then:

$$z = e^{i\theta}, \quad dz = ie^{i\theta}\,dz$$

$$f(z) = \cosh\left(e^{i\theta} + e^{-i\theta}\right) = \cosh(2\cos\theta)$$

$$z^{n+1} = e^{i(n+1)\theta}, \quad z^{n-1} = e^{i(n-1)\theta}$$

(4) $$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cosh(2\cos\theta)\, e^{-n\theta i}\,d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos n\theta \cosh(2\cos\theta)\,d\theta$$
$$- i\,\frac{1}{2\pi}\int_{-\pi}^{\pi} \sin n\theta \cosh(2\cos\theta)\,d\theta$$

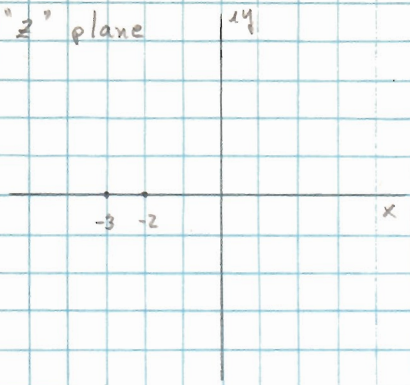
(5) $$b_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cosh(2\cos\theta)\, e^{in\theta}\,d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos n\theta \cosh(2\cos\theta)\,d\theta$$
$$+ i\,\frac{1}{2\pi}\int_{-\pi}^{\pi} \sin n\theta \cosh(2\cos\theta)\,d\theta$$

(6) $\int_{-\pi}^{\pi} \sin n\theta \cosh(2\cos\theta)\,d\theta = 0$ because the integrand is an odd function in θ.

(7) $\therefore \quad a_n = b_n = \frac{1}{\pi}\int_0^{\pi} \cos n\theta \cosh(2\cos\theta)\,d\theta$

and $f(z) = \cosh\left(z + z^{-1}\right) = a_0 + \sum_1^\infty a_n \left(z + z^{-1}\right)$

23. (1) $f(z) = \dfrac{z^2-1}{(z+2)(z+3)}$ has singularities at $z = -3$, $z = -2$.



(2) In the region $|z| < 2$, $f(z)$ is analytic and may be expanded in a Taylor series about any point within. For generality we chose the point a, $|a| < 2$

(3) $$f(z) = \frac{z^2-1}{(z+2)(z+3)} = \frac{k_1}{z+2} + \frac{k_2}{z+3} + 1$$

$$= \frac{3}{z+2} - \frac{8}{z+3} + 1$$

(4) $$f(z) = 1 + \sum_1^\infty a_n (z-a)^n \Rightarrow 1 + \sum_1^\infty (-1)^{n+1}\left[\frac{3}{2^n} - \frac{8}{3^n}\right] z^n$$

(5) $$a_n = \frac{f^{(n)}(a)}{n!} \; ; \quad f^{(n)}(z) = (-1)^n \, n! \left[\frac{3}{(z+2)^{n+1}} - \frac{8}{(z+3)^{n+1}}\right] ; \quad n \neq 0$$

(6) $$a_n = (-1)^{n+1}\left[\frac{3}{(a+2)^n} - \frac{8}{(a+3)^n}\right].$$ For the region $|z| < 2$, we make $a = 0$ to include the whole region

In the rest of the regions, it is impossible to find a Taylor series which will represent the function over the entire region since the radius of convergence will be the distance from a to the nearest singularity. For example, we might choose $a = 2.5$ to represent the function in the region $2 < |z| < 3$, however, the radius of convergence is only .5 and thus the region of convergence of the Taylor series does not cover the region $2 < |z| < 3$. Therefore, we must form our series in terms of the more general Laurent expansion.

(7) In the region $2 < |z| < 3$, we may proceed as follows:

$$f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

Examine $$\frac{3}{z+2} = \frac{3}{z}\left[\frac{1}{1+\left(\frac{2}{z}\right)}\right] = \frac{3}{z}\sum_{n=0}^\infty (-1)^{n+1}\left(\frac{2}{z}\right)^n$$

we see that this expansion is valid for $\left|\frac{2}{z}\right| < 1$ or for $|z| > 2$

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Problem 23
Continued

(8) Examine $\dfrac{8}{z+3} = \dfrac{8}{3}\left[\dfrac{1}{\left(\frac{z}{3}\right)+1}\right] = \dfrac{8}{3}\sum_{n=0}^{\infty}(-1)^{n+1}\left(\dfrac{z}{3}\right)^n$

which holds for $\left|\frac{z}{3}\right| < 1$, or $|z| < 3$.

(9) Then, for $2 < |z| < 3$,

$$f(z) = 1 + \frac{3}{z}\sum_{n=0}^{\infty}(-1)^{n+1}\left(\frac{2}{z}\right)^n - \frac{8}{3}\sum_{n=0}^{\infty}(-1)^{n+1}\left(\frac{z}{3}\right)^n$$

which is a Laurent expansion in the region and is therefore <u>the</u> Laurent expansion in the region.

(10) In the region $|z| > 3$, we may use somewhat the same procedure as before.

we have $\dfrac{3}{z+2} = \dfrac{3}{z}\left[\dfrac{1}{1+\left(\frac{2}{z}\right)}\right] = \dfrac{3}{z}\sum_{n=0}^{\infty}(-1)^{n+1}\left(\dfrac{2}{z}\right)^n$; $|z| > 2$

(11) $\dfrac{8}{z+3} = \dfrac{8}{z}\left[\dfrac{1}{1+\left(\frac{3}{z}\right)}\right] = \dfrac{8}{z}\sum_{n=0}^{\infty}(-1)^{n+1}\left(\dfrac{3}{z}\right)^n$; $|z| > 3$

(12) $\therefore$ $f(z) = 1 + \dfrac{3}{z}\sum_{n=0}^{\infty}(-1)^{n+1}\left(\dfrac{2}{z}\right)^n - \dfrac{8}{z}\sum_{n=0}^{\infty}(-1)^{n+1}\left(\dfrac{3}{z}\right)^n$

which is a Laurent expansion in the region $|z| > 3$ and is therefore <u>the</u> Laurent expansion in the region.

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25. (1) Given: $f(z)$ regular, $|z| < R'$, C is circle $|z| = R$ $(R < R')$
and $0 < |\alpha| < R$

Show: $$f(\alpha) = \frac{1}{2\pi i}\int_C \left\{ \frac{R^2\alpha - z^2\bar{\alpha}}{z(z-\alpha)(R^2 - z\bar{\alpha})} + \frac{1}{z} \right\} f(z)\,dz$$

(2) Consider: $$\frac{R^2\alpha - z^2\bar{\alpha}}{z(z-\alpha)(R^2 - z\bar{\alpha})} = \frac{-\frac{1}{\bar{\alpha}}(R^2\alpha - z^2\bar{\alpha})}{z(z-\alpha)\left(z - \frac{R^2}{\bar{\alpha}}\right)}$$

We expand this by partial fractions after the famous method of Heaviside:

$$\frac{-\frac{1}{\bar{\alpha}}(R^2\alpha - z^2\bar{\alpha})}{z(z-\alpha)\left(z - \frac{R^2}{\bar{\alpha}}\right)} = \frac{K_1}{z} + \frac{K_2}{z-\alpha} + \frac{K_3}{z - \frac{R^2}{\bar{\alpha}}}$$

(3) $$K_1 = \left.\frac{-\frac{1}{\bar{\alpha}}(R^2\alpha - z^2\bar{\alpha})}{(z-\alpha)\left(z - \frac{R^2}{\bar{\alpha}}\right)}\right]_{z\to 0} = -1$$

$$K_2 = \left.\frac{-\frac{1}{\bar{\alpha}}(R^2\alpha - z^2\bar{\alpha})}{z\left(z - \frac{R^2}{\bar{\alpha}}\right)}\right]_{z\to \alpha} = +1$$

$$K_3 = \left.\frac{-\frac{1}{\bar{\alpha}}(R^2\alpha - z^2\bar{\alpha})}{z(z-\alpha)}\right]_{z\to \frac{R^2}{\bar{\alpha}}} = +1$$

(4) Substituting in the integrand above:

$$\frac{1}{2\pi i}\int_C \left\{ \quad \right\} f(z)\,dz = \frac{1}{2\pi i}\int_C \left\{ -\frac{1}{z} + \frac{1}{z-\alpha} + \frac{1}{z - \frac{R^2}{\bar{\alpha}}} + \frac{1}{z} \right\} f(z)\,dz$$

$$= \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z-\alpha} + \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z - \frac{R^2}{|\alpha|^2}\alpha}$$

Now the point $\frac{R^2}{|\alpha|^2}\alpha$ is outside the curve C; therefore $\frac{f(z)}{z - \frac{R^2}{|\alpha|^2}\alpha}$ is analytic in the region concerned and thus its integral vanishes.

The first integral on the RHS is merely Cauchy's integral equal to $f(\alpha)$. Therefore:

(5) $$\frac{1}{2\pi i}\int_C \left\{ \frac{R^2\alpha - z^2\bar{\alpha}}{z(z-\alpha)(R^2 - z\bar{\alpha})} + \frac{1}{z} \right\} f(z)\, dz = f(\alpha) \quad ; \quad \text{QED.}$$

(6) For $0 < r < R$, when C is the curve $|z| = R$ and $\alpha = re^{i\theta}$,

$$z = Re^{i\varphi}, \quad dz = iRe^{i\varphi}d\varphi, \quad \frac{dz}{z} = i\,d\varphi$$

(7) $$(z-\alpha)(R^2 - z\bar{\alpha}) = (Re^{i\varphi} - re^{i\theta})(R^2 - rRe^{i(\varphi-\theta)})$$

$$= R^3e^{i\varphi} - rR^2e^{i(2\varphi-\theta)} - R^2re^{i\theta} + r^2Re^{i\varphi}$$

$$= Re^{i\varphi}\left[R^2 - rRe^{i(\varphi-\theta)} - Rre^{i(\theta-\varphi)} + r^2\right]$$

$$= Re^{i\varphi}\left(R^2 + r^2 - 2rR\cos\{\theta - \varphi\}\right)$$

(8) $$R^2\alpha - z^2\bar{\alpha} = R^2re^{i\theta} - R^2e^{i2\varphi}\cdot re^{-i\theta}$$

$$= R^2re^{i\varphi}\left[e^{i(\theta-\varphi)} - e^{i(\varphi-\theta)}\right]$$

$$= R^2re^{i\varphi}\cdot 2i\sin(\theta - \varphi)$$

Substituting in above:

(9) $$f(re^{i\theta}) = \frac{1}{2\pi i}\int_{-\pi}^{\pi}\left\{ \frac{2i R^2re^{i\varphi}\sin(\theta-\varphi)}{Re^{i\varphi}\cdot Re^{i\varphi}(R^2+r^2-2rR\cos\{\theta-\varphi\})} + \frac{1}{Re^{i\varphi}} \right\} Rf(Re^{i\varphi})\cdot ie^{i\varphi}\,d\varphi$$

$$= \frac{i}{\pi}\int_{-\pi}^{\pi}\left\{ \frac{Rr\sin(\theta-\varphi)}{R^2+r^2-2rR\cos\{\theta-\varphi\}} - \frac{i}{2} \right\} f(Re^{i\varphi})\,d\varphi$$

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26. (1) We wish to expand $\exp \zeta$ where $\zeta = \frac{1}{2} z(\omega - \frac{1}{\omega})$. Now $\exp \zeta$ is an integral function of ζ whereas ζ is an analytic function of ω whose only singularity is the origin. Hence, since we have an integral function of an analytic function,

$$\exp\left\{\frac{1}{2} z\left(\omega - \frac{1}{\omega}\right)\right\}$$

is an analytic function of ω regular in the annulus $R \leq |\omega| \leq R'$ no matter how small R or how large R'. See Copson, P.77, Example I.

(2) We will expand in a Laurent expansion of the form

$$\sum_{n=0}^{\infty} a_n \omega^n + \sum_{n=1}^{\infty} b_n \omega^{-n} = \sum_{n=-\infty}^{\infty} \omega^n J_n(z)$$

where $a_n = J_n(z)$; $b_n = J_{-n}(z)$

(3) $$J_n(z) = \frac{1}{2\pi i} \int_C \frac{\exp\{\frac{1}{2} z (\omega - \omega^{-1})\}\, d\omega}{\omega^{n+1}}$$

(4) Let C be the circle $|\omega| = 1$ in the annulus $R \leq |\omega| \leq R'$. Then

$$\omega = e^{i\theta}, \quad \omega - \omega^{-1} = 2i \sin\theta, \quad d\omega = i e^{i\theta} d\theta$$

$$\exp\left\{\frac{1}{2} z(\omega - \omega^{-1})\right\} = \exp(i z \sin\theta)$$

$$J_n(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{iz\sin\theta}\, e^{i\theta}}{e^{i(n+1)\theta}} d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(z\sin\theta - n\theta)} d\theta$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(z\sin\theta - n\theta)\, d\theta + i \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin(z \sin\theta - n\theta)\, d\theta$$

The second integral vanishes because its integrand is odd and the first integral is doubled because its integrand is even.

II (5) $\therefore$ $$J_n(z) = \frac{1}{\pi} \int_0^{\pi} \cos(n\theta - z\sin\theta)\, d\theta$$

$$J_{-n}(z) = \frac{1}{\pi} \int_0^{\pi} \cos(n\theta + z \sin\theta)\, d\theta$$

Equation (5) is well known to generating Bessel functions of the first kind.

Let us now consider the original expression, viz.,

$$\exp\left\{\frac{z}{2}\left(\omega - \frac{1}{\omega}\right)\right\} = \sum_{-\infty}^{\infty} \omega^n J_n(z)$$

where the exponential is thought of as the generating function of Bessel functions. Let us expand it in a power series.

(6) $$\exp\left\{\frac{z}{2}\omega\right\} = \sum_{n=0}^{\infty} \frac{\left(\frac{z}{2}\right)^n}{n!}\omega^n$$

$$\exp\left\{-\frac{z}{2}\frac{1}{\omega}\right\} = \sum_{m=0}^{\infty} \frac{\left(\frac{z}{2}\right)^m}{m!}\omega^{-m}(-1)^m$$

(7) $$\exp\left\{\frac{z}{2}(\omega-\omega^{-1})\right\} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{m+n}}{n!\,m!}(-1)^m\omega^{n-m} = \sum_{-\infty}^{\infty} J_n(z)\,\omega^n$$

$$= \sum_{n=0}^{\infty} J_n(z)\,\omega^n + \sum_{n=1}^{\infty} J_{-n}(z)\,\omega^{-n}$$

To find $J_n(z)$ we equate coefficients of ω^n on right and left. To get completely ω^n on the left, we make the substitution:

$n \to n+m$

and we get $$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2m+n}}{(n+m)!\,m!}(-1)^m\omega^n = \sum_{n=0}^{\infty} J_n(z)\,\omega^n$$

III $\therefore$ $$J_n(z) = \sum_{m=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2m+n}}{(n+m)!\,m!}(-1)^m$$

To find $J_{-n}(z)$; we make $m \to n+m$ on the left and get

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2n+m}}{n!\,(m+n)!}(-1)^{m+n}\omega^{-m}$$

We now make $n \Leftrightarrow m$ and get finally:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2m+n}}{(m)!\,(m+n)!}(-1)^{m+n}\omega^{-n} = \sum_{n=1}^{\infty} J_{-n}(z)\,\omega^{-n}$$

I or $$J_{-n}(z) = \sum_{m=0}^{\infty} (-1)^{m+n}\frac{\left(\frac{z}{2}\right)^{2m+n}}{(m)!\,(m+n)!} = (-1)^n\sum_{m=0}^{\infty}(-1)^m\frac{\left(\frac{z}{2}\right)^{2m+n}}{(m)!\,(m+n)!}$$

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$$= (-1)^n J_n(z), \quad \text{QED.}$$

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27. (1) Find: zeros of $e^{\frac{1}{z}} - c$, $c = k e^{i\gamma}$, $-\pi < \gamma \leq \pi$

(2) Let $f(z) = e^{\frac{1}{z}} - c$

when $f(z) = 0$, $e^{\frac{1}{z}} = c$

(3) $\therefore \frac{1}{z} = \text{Log } c = \ln k + i\gamma + i2\pi n$

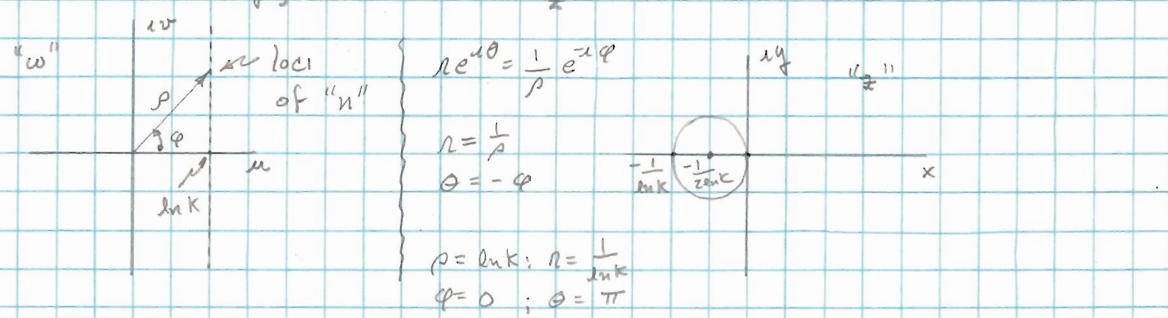
$= \ln k + i\gamma$ for the principal values of Log c

(4) $z = \dfrac{1}{\ln k + i\gamma} = \dfrac{\ln k - i\gamma}{\ln^2 k + \gamma^2} = \dfrac{\ln k - i(\gamma + 2\pi n)}{\ln^2 k + (\gamma + 2\pi n)^2}$

(5) $\therefore \; x = \dfrac{\ln k}{\ln^2 k + (\gamma + 2\pi n)^2}, \quad y = \dfrac{-(\gamma + 2\pi n)}{\ln^2 k + (\gamma + 2\pi n)^2}$

describes analytically the position of the zeros of the function in terms of the parameter n.

(6) Geometrically; Let $\omega = \frac{1}{z} = \ln k + i(\gamma + 2\pi n)$:



Thus we see that the zeros of $f(z)$ lie on a circle in the "z" plane with center at $-\frac{1}{2\ln k}$ and of the same radius, and the location of which are given by equation (5).

(7) We now invoke Picard's Theorem on $e^{1/z}$ which has an isolated essential singularity at the origin as is evident from its obvious Laurent expansion and therefore attains any given value in $0 < |z| < r$, except zero, no matter how small r.

Therefore, one of these values must make $f(z)$ (above) zero in $0 < |z| < r$, no matter how small r, for any value of c, $c \neq 0$. QED.

28. i. (1) We first assert that if $f(z)$ has a simple pole at $z = \alpha$

$$b_1 = \lim_{z \to \alpha} \{(z-\alpha) f(z)\}$$ which may be needed later.

Proof: $$f(z) = b_1 (z-\alpha)^{-1} + \sum_0^\infty a_n (z-\alpha)^n$$

$$(z-\alpha) f(z) = b_1 + \sum_0^\infty a_n (z-\alpha)^{n+1}$$

$$\lim_{z \to \alpha} \{(z-\alpha) f(z)\} = b_1$$

(2) $$f_1(z) = \frac{\cot \pi z}{(z-1)^2} = \frac{\cos \pi z}{(z-1)^2 \sin \pi z}$$

The only singularities are zeroes of the denominator which are at $z = 1$ and at $z = 0, \pm 1, \pm 2, \pm 3, \ldots$
Thus we have simple poles at $z = 0, -1, \pm 2, \pm 3, \ldots$
and a triple pole at $z = 1$.

ii. (1) $f_2(z) = z \operatorname{cosec} z = \dfrac{z}{\sin z}$. Now the only singularities are at the zeros of the denominator $z = n\pi$, if $z \neq \infty$, with $n = \pm 1, \pm 2, \pm 3, \ldots$, which all give simple poles.

The point $z = 0$ is not a pole because $f_2(0) = 1$ as was shown in a previous example.

iii. (1) $f_3(z) = \dfrac{z^4}{(c^2+z^2)^4}$. If we exclude the point at ∞, the only poles of the function are the zeros of $z^2 + c^2$ or when:

$z = \pm i c$ with c real, or at $+c$ and $-c$ on the imaginary axis.

$\therefore$ $f_3(z)$ has poles of order 4 at $+ic$ and $-ic$

iv. (1) $f_4(z) = \dfrac{1}{z(e^z - 1)}$. Again the poles are the zeros of the denominator.

(2) There is a pole at $z = 0$.

(3) $e^z = 1$, $z = \operatorname{Log} 1 = i 2\pi n$, $n = 0, \pm 1, \pm 2, \ldots$

(4) $\therefore$, there is a double pole at the origin with simple poles at the points $2\pi n$, $n = \pm 1, \pm 2, \pm 3, \ldots$ on the axis of imaginaries.

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29. i. (1) We now make use of the "little theorem" proved at the beginning of problem 28, viz:

If $f(z)$ has a simple pole at $z=\alpha$, its residue is given by:

$$\lim_{z \to \alpha} \{(z-\alpha) f(z)\}$$

(2) We have for the residues of the simple poles of $f_1(z)$, $n \neq 1$

$$\lim_{z \to n} \frac{(z-n) \cot \pi z}{(z-1)^2} = \lim_{\zeta \to 0} \frac{\zeta \cot \pi (\zeta + n)}{(\zeta+n-1)^2}, \quad \zeta = z-n$$

$$= \lim_{\zeta \to 0} \frac{\zeta \cos \pi (\zeta+n)}{(\zeta+n-1)^2 \sin \pi (\zeta + n)} = \frac{1}{\pi (n-1)^2}$$

from an evaluation of the form $\frac{\zeta}{\sin \pi \zeta}$ of an earlier problem.

(3) For the pole at $z=1$, we expand in a Laurent series about $z=1$; writing $\zeta = z-1$

$$\frac{\cot \pi z}{(z-1)^2} = \frac{\cot \pi (\zeta+1)}{\zeta^2} = \frac{\cot \pi \zeta}{\zeta^2} = \frac{1}{\pi \zeta^3} \left(1 - \tfrac{1}{3}\pi^2 \zeta^2 - \tfrac{1}{45}\pi^4 \zeta^4 + \ldots\right)$$

(4) $\therefore$ the residue at the triple pole $z=1$ is $-\frac{1}{3}\pi$

ii. (1) Since $f_2(z)$ has only simple poles, we may apply our "little theorem" at $z = n\pi$, $n \neq 0$

(2) $$\lim_{z \to n\pi} (z - n\pi) z \operatorname{cosec} z = \lim_{z \to n\pi} \frac{(z-n\pi) z}{\sin z} = (-1)^n (n\pi)$$

from the previous results of problem 9.

(3) $\therefore$ our residues at the poles $z = n\pi$, $n \neq 0$ are $(-1)^n (n\pi)$

III. (1) $$f_3(z) = \frac{z^4}{(z^2+c^2)^4} = \frac{z^4}{(z+ic)^4\,(z-ic)^4}$$

(2) To find the residue at $z = ic$, we expand $\dfrac{z^4}{(z+ic)^4}$ in a Taylor series about $z = ic$ and then multiply by $\dfrac{1}{(z-ic)^4}$ to get the Laurent series about $z = ic$.

(3) However, this is tedious and there may be another way. Let $f(z)$ have a pole of order m at $z = \alpha$, then

$$f(z) = \frac{b_m}{(z-\alpha)^m} + \frac{b_{m-1}}{(z-\alpha)^{m-1}} + \dots + \frac{b_1}{z-\alpha} + \dots$$

Form: $\varphi(z) = (z-\alpha)^m f(z) = b_m + b_{m-1}(z-\alpha) + \dots + b_1 (z-\alpha)^{m-1} + \dots$

which is merely a Taylor series of $\varphi(z)$ about $z = \alpha$ whose coefficients are: $a_n = \dfrac{f^{(n)}(\alpha)}{n!}$

with $$b_1 = \frac{1}{(m-1)!} f^{(m-1)}(\alpha)$$

or: $$b_1 = \frac{1}{(m-1)!} \lim_{z\to\alpha} \frac{d^{m-1}}{dz^{m-1}} \left\{ (z-\alpha)^m f(z) \right\}$$

which we notice reduces to our "little theorem" for simple poles.

(4) For the pole at $z = ic$: $\dfrac{d^3}{dz^3}\left\{ \dfrac{z^4}{(z+ic)^4} \right\}$

$$\frac{d}{dz}\{\ \} = \frac{4z^3(z+ic)^4 - 4z^4(z+ic)^3}{(z+ic)^8} = \frac{4z^3(ic)}{(z+ic)^5}$$

$$\frac{d^2}{dz^2}\{\ \} = 4(ic)\frac{\left[3z^2(z+ic)^5 - 5z^3(z+ic)^4\right]}{(z+ic)^{10}} = \frac{4(ic)\,z^2\left[-2z+3ic\right]}{(z+ic)^6}$$

$$\frac{d^3}{dz^3}\{\ \} = 4(ic)\frac{\left[\{-6z^2+6icz\}(z+ic)^6 - 6(z+ic)^5(-2z^3+3icz^2)\right]}{(z+ic)^{12}}$$

$$= 24(ic)\,z\,\frac{\left[\{-z+ic\}(z+ic) + 2z^2 - 3icz\right]}{(z+ic)^7}$$

$$= 24(ic)\,z\,\frac{\left[-z^2 + 2icz + (ic)^2 + 2z^2 - 3icz\right]}{(z+ic)^7}$$

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29 continued.

(4) $\dfrac{d^3}{dz^3}\{\ \} = \dfrac{24(ic)z\left[z^2 - icz + (ic)^2\right]}{(z+ic)^7}$

(5) $\lim\limits_{z\to ic} \dfrac{d^3}{dz^3}\{\ \} = \dfrac{24(ic)^4}{(2ic)^7} = \dfrac{3}{2^4(ic)^3} = \dfrac{3i}{2^4c^3}$

(6) Residue : $\dfrac{3i}{3!\,2^4c^3}$

(7) For the pole at $z = -ic$:

$$\frac{d^3}{dz^3} = \frac{24(-ic)z\left[z^2 + icz + (-ic)^2\right]}{(z-ic)^7}$$

(8) $\lim\limits_{z\to -ic} \dfrac{d^3}{dz^3}\{\ \} = \dfrac{24(ic)^4}{-(2ic)^7} = \dfrac{-3i}{2^4c^3}$

(9) Residue : $\dfrac{-3i}{3!\,2^4c^3}$

iv. (1) $f_4(z) = \dfrac{1}{z(e^z-1)}$; double pole at $z=0$, simple poles at $z = 2\pi n i$,
$n = \pm1, \pm2, \pm3, \ldots$

(2) For the poles at $z = 2n\pi i$: $\lim\limits_{z\to 2n\pi i} \dfrac{(z-2n\pi i)}{z(e^z-1)}$

$= \lim\limits_{z\to 2n\pi i} \dfrac{1}{(e^z-1)+ze^z} = \dfrac{1}{2n\pi i}$ which are the residues.

(3) For the double pole at $z=0$

$$\frac{d}{dz}\left\{z^2 \cdot \frac{1}{z(e^z-1)}\right\} = \frac{d}{dz}\left\{\frac{z}{e^z-1}\right\} = \frac{e^z-1-ze^z}{(e^z-1)^2}$$

(4) $\lim\limits_{z\to 0}\left|\dfrac{d}{dz}\{\ \}\right. = \dfrac{e^z - ze^z - e^z}{2(e^z-1)e^z} = \dfrac{-z}{2(e^z-1)} = \dfrac{-1}{2e^z} = -\dfrac{1}{2}$

(5) The residue at $z=0$ is $-\frac{1}{2}$

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Problems 30, 31, 32

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30. i. (1) $f_1(z) = \dfrac{\cot \pi z}{(z-1)^2}$; make $z \to \frac{1}{\omega}$

(2) $f_1\left(\frac{1}{\omega}\right) = \dfrac{\cot \pi \left(\frac{1}{\omega}\right)}{\left(\frac{1}{\omega}-1\right)^2} = \dfrac{\cot \pi \left(\frac{1}{\omega}\right)}{\omega^{-2}(\omega-1)^2}$

$= \dfrac{\cot \pi\left(\frac{1}{\omega}\right)}{1 - \frac{2}{\omega} + \frac{1}{\omega^2}} = \dfrac{1}{\left(\frac{1}{\omega^2} - \frac{2}{\omega} + 1\right) \tan \pi \left(\frac{1}{\omega}\right)}$

$= \dfrac{1}{\left(\frac{1}{\omega^2} - \frac{2}{\omega} + 1\right)\left(\frac{\pi}{\omega} + \frac{\pi^3}{3\omega^3} + \frac{2\pi^5}{15\omega^5} + \cdots\right)} = \dfrac{1}{F(\omega)}$

(3) The principal part of $F(\omega)$ is non-terminating, thus it has an isolated essential singularity at $\omega = 0$.

Therefore $\frac{1}{F(\omega)}$ also has an isolated essential singularity at $\omega = 0$. See Copson, p. 79. It is isolated because $\omega = 0$ is the limit point of zeros of $\frac{1}{F(\omega)}$.

Thus $f_1(z)$ has an ~~isolated~~ essential singularity at the point at ∞.

This means that it is non-isolated.

ii. (1) $f_2(z) = z \operatorname{cosec} z$; make $z \to \frac{1}{\omega}$

(2) $f_2\left(\frac{1}{\omega}\right) = \dfrac{1}{\omega \sin \frac{1}{\omega}}$

$f_2\left(\frac{1}{\omega}\right)$ has a non-isolated essential singularity at $\omega = 0$ because it is the limit point of the poles of $f_2\left(\frac{1}{\omega}\right)$, $\omega = \frac{1}{n\pi}$, $n = \pm 1, \pm 2, \ldots$

$\therefore$ $f_2(z)$ has a non-isolated essential singularity at the point at ∞.

iii. (1) $f_3(z) = \dfrac{z^4}{(c^2+z^2)^4}$; make $z \to \frac{1}{\omega}$; $f_3\left(\frac{1}{\omega}\right) = \dfrac{\omega^4}{(c^2\omega^2+1)^4}$

(2) $f_3\left(\frac{1}{\omega}\right)$ has a zero of order 4 at $\omega = 0$,

therefore $f_3(z)$ has a zero of order 4 at the point at ∞

IV. (1) $f_4(z) = \dfrac{1}{z(e^{z}-1)}$; put $z \to \frac{1}{\omega}$; $f_4(\frac{1}{\omega}) = \dfrac{1}{\omega^{-1}(e^{1/\omega}-1)}$

(2) $$f_4\left(\frac{1}{\omega}\right) = \frac{1}{\frac{1}{\omega}\left(1+\frac{1}{\omega}+\frac{1}{2}\left(\frac{1}{\omega}\right)^2+\frac{1}{3!}\left(\frac{1}{\omega}\right)^3+\cdots -1\right)}$$

$$= \frac{1}{\left(\frac{1}{\omega}\right)^2+\frac{1}{2}\left(\frac{1}{\omega}\right)^3+\frac{1}{3!}\left(\frac{1}{\omega}\right)^4+\cdots}$$

(3) Because the principal part of $\dfrac{1}{f_4(\frac{1}{\omega})}$ does not terminate, $\dfrac{1}{f_4(\frac{1}{\omega})}$ has an isolated ~~essential~~ singularity at $\omega = 0$.

Using the same reasoning as before, $f_4(z)$ has an isolated essential singularity at the point at ∞.

Throughout this problem, we have used the fact that if $\dfrac{1}{f(z)}$ has an essential singularity at $z = a$, so also does $f(z)$. This is easy to show.

Suppose $f(z) = \sum_{1}^{\infty} b_n (z-a)^n$, then by long division, $\dfrac{1}{f(z)} = \dfrac{1}{\sum_{-\infty} b_n (z-a)^n} = \sum_{-\infty}^{\infty} c_n (z-a)^n$

or still with a non-terminating principal part or an essential singularity at $z = a$.

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Problems
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31. 1. (1) $f(z) = \cot z = \dfrac{1}{\tan z} = \dfrac{1}{z + \frac{z^3}{3} + \frac{2z^5}{15} + \cdots}$

(2)

$$\begin{array}{r|l} & \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} \\ \hline x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots & 1 \\ & 1 + \frac{x^2}{3} + \frac{2x^4}{15} \\ \hline & -\frac{x^2}{3} - \frac{2x^4}{15} \\ & -\frac{x^2}{3} - \frac{x^4}{9} \\ \hline & -\frac{1}{45}x^4 \end{array}$$

(3) $f(z) = 1/z - \frac{z}{3} - \frac{z^3}{45} + \cdots$

with a simple pole at $z=0$ with residue 1.

11. (1) $f(z) = \operatorname{cosec}^2 z \log(1-z) = \dfrac{\log(1-z)}{\sin^2 z}$

(2) $\sin z = z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - \cdots$

(3) $\log(1-z) = -(z + \frac{1}{2}z^2 + \frac{1}{3}z^3 + \cdots)$

$$\begin{array}{r|l} & -1/z - 1/2 \\ \hline z^2 - \frac{1}{3}z^4 + \frac{4}{90}z^6 & -z - \frac{1}{2}z^2 - \frac{1}{3}z^3 \\ & -z + 0 + \frac{1}{3}z^3 \\ \hline & -\frac{1}{2}z^2 - \frac{2}{3}z^3 \\ & -\frac{1}{2}z^2 + 0 \end{array}$$

$$\begin{array}{r} z - \frac{z^3}{3!} + \frac{z^5}{5!} \\ z - \frac{z^3}{3!} + \frac{z^5}{5!} \\ \hline z^2 - \frac{z^4}{3!} + \frac{z^6}{5!} \\ - \frac{z^4}{3!} + \frac{z^6}{(3!)^2} - \frac{z^8}{3!\,5!} \\ + \frac{z^6}{5!} - \frac{z^8}{3!\,5!} \\ \hline z^2 - \frac{2z^4}{3!} + \left(\frac{2}{5!} + \frac{1}{(3!)^2}\right) z^6 \end{array}$$

$= z^2 - \frac{1}{3}z^4 + \frac{4}{90}z^6$

(4) $\therefore f(z) = -\frac{1}{z} - \frac{1}{2} + \cdots$

with a simple pole at $z=0$ and with residue -1.

31. (1) $f(z) = \dfrac{z}{\sin z - \tan z}$

(2) $\sin z = z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - \dfrac{z^7}{7!} + \cdots$

$\tan z = z + \dfrac{z^3}{3} + \dfrac{2z^5}{15} + \dfrac{17z^7}{315} + \cdots$

(3) $\sin z - \tan z = \left(-\dfrac{1}{3} - \dfrac{1}{6}\right) z^3 + \left(\dfrac{1}{120} - \dfrac{2}{15}\right) z^5 + \left(-\dfrac{1}{5040} - \dfrac{17}{315}\right) z^7 + \cdots$

$= -\dfrac{1}{2} z^3 - \dfrac{1}{8} z^5 - \dfrac{273}{5040} z^7 + \cdots$

$$\begin{array}{r|l} & -2/z^2 + \frac{1}{2} \\ \hline -\frac{1}{2} z^3 - \frac{1}{8} z^5 - \frac{273}{5040} z^7 & z \\ & z + \frac{1}{4} z^3 + \frac{273}{2520} z^5 \\ \hline & -\frac{1}{4} z^3 - \frac{273}{2520} z^5 \\ & -\frac{1}{4} z^3 - \frac{1}{16} z^5 \\ \hline \end{array}$$

(4) $\therefore\ f(z) = -\dfrac{2}{z^2} + \dfrac{1}{2} + \cdots$

$\therefore$ $f(z)$ has a double pole at $z=0$, with residue 0.

In this problem, we hold that all series converge absolutely in their areas of interest such that the operations of addition, multiplication, and division are legitimate.

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Problems
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32. (1) It is reasonable to hope that the function will take the form:

$$f(z) = \frac{\varphi(z)}{(z+1)(z-2)^2}$$

where $\varphi(z)$ is a regular function.
We reason this from the Theorem on rational fractions since it is given that $f(z)$ has no essential singularities anywhere.

(2) We hold that if the <u>only</u> singularities of $f(z)$ are those given, then $f(z)$ has no singularity at the point at infinity. This can only be true if the degree of the numerator is equal to or less than the degree of the denominator. Taking the more general case as being the condition of equal degree, we have by long division and a Heaviside expansion:

$$f(z) = K_0 + \frac{K_1}{z+1} + \frac{K_2}{z-2} + \frac{K_3}{(z-2)^2}$$

(3) It is a trivial application of the theorem proved in problem 29, viz;

$$(\text{Residue at } z=a) = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} \left\{ (z-a)^m f(z) \right\},$$

to show that K_1 and K_2 are the residues at $z=-1$ and $z=2$ respectively. As a matter of fact, this theorem is precisely the way one calculates the coefficients of the first degree terms in a partial fraction expansion.

(4) $\therefore$ $$f(z) = K_0 + \frac{1}{z+1} + \frac{2}{z-2} + \frac{K_3}{(z-2)^2}$$

$$f(0) = \frac{7}{4} = K_0 + 1 - 1 + \frac{K_3}{4} = K_0 + \frac{K_3}{4}$$

$$f(1) = \frac{5}{2} = K_0 + \frac{1}{2} - 2 + K_3 = K_0 + K_3 - \frac{3}{2}$$

$$\left.\begin{aligned} 4K_0 + K_3 &= 7 \\ K_0 + K_3 &= 4 \end{aligned}\right\} \quad 3K_0 = 3, \quad K_0 = 1, \quad K_3 = 3$$

(5) $f(z) = 1 + \frac{1}{z+1} + \frac{2}{z-2} + \frac{3}{(z-2)^2}$

$= \frac{z^3 - 3z + 7}{(z+1)(z-2)^2}$

(6) Consider: $\frac{1}{z+1} = \frac{1}{z} \frac{1}{1+\frac{1}{z}} = \frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{-1}{z}\right)^n$, for $\frac{1}{z} < 1$, $z > 1$

Consider: $\frac{2}{z-2} = \frac{1}{\frac{z}{2} - 1} = - \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n$, for $\frac{z}{2} < 1$, $z < 2$

Consider: $\frac{3}{(z-2)^2} = \frac{3}{4} \frac{1}{(\frac{z}{2} - 1)^2} = \frac{3}{4} \sum_{n=0}^{\infty} (n+1) \left(\frac{z}{2}\right)^n$, for $\frac{z}{2} < 1$, $z < 2$

(7) $\therefore f(z) = 1 + \frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{-1}{z}\right)^n - \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n + \frac{3}{4} \sum_{n=0}^{\infty} (n+1) \left(\frac{z}{2}\right)^n$

$= 1 + \sum_{n=0}^{\infty} \left\{ \frac{3}{4}(n+1) - 1 \right\} \left(\frac{z}{2}\right)^n + \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{z}\right)^{n+1}$

$= 1 + \frac{1}{4} \sum_{n=0}^{\infty} \{3n - 1\} \left(\frac{z}{2}\right)^n + \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{1}{z}\right)^n$

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Problems 33, 34

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33. (1) Let a_p be any zero of order s_p of $f(z)$. Then:

$$f(z) = (z-a_p)^{s_p}\varphi(z)$$ where $\varphi(z)$ is regular at and within a small neighborhood of a_p, and non-zero.

(2) $$f'(z) = s_p(z-a_p)^{s_p-1}\varphi(z) + (z-a_p)^{s_p}\varphi'(z)$$

(3) $$\frac{f'(z)}{f(z)} = \frac{s_p}{z-a_p} + \frac{\varphi'(z)}{\varphi(z)}$$

(4) $$z\frac{f'(z)}{f(z)} = \frac{z\,s_p}{z-a_p} + \frac{z\varphi'(z)}{\varphi(z)}$$

$$= s_p + \frac{a_p s_p}{z-a_p} + \frac{z\varphi'(z)}{\varphi(z)}$$

There fore the function $\frac{z f'(z)}{f(z)}$ has a residue $a_p s_p$ at $z = a_p$ considering $\frac{z\varphi'(z)}{\varphi(z)}$ to be regular in the neighborhood of a_p. Clearly, such is the behaviour of all such points that are zeroes of $f(z)$ and residues of this type exist within the contour c.

(5) Let b_k be any pole of order r_k of $f(z)$. Then:

$$f(z) = (z-b_k)^{-r_k}\psi(z)$$ where $\psi(z)$ is regular at and in the neighborhood of b_k, and non-zero.

(6) $$f'(z) = -r_k(z-b_k)^{-r_k-1}\psi(z) + (z-b_k)^{-r_k}\psi'(z)$$

(7) $$\frac{z f'(z)}{f(z)} = \frac{-r_k z}{z-b_k} + \frac{z\psi'(z)}{\psi(z)} = -r_k - \frac{r_k b_k}{z-b_k} + \frac{z\psi'(z)}{\psi(z)}$$

Thus $\frac{z f'(z)}{f(z)}$ has a residue $-r_k b_k$ at $z = b_k$ and this is the case for all poles of $f(z)$ within the closed contour c.

(8) Invoking Cauchy's Residue Theorem: $\int_c f(z)\,dz = 2\pi i \sum R$,

$$\frac{1}{2\pi i}\int_c \frac{z f'(z)}{f(z)}\,dz = \sum_{p=1}^{m} a_p s_p - \sum_{k=1}^{n} b_k r_k$$

34. (1) We wish to locate the roots, or zeros, of

$$F(z) = z^4 + z^3 + 4z^2 + 2z + 3 = 0$$

(2) Let $z \to x$: $x^4 + x^3 + 4x^2 + 2z + 3 = 0$

By Descartes' Rule, there are no real +roots as there are no variations in the signs of the real coefficients

(3) Let $z \to -x$: $x^4 - x^3 + 4x^2 - 2x + 3 = 0$

By Descarte's Rule, there are either 2 negative real roots or no negative real roots.

For $0 < x < 1$, $x^4 - x^3 + 4x^2 > 0$, $-2x + 3 > 0$.

For $x > 1$, $x^4 - x^3 > 0$, $4x^2 - 2x + 3 > 0$, so that there are no real negative roots.

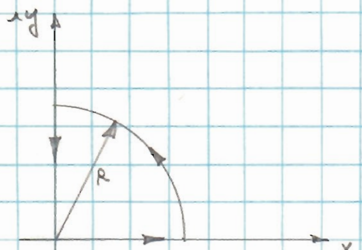
(4) Let $z \to iy$: $y^4 - iy^3 - 4y^2 + 2iy + 3 = 0$

$y^4 - 4y^2 + 3 = 0$: $y^2 = 2 \pm \frac{1}{2}[16-12]^{1/2} = 3, 1$; $y = 1, \sqrt{3}$

$y(-y^2 + 2) = 0$: $y = 0$, $y = \sqrt{2}$

Thus there is no y for which the real and imaginary parts vanish together and there are no purely imaginary roots.

(5) We now consider $\Delta \arg F(z)$ around the first quadrant of the complex plane with a path of very large radius R.



Along x axis, $\Delta \arg F(z) = 0$

Around C: $Re^{i\theta}$, $\Delta \arg F(z)$

$= \Delta \arg(R^4 e^{i4\theta}) + \Delta \arg(1 + O[R^{-1}])$

$= 2\pi$ as $R \to \infty$ from Rouche's theorem as there are no zeros on the real or negative axis or at ∞.

(6) On the y axis, $\Delta \arg F(z) = \Delta \arctan\left(\frac{-y^3 + 2y}{y^4 - 4y^2 + 3}\right)$

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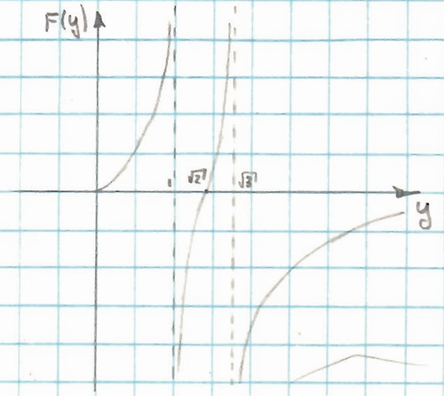
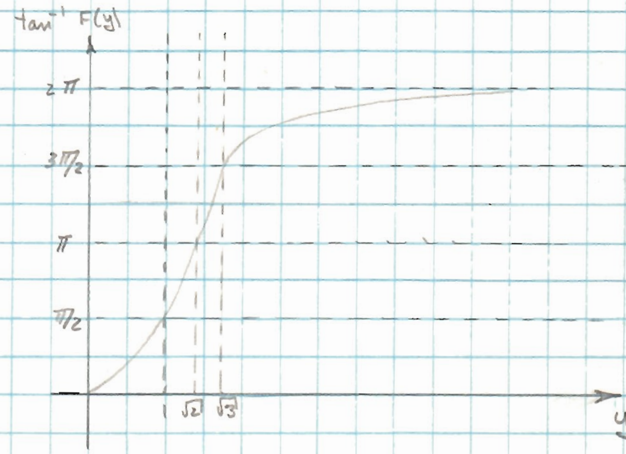
34. Continued

(7) y : ∞ $\sqrt{3}$ $\sqrt{2}$ $\sqrt{1}$ 0

$F(y) = \dfrac{-y^3 + 2y}{y^4 - 4y^2 + 3}$: 0 to $-\infty$, $+\infty$ to 0, 0 to $-\infty$, $+\infty$ to 0

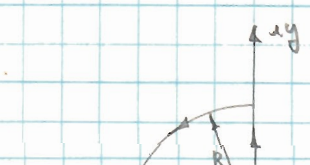




Thus it is seen that the change in arg $F(z)$ comming along the imaginary axis from ∞ to 0 is -2π.

(8) $\therefore$ $\Delta_c \arg F(z) = 0 + 2\pi - 2\pi = 0$, then $N = \frac{1}{2\pi} \Delta_c \arg F(z) = 0$

or there are no roots in the first quadrant. Since it is known that no roots are real, their must be four complex roots occuring in two pairs of complex conjugates. Since there are no roots in the first quadrant it follows that there are no roots in the fourth quadrant and the two pairs of complex conjugates are in the second and third quadrants. This is easily shown:



Second Quadrant :

$\Delta_c \arg F(z) = 2\pi + 2\pi + 0$

$= 4\pi$

(2π: along y from 0 to ∞; 2π: around the circle of radios $R \to \infty$; 0: along x from ∞ to 0)

$N = \dfrac{4\pi}{2\pi} = 2$ or

there are two roots in the second quadrant. That their complex conjugates are in the third quadrant follows immediately.

Problem 35

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35. (1) Given: $w = \cosh^{-1} z$

(2) $z = \cosh w = \dfrac{e^{w} + e^{-w}}{2}$; $2z = e^{w} + e^{-w}$

(3) $e^{2w} - 2ze^{w} + 1 = 0$

(4) $e^{w} = z \pm \frac{1}{2}(4z^2 - 4)^{1/2} = z \pm (z^2-1)^{1/2}$

(5) $w = \mathrm{Log}\left\{ z \pm (z^2-1)^{1/2} \right\}$

We take the positive root for convenience at this time.

$w = \mathrm{Log}\left\{ z + (z^2-1)^{1/2} \right\}$

knowing that two values exist.

(6) Let $\zeta = z + (z^2-1)^{1/2}$, then $\mathrm{Log}\,\zeta = \log \zeta + 2n\pi i$

Thus $\mathrm{Log}\,\zeta$ has infinitely many branches. The usual branch points are at $\zeta = 0$ and ∞. However, there is no z for which $\zeta = 0$, hence we proceed to the branch points of ζ.

(7) $\zeta = z + f$, $f = (z^2-1)^{1/2}$, $f^2 = (z+1)(z-1)$, Therefore:

"z" plane

Thus the branch points of f are the branch points of ζ and the heretofore single valued function $\log \zeta$.

(9) Let f_1 be one branch of f and f_2 be the other with:

$f_1 = -f_2$; $f_1^2 = f_2^2 = z^2 - 1$, $f_1 f_2 = 1 - z^2$

(10) $\mathrm{Log}\left[z + f_1\right] + \mathrm{Log}\left[z + f_2\right] = \log\left[z + f_1\right]\left[z + f_2\right] + 2n\pi i$

(11) $(z + f_1)(z + f_2) = z^2 + z(f_1 + f_2) + f_1 f_2 = z^2 + 1 - z^2 = 1$

(12) $\log 1 = 0$

(13) $\mathrm{Log}\,(z + \mathcal{L}_2) = -\,\mathrm{Log}\,(z + \mathcal{L}_1) + 2n\pi i = -\log\,(z + \mathcal{L}_1) + 2n\pi i$

(14) $\mathrm{Log}\,(z + \mathcal{L}_1) = \log\,(z + \mathcal{L}_1) + 2n\pi i$

It is plain that if the branch w_1 is called $\log\,(z + \mathcal{L}_1)$ then the other branches are given by:

$$2n\pi i \pm \log\,(z + \mathcal{L}_1) = 2n\pi i \pm w_1$$

The Riemann surface consists of overlapping planes, one side of the cut in each plane joined to the opposite side of the cut in the other. The function is regular in each plane as long as one doe not pass thru the cuts from +1 and -1 to ∞.

(15) Now: $w = \cosh^{-1} z$; $z = \cosh w$; $\cosh^2 w - \sinh^2 w = 1$

$$z^2 - \sinh^2 w = 1 \quad ; \quad \sinh w = \pm \left[z^2 - 1\right]^{1/2}$$

(16) $\dfrac{dz}{dw} = \sinh w = \pm \left[z^2 - 1\right]^{1/2}$; $\dfrac{dw}{dz} = \pm \left[z^2 - 1\right]^{-1/2}$

$$= \pm i \left[1 - z^2\right]^{-1/2} = \pm i \left[1 + \frac{1}{2} z^2 + \frac{\frac{1}{2}\cdot\frac{3}{2}}{2!} z^4 + \dots\right] ; \; |z| < 1.$$

(17) Integrating term by term:

$$w = \text{constant} \pm i \left[z + \frac{1}{2}\frac{z^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{z^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{z^7}{7} + \dots\right]$$

(18) We wish to find the series for the branch where Can't $w \cancel{=} 0$ for $z = 0$ inside $|z| = 1$. Take $\sinh w = \left[z^2 - 1\right]^{1/2} = i$ at $z = 0$ so that + sign is used and the constant will be zero for $w = 0$ at $z = 0$:

$\left(\frac{2n+1}{2}\pi - (-1)^n\right) x$

$$w_1 = i \left[z + \frac{1}{2}\frac{z^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{z^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{z^7}{7} + \dots\right]$$

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36. (1) Given: $f(z) = z - \frac{1}{2} z^2 + \frac{1}{3} z^3 - \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} z^n}{n}$

$$= \mathcal{Log}\,(z+1) \quad , \quad |z| < 1$$

(2) The function $\mathcal{Log}\,\zeta$ is infinitely many valued and has branch points at $\zeta = 0, \infty$ or, in this case, $z = -1, \infty$. Thus the above Taylor series has a region of convergence completely in one branch. Thus we may write in the region of convergence:

$$f(z) = \log(z+1) \quad , \quad |z| < 1 \quad , \text{ taking } n = 0 \text{ for this branch}$$

(3) We now search for a series, which will continue $f(z)$ outside its present region of convergence. Try:

$$z \rightarrow \frac{z - \alpha}{1 + \alpha} \; ; \quad \therefore \quad z + 1 \rightarrow \frac{z - \alpha}{1 + \alpha} + 1 = \frac{z+1}{1+\alpha}$$

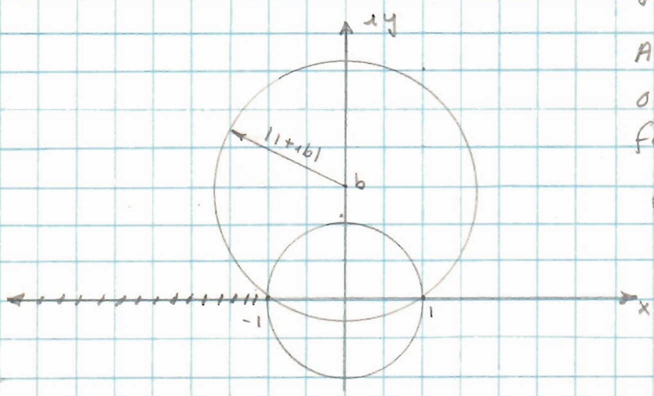
and:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left(\frac{z - \alpha}{1 + \alpha} \right)^n = \mathcal{Log} \left(\frac{z+1}{1+\alpha} \right) \; ; \quad |z - \alpha| < |1 + \alpha|$$

(4) How do make this equal to $f(z)$ in their overlap region? Add $\mathcal{Log}\,(1+\alpha)$:

$$\therefore \quad f_1(z) = \mathcal{Log}\,(1+\alpha) + \mathcal{Log} \left(\frac{z+1}{1+\alpha} \right) = \mathcal{Log}\,(z+1) = f(z)$$

(5) Take $\alpha = ib$, then the radius of convergence for the continuation is $|1 + ib|$. It is seen that we will alway intersect at $z = \pm 1$:



As we desire $f_1(z) = f(z)$ in their overlap region and we have taken $f(z)$ for $|z| < 1$ in the zeroeth branch:

$$f_1(z) = \log(1+\alpha) + \log\left(\frac{z+1}{1+\alpha}\right) = \log(z+1)$$
$$\equiv f(z)$$

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(6) Thus, for $\alpha = ib$: $f_1(z) = \log(1+ib) + \log\left(\frac{z+1}{1+ib}\right)$

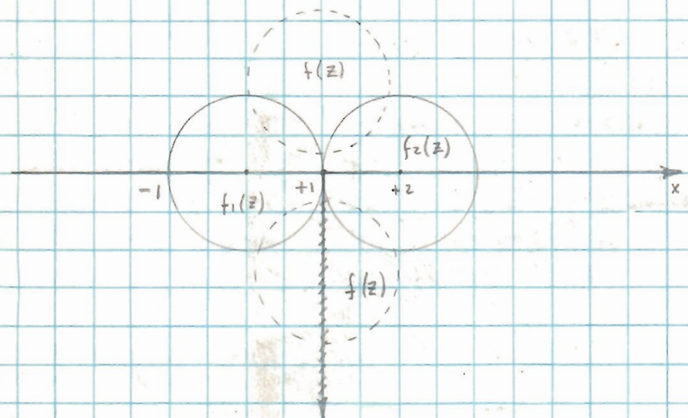
$$= \frac{1}{2} \ln(1+b^2) + i \arctan b + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left(\frac{z - ib}{1 + ib} \right)$$

37. (1) Given: $f_1(z) = z + \frac{1}{2}z^2 + \frac{1}{3}z^3 + \cdots = \sum_{n=1}^{\infty} \frac{z^n}{n}$

$$f_2(z) = i\pi - (z-2) + \frac{1}{2}(z-2)^2 - \frac{1}{3}(z-2)^3 + \cdots = i\pi - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(z-2)^n$$

(2) The regions of convergence are obviously:

$f_1(z)$: $|z| < 1$; $f_2(z)$: $|z-2| < 1$



If $f_1(z)$, $f_2(z)$ are analytic continuations of some function $f(z)$, two possible ways $f_1(z)$ and $f_2(z)$ are connected are shown. If $z = 1$ is neither a branch point or a singularity both continuations are equivalent. At any rate $f_1(z) = f(z) = f_2(z)$ provided a specific continuation is stipulated.

(3) We now examine the sum of $f_1(z)$:

$$f_1(z) = -\left[(-z) - \frac{1}{2}(-z)^2 + \frac{1}{3}(-z)^3 - \cdots\right] = -\log(1-z)$$

Now $\log \zeta$ has $\zeta = 0, \infty$ for branch points or for $\zeta = 0$, $1 - z = 0$, $z = 1$ is a branch point and also $z = \infty$. We make the cut as shown in the diagram. Since we want $f_1(z) = f_2(z) = f(z)$, we must now continue thru the upper $f(z)$ and $f(z)$, $f_2(z)$ will be equal to the principal value of $f_1(z)$ since the cut is now outside their circles of convergence.

(4) $\therefore$ $f_1(z) = -\log(1-z)$; $f(z) = -\log(1-z)$ and $f_2(z) = -\log(1-z)$ if $f_1(z)$, $f_2(z)$ are continuations of $f(z)$.

(5) $f_2(z) = i\pi - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(z-2)^n$

Let $\zeta = z - 2$, $f_2(z) = i\pi - \log(\zeta+1) = i\pi - \log(z-1)$

$= i\pi - \log(-1) - \log(1-z)$

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(6) $\log(-1) = i \arctan\frac{0}{-1} = i\pi$, $\therefore$ $f_2(z) = -\log(1-z)$

$= f_1(z) = f(z)$

Thus $f_1(z)$, $f_2(z)$ are analytic continuations $f(z)$ when it is shown with the cut as taken above. The cut could be made positive and $f(z)$ the lower region and the same results would be had.

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NB: The first part of question 38 referring to example 38 was not done on the advice of the grader as it was too confusing.

38. (1) Evaluate: $I = \int_0^{\pi} \frac{\cos^2 3\theta \; d\theta}{1 - 2p\cos 2\theta + p^2}$, $0 < p < 1$

(2) Let $\varphi = 2\theta$, $2d\theta = d\varphi$

$$\therefore \quad I = \frac{1}{2}\int_0^{2\pi} \frac{\cos^2 3/2\,\varphi \; d\varphi}{1 - 2p\cos\varphi + p^2}$$

Let $z = e^{i\varphi}$, $d\varphi = \frac{dz}{iz}$, $\cos\varphi = \frac{z^2+1}{2z}$

$$\cos 3/2\,\varphi = \frac{e^{i 3/2 \varphi} + e^{-i 3/2 \varphi}}{2} = \frac{1}{2}\left(z^{3/2} + \frac{1}{z^{3/2}}\right) = \frac{z^3+1}{2\, z^{3/2}}$$

$$\cos^2 3/2\,\varphi = \frac{z^6 + 2z^3 + 1}{4z^3}$$

(3) $$1 - 2p\cos\varphi + p^2 = \frac{z - p z^2 - p + p^2 z}{z}$$

$$= \frac{-p\left(z^2 - \left\{\frac{p^2+1}{p}\right\} z + 1\right)}{z}$$

(4) On $C = |z| = 1$: $$J = I(z) = \frac{i}{8p}\int_C \frac{(z^6 + 2z^3 + 1)\, dz}{z^3(z^2 - az + 1)}$$

where $a = \frac{p^2+1}{p} > 1$

(5) Dividing out the integrand:

$$\begin{array}{r|l} & z + a \\ \hline z^5 - az^4 + z^3 & z^6 + 2z^3 + 1 \\ & z^6 - az^5 + z^4 \\ \hline & az^5 - z^4 + 2z^3 + 1 \\ & az^5 - a^2 z^4 + az^3 \\ \hline & (a^2-1)z^4 + (2-a)z^3 + 1 \end{array}$$

$$\therefore \quad J_I = z + a + \frac{(a^2-1)z^4 + (2-a)z^3 + 1}{z^3(z^2 - az + 1)}$$

(6) The first two terms in the reduced integrand vanish by Cauchy's Theorem, so we need only be concerned with the remaining improper fraction. Factoring the denominator:

$$z = \frac{(p^2+1)}{2p} \pm \frac{1}{2p}\left[p^4 + 2p^2 + 1 - 4p^2\right]^{1/2} = \frac{(p^2+1) \pm (p^2-1)}{2p}$$

$$= p, \frac{1}{p} \qquad z^2 - az + 1 = (z - p)\left(z - \frac{1}{p}\right)$$

However, the point $\frac{1}{p}$ is outside the circle $|z| = 1$ and thus its residue need not be evaluated.

(7) Now: $$J = \frac{i}{8p}\int_c \frac{(a^2-1)z^4 + (2-a)z^3 + 1}{z^3(z-p)(z-\frac{1}{p})}\,dz$$

(8) $$\frac{(a^2-1)z^4 + (2-a)z^3 + 1}{z^3(z-p)(z-\frac{1}{p})} = \frac{K_1}{z^3} + \frac{K_2}{z^2} + \frac{K_3}{z} + \frac{K_4}{z-p} + \frac{K_5}{z-\frac{1}{p}}$$

We are interested in K_3 and K_4:

$$K_4: \quad \lim_{z\to p} \frac{(a^2-1)z^4 + (2-a)z^3 + 1}{z^3(z-1/p)} = \frac{(a^2-1)p^4 + (2-a)p^3 + 1}{p^2(p^2-1)}$$

$$= \frac{(p^3+1)^2}{p^2(p^2-1)}$$

(9) We can find K_3 by long division (this is only one way):

$$\begin{array}{r|l} & 1/z^3 + \frac{a}{z^2} + (a^2-1)\frac{1}{z} \\ \hline z^3 - az^4 + z^5 & 1 + (2-a)z^3 + (a^2-1)z^4 \\ & 1 - az + z^2 \\ \hline & +az - z^2 + (2-a)z^3 + (a^2-1)z^4 \\ & az - a^2z^2 \\ \hline & (a^2-1)z^2 \end{array}$$

$$\therefore\ K_3 = (a^2-1) = \frac{p^4+p^2+1}{p^2}$$

(10) $$\sum R = K_3 + K_4 = \frac{1}{p^2}\left[\frac{(p^4+p^2+1)(p^2-1) + (p^3+1)^2}{p^2-1}\right]$$

$$= 2p\,\frac{(p^3+1)}{(p^2-1)} = \frac{(p+1)(p^2-p+1)}{(p+1)(p-1)}\cdot 2p = 2p\left[\frac{p^2-p+1}{p-1}\right]$$

(11) $$J = \frac{i}{8p}\cdot 2\pi i \sum R = -\frac{\pi}{2}\left[\frac{p^2-p+1}{p-1}\right]$$

(12) $$\therefore \int_0^{\pi} \frac{\cos^2 3\theta\, d\theta}{1-2p\cos 2\theta + p^2} = \frac{\pi}{2}\left[\frac{p^2-p+1}{1-p}\right]; \quad 0 < p < 1$$

Problems 38, 39
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39. (1) Evaluate: $I = \int_0^{2\pi} \cos^n\theta \, d\theta$ for all $n \neq 0$

(2) Let $z = e^{i\theta}$, $d\theta = \frac{dz}{iz}$, $\cos\theta = \frac{z^2+1}{2z} = \frac{1}{2}\left(z + \frac{1}{z}\right)$

and C be the unit circle $|z| = 1$

(3) $J = I(z) = \frac{-i}{2^n}\int_C \frac{(z^2+1)^n}{z^{n+1}}\,dz = \frac{-i}{2^n}\int_C \left(z+\frac{1}{z}\right)^n \frac{dz}{z}$

(4) We expand $\left(z+\frac{1}{z}\right)^n$ in a binomial expansion:

$$\left(z+\frac{1}{z}\right)^n = \binom{n}{0}z^n + \binom{n}{1}z^{n-1}\cdot\frac{1}{z} + \cdots + \binom{n}{k}z^{n-k}\cdot\frac{1}{z^k} + \cdots + \binom{n}{n}\frac{1}{z^n}$$

with $\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$; $k = 1, 2, 3, \ldots$

(5) We now look for the constant term as when this is multiplied by $\frac{1}{z}$, it will yield the residue of the integrand.

Clearly, this term occurs for $n - 2k = 0$, or, $k = \frac{n}{2}$.

As n and k are integers, it is clear that no constant term, and hence no residue, exists for n odd. That is, k is only an integer for n even.

$\therefore \int_0^{2\pi} \cos^n\theta\,d\theta = \frac{-i}{2^n}\int_C \left(z+\frac{1}{z}\right)^n \frac{dz}{z} = 0$ for n odd

(6) For n even: $\binom{n}{\frac{n}{2}} = \frac{n(n-1)(n-2)\cdots\left(\frac{n+2}{2}\right)}{\left(\frac{n}{2}\right)!}$

(7) $\left(\frac{n}{2}\right)! = \left(\frac{n}{2}\right)\left(\frac{n-2}{2}\right)\left(\frac{n-4}{2}\right)\cdots = 2^{-n/2}(n)(n-2)(n-4)\cdots$

$= 2^{-n/2}\cdot 2\cdot 4\cdot 6\cdot 8\cdots n$, as n is even

(8) $$n(n-1)(n-2)\cdots\left(\frac{n+2}{2}\right) = \frac{n!\ \left(\frac{n+2}{2}\right)}{\left(\frac{n+2}{2}\right)!} = \frac{n!}{\left(\frac{n}{2}\right)!}$$

$$= \frac{1\cdot 2\cdot 3\cdot 4\cdots(n-1)\,n}{2^{-n/2}\cdot 2\cdot 4\cdot 6\cdots(n-4)(n-2)(n)} = 2^{n/2}\cdot 1\cdot 3\cdot 5\cdot 7\cdots(n-1)$$

(9) $$\therefore\ \Sigma R = \binom{n}{\frac{n}{2}} = 2^{n}\cdot\frac{1\cdot 3\cdot 5\cdot 7\cdots(n-1)}{2\cdot 4\cdot 6\cdot 8\cdots n}\ ;\quad n \text{ even}$$

(10) $$\frac{i}{2^n}\int_C\left(z+\frac{1}{z}\right)^n\frac{dz}{z} = \frac{-i}{2^n}\cdot 2\pi i\ \Sigma R$$

$$= \int_0^{2\pi}\cos^n\theta\,d\theta = \frac{1\cdot 3\cdot 5\cdot 7\cdots(n-1)}{2\cdot 4\cdot 6\cdot 8\cdots n}\,2\pi\ ,\quad n \text{ even}$$

(11) For the case of $n=0$, we can write:

$$\int_0^{2\pi}\cos^n\theta\,d\theta = \frac{n!}{\left[\left(\frac{n}{2}\right)!\right]^2}\,2\pi$$

Taking $0! \equiv 1$, $\int_0^{2\pi}\cos^n\theta\,d\theta = \int_0^{2\pi}d\theta = 2\pi$ (obviously!)

Problems
40, 41, 42

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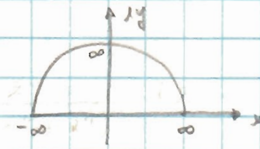
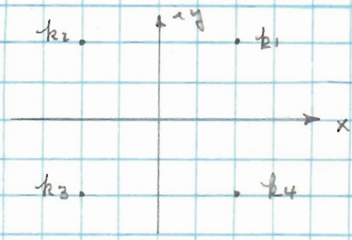
40. (1) Show $\displaystyle\int_0^\infty \frac{x^6\,dx}{(x^4+a^4)^2} = \frac{3\sqrt{2}\,\pi}{16\,a}$, $a>0$.

(2) Consider $J = \displaystyle\int_C \frac{z^6\,dz}{(z^4+a^4)^2}$; C:



(3) $z^4+a^4=0$; $z^4=-a^4=a^4e^{i\pi}$; $z = ae^{i\pi/4}, ae^{i3\pi/4}, ae^{i5\pi/4}, ae^{i7\pi/4}$

$\qquad\qquad = k_1,\ k_2,\ k_3,\ k_4$



$I(z)$ has no poles on the real axis.
$z\,I(z) \to 0$ as $|z|\to\infty$

Since $I(z)$ is a rational function of z, with the degree of $D(z) > N(z)$ by two.

Thus the conditions are satisfied for evaluating (2) on the contour of a semicircle of infinite radius enclosing the upper half-plane which will give:

$$\int_{-\infty}^{\infty} I(x)\,dx = 2\pi i \sum R_+$$

(4) There are only two double poles in the upper-half plane $z = k_1, k_2$,

$$\frac{1/2\ z^6}{(z^4+a^4)^2} = \frac{K_1'}{(z-a^{i\pi/4})^2} + \frac{K_1}{(z-a^{i\pi/4})} + \frac{K_2'}{(z-a^{i3\pi/4})^2} + \frac{K_2}{(z-a^{i3\pi/4})} + \cdots$$

(5) We now use the fact that: $R = \dfrac{1}{(m-1)!}\lim\limits_{z\to\alpha}\dfrac{d^{m-1}}{dz^{m-1}}\left\{(z-\alpha)^m f(z)\right\}$

Let α be any one of the pole positions, then $\alpha^4=-a^4$

(6) $$R = \frac{1}{2}\lim_{z\to\alpha}\frac{d}{dz}\left\{\frac{(z-\alpha)^2 z^6}{(z^4-\alpha^4)^2}\right\} = \frac{1}{2}\lim_{z\to\alpha}\frac{d}{dz}\left\{\frac{z^6}{(z^2+\alpha^2)^2(z+\alpha)^2}\right\}$$

$$= \frac{1}{2}\lim_{z\to\alpha}\left\{\frac{6z^5(z^2+\alpha^2)^2(z+\alpha)^2 - z^6\left[2(z^2+\alpha^2)^2(z+\alpha)+4z(z^2+\alpha^2)(z+\alpha)^2\right]}{(z^2+\alpha^2)^4(z+\alpha)^4}\right\} = \frac{-3\alpha^3}{16a^4}$$

(7) If we stipulate $a>0$, then we must take $\alpha = ae^{i\pi/4}, ae^{i3\pi/4}$

$$\therefore\ 2\sum R_+ = -\frac{3a^3}{16a^4}\left\{e^{i3\pi/4}+e^{i9\pi/4}\right\} = \frac{-3}{16a}\left\{e^{i\pi/4}-e^{-i\pi/4}\right\}$$

$$= \frac{-3\cdot 2i}{16a}\sin\pi/4 = \frac{-3i\sqrt{2}}{16a}$$

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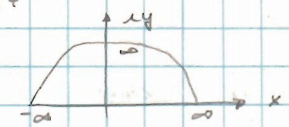
(8) $$\therefore\ \int_0^\infty \frac{x^6\,dx}{(x^4+a^4)^2} = \frac{1}{2}\cdot 2\pi i\cdot\frac{-3i\sqrt{2}}{16a} = \frac{3\pi\sqrt{2}}{16a}, \quad a>0.$$

41. (1) show $\displaystyle\int_0^\infty \frac{x^3 \sin mx}{x^4+a^4}\,dx = \frac{1}{2}\pi e^{-ma/\sqrt{2}} \cos\frac{ma}{\sqrt{2}}$; $m>0,\ a>0$

(2) Consider: $\displaystyle J = \int_0^\infty \frac{x^3 e^{mix}}{x^4+a^4}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^3 e^{mix}}{x^4+a^4}\,dx$

$\displaystyle = \frac{1}{2}\int_C I(z)\, e^{miz}\,dz$; C:



(3) Now $I(z) = \dfrac{N(z)}{D(z)}$ with the degree of $D(z) > N(z)$; and

$I(z) \to 0$ as $|z| \to \infty$

Thus, in the infinite semicircle of the upper half-plane,

$$\int_{-\infty}^{\infty} I(x)\, e^{mix}\,dx = 2\pi i \sum R_+ \left(I(z) e^{miz} \right)$$

from considerations of Jordan's Lemma (Lecture Notes, Phillips, p124)

(4) The poles of $I(z)\,e^{miz}$ are at, from example 40,

$$z = a e^{i\pi/4},\ a e^{i3\pi/4},\ a e^{i5\pi/4},\ a e^{i7\pi/4}$$

(5) $\displaystyle R = \lim_{z\to\alpha} \left\{ (z-\alpha)\, I(z)\, e^{miz} \right\}$

Let one of the poles be at α, then $\alpha^4 = -a^4$

$$\therefore\ R = \lim_{z\to\alpha}\left\{\frac{z^3(z-\alpha)\, e^{miz}}{z^4-\alpha^4}\right\} = \lim_{z\to\alpha}\left\{\frac{z^3 e^{miz}}{(z^2+\alpha^2)(z+\alpha)}\right\} = \frac{e^{mi\alpha}}{4}$$

(6) We now choose $a > 0$ so that the proper poles are $\alpha = a e^{i\pi/4},\ a e^{i3\pi/4}$.
$= a e^{i\pi/4},\ -a e^{-i\pi/4}$

(7) $\displaystyle \therefore\ \sum R_+ = \frac{1}{4}\left\{ \exp\left[mi\, a e^{i\pi/4}\right] + \exp\left[-mi\, a e^{-i\pi/4}\right] \right\}$

$\displaystyle = \frac{1}{4}\left\{ e^{mia/\sqrt{2}}\, e^{-ma/\sqrt{2}} + e^{-mia/\sqrt{2}}\, e^{-ma/\sqrt{2}} \right\} = \frac{1}{2} e^{-ma/\sqrt{2}} \cos\frac{ma}{\sqrt{2}}$

(8) $\displaystyle \therefore\ \int_0^\infty \frac{x^3 e^{mix}}{x^4+a^4}\,dx = \frac{1}{2}\cdot 2\pi i \cdot \frac{1}{2} e^{-ma/\sqrt{2}} \cos\frac{ma}{\sqrt{2}}$

$\displaystyle = i\,\pi/2\; e^{-ma/\sqrt{2}} \cos\frac{ma}{\sqrt{2}}$

(9) $\displaystyle \int_0^\infty \frac{x^3 \sin mx}{x^4+a^4}\,dx = \mathrm{Im}\int_0^\infty \frac{x^3 e^{mix}}{x^4+a^4}\,dx = \frac{\pi}{2} e^{-\frac{ma}{\sqrt{2}}} \cos\frac{ma}{\sqrt{2}}$,

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$m>0,\ a>0$

Problems 40, 41, 42
Continued

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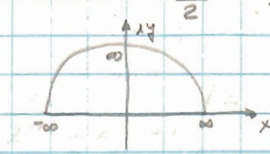
42.1. (1) Show $\int_0^\infty \frac{(1+x^2)\cos ax}{1+x^2+x^4}\,dx = \frac{\pi}{\sqrt{3}}\, e^{-\sqrt{3}\,a/2} \cos\frac{a}{2}$; $a>0$

(2) consider: $\int_C \frac{(1+z^2)\, e^{aiz}}{z^4+z^2+1}\,dz$: C:



(3) $z^4+z^2+1=0$: $z^2 = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1-4} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$; $-\frac{1}{2} - i\frac{\sqrt{3}}{2}$

$$= e^{i4\pi/3},\ e^{i2\pi/3}$$

$z = e^{i4\pi/3}, e^{i7\pi/3}, e^{i2\pi/3}, e^{i5\pi/3}$ are poles

and $z = \underbrace{e^{i2\pi/3}}_{-e^{-i\pi/3}}, \underbrace{e^{i7\pi/3}}_{e^{i\pi/3}}$ are in the upper half plane.

(4) No poles on real axis; $a>0$, $I(z)\to 0$ as $|z|\to\infty$; $D(z) > N(z)$ by >1.
$\therefore$ from Jordan's Lemma considerations:

$$\int_{-\infty}^{\infty} I(x)\, e^{aix}\,dx = 2\pi i \sum R_+ \left[I(z)\, e^{aiz} \right]$$

(5) $R_1 = \lim_{z\to -e^{-i\pi/3}} \left\{ \frac{(1+z^2)\, e^{aiz}\,(z+e^{-i\pi/3})}{z^4+z^2+1} \right\}$

the poles are at $z = -e^{i\pi/3}, e^{i\pi/3}, -e^{-i\pi/3}, e^{-i\pi/3}$

$\therefore R_1 = \lim_{z\to -e^{-i\pi/3}} \left\{ \frac{(1+z^2)\, e^{aiz}}{(z+e^{i\pi/3})(z-e^{i\pi/3})(z-e^{-i\pi/3})} \right\}$

$$= \frac{(1+e^{-i2\pi/3})\exp\left[-ai e^{-i\pi/3}\right]}{2i\sin\pi/3 \cdot -2\cos\pi/3 \cdot -2e^{-i\pi/3}} = \frac{1}{2i}\,\frac{\exp\left[-ai e^{-i\pi/3}\right]}{\sqrt{3}}$$

(6) $R_2 = \lim_{z\to e^{i\pi/3}} \left\{ \frac{(1+z^2)\, e^{aiz}}{(z+e^{i\pi/3})(z+e^{-i\pi/3})(z-e^{-i\pi/3})} \right\} = \frac{(1+e^{i2\pi/3})\exp\left[ai e^{i\pi/3}\right]}{2e^{i\pi/3}\cdot 2\cos\pi/3 \cdot 2i\sin\pi/3}$

$$= \frac{1}{2i}\,\frac{\exp\left[ai e^{i\pi/3}\right]}{\sqrt{3}}$$

(7) $\frac{1}{2}\cdot 2\pi i \sum R_+ \quad \frac{\pi}{2\sqrt{3}} \left\{ e^{\frac{ai}{2}} e^{-\frac{\sqrt{3}}{2}a} + e^{-\frac{ai}{2}} e^{-\frac{\sqrt{3}}{2}a} \right\} = \frac{\pi}{\sqrt{3}}\, e^{-\frac{\sqrt{3}}{2}a} \cos\frac{a}{2}$

(8) $\therefore \int_0^\infty \frac{(1+x^2)\cos ax}{x^4+x^2+1}\,dx = \mathrm{Re}\left\{\frac{1}{2}\cdot 2\pi i \sum R_+\right\} = \frac{\pi}{\sqrt{3}}\, e^{-\frac{\sqrt{3}}{2}a} \cos\frac{a}{2}$,

$a>0$.

42 ii (1) Show $\int_0^\infty \frac{x \sin ax}{x^4+x^2+1}\,dx = \frac{\pi}{\sqrt{3}} e^{-\frac{\sqrt{3}}{2}a} \sin\frac{a}{2}$, $a>0$.

(2) Consider: $\int_c \frac{z\, e^{aiz}}{z^4+z^2+1}\,dz$: $I(z) \to 0$ as $|z|\to\infty$, $a>0$, poles as before.

$\therefore \int_{-\infty}^{\infty} I(x)\, e^{aix}\,dx = 2\pi i \sum R_+ \left[I(z)\, e^{aiz} \right]$, from previous work.

taking residues at $z = -e^{-i\pi/3}, e^{i\pi/3}$

(3) $R_1 = \lim_{z\to -e^{-i\pi/3}} \left\{ \frac{z\, e^{aiz}}{(z+e^{i\pi/3})(z-e^{i\pi/3})(z-e^{-i\pi/3})} \right\} = \frac{1}{2} \frac{\exp[ai e^{-i\pi/3}]}{2i \cdot \frac{\sqrt{3}}{2} \cdot -1}$

$= -\frac{1}{2\sqrt{3}\, i} e^{-\frac{ai}{2}} e^{-\frac{\sqrt{3}}{2}a}$

(4) $R_2 = \lim_{z\to e^{i\pi/3}} \left\{ \frac{z\, e^{aiz}}{(z+e^{i\pi/3})(z+e^{-i\pi/3})(z-e^{-i\pi/3})} \right\} = \frac{1}{2} \frac{\exp[ai e^{i\pi/3}]}{2 \cdot \frac{1}{2} \cdot 2i \frac{\sqrt{3}}{2}}$

$= \frac{1}{2\sqrt{3}\, i} e^{\frac{ai}{2}} e^{-\frac{\sqrt{3}}{2}a}$

(5) $\frac{1}{2} \cdot 2\pi i \sum R_+ = \frac{\pi}{2\sqrt{3}} e^{-\frac{\sqrt{3}}{2}a} \left\{ e^{i\frac{a}{2}} - e^{-i\frac{a}{2}} \right\}$

$= i \frac{\pi}{\sqrt{3}} e^{-\frac{\sqrt{3}}{2}a} \sin\frac{a}{2}$

(6) $\int_0^\infty \frac{x \sin ax}{x^4+x^2+1}\,dx = \operatorname{Im}\left\{ \frac{1}{2} \cdot 2\pi i \sum R \right\} = \frac{\pi}{\sqrt{3}} e^{-\frac{\sqrt{3}}{2}a} \sin\frac{a}{2}$

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$z = Re^{i\theta}$
$dz = iRe^{i\theta} d\theta$

Problems 46, 47, 48

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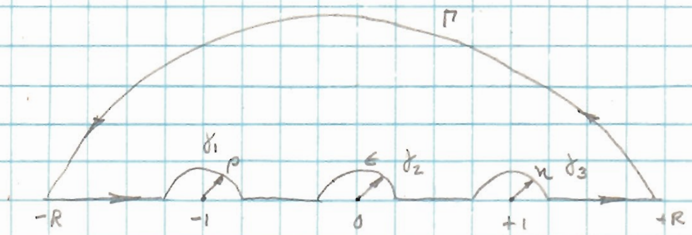
46. (1) Show $\frac{1}{\pi}\int_0^\infty \frac{x^b - x^{-b}}{x^2-1}\,dx = \tan\frac{1}{2}\pi b \; ; \quad -1 < b < 1$

(2) Consider $\int_c \frac{z^b}{z^2-1}\,dz$ where c is:



(3) $$\int_\epsilon^{1-\eta} \frac{x^b\,dx}{x^2-1} + \int_{\gamma_3} \frac{z^b}{z^2-1}\,dz + \int_{1+\eta}^{R} \frac{x^b}{x^2-1}\,dx + \int_\Gamma \frac{z^b}{z^2-1}\,dz + \int_{-R}^{-1-\rho} \frac{x^b}{x^2-1}\,dx + \int_{\gamma_2} \frac{z^b}{z^2-1}\,dz + \int_{-1+\rho}^{-\epsilon} \frac{x^b}{x^2-1}\,dx + \int_{\gamma_1} \frac{z^b}{z^2-1}\,dz = 0$$

(3) $$\left|\int_\Gamma \frac{z^b}{z^2-1}\,dz\right| = \left|\int_0^\pi \frac{i R^{b+1} e^{i(b+1)\theta}}{R^2 e^{i2\theta} - 1}\,d\theta\right| \le \frac{R^{b+1}}{R^2-1}\int_0^\pi d\theta = \frac{R^{b+1}\pi}{R^2-1}$$

$\to 0$ as $R \to \infty$ if $b+1 < 2$ or $b < 1$.

(4) $$\left|\int_{\gamma_2} \frac{z^b}{z^2-1}\,dz\right| = \left|\int_\pi^0 \frac{i\epsilon^{b+1} e^{i(b+1)\theta}}{\epsilon^2 e^{i2\theta} - 1}\,d\theta\right| \le \frac{\pi\epsilon^{b+1}}{1-\epsilon^2} \to 0$$

as $\epsilon \to 0$ if $b+1 > 0$ or $b > -1$

(5) $$\int_{\gamma_1} \frac{z^b}{z^2-1}\,dz = -\pi i\, R(z=-1) \; ; \quad \int_{\gamma_3} \frac{z^b}{z^2-1}\,dz = -\pi i\, R(z=+1)$$

(6) $$\lim_{z\to 1}\left\{\frac{z^b}{z+1}\right\} = -\frac{1}{2} \; ; \quad \lim_{z\to -1}\left\{\frac{z^b}{z-1}\right\} = +\frac{1}{2}(-1)^b$$

$$= +\frac{1}{2}e^{i\pi b}$$

(7) Let $R \to \infty$; $\rho, \epsilon, \eta \to 0$: $$P\int_0^\infty \frac{x^b}{x^2-1}\,dx - \frac{1}{2}\pi i + P\int_{-\infty}^0 \frac{x^b}{x^2-1}\,dx + \frac{1}{2}e^{i\pi b}\pi i = 0$$

Now $$\int_{-\infty}^0 \frac{x^b}{x^2-1}\,dx = -\int_\infty^0 \frac{(-1)^b x^b}{x^2-1}\,dx = \int_0^\infty \frac{e^{i\pi b}x^b}{x^2-1}\,dx$$

(8) $$\therefore \quad P\int_0^\infty \frac{x^b}{x^2-1}\,dx = \frac{\frac{1}{2}\pi i\,(1 - e^{i\pi b})}{1 + e^{i\pi b}}$$

(9) $\quad P\int_0^\infty \frac{x^b}{x^2-1}\,dx = -\frac{1}{2}\pi i \frac{\left(e^{i\pi\frac{b}{2}} - e^{-i\pi\frac{b}{2}}\right)}{e^{i\pi\frac{b}{2}} + e^{-i\pi b/2}} = \frac{1}{2}\pi \tan\frac{1}{2}\pi b$

(10) $\quad$ Let $b \to -b$: $\quad \int_\Gamma \to 0$ as $R \to \infty$ if $-b+1 < 2$

or $b > -1$

$\int_{\gamma_2} \to 0$ as $\epsilon \to 0$ if $-b+1 > 0$ or $b < 1$.

and $e^{i\pi b} \longrightarrow e^{-i\pi b}$ everywhere, thus

$\tan\frac{1}{2}\pi b \to \tan\frac{1}{2}\pi(-b) = -\tan\frac{1}{2}\pi b$

(11) $\quad \therefore P\int_0^\infty \frac{x^{-b}}{x^2-1}\,dx = -\frac{1}{2}\pi \tan\frac{1}{2}\pi b$

(12) $\quad \frac{1}{\pi} P\int_0^\infty \frac{x^b - x^{-b}}{x^2-1}\,dx = \tan\frac{1}{2}\pi b \quad ; (-1 < b < 1)$

$\lim_{x\to\pm1} \frac{x^b - x^{-b}}{x^2-1} = \lim_{x\to\pm1}\left\{\frac{b x^{b-1} + b x^{-b-1}}{2x}\right\} = b, \quad \frac{b(-1)^b + b(-1)^{-b}}{2}$

so that the principle value is no longer needed.

(13) $\quad \therefore \frac{1}{\pi}\int_0^\infty \frac{x^b - x^{-b}}{x^2-1}\,dx = \tan\frac{1}{2}\pi b \quad ; (-1 < b < 1)$

(14) $\quad$ Let $x = e^{\pi y}$, $b = \frac{a}{\pi}$; $x^b = e^{ay}$, $x^{-b} = e^{-ay}$, $x^2 = e^{2\pi y}$

$dx = \pi e^{\pi y}\,dy$. substituting in above;

$\frac{1}{\pi}\int_{-\infty}^\infty \frac{e^{ay} - e^{-ay}}{e^{2\pi y} - 1}\cdot\pi e^{\pi y}\,dy = \int_{-\infty}^\infty \frac{e^{ay} - e^{-ay}}{e^{\pi y} - e^{-\pi y}}\,dy = \int_{-\infty}^\infty \frac{\sinh ay}{\sinh \pi y}\,dy$

$= \tan\frac{1}{2}a \quad , (-\pi < a < \pi)$

(16) $\quad \int_0^\infty \frac{\sinh ay}{\sinh \pi y}\,dy = \frac{1}{2}\int_{-\infty}^\infty \frac{\sinh ay}{\sinh \pi y}\,dy = \frac{1}{2}\tan\frac{1}{2}a \; ; (-\pi < a < \pi)$

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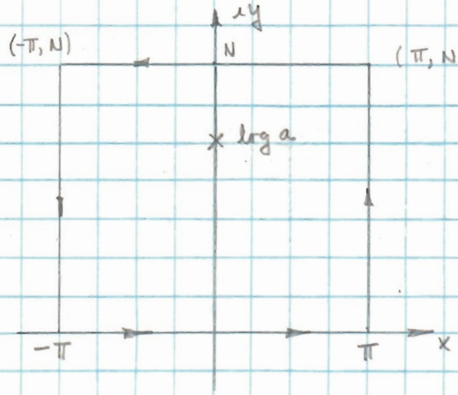
47\. (1) Show $\displaystyle\int_0^{\pi} \frac{x \sin x}{1 - 2a\cos x + a^2}\,dx = \frac{\pi}{a} \log \frac{1+a}{a}$

(2) Consider $\displaystyle\int_C \frac{z}{a - e^{-iz}}\,dz$ where C is:



The poles of the integrand are at

$e^{-iz} = a$; $-iz = \log a + i\,2N\pi$

or $z = -2N\pi + i\log a$; $N = 0, \pm 1, \pm 2, \pm 3, \cdots$

For $N > 0$, the poles always are outside the contour. For $N = 0$ and $a > 1$, the pole is inside the contour.

(3) $\displaystyle \lim_{z \to i\log a} \left\{ \frac{(z - i\log a)\,z}{a - e^{-iz}} \right\} = \lim_{z \to i\log a} \left\{ \frac{2z - i\log a}{i e^{-iz}} \right\} = \frac{\log a}{a}$

(4) $\displaystyle\int_{-\pi}^{\pi} \frac{x}{a - e^{-ix}}\,dx + \int_0^N \frac{i(\pi + iy)}{a - e^{-i(\pi + iy)}}\,dy$

$\displaystyle + \int_{\pi}^{-\pi} \frac{(x + iN)}{a - e^{-i(x+iN)}}\,dx + \int_N^0 \frac{i(-\pi + iy)}{a - e^{-i(-\pi+iy)}}\,dy = \frac{\log a}{a} \cdot 2\pi i$

(5) $\displaystyle\int_0^N \frac{i(\pi + iy)}{a + e^y}\,dy + \int_N^0 \frac{i(-\pi + iy)}{a + e^y}\,dy = \int_0^N \frac{2i\pi}{a + e^y}\,dy$

make the substitution $u = a + e^y$, $du = +e^y\,dy = (u - a)\,dy$

$\displaystyle\int \frac{du}{u(u-a)} = -\frac{1}{a}\int \frac{du}{u} + \frac{1}{a}\int \frac{du}{u - a} = -\frac{1}{a}\ln u + \frac{1}{a}\ln(u - a)$

$\displaystyle = \frac{1}{a} \ln \frac{(u-a)}{u}$

(6) $\displaystyle\int_0^N \frac{2i\pi}{a + e^y}\,dy = \frac{2i\pi}{a} \ln\left(\frac{+e^y}{a + e^y}\right)\Bigg]_0^N = \frac{2i\pi}{a} \ln \frac{e^N}{e^N + a}$

$\displaystyle - \frac{2i\pi}{a} \log \frac{1}{1+a} \longrightarrow \frac{2i\pi}{a} \log(1+a)$ as $N \to \infty$

(7) $$\left| \int_{\pi}^{-\pi} \frac{x+iN}{a-e^{-ix+N}}\,dx \right| = \left| \int_{\pi}^{-\pi} \frac{x+iN}{a-e^{N}\cos x + i\sin x\cdot e^{N}}\,dx \right|$$

$$\leq \int_{\pi}^{-\pi} \frac{\sqrt{x^2+N^2}\,dx}{\left[(a-e^{N}\cos x)^2+e^{2N}\sin^2 x\right]^{1/2}} \approx \int_{\pi}^{-\pi} \frac{\sqrt{x^2-N^2}}{e^{N}}\,dx \Rightarrow 0$$ as $N\to\infty$ because the exponential always wins.

(8) $$\therefore \int_{-\pi}^{\pi} \frac{x}{a-e^{-ix}}\,dx = \frac{2\pi i}{a}\left(\log\frac{a}{1+a}\right)$$

(9) Consider $\int_c \frac{z}{a-e^{iz}}\,dz$ around the same contour except reflected across the real axis. The pole within the contour is then at $-i\log a$.

(10) $$\int_{-\pi}^{\pi} \frac{x}{a-e^{ix}}\,dx + \int_{0}^{-N} \frac{i(\pi+iy)\,dy}{a+e^{-y}} + \underbrace{\int_{\pi}^{-\pi} \frac{(x-iN)\,dx}{a-e^{ix+N}}}_{=0 \text{ as } N\to\infty \text{ as above}}$$

$$+ \int_{-N}^{0} \frac{i(\pi+iy)\,dy}{a+e^{-y}} = -\frac{\log a}{a}$$

(11) $$2\pi i\int_{0}^{N} \frac{dy}{a+e^{-y}} = -\frac{2\pi i}{a}\ln\left(\frac{+e^{-y}}{a+e^{-y}}\right)\Bigg]_{0}^{N} = \frac{2\pi i}{a}\log\frac{1}{a+1}\ ;\ N\to\infty$$

$u = a+e^{-y}$

$du = -e^{-y}\,dy = -(u-a)\,dy$

$$= -\frac{2\pi i}{a}\log(a+1)$$

(12) $$\therefore \int_{-\pi}^{\pi} \frac{x}{a-e^{ix}}\,dx = -\frac{2\pi i}{a}\log\left(\frac{a}{a+1}\right)$$

(13) Subtract (8) from (12):

$$\int_{-\pi}^{\pi} \frac{x\,(e^{ix}-e^{-ix})\,dx}{a^2-2(e^{ix}+e^{-ix})a+1} = \frac{4\pi i}{a}\log\left(\frac{a+1}{a}\right)$$

or $$\int_{-\pi}^{\pi} \frac{x\sin x}{a^2-2a\cos x+1}\,dx = \frac{2\pi}{a}\log\left(\frac{a+1}{a}\right)$$

$$\int_{0}^{\pi} \frac{x\sin x}{a^2-2a\cos x+1}\,dx = \frac{\pi}{a}\log\left(\frac{a+1}{a}\right)\ ;\ a>1.$$

Problems 46, 47, 48
Continued

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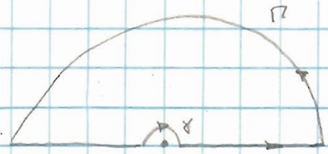
48. (1) show $P\int_0^\infty \frac{dx}{1-x^5} = \frac{2\pi}{25}\left(\sin \pi/5 + 3\sin\frac{2\pi}{5}\right)$

(2) Consider $\int_C \log z \, g(z)\, dz$ where C is:



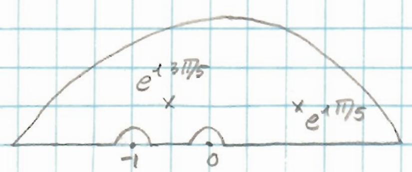
It was shown in lecture that if $|z \log z \, g(z)| \to 0$ when $|z| \to \infty, 0$, Then the following relation holds:

(3) $P\int_0^\infty \log x\left[g(x)+g(-x)\right]dx + i\pi\int_0^\infty g(-x)\,dx = 2\pi i \sum R^+\left[\log z \, g(z)\right]$

(4) Consider $\int_C \frac{\log z \, dz}{z^5+1}$: we see $|z \log z \, g(z)| \to 0$ when $|z| \to \infty, 0$, that is, it was shown that the lowest power of $D(z)$ must be <1 and the highest >1.

(5) $z^5+1=0$, $z = e^{i\left(\frac{\pi+2n\pi}{5}\right)}$, $n=0,1,2,3,4$

$z = e^{i\pi/5}, e^{i3\pi/5}, e^{i\pi}, e^{i7\pi/5}, e^{i9\pi/5}$



We must modify (3) by adding in 1/2 the residue at -1 to the RHS $\sum R$. When we have poles on the real axis, we must write:

(6) $P\int_0^\infty \log x\left[g(x)+g(-x)\right]dx + i\pi P\int_0^\infty g(-x)\,dx = 2\pi i\sum R^+ + \pi i \sum R_0$

because there may be singularities on the real axis.

(7) Let α be one of the poles such that $\alpha^5 = -1$

$$\lim_{z\to\alpha}\left\{\frac{(z-\alpha)\log z}{z^5+1}\right\} = \lim_{z\to\alpha}\left\{\frac{\log z + \frac{z-\alpha}{z}}{5z^4}\right\}$$

$$= \frac{\log\alpha}{5\alpha^4} = \frac{-\alpha\log\alpha}{5}$$

(8) $\sum R^+ = -\left[e^{i\pi/5}\cdot i\frac{\pi}{25} + e^{i3\pi/5}\cdot i\frac{3\pi}{25}\right]$, $\sum R_0 = -e^{i\pi}\cdot\frac{i\pi}{5}$
$= \frac{i\pi}{5}$

(9) $$2\pi i \sum R^{+} + \pi i \sum R_0 = \frac{2\pi^2}{25}\left[\cos\frac{\pi}{5} + i \sin\frac{\pi}{5} + 3\cos\frac{3\pi}{5} + 3i \sin\frac{3\pi}{5}\right]$$

$$-\frac{\pi^2}{5}$$

(10) Now $$g(x) + g(-x) = \frac{1}{x^5+1} + \frac{1}{1-x^5} = \frac{1-x^5+x^5+1}{1-x^{10}}$$

$$= \frac{2}{1-x^{10}}$$

We substitute in (6) and equate real and imaginary parts with (9):

(11) $$P\int_0^{\infty} \frac{2\log x}{1-x^{10}}\,dx = \frac{2\pi^2}{25}\left[\cos\frac{\pi}{5} + 3\cos\frac{3\pi}{5} - \frac{5}{2}\right]$$

$$\pi P\int_0^{\infty} \frac{dx}{1-x^5}\,dx = \frac{2\pi^2}{25}\left[\sin \pi/5 + 3\sin 3\pi/5\right]$$

(12) Now: $$\left.\begin{aligned}\sin\theta &= \sin(\pi-\theta)\\ \cos\theta &= -\cos(\pi-\theta)\end{aligned}\right\} \quad \begin{aligned}\sin 3\pi/5 &= \sin 2\pi/5\\ \cos 3\pi/5 &= -\cos 2\pi/5\end{aligned}$$

(13) $$P\int_0^{\infty} \frac{dx}{1-x^5}\,dx = \frac{2\pi}{25}\left[\sin\pi/5 + 3\sin 2\pi/5\right]$$

$$P\int_0^{\infty} \frac{\log x}{x^{10}-1}\,dx = \frac{\pi^2}{25}\left[\frac{5}{2} + 3\cos 2\pi/5 - \cos\pi/5\right]$$

The principle value in the first equation is taken because of the singularity at $x=1$.

Problems 49, 50, 51

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$\frac{29}{30}$

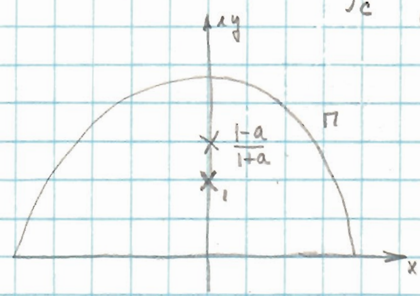
49. (1) show $$\int_0^{\frac{1}{2}\pi} \frac{a \sin 2\theta \;\theta\, d\theta}{1 - 2a \cos 2\theta + a^2} = \frac{\pi}{4}\log(1+a) \quad ; \; -1 < a < 1$$
$$= \frac{\pi}{4}\log\left(1+\frac{1}{a}\right) \quad ; \; |a| > 1$$

(2) Consider the substitution $x = \tan\theta$.

Then $dx = \sec^2\theta \, d\theta$; $\cos 2\theta = \frac{1-x^2}{1+x^2}$; $\sin 2\theta = \frac{2x}{x^2+1}$

and:
$$I = \int_0^\infty \frac{2ax \tan^{-1}x \; dx}{(x^2+1)\left\{(1+a)^2 x^2 + (1-a)^2\right\}}$$
$$= \frac{2a}{(a+1)^2} \int_0^\infty \frac{x \tan^{-1}x \; dx}{(x^2+1)\left\{x^2 + \left(\frac{1-a}{1+a}\right)^2\right\}}$$

(3) Consider: $$\int_c \frac{z \log(1-iz) \; dz}{(z^2+1)\left\{z^2 + \left(\frac{1-a}{1+a}\right)^2\right\}}$$, where C is:



The integrand has poles in the upper half plane at $z = i$ and $z = i\left(\frac{1-a}{1+a}\right)$ if $-1 < a < 1$

and at $z = i\left(\frac{a-1}{1+a}\right)$ if $|a| > 1$

The branch point at $z = -i$ is happily outside the contour. The integral round Γ vanishes as $|z| \to \infty$.

(4) $$\int_{-R}^{R} \frac{x \log(1-ix)\, dx}{(x^2+1)\left\{x^2 + \left(\frac{1-a}{1+a}\right)^2\right\}} + \int_0^\pi \frac{Re^{i\theta} \log(1-ie^{i\theta}\cdot R)\cdot iRe^{i\theta}\, d\theta}{(R^2e^{i\theta}+1)\left\{R^2e^{i\theta} + \left(\frac{1-a}{1+a}\right)^2\right\}}$$
$$= 2\pi i \sum R^+$$

(5) $$\left|\int_0^\pi \{\;\}\, d\theta\right| \leq \int_0^\pi \frac{R^2 |\log|1-ie^{i\theta}\cdot R||\; d\theta}{\left[R^2-1\right]\left[R^2 - \left(\frac{1-a}{1+a}\right)^2\right]} \longrightarrow 0 \quad \text{as } |z| = R \to \infty.$$

(6) $$R_1 = \lim_{z\to i} \left\{\frac{z \log(1-iz)}{(z+i)\left\{z^2 + \left(\frac{1-a}{1+a}\right)^2\right\}}\right\} = \frac{\frac{1}{2}\log 2}{-1 + \left(\frac{1-a}{1+a}\right)^2}$$

$$= -\frac{(1+a)^2 \log 2}{8a}$$

(7) For $|a|<1$: $\quad R_2 = \lim\limits_{z \to i\left(\frac{1-a}{1+a}\right)} \left\{ \dfrac{z \log(1-iz)}{(z^2+1)\left(z + i\left[\frac{1-a}{1+a}\right]\right)} \right\}$

$$= \frac{\frac{1}{2}\log\left(\frac{2}{1+a}\right)}{-\left[\frac{1-a}{1+a}\right]^2 + 1} = \frac{(a+1)^2}{8a} \log \frac{2}{1+a}$$

(8) For $|a|>1$: $\quad R_2 = \lim\limits_{z \to i\left(\frac{a-1}{1+a}\right)} \left\{ \dfrac{z \log(1-iz)}{(z^2+1)\left(z + i\left[\frac{a-1}{1+a}\right]\right)} \right\}$

$$= \frac{1/2 \log\left(\frac{2a}{1+a}\right)}{-\left[\frac{a-1}{1+a}\right]^2 + 1} = \frac{(a+1)^2}{8a} \log \frac{2a}{1+a}$$

(9) For $|a|<1$: $\quad 2\pi i \sum R = \dfrac{(1+a)^2}{8a} \log \dfrac{1}{a+1} \cdot 2\pi i$

For $|a|>1$: $\quad 2\pi i \sum R = \dfrac{(1+a)^2}{8a} \log \dfrac{a}{a+1} \cdot 2\pi i$

(10) $\therefore$
$$\int_{-\infty}^{\infty} \frac{x \log(1-ix)\, dx}{(x^2+1)\left(x^2+\left\{\frac{1-a}{a+1}\right\}^2\right)} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x \log(1+x^2)\, dx}{(x^2+1)\left(x^2+\left\{\frac{1-a}{1+a}\right\}^2\right)}$$
$$- i \int_{-\infty}^{\infty} \frac{x \tan^{-1}x\, dx}{(x^2+1)\left(x^2+\left\{\frac{1-a}{1+a}\right\}^2\right)} = \frac{i\pi\,(1+a)^2}{4a} \log \frac{1}{a+1}, \quad |a|<1$$
$$= \frac{i\pi\,(1+a)^2}{4a} \log \frac{a}{a+1}, \quad |a|>1$$

(11) $\therefore$
$$\int_0^{\infty} \frac{x \tan^{-1}x\, dx}{(x^2+1)\left(x^2+\left\{\frac{1-a}{a+1}\right\}^2\right)} = \frac{\pi(1+a)^2}{8a} \log(a+1), \quad |a|<1$$
$$= \frac{\pi(1+a)^2}{8a} \log\left(1+\frac{1}{a}\right), \quad |a|>1$$

(12)
$$\int_0^{\pi/2} \frac{a \sin 2\theta\ \theta\, d\theta}{1 - 2a\cos 2\theta + a^2} = \frac{2a}{(a+1)^2} \int_0^{\infty} \frac{x \tan^{-1}x\, dx}{(x^2+1)\left(x^2+\left\{\frac{1-a}{a+1}\right\}^2\right)}$$
$$= \frac{\pi}{4} \log(a+1), \quad |a|<1$$
$$= \frac{\pi}{4} \log\left(1+\frac{1}{a}\right), \quad |a|>1$$

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Problems 49, 50, 51
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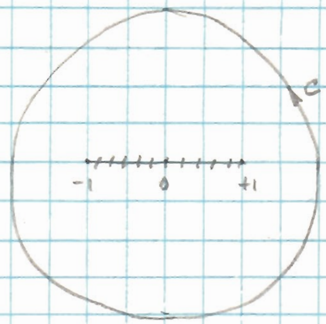
50. (1) show $\oint_c z^2 \log \frac{z+1}{z-1}\, dz = \frac{4}{3}\pi i$ where c is $|z| = 2$

(2) $z = 2e^{i\theta}$

(3) The integrand has branch points at $z = \pm 1$



(4) Consider integrating by parts:

$$\int v\, du = uv - \int u\, dv$$

Let $v = \log \frac{z+1}{z-1}$; $du = z^2 dz$

$dv = \left(\frac{z-1}{z+1}\right)\left(\frac{-2}{(z-1)^2}\right) dz$; $u = \frac{z^3}{3}$

$= \frac{-2}{z^2-1}\, dz$

(5) $\therefore$ $$\int z^2 \log \frac{z+1}{z-1}\, dz = \frac{z^3}{3}\log\left(\frac{z+1}{z-1}\right) + \frac{2}{3}\int \frac{z^3}{z^2-1}\, dz$$

$$= \frac{z^3}{3}\log\left(\frac{z+1}{z-1}\right) + \frac{2}{3}\int z\, dz + \frac{2}{3}\int \frac{z}{z^2-1}\, dz$$

(6) $$\oint_c z^2 \log\left(\frac{z+1}{z-1}\right) dz = \frac{8}{3} e^{i3\theta} \log\left(\frac{2e^{i\theta}+1}{2e^{i\theta}-1}\right)\Bigg]_0^{2\pi}$$

$$+ \frac{2}{3}\oint_c z\, dz + \frac{2}{3}\oint_c \frac{z\, dz}{z^2-1}$$

(7) $\oint_c z\, dz = 0$, by Cauchy's theorem.

(8) $\frac{8}{3}\log 3 - \frac{8}{3}\log 3 = 0$

(9) $\oint_c \frac{z\, dz}{z^2-1} = 2\pi i \sum R$

(10) $R_1 = \lim_{z\to 1} \frac{z}{z+1} = \frac{1}{2}$; $R_2 = \lim_{z\to -1}\frac{z}{z-1} = +\frac{1}{2}$

(11) $\therefore$ $\frac{2}{3}\oint_c \frac{z\, dz}{z^2-1} = \frac{4\pi i}{3}$

(12) $\therefore$ $\oint_c z^2 \log\left(\frac{z+1}{z-1}\right) dz = \frac{4\pi i}{3}$

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51. a. (1) Evaluate $\int_C \frac{dz}{z^4+1}$ where C is the ellipse $x^2 - xy + y^2 + x + y = 0$

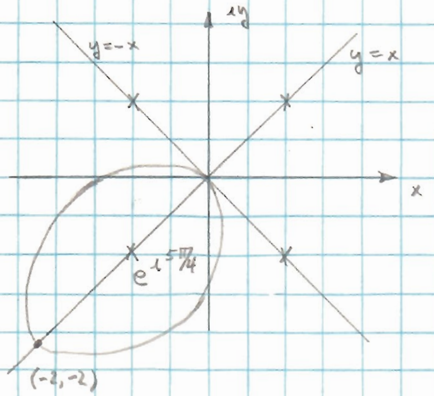
(2) The poles of the integrand are at $z = e^{i\left(\frac{\pi + 2n\pi}{4}\right)}$; $n = 0, 1, 2, 3$

or $z = e^{i\pi/4}, \ e^{i3\pi/4}, \ e^{i5\pi/4}, \ e^{i7\pi/4}$

That is, they lie on a circle of radius one. The cartesian equations of the lines passing thru the poles are $y = x$, $y = -x$. Substituting above:

$x^2 - x^2 + x^2 + x + x = 0$; $x^2 + 2x = 0$; $x = 0, -2$; $y = 0, -2$

$x^2 + x^2 + x^2 + x - x = 0$; $x = 0$; $y = 0$



$x^2 - xy + y^2 + x + y = 0$

$x = 0$: $y^2 + y = 0$, $y = 0, -1$

$y = 0$: $x^2 + x = 0$, $x = 0, -1$

Thus we take only the residue at $z = e^{i5\pi/4}$.

Let α be any pole such that $\alpha^4 = -1$:

$$\lim_{z\to\alpha} \left\{ \frac{z-\alpha}{z^4+1} \right\} = \frac{1}{4\alpha^3} = -\frac{\alpha}{4}$$

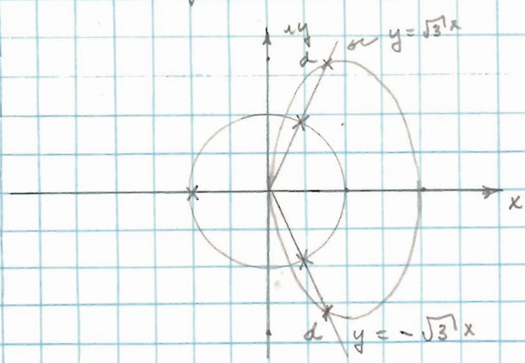
(3) $\therefore \int_C \frac{dz}{z^4+1} = -\frac{2\pi i}{4} e^{i\frac{5\pi}{4}} = -\frac{\pi i}{2} e^{i\pi} e^{i\pi/4} = \frac{i\pi}{2}\left\{\cos\frac{\pi}{4} + i\sin\pi/4\right\}$

$= \frac{i\pi\sqrt{2}}{4} - \frac{\pi\sqrt{2}}{4} = \frac{\pi\sqrt{2}}{4}\{-1 + i\}$

b. (1) Take $\int_C \frac{dz}{z^3+1}$ where C is the ellipse $2x^2 + y^2 = 4x$

or $2(x^2 - 2x) + y^2 = 0$; $2(x-1)^2 + y^2 = 2$; $(x-1)^2 + \frac{y^2}{2} = 1$

(2) The poles are at $z = e^{i\pi/3}, \ e^{i\pi}, \ e^{i5\pi/3}$



$y = 0$: $x = 0, 2$

$x = 0$: $y = 0$

$y = \pm\sqrt{3}\,x$; $2x^2 - 4x + 3x^2 = 0$, $5x - 4 = 0$

$x = \frac{4}{5}, 0$

$y = \pm\sqrt{3}\,\frac{4}{5}$

$d = \frac{4}{5}\left[1 + 3\right]^{1/2} = \frac{8}{5} > 1$

(3) Let α be any pole such that $\alpha^3 = -1$: $\lim_{z\to\alpha} \frac{z-\alpha}{z^3+1} = \frac{1}{3\alpha^2} = -\frac{\alpha}{3}$

(4) $\int_C \frac{dz}{z^3+1} = -\frac{2\pi i}{3}\left[e^{\cancel{i\pi/3}} \cancel{+} e^{i2\pi} e^{-i\pi/3}\right] = -\frac{2\pi i}{3}\left[2\cos\pi/3\right] = \cancel{-\frac{\pi i}{6}}$

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Problems
52, 53, 54

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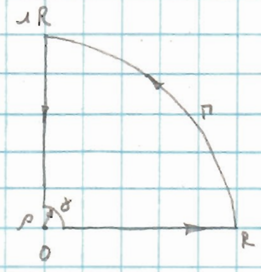
52. (1) Show $\displaystyle\int_0^\infty \frac{\cos x}{\sqrt{x}}\,dx = \int_0^\infty \frac{\sin x}{\sqrt{x}}\,dx = \sqrt{\frac{\pi}{2}}$

(2) Consider $\displaystyle\int_c z^{a-1} e^{imz}\,dz$ where c is the first quadrant.



(3) $$\int_\rho^R x^{a-1} e^{imx}\,dx + \int_\Gamma z^{a-1} e^{imz}\,dz + i\int_R^\rho (iy)^{a-1} e^{-my}\,dy + \int_\gamma z^{a-1} e^{imz}\,dz = 0$$

(4) $$\left|\int_\Gamma z^{a-1} e^{imz}\,dz\right| = \left|\int_0^{\pi/2} iRe^{i\theta}\cdot R^{a-1} e^{i(a-1)\theta} e^{imR\cos\theta} e^{-mR\sin\theta}\,d\theta\right|$$

$$\leq \left|R^a \int_0^{\pi/2} e^{-mR\sin\theta}\,d\theta\right| \leq \left|R^a \int_0^{\pi/2} e^{-2R\theta m/\pi}\,d\theta\right|$$

(since $\frac{\sin\theta}{\theta} \geq \pi/2$ in $0 \leq \theta \leq \frac{1}{2}\pi$)

$$= \frac{\pi R^{a-1}}{m}\left[e^{-mR} - 1\right] \to 0 \text{ as } R \to \infty, \text{ if } a < 1,\ m > 0.$$

(5) $$\left|\int_\gamma z^{a-1} e^{imz}\,dz\right| = \left|\rho^a \int_{\pi/2}^0 e^{-\rho m\sin\theta}\,d\theta\right| \to 0 \text{ as } \rho \to 0, \text{ if } a > 0.$$

(6) $$\therefore \int_0^\infty x^{a-1} e^{imx}\,dx - i e^{i\frac{\pi}{2}(a-1)} \int_0^\infty y^{a-1} e^{-my}\,dy = 0$$

(7) Let $u = my$, $du = m\,dy$: $\displaystyle\int_0^\infty y^{a-1} e^{-my}\,dy = \int_0^\infty \left(\frac{u}{m}\right)^{a-1} e^{-u}\,\frac{du}{m}$

$= m^{-a} \displaystyle\int_0^\infty u^{a-1} e^{-u}\,du = m^{-a}\,\Gamma(a)$ by definition of the Gamma function.

(8) $$\therefore \int_0^\infty x^{a-1} e^{imx}\,dx = e^{i\frac{\pi}{2}a}\, m^{-a}\,\Gamma(a)$$

(9) or $$\int_0^\infty x^{a-1}\cos mx\,dx = m^{-a}\,\Gamma(a)\cos\frac{\pi a}{2}$$

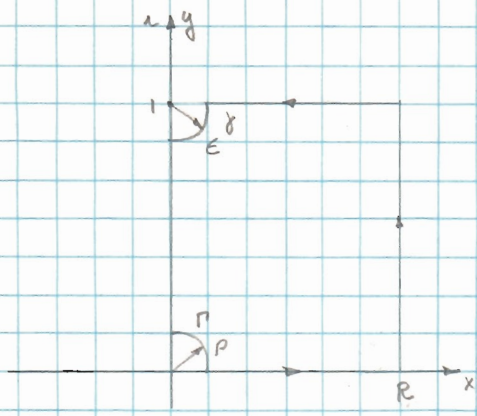
$$\int_0^\infty x^{a-1}\sin mx\,dx = m^{-a}\,\Gamma(a)\sin\frac{\pi a}{2}$$

(10) In this problem; $0 < a = 1/2 < 1$, $m = 1$, $\therefore \sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$
$m^{-a} = 1$:

$\therefore \Gamma(a) = \Gamma(\frac{1}{2}) = \sqrt{\pi}$: thus $\displaystyle\int_0^\infty \frac{\cos x}{\sqrt{x}}\,dx = \int_0^\infty \frac{\sin x}{\sqrt{x}} = \sqrt{\frac{\pi}{2}}$

53\. (1) Show $\displaystyle\int_0^\infty \frac{\sin ax}{e^{2\pi x}-1}\,dx = \frac{1}{4}\coth\frac{1}{2}a - \frac{1}{2a} \quad (a>0).$

(2) Consider $\displaystyle\int_C \frac{e^{iaz}}{e^{2\pi z}-1}\,dz$ where C is:



(3) $e^{2\pi z} = 1, \quad 2\pi z = \log 1 = i2n\pi$

$\therefore$ poles at $z = in, \; n = 0, 1, 2, 3, \ldots$

(4) $$\int_\rho^R \frac{e^{iax}}{e^{2\pi x}-1}\,dx + \int_0^1 \frac{e^{ia(R+iy)}}{e^{2\pi(R+iy)}-1}\,i\,dy$$

$$+ \int_R^\varepsilon \frac{e^{ia(x+i)}}{e^{2\pi(x+i)}-1}\,dx + \int_{1-\varepsilon}^\rho \frac{e^{-ay}}{e^{i2\pi y}-1}\,i\,dy$$

$$+ \int_\Gamma \frac{e^{iaz}}{e^{2\pi z}-1}\,dz + \int_\gamma \frac{e^{iaz}}{e^{2\pi z}-1}\,dz = 0$$

(5) Theorem: If b is the residue of a simple pole of $f(z)$ at a and if C is the arc $\theta_1 \le \arg(z-a) \le \theta_2$ of the circle $|z-a| = r$, then:

$$\lim_{r\to 0} \int_C f(z)\,dz = ib(\theta_2 - \theta_1)$$

The proof is in Phillips, p. 117. This theorem has been used before and will be used now:

(6) $$\int_\Gamma \frac{e^{iaz}}{e^{2\pi z}-1}\,dz \to ib\left(0 - \pi/2\right) = -i\frac{\pi}{2}b \quad \text{as } \rho\to 0$$

$iR(\theta)$

$$b = \lim_{z\to 0}\left\{\frac{z\,e^{iaz}}{e^{2\pi z}-1}\right\} = \lim_{z\to 0}\left\{\frac{1}{2\pi e^{2\pi z}}\right\} = \frac{1}{2\pi}$$

$$\therefore \int_\Gamma \to -i\frac{1}{4} \quad \text{as } \rho \to 0$$

(7) $$\int_\gamma \to ib\left(-\pi/2 - 0\right) = -i\frac{\pi}{2}b \quad \text{as } \varepsilon\to 0$$

$$b = \lim_{z\to i}\left\{\frac{z\,e^{iaz}}{e^{2\pi z}-1}\right\} = \frac{1}{2\pi}e^{-a}, \;\therefore \int_\gamma \to -i\frac{1}{4}e^{-a} \quad \text{as } \varepsilon\to 0$$

for $a>0$

(8) $$\left|\frac{e^{iaR}\,e^{-ay}}{e^{2\pi R}\,e^{i2\pi y}-1}\right| \le \frac{e^{-ay}}{1-e^{2\pi R}} \to 0 \quad \text{as } R\to\infty$$

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53. Continued:

(9) $$-\int_0^1 \frac{e^{-ay}}{e^{i2\pi y}-1}\,dy = -\frac{1}{2}\int_0^1 \frac{e^{-ay}\left(\cos 2\pi y - 1 - i\sin 2\pi y\right)}{1-\cos 2\pi y}\,dy$$

$$= \frac{1}{2}\int_0^1 e^{-ay}\,dy + \frac{i}{2}\int_0^1 e^{-ay}\cot \pi y\,dy$$

$$= -\frac{1}{2a}\left[e^{-a}-1\right] + \frac{i}{2}\int_0^1 e^{-ay}\cot \pi y\,dy$$

(10) Now, taking imaginary parts of (4):

$$(1-e^{-a})\int_0^\infty \frac{\sin ax\,dx}{e^{2\pi x}-1} + \frac{1}{2a}\left[1-e^{-a}\right] - \frac{1}{4}\left(1+e^{-a}\right) = 0$$

(11) $$\therefore \int_0^\infty \frac{\sin ax}{e^{2\pi x}-1}\,dx = \frac{1}{4}\coth\frac{a}{2} - \frac{1}{2a} \quad ; \quad a>0$$

---

54. i. (1) $$\frac{z}{e^z-1} + \frac{z}{2} = \frac{2z + ze^z - z}{2(e^z-1)} = \frac{z}{2}\left(\frac{e^z+1}{e^z-1}\right)$$

$$= \frac{z}{2}\left(\frac{e^{z/2}+e^{-z/2}}{e^{z/2}-e^{-z/2}}\right) = \frac{1}{2}z\coth\frac{1}{2}z$$

(2) $$\frac{1}{2}z\coth\frac{1}{2}z = \frac{1}{2}z\left(\frac{\cosh 1/2\, z}{\sinh 1/2\, z}\right)$$

Let $z \to -z$, then clearly $-\frac{1}{2}z\coth(-\frac{1}{2}z) = \frac{1}{2}z\coth\frac{1}{2}z$ and is thus an even function of z.

ii. (1) $$\frac{z}{e^z-1} = \frac{1}{2}z\coth\frac{1}{2}z - \frac{1}{2}z$$

(2) We expand $\frac{1}{2}z\coth\frac{1}{2}z = f(z)$ in a Maclaurin series noting that only even powers of z will appear as $f(z)$ is even.

(3) Now $$\frac{z}{2}\coth\frac{1}{2}z = \frac{z}{2}\,\frac{\cosh\frac{1}{2}z}{\sinh\frac{1}{2}z} = \frac{\frac{z}{2} + \frac{(z/2)^3}{2!} + \frac{(z/2)^5}{4!} + \cdots}{\frac{z}{2} + \frac{(z/2)^3}{3!} + \frac{(z/2)^5}{5!} + \cdots}$$

providing $|z| < 2\pi$, otherwise $\sinh\frac{1}{2}(i2\pi) = 0$ with disaster.

(4) $$\frac{z}{2}\coth\frac{z}{2} = \sum_{n=0}^{\infty} f^{(2n)}(0)\,\frac{z^{2n}}{(2n)!}$$

from the discussion in equation (2).

(5)
$$\begin{array}{r|l} & 1 + \frac{1}{3}\left(\frac{z}{2}\right)^2 - \frac{1}{45}\left(\frac{z}{2}\right)^4 + \cdots \\ \hline \frac{z}{2} + \left(\frac{z}{2}\right)^3/3! + \left(\frac{z}{2}\right)^5/5! & \frac{z}{2} + \left(\frac{z}{2}\right)^3/2! + \left(\frac{z}{2}\right)^5/4! + \cdots \\ & \frac{z}{2} + \left(\frac{z}{2}\right)^3/3! + \left(\frac{z}{2}\right)^5/5! \\ \hline & \left(\frac{1}{2!}-\frac{1}{3!}\right)\left(\frac{z}{2}\right)^3 + \left(\frac{1}{4!}-\frac{1}{5!}\right)\left(\frac{z}{2}\right)^5 \\ & \frac{1}{3}\left(\frac{z}{2}\right)^3 + \frac{1}{3!\,3}\left(\frac{z}{2}\right)^5 \\ \hline & -\frac{1}{45}\left(\frac{z}{2}\right)^5 \end{array}$$

(6) It is apparent that the series will alternate from this point. therefore we have:

$$\frac{z}{2}\coth\frac{z}{2} = 1 + \sum_{n=1}^{\infty}(-1)^{n-1}\left|f^{(2n)}(0)\right|\frac{z^{2n}}{(2n)!}$$

$$= 1 + \sum_{n=1}^{\infty}(-1)^{n-1} B_n \frac{z^{2n}}{(2n)!}$$

(7) $$\frac{z}{e^z-1} = 1 - \tfrac{1}{2}z + \sum_{n=1}^{\infty}(-1)^{n-1} B_n \frac{z^{2n}}{(2n)!}$$

where $B_1 = 2\cdot\frac{1}{3}\cdot\frac{1}{4} = \frac{1}{6}$

It looks as though the coefficients B_n are the Bernoullian numbers!

iii (1) Consider $f(z) = \dfrac{z}{e^z-1}$: this is a meromorphic function with simple poles with none at the origin.

These poles are at $e^z = 1$, $z = \log 1 = i2p\pi$

$p = \pm1, \pm2, \pm3, \ldots$

thus the poles are ordered.

The residues at these poles are:

$$R = \lim_{z\to i2p\pi}\left\{\frac{(z-i2p\pi)\,z}{e^z-1}\right\} = i2p\pi \lim_{z\to i2p\pi}\left\{\frac{1}{e^z}\right\}$$

$$= i2p\pi$$

(2) Consider the integral $I = \dfrac{1}{2\pi i}\displaystyle\int_{C_m}\frac{f(w)}{w^2(w-z)}\,dw$

with residues $\dfrac{b_p}{a_p^2(a_p-z)}$; at $w=z$, $\dfrac{f(z)}{z^2}$;

and at $w=0$, $-\dfrac{f(0)}{z^2} - \dfrac{f'(0)}{z}$

Problems 52, 53, 54
Continued

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Problem 54
Continuation:

iii (2) Therefore:

$$\frac{1}{2\pi i}\int_{C_m} \frac{f(w)}{w^2(w-z)}\,dw = \frac{f(0)}{z^2} + \frac{f'(0)}{z} + \sum_{p=1}^{m} \frac{b_p}{a_p^2(z-a_p)} - \frac{f(z)}{z^2}$$

(3) $$\left| \int \frac{\frac{w}{e^w-1}}{w^2(w-z)}\,dz \right| \le \frac{L_m}{(2\pi)^2 R_m^2 (R_m - |z|)} \max |f(w)|$$

$$= \frac{2\pi R_m \cdot R_m}{(2\pi)^2 R_m^2 (R_m - |z|)}$$ if we choose as a contour a circle of radius R_m

$\rightarrow 0$ as $m \rightarrow \infty$

or $$f(z) = f(0) + z f'(0) + z^2 \sum_{p=1}^{\infty} \frac{b_p}{a_p^2 (z - a_p)}$$

(4) Now $f(0) = 1$

$$\lim_{z \to 0} f'(z) = \frac{e^w - 1 - we^w}{(e^w-1)^2} \longrightarrow \frac{e^w - e^w - we^w}{2(e^w-1)e^w} = -\frac{1}{2}$$

(5) Substituting:

$$\frac{z}{e^z-1} = 1 - \frac{1}{2}z + z^2 \left\{ \sum_{p=1}^{\infty} \frac{(2p\pi i)}{(2p\pi i)^2} \frac{1}{(z - i2\pi p)} + \sum_{p=1}^{\infty} \frac{(-2p\pi i)}{(-2p\pi i)^2} \frac{1}{(z + i2\pi p)} \right\}$$

$$= 1 - \frac{1}{2}z + z^2 \sum_{p=1}^{\infty} \frac{2}{z^2 + 4\pi^2 p^2} \quad ; \quad z \neq 2p\pi i$$

iv (1) Consider $$\frac{x}{1+x} = x - x^2 + x^3 - \cdots + (-1)^{m-1} x^m + \frac{(-1)^m x^{m+1}}{1+x}$$

and $$\frac{z^2}{z^2 + 4\pi^2 p^2} = \frac{\left(\frac{z^2}{4\pi^2 p^2}\right)}{\left(\frac{z^2}{4\pi^2 p^2}\right) + 1}$$

$$= \left(\frac{z^2}{4\pi^2 p^2}\right) - \left(\frac{z^2}{4\pi^2 p^2}\right)^2 + \cdots + (-1)^{n-1} \frac{z^{2n}}{(2\pi p)^{2n}} +$$

(2) Therefore, the coefficient of the z^{2n} term is:

$$(-1)^{n-1}\, 2 \sum_{p=1}^{\infty} \frac{1}{(2\pi p)^{2n}}$$

(3) We now compare with the identical term in the Taylor series expansion and get:

$$\frac{B_n}{(2n)!} = \frac{2}{(2\pi)^{2n}} \sum_{p=1}^{\infty} \frac{1}{p^{2n}} = \frac{2}{(2\pi)^{2n}} S_{2n}$$

where $S_k = \sum_{p=1}^{\infty} \frac{1}{p^k}$, $k > 1$

Problems 55, 56, 57

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55. (1) It is shown in the statement of the problem that if one writes:

$$\frac{x}{e^x - 1} = 1 + A_1 x + A_2 \frac{x^2}{2!} + \dots + \frac{A_N x^n}{n!} + \dots$$

where $A_{2n} = (-1)^{n-1} B_n$, $A_{2n+1} = 0$, $A_1 = -\frac{1}{2}$

we can symbolize A_n by A^n and thus write:

$$\frac{x}{e^x - 1} = e^{Ax} \quad ; \quad x = e^{(A+1)x} - e^{Ax}$$

which upon equating coefficients gives

$$A + 1 - A = 1$$

$$(A+1)^2 - A^2 = 0$$

$$(A+1)^n - A^n = 0$$

(2) For B_1: $(A+1)^3 - A^3 = 0$

or $A_3 + 3A_2 + 3A_1 + 1 - A_3 = 0$; $3A_2 - \frac{3}{2} + 1 = 0$, $A_2 = \frac{1}{6}$

$A_2 = B_1 = \frac{1}{6}$, $\therefore$ $B_1 = \frac{1}{6}$

(3) Consider $f(z) = z^n (z-1)^{n+1}$:

Form $f(z+1) - f(z) = z^n \left[z(z+1)^n - (z-1)^{n+1} \right]$ and expand in a binomial series:

$$z^{2n+1} + \binom{n}{1} z^{2n} + \binom{n}{2} z^{2n-1} + \binom{n}{3} z^{2n-2} + \dots + \binom{n}{2m} z^{2n-2m+1}$$

$$+ \binom{n}{2m+1} z^{2n-2m} + \dots + z^{n+1}$$

$$- z^{2n+1} + \binom{n+1}{1} z^{2n} - \binom{n+1}{2} z^{2n-1} + \binom{n+1}{3} z^{2n-2}$$

$$- \dots + (-1)^{2m-1} \binom{n+1}{2m} z^{2n-2m+1} + (-1)^{2m} \binom{n+1}{2m+1} z^{2n-2m} + \dots \pm z^n$$

(4) Combining terms:

$$\underbrace{\left[\binom{n}{1}+\binom{n+1}{1}\right]}_{2n+1} z^{2n} + \underbrace{\left[\binom{n}{2}-\binom{n+1}{2}\right]}_{\frac{n(n-1)-(n+1)n}{2!}} z^{2n-1} + \underbrace{\left[\binom{n}{3}+\binom{n+1}{3}\right]}_{\frac{n(n-1)(n-2)+(n+1)(n)(n-1)}{3!}} z^{2n-2} + \cdots$$

$$+ \underbrace{\left[\binom{n}{2m}-\binom{n+1}{2m}\right]}_{\frac{n(n-1)\cdots(n-2m+1)\,-\,(n+1)(n)\cdots(n-2m+2)}{(2m)!}} z^{2n-2m+1} + \underbrace{\left[\binom{n}{2m+1}+\binom{n+1}{2m+1}\right]}_{\frac{n(n-1)\cdots(n-2m)\,+\,(n+1)(n)\cdots(n-2m+1)}{(2m+1)!}} z^{2n-2m} + \cdots + z^{n+1} \pm z^{n}$$

or $$(2n+1)\,z^{2n} - \frac{2n}{2!}\,z^{2n-1} + \frac{(2n-1)(n)(n-1)}{3!}\,z^{2n-2} - \cdots$$

$$\cdots - \frac{(2m)(n)(n-1)\cdots(n-2m+2)}{(2m)!}\,z^{2n-2m+1} + \frac{(2n-2m+1)(n)(n-1)\cdots(n-2m+1)}{(2m+1)!}\,z^{2n-2m} + \cdots$$

(5) Now substitute $A \to z$, bringing down power, forming $f(A+1)-f(A)=0$, recalling that:

$$A_{2n} = (-1)^{n-1} B_n \quad \text{and} \quad A_{2n+1} = 0$$

$$(2n+1)B_n - \frac{(2n-1)(n)(n-1)}{3!}B_{n-1} + \cdots + (-1)^m \frac{(2n-2m+1)(n)(n-1)\cdots(n-2m+1)}{(2m+1)!}B_{n-m}$$

$$\pm B_{\frac{n+1}{2}} \pm B_{\frac{n}{2}} = 0$$, the end terms being decided by whether or not n is odd or even.

(6) For $n=2$: $5B_2 - B_1 = 0$; $B_2 = \frac{B_1}{5} = \frac{1}{30}$

For $n=3$: $7B_3 - 5B_2 = 0$; $B_3 = \frac{5}{7}B_2 = \frac{1}{42}$

For $n=4$: $9B_4 - 14B_3 + B_2 = 0$; $9B_4 - \frac{1}{3} + \frac{1}{30}$; $B_4 = \frac{1}{30}$

For $n=5$: $11B_5 - 30B_4 + 7B_3 = 0$; $11B_5 - 1 + \frac{1}{6} = 0$, $B_5 = \frac{5}{66}$

(7) From part iv of problem 54 or the result of problem 57:

$$\sum_{p=1}^{\infty} \frac{1}{p^{2m}} = \frac{(2\pi)^{2m} B_m}{2\,(2m)!}$$

For $m=1$: $$\sum_{p=1}^{\infty}\frac{1}{p^2} = \frac{(2\pi)^2\cdot\frac{1}{6}}{2\cdot 2} = \frac{\pi^2}{6}$$

For $m=2$: $$\sum_{p=1}^{\infty}\frac{1}{p^4} = \frac{16\pi^4\cdot\frac{1}{30}}{2\cdot4\cdot3\cdot2} = \frac{\pi^4}{90}$$

For $m=3$: $$\sum_{p=1}^{\infty}\frac{1}{p^6} = \frac{64\pi^6\cdot\frac{1}{42}}{2\cdot6\cdot5\cdot4\cdot3\cdot2} = \frac{\pi^6}{945}$$

For $m=4$: $$\sum_{p=1}^{\infty}\frac{1}{p^8} = \frac{256\pi^8\cdot\frac{1}{30}}{2\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2} = \frac{\pi^8}{9450}$$

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Problems 55, 56, 57
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56. i (1) It was shown in problem 54 that:

$$\frac{z}{e^z-1} + \frac{1}{2}z = \frac{1}{2}z \coth \frac{1}{2}z = 1 + \sum_{n=1}^{\infty} (-1)^{n-1} B_n \frac{z^{2n}}{(2n)!}$$

(2) Make the substitution: $z = i2x$,

$$ix \coth ix = 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} B_n (2)^{2n} x^{2n} (i)^{2n}}{(2n)!}$$

or $$x \cot x = 1 - \sum_{n=1}^{\infty} \frac{B_n (2)^{2n} x^{2n}}{(2n)!}, \quad |x| < \pi$$

ii (1) Now $\cot 2x = \dfrac{\cot^2 x - 1}{2 \cot x}$ from CRC tables ; $2 \cot 2x = (\cot^2 x - 1) \tan x$

$= \dfrac{1}{\tan x} - \tan x$; $\therefore \tan x = \cot x - 2 \cot 2x$

or $x \tan x = x \cot x - 2x \cot 2x$

(2) From part i:

$$x \tan x = 1 - \sum_{n=1}^{\infty} \frac{B_n (2)^{2n} x^{2n}}{(2n)!} - 1 + \sum_{n=1}^{\infty} \frac{B_n (2)^{2n} (2)^{2n} x^{2n}}{(2n)!}$$

$$= \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1) B_n x^{2n}}{(2n)!}$$

(3) $$\therefore \tan x = \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1) B_n x^{2n-1}}{(2n)!}, \quad |x| < \frac{\pi}{2}$$

iii (1) Now $\tan \frac{1}{2}x = \dfrac{1 - \cos x}{\sin x}$, from CRC tables

$= \dfrac{\sin x}{1 + \cos x}$; $\cot \frac{1}{2}x = \dfrac{1 + \cos x}{\sin x} = \operatorname{cosec} x + \cot x$

$\therefore x \operatorname{cosec} x = 2 \cdot \frac{x}{2} \cot \frac{1}{2}x - x \cot x$

(2) from part i:

$$x \operatorname{cosec} x = 2 - 2\sum_{n=1}^{\infty} \frac{B_n\, x^{2n}}{(2n)!} - 1 + \sum_{n=1}^{\infty} \frac{B_n\, 2^{2n}\, x^{2n}}{(2n)!}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{B_n\,(2^{2n}-2)\, x^{2n}}{(2n)!}, \quad |x| < \pi$$

iv. (1) Recall: $\int \cot x\, dx = \log \sin x$

(2) From i:

$$\cot x = \frac{1}{x} - \sum_{n=1}^{\infty} \frac{2^{2n}\, B_n\, x^{2n-1}}{(2n)!}$$

(3) $$\int \cot x\, dx = \log \sin x = \log x - \sum_{n=1}^{\infty} \frac{2^{2n}\, B_n\, x^{2n}}{2n\,(2n)!}$$

(4) or: $$\log \frac{\sin x}{x} = -\sum_{n=1}^{\infty} \frac{2^{2n-1}\, B_n\, x^{2n}}{n\,(2n)!}$$

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57. (1) Consider $\int_c \frac{\pi \cot \pi z}{z^{2m}} dz = 2\pi i \sum R$

(2) The integrand has simple poles at $z = n$, $n = \pm 1, \pm 2, \ldots$

and a pole of order $2m+1$ at $z = 0$

The residue at the simple poles is:

$$\lim_{z \to n} \left\{ \frac{(z-n)\pi \cot \pi z}{z^{2m}} \right\} = \frac{1}{n^{2m}} \lim_{z \to n} \left\{ \frac{\pi (z-n)}{\tan \pi z} \right\}$$

$$= \frac{1}{n^{2m}} \lim_{z \to n} \left\{ \frac{\pi}{\pi \sec^2 \pi z} \right\} = \frac{1}{n^{2m}}$$

(3.) The sum of all the residues at these poles within C is:

$$\sum_{n=-p}^{p}{}' \frac{1}{n^{2m}} = 2 \sum_{n=1}^{p} \frac{1}{n^{2m}} \quad \text{as } \frac{1}{n^{2m}} \text{ is even.}$$

(4) Now it was shown in the previous problem that:

$$x \cot x = 1 - \sum_{n=1}^{\infty} \frac{2^{2n} B_n}{(2n)!} x^{2n} \qquad |x| < \pi$$

or

$$\pi z \cot \pi z = 1 - \sum_{n=1}^{\infty} \frac{2^{2n} B_n}{(2n)!} (\pi z)^{2n}$$

Then:

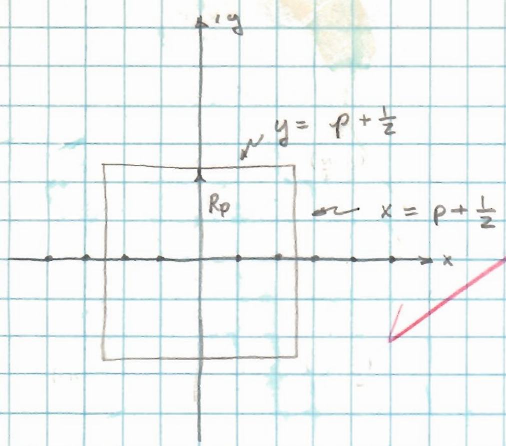
$$\frac{\pi \cot \pi z}{z^{2m}} = \frac{1}{z^{2m+1}} - \sum_{n=1}^{\infty} \frac{2^{2n} B_n}{(2n)!} \pi^{2n} z^{2n-2m-1}$$

and the residue at $z = 0$ is when $n = m$ and is:

$$- \frac{(2\pi)^{2m} B_m}{(2m)!}$$

(5) Now consider the contour to be the square with corners:

$$(p + \tfrac{1}{2})(\pm 1 \pm i)$$



(6) $$\left| \int_C \frac{\pi \cot \pi z \, dz}{z^{2m}} \right| \le \frac{\pi L}{R_p^{2m}} \, |\cot \pi z|_{max}$$

$$= \frac{8\pi R_p}{R_p^{2m}} \, |\cot \pi z|_{max}$$

To show that $\cot \pi z$ is bounded on C, it is sufficient to show that $\sin \pi z$ does not vanish on C.

Consider: $|\sin \pi z| = (\cosh^2 \pi y - \cos^2 x\pi)^{1/2}$

$= \cosh \pi y \ge 1$ on $x = p + \frac{1}{2}$

and $|\sin \pi z| = (\cosh^2 \pi(p+\frac{1}{2}) - \cos^2 \pi x)^{1/2} \ge 0$ on $y = p + \frac{1}{2}$

$\therefore$ $|\sin \pi z|$ does not vanish on C.

and $\left| \int_C \right| \to 0$ as $p \to \infty$

(7) Then $$0 = 2 \sum_{n=1}^{\infty} \frac{1}{n^{2m}} - \frac{(2\pi)^{2m} B_m}{(2m)!}$$

or $$\sum_{n=1}^{\infty} \frac{1}{n^{2m}} = \frac{(2\pi)^{2m} B_m}{2\,(2m)!}$$

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Problems
58, 60, 61

$30/30$

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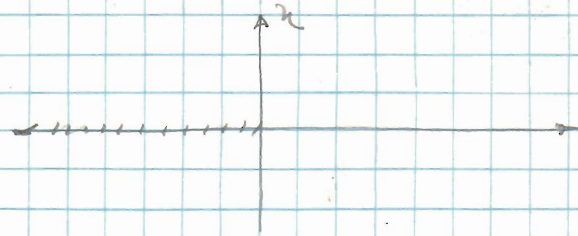
58. a. (1) $w = \varphi + i\psi$, with $\varphi = \beta x^{1/2}$ on $y = 0,\ x \geq 0$.

(2) Consider $\zeta = z^{1/2} = \xi + i\eta$



$|\text{grad}\ \varphi| = \left|\dfrac{dw}{dz}\right|$

$= \dfrac{\beta}{2r^{1/2}} \to 0$ as $r \to \infty$

(3) $w = \beta\zeta = \beta z^{1/2}$, $\therefore\ \varphi = \beta\xi$

(4) $\zeta = z^{1/2} = (x + iy)^{1/2} = \sqrt[4]{x^2+y^2}\ \exp\left\{i\,\tfrac{1}{2}\tan^{-1}\dfrac{y}{x}\right\}$

$\sqrt[4]{x^2+y^2}\cos\tfrac{1}{2}\theta + i\sqrt[4]{x^2+y^2}\sin\tfrac{1}{2}\theta$

(5) $\xi = \sqrt[4]{x^2+y^2}\left\{\dfrac{1+\cos\theta}{2}\right\}^{1/2} = \sqrt[4]{x^2+y^2}\left\{\dfrac{1+\dfrac{x}{\sqrt{x^2+y^2}}}{2}\right\}^{1/2}$

$= \left\{\dfrac{\sqrt{x^2+y^2}+x}{2}\right\}^{1/2}$

(6) $\varphi = \beta\left\{\dfrac{\sqrt{x^2+y^2}+x}{2}\right\}^{1/2}$ which is a solution and satisfies Laplace's equation because w is analytic in the z plane (after cut).

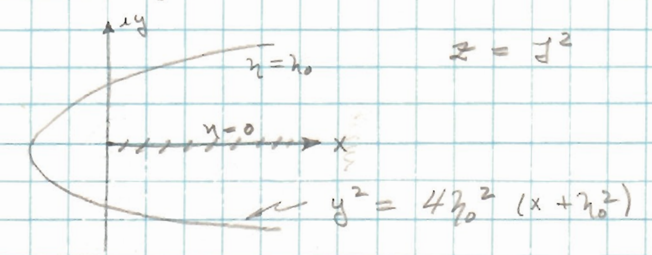
---

b. (1) $w = \varphi + i\psi$ with

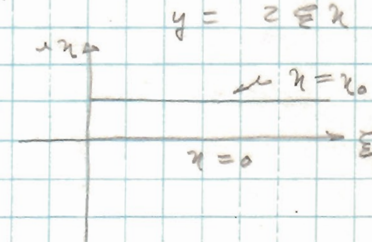
$\psi = Uy$ on the curve in the z plane:

$y^2 = 4\eta_0^2\,(x + \eta_0^2)$

(2) Evidently $\zeta = z^{1/2} = \xi + i\eta$, with $x = \xi^2 - \eta^2$
$y = 2\xi\eta$



$z = \zeta^2$



(3) on the curve, x is given by $x = \xi^2 - \eta^2$
Then $y^2 = 4\eta_0^2\xi^2$, or $y = 2\eta_0\xi$

(4) Therefore $\psi = Uy = 2\eta_0 U \xi$

(5) Since $\zeta = z^{1/2}$, $\xi = \left\{ \dfrac{\sqrt{x^2+y^2} + x}{2} \right\}^{1/2}$

(6) $\psi = 2\eta_0 U \left\{ \dfrac{\sqrt{x^2+y^2} + x}{2} \right\}^{1/2}$

(7) we can con construct ω such that $\psi = 2\eta_0 U \xi$

$$\omega = \varphi + i\psi = i 2\eta_0 U \xi - 2\eta_0 U \eta$$

$$= i 2\eta_0 U \{ \xi + i\eta \} = i 2\eta_0 U z^{1/2}$$

(8) Thus (6) is a solution as it satisfies the boundary conditions and Laplaces equation because ω is a regular function in the cut z plane.

(9) $|\operatorname{grad} \psi| = \left|\dfrac{d\omega}{dz}\right| = O\left(\dfrac{1}{r^{1/2}}\right) \to 0$ as $r \to \infty$

Problems
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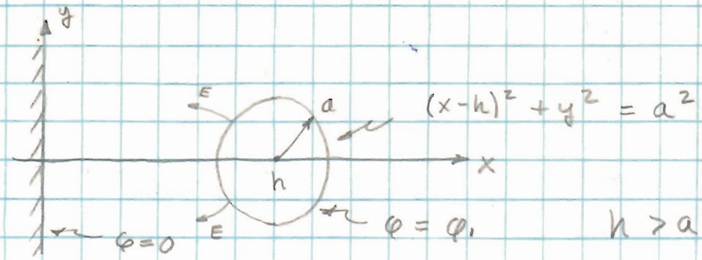
60.



(1) $\omega = \varphi + i\psi$

(2) We have for the defmition of capacitance:

$$C = -\frac{1}{4\pi} \frac{[\psi]}{\varphi_1}$$

(3) Choose the transformation $\omega = 2q \log \frac{z+\lambda}{z-\lambda}$ why?

where λ is a real constant.

(4) Consider the bilinear transformation $\zeta = \frac{z+\lambda}{z-\lambda}$ operating on the region of the z plane:

$$|z-h| > a \;, \quad \operatorname{Re} z \geq 0$$

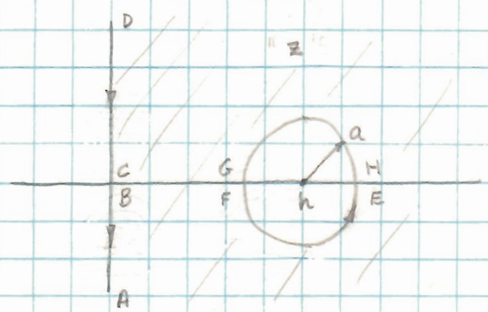
(5) $\zeta = \frac{az+b}{cz+d}$; $\zeta = \frac{z+\lambda}{z-\lambda}$

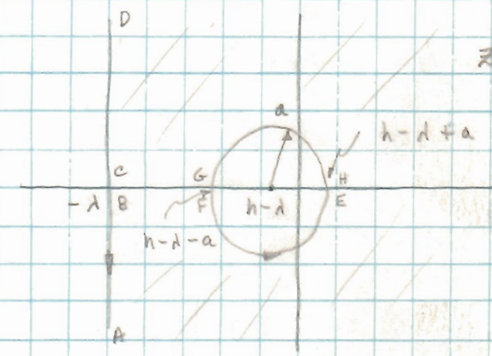
$z' = cz + d$; $z' = z - \lambda$

$z'' = 1/z'$; $z'' = 1/z'$

$\zeta = \left(\frac{a}{c}\right) + \frac{(bc-ad)}{c} z''$; $\zeta = 1 + 2\lambda z''$

(6)

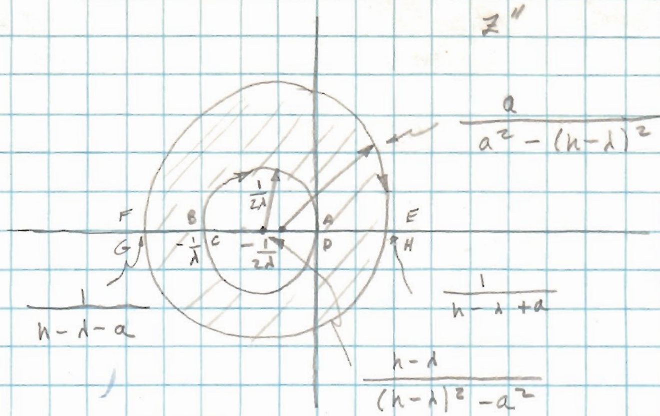


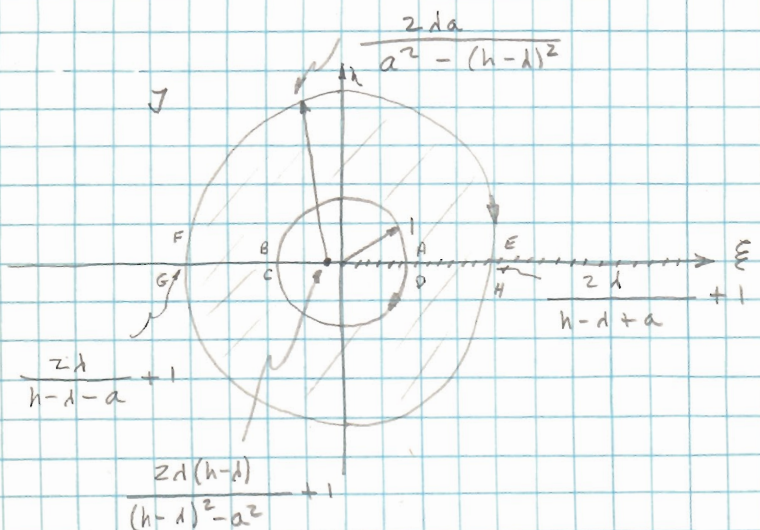


(7) $z'' = \frac{1}{z'}$

$r'' = \frac{1}{r'}$

$\theta'' = -\theta'$





For the purpose of forming ∥ lines in the ω-plane we make the circles concentric by putting

$$\frac{2\lambda(h-\lambda)}{(h-\lambda)^2 - a^2} + 1 = 0$$

or $\lambda^2 = h^2 - a^2$

$$\lambda = \pm\sqrt{h^2 - a^2}$$

(8) We then have for the radius of the outer concentric circle:

$$R = \sqrt{\xi^2 + \eta^2} = \frac{\pm a\sqrt{h^2 - a^2}}{\pm h\sqrt{h^2-a^2} - (h^2 - a^2)}$$

$$= \frac{1}{\frac{h}{a} \mp \sqrt{\left(\frac{h}{a}\right)^2 - 1}}$$

(9) Now, $\omega = \varphi + i\psi = 2q \log \zeta = 2q \log R + i\,2q \tan^{-1}\frac{\eta}{\xi}$

in the cut ζ plane.

$$\varphi = 2q \log R = -2q \log\left\{\frac{h}{a} \mp \sqrt{\left(\frac{h}{a}\right)^2 - 1}\right\}$$

$= -2q \cosh^{-1}\left(\frac{h}{a}\right)$ for the outer circle

$\varphi = 2q \log 1 = 0$ for the inner circle.

$\psi = 2q \tan^{-1}\frac{\eta}{\xi}$. As one describes an arc around the cut plane, ψ goes from $-4q\pi$ to 0.

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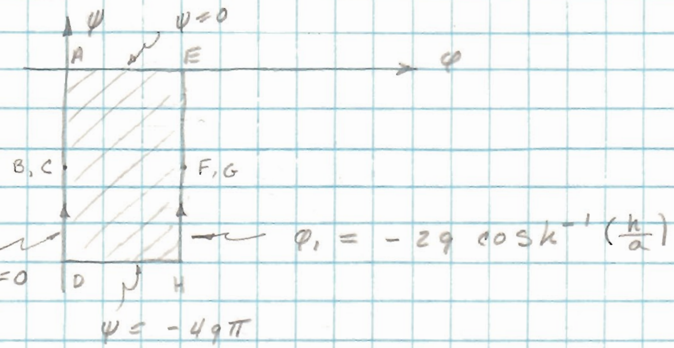
(10) We have in the w plane:



(11) $$C = -\frac{1}{4\pi}\,\frac{[\psi]}{\varphi_1}$$

$[\psi]$ around the cylindrical conductor is $0-(-4q\pi)$ $= 4q\pi$

$$\therefore \quad C = \left[2\cosh^{-1}\left(\frac{h}{a}\right)\right]^{-1} \quad \text{taking } K=1.$$

(12) We recall the power series expansion of $\cosh^{-1} u$:

$$\cosh^{-1} u = \log 2u - \frac{1}{2}\cdot\frac{1}{2u^2} - \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{4u^2} - \cdots, \quad u^2 \gg 1$$

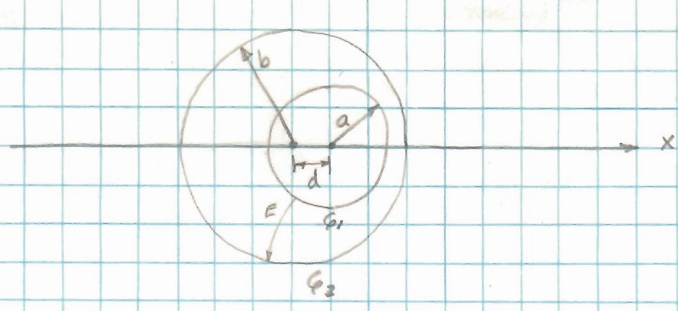
(13) If $\frac{h}{a} \gg 1$, it is clear that, to the second order

$$\cosh^{-1}\left(\frac{h}{a}\right) \approx \log\frac{2h}{a} - \frac{a^2}{4h^2}$$

and $$C \approx \left[2\log\frac{2h}{a} - \frac{a^2}{2h^2}\right]^{-1}, \quad \frac{h}{a} \gg 1$$

61. (1)

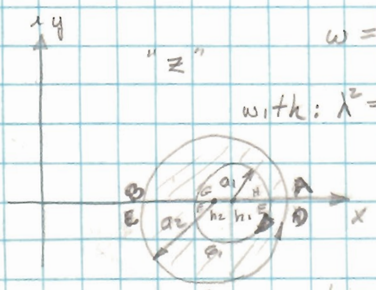
1D

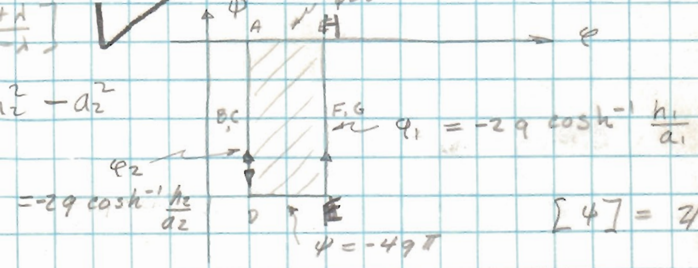


It is obvious that problem 60 was just a special case of this problem as the imaginary axis is merely a circle of infinite radius. When the outer circle has a finite radius, we can make it concentric with the inner in the ζ plane as done previously and we will have a line of constant φ removed from the imaginary axis of the ω plane. We can then write by inspection:

$$\omega = 2q \log\left[\frac{z+\lambda}{z-\lambda}\right]$$

with: $\lambda^2 = h_1^2 - a_1^2 = h_2^2 - a_2^2$





$[\psi] = 4q\pi$

$|h_1 - h_2| \equiv d$

(2) $$C = \frac{1}{4\pi}\frac{[\psi]}{\varphi_2 - \varphi_1} = \left[-2\cosh^{-1}\left(\frac{h_2}{a_2}\right) + 2\cosh^{-1}\left(\frac{h_1}{a_1}\right)\right]^{-1}$$

(3) $$-\cosh^{-1}\left(\frac{h_2}{a_2}\right) + \cosh^{-1}\left(\frac{h_1}{a_1}\right) = \log\left\{\frac{\left(\frac{h_1}{a_1}\right) \pm \sqrt{\left(\frac{h_1}{a_1}\right)^2 - 1}}{\left(\frac{h_2}{a_2}\right) \pm \sqrt{(h_2/a_2)^2 - 1}}\right\}$$

(4) $$\{\ \} = \frac{h_1 h_2}{a_1 a_2} - \sqrt{\frac{(h_1^2 - a_1^2)(h_2^2 - a_2^2)}{a_1^2 a_2^2}} \pm \left(\frac{h_2}{a_2}\right)\sqrt{\frac{h_1^2 - a_1^2}{a_1^2}} \mp \left(\frac{h_1}{a_1}\right)\sqrt{\frac{h_2^2 - a_2^2}{a_2^2}}$$

$$= \frac{1}{a_1 a_2}\left\{h_1 h_2 - \lambda^2 \pm h_2\lambda \mp h_1\lambda\right\}$$

$$= \frac{1}{a_1 a_2}\left\{\underbrace{h_1 h_2 - h_1^2}_{h_1(h_2 - h_1) = -h_1 d = -\frac{1}{2}(d^2 - a_2^2 + a_1^2)} + a_1^2 + d\sqrt{h_1^2 - a_1^2}\right\}; \quad h_1 = \frac{d^2 - a_2^2 + a_1^2}{2d}$$

$$= \left\{\frac{a_1^2 + a_2^2 - d^2}{2a_1 a_2} + \left[\frac{h_1^2 d^2 - a_1^2 d^2}{a_1^2 a_2^2}\right]^{1/2}\right\}; \quad [\ ]^{1/2} = \left[\frac{d^4 - 2a_2^2 d^2 + 2a_1^2 d^2 - 4a_1^2 a_2^2 + a_2^4 + a_1^4 - 4a_1^2 d^2}{4a_1^2 a_2^2}\right].$$

$$\therefore [\ ]^{1/2} = \left[\left(\frac{a_1^2 + a_2^2 - d^2}{2a_1 a_2}\right)^2 - 1\right]^{1/2}$$

(5) $$\cosh^{-1}\left(\frac{h_1}{a_1}\right) - \cosh^{-1}\left(\frac{h_2}{a_2}\right) = \log\left\{\left(\frac{a_1^2 + a_2^2 - d^2}{2a_1 a_2}\right) + \left[\left(\frac{a_1^2 + a_2^2 - d^2}{2a_1 a_2}\right)^2 - 1\right]^{1/2}\right\}$$

$$= \cosh^{-1}\left\{\frac{a_1^2 + a_2^2 - d^2}{2a_1 a_2}\right\}$$, in our problem $a_2 = b$, $a_1 = a$

(6) $$\therefore\ C = \left[2\cosh^{-1}\left\{\frac{a^2 + b^2 - d^2}{2ab}\right\}\right]^{-1}.$$

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59. (1) Given $z = e^{\zeta}$, $\text{Log}\, z = \zeta$

(2) $e^{\zeta} = e^{\xi + i\eta} = e^{\xi}(\cos\eta + i\sin\eta)$

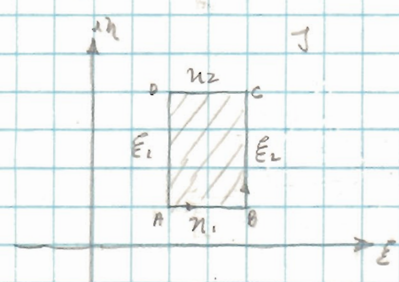
or $x = e^{\xi}\cos\eta$
$y = e^{\xi}\sin\eta$

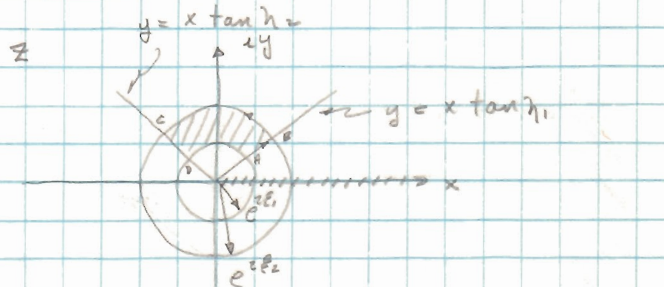
(3) For ξ = constant $= \xi_0$, $x^2 + y^2 = e^{2\xi_0}$ (circles)

For η = constant $= \eta_0$, $\frac{x}{y} = \tan\eta_0$ (radial lines)

(4) $\text{Log}\, z = \log r + i(\theta + 2n\pi)$, $n = 0, \pm 1, \pm 2, \ldots$

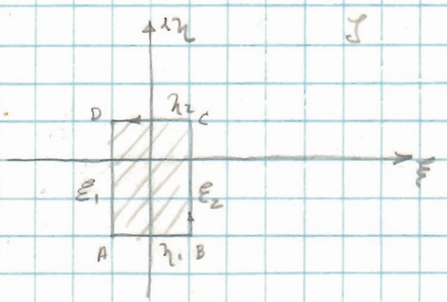
What branch we choose for the principle value will depend on the positions of ξ_0, η_0.

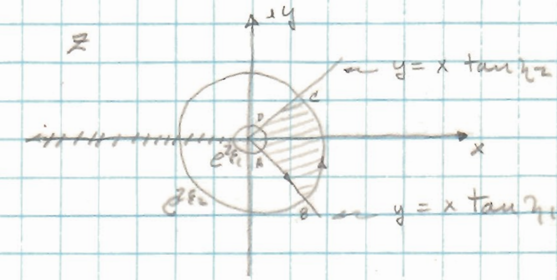




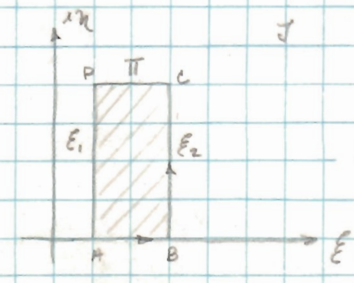
We stipulate $\eta_2 - \eta_1 \leq 2\pi$

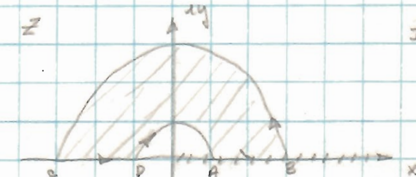
We cut the z plane to keep e^{ζ} single-valued





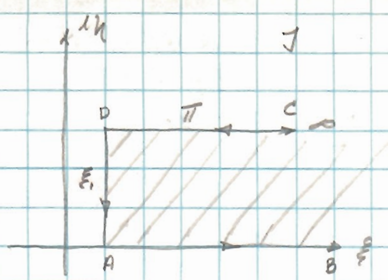
One places the cut such that it does not pass thru the region mapped into the z plane, thereby keeping the one-to-one correspondence between planes

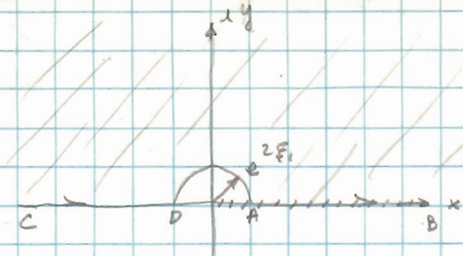




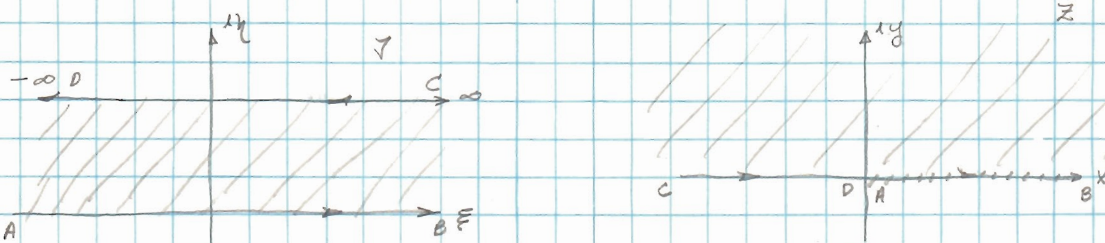
Inner radius $= e^{2\xi_1}$
Outer $= e^{2\xi_2}$

Cut could be to $-\infty$ also.



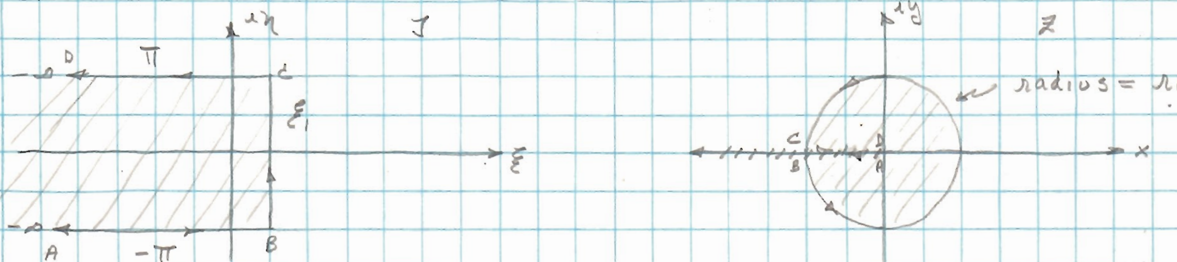


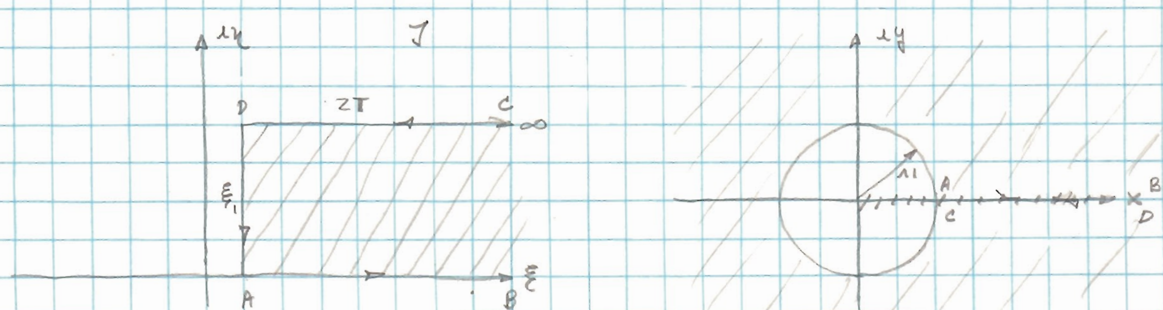
Goes into upper half of z plane

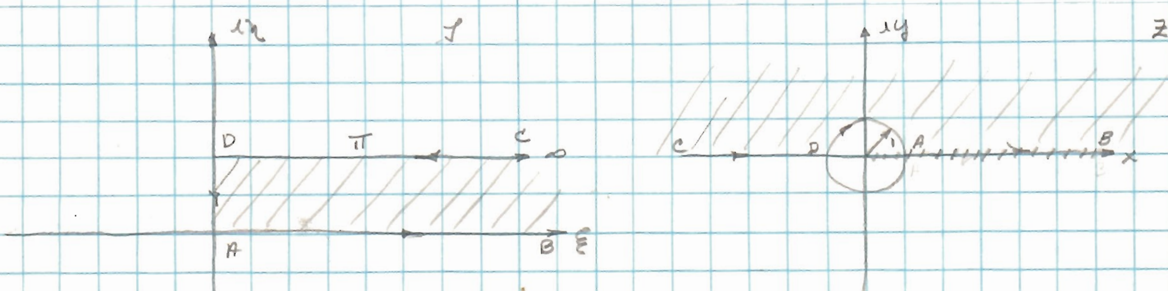


One must be sure in choosing the branch of Log z that the cut does not pass thru the mapped region from the ζ plane.

The $\xi_1 = \log r_1$, $\eta = \theta$







The transformation $z = e^{\zeta}$ maps the strip $\xi > 0$, $0 < \eta \leq \pi$ into the upper half of the z plane outside the unit semi-circle.

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(5) $z = \cosh \zeta = \frac{1}{2} e^{\zeta} + \frac{1}{2} e^{-\zeta}$

$= \frac{1}{2} e^{\xi} (\cos\eta + i \sin\eta) + \frac{1}{2} e^{-\xi} (\cos\eta - i \sin\eta)$

$= \cosh\xi \cos\eta + i \sinh\xi \sin\eta$

$x = \cosh\xi \cos\eta \; , \quad y = \sinh\xi \sin\eta$

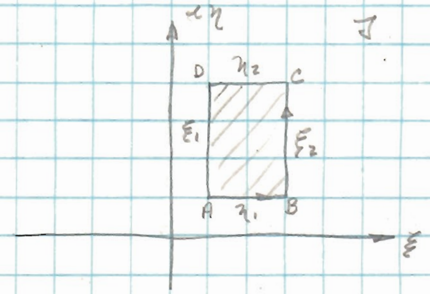
$\zeta = \cosh^{-1} z$

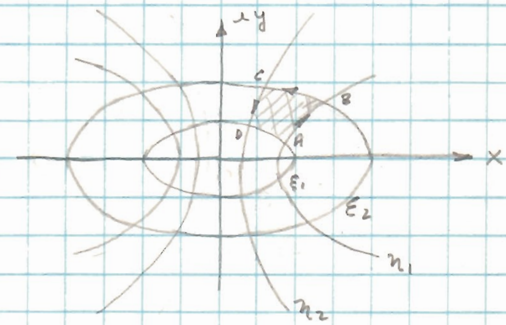
$= \log\left[z + (z^2 - 1)^{1/2} \right]$

Branch points at $\pm 1, \infty$

(6) For $\xi = \xi_0$: $\dfrac{x^2}{\cosh^2 \xi_0} + \dfrac{y^2}{\sinh^2 \xi_0} = 1$

For $\eta = \eta_0$: $\dfrac{x^2}{\cos^2 \eta_0} - \dfrac{y^2}{\sin^2 \eta_0} = 1$

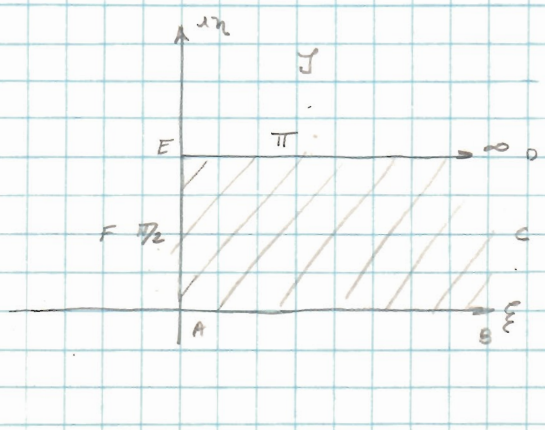


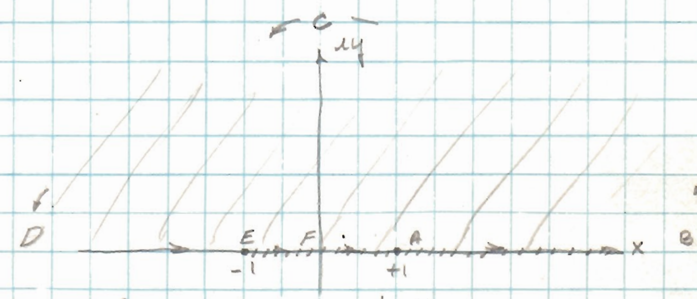


(7) $a^2 = \cosh^2 \xi_0 \; , \; b^2 = \sinh^2 \xi_0$ for ellipse
$c^2 = a^2 - b^2 = 1 \; , \; c = \pm 1$ and are focii

As $\xi_0 \to 0, \quad a \to \pm 1 \; , \; b \to 0$
As $\xi_0 \to +\infty, \quad a \to \pm\infty \; , \; b \to \pm\infty$

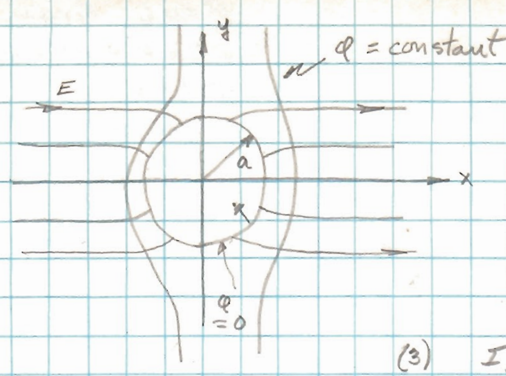
(8) For hyperbola : $a^2 = \cos^2 \eta_0 \; , \; b^2 = \sin^2 \eta_0$
As $\eta_0 \to 0, \quad a \to \pm 1 \; , \; b \to 0$
As $\eta_0 \to \pi, \quad a \to \mp 1, \; b \to 0$, passes $\pi/2$ on way





Thus $z = \cosh \zeta$ maps the given open rectangle into the upper half of the z plane.

62.



(1) We know from elementary electrostatics that E lines terminate on and emanate from uncharged conductors.

(2) $\nabla^2 \varphi = 0$, $E = -\operatorname{grad} \varphi$
$= -\dfrac{\partial \varphi}{\partial x}$

(3) If $w = \varphi + i\psi$, $\dfrac{\partial \varphi}{\partial x} = \dfrac{\partial \psi}{\partial y}$, $\dfrac{\partial \varphi}{\partial y} = -\dfrac{\partial \psi}{\partial x}$

(4) $$\frac{dw}{dz} = \frac{\partial \varphi}{\partial x} + i\frac{\partial \psi}{\partial x} = \frac{\partial \varphi}{\partial x} - i\frac{\partial \varphi}{\partial y}$$

(5) $\dfrac{\partial \varphi}{\partial x} = -E$, $\dfrac{\partial \varphi}{\partial y} = 0$ in the unperturbed field.

(6) $\therefore$ $\dfrac{dw}{dz} = -E$ or $w = -Ez$

(7) We want to include the effect of the conductor which we know must vanish at $|z| = \infty$.
We then write:

$$w = w' - Ez$$

and choose $\varphi' \to o\left(\frac{1}{r}\right)$ as $r \to \infty$, $\varphi' = \dfrac{A}{x}$

or $w' = \dfrac{A}{z}$

(8) $w = \dfrac{A}{z} - Ez$, where $z = r e^{i\theta}$

(9) $\varphi = \left(\dfrac{A}{r} - Er\right)\cos\theta$, $\psi = -\left(\dfrac{A}{r} + Er\right)\sin\theta$

(10) The boundary condition is: $\varphi = 0$ at $r = a$:

$Ea = \dfrac{A}{a}$, $A = a^2 E$

(11) Finally: $$w = E\left(\frac{a^2}{z} - z\right)$$

$$\varphi = E\left(\frac{a^2}{r} - r\right)\cos\theta$$

$$\psi = -E\left(\frac{a^2}{r} + r\right)\sin\theta$$

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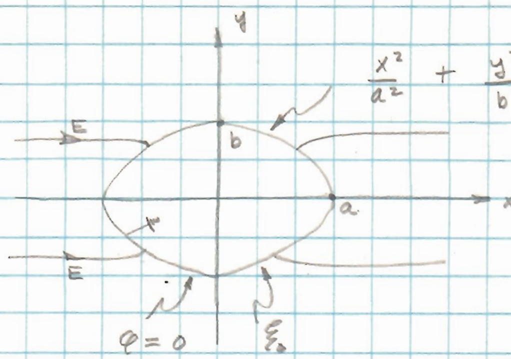
63.

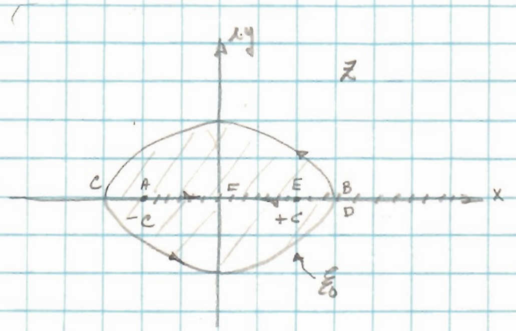


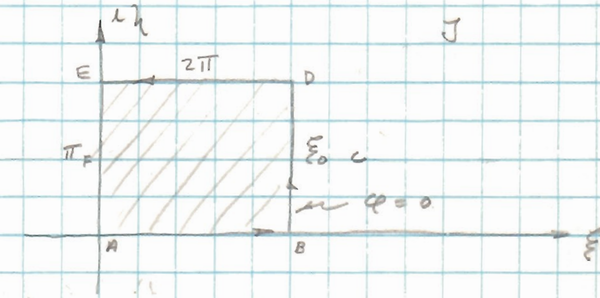
(1) It is clear that the following equations can be taken over from problem 62:

$w = \varphi + i\psi$, $w' = \varphi' + i\psi'$
$= -Ez$ for the unperturbed field. In the perturbed field:
$w = w' - Ez$

(2) It will prove usefull in satisfying the initial conditions to transform the problem to the ζ plane via:

$z = c \cosh \zeta$; $a = c \cosh \xi_0$, $b = c \sinh \xi_0$
$= c(\cosh\xi \cos\eta + i \sinh\xi \sin\eta)$
$a^2 - b^2 = c^2$





(3) We now have $w = w' - cE \cosh \zeta$

We want φ' to vanish as x get large or as ξ becomes large. However, we are restricted by the cut in the range of η, so we must choose $w'(\zeta)$ such that we stay in the strip.

Take $w' = Be^{-\zeta} = Be^{-\xi}(\cos\eta - i\sin\eta)$

$|e^{-\zeta}| = e^{-\xi} \to 0$ as $\xi \to \infty$

(4) $\therefore$ $w = Be^{-\zeta} - cE\cosh\zeta$

$\varphi = (Be^{-\xi} - cE\cosh\xi)\cos\eta$

$\psi = -(Be^{-\xi} + cE\sinh\xi)\sin\eta$

(5) We must now satisfy the boundary condition $\varphi = 0$:

Note: $e^{-\zeta} = \cosh \zeta - \sinh \zeta = z - [z^2 - 1]^{1/2}$

(6) $$0 = B e^{-\xi_0} - cE \cosh \xi_0$$

$$B = \frac{cE \cosh \xi_0}{e^{-\xi_0}}$$

$$e^{-\xi_0} = \cosh \xi_0 - \sinh \xi_0 = \frac{a-b}{c}$$

$$B = \frac{acE}{a-b}$$

(7) $\therefore$ $$w = cE \left\{ \frac{a}{a-b} e^{-\zeta} - \cosh \zeta \right\}$$

$$\varphi = cE \left\{ \frac{a}{a-b} e^{-\xi} - \cosh \xi \right\} \cos \eta$$

$$\psi = -cE \left\{ \frac{a}{a-b} + \sinh \xi \right\} \sin \eta$$

Problems
43, 44, 45

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43. 1. (1) show $\displaystyle\int_0^\infty \frac{x^{a-1}}{x^2+x+1}\,dx = \frac{2\pi}{\sqrt{3}}\,\frac{\cos\left(\frac{\pi+2\pi a}{6}\right)}{\sin\pi a}$; $0<a<2$

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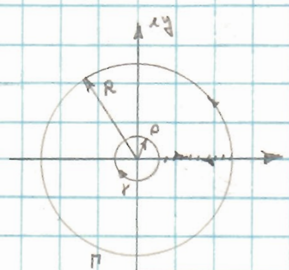
(2) consider $\displaystyle\int_C z^{a-1} I(z)\,dz$:

take $z^a I(z) \to 0$ as $|z| \to \infty$ for $a < 2$
and $z^a I(z) \to 0$ as $|z| \to 0$ for $a > 0$ with $n<m$

(3) Choose as C:



As $R\to\infty$, and $\rho\to 0$, the integrals around the contours Γ and γ vanish. We are left with; assuming no poles on the positive real axis:

$$\int_0^\infty x^{a-1} I(x)\,dx + \int_\infty^0 x^{a-1} e^{i2\pi(a-1)} I(x)\,dx = 2\pi i \sum R$$

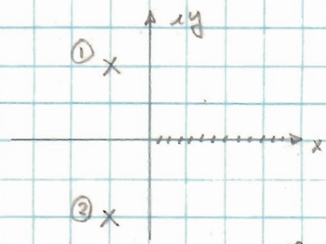
as the origin is a branch point of z^{a-1}

(4) Since $e^{2\pi i(a-1)} = e^{i2\pi a}$; $\displaystyle\int_0^\infty x^{a-1} I(x)\,dx = \frac{2\pi i \sum R}{1-e^{i2\pi a}}$

$$= \frac{-\pi e^{-i\pi a}}{\sin\pi a} \sum R\left[z^{(a-1)} I(z)\right]$$

(5) Consider $\displaystyle\int_C \frac{z^{a-1}}{z^2+z+1}\,dz$ which meets the above requirements for $0<a<2$.

(6) $z^2+z+1=0$: $z = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2} = e^{i2\pi/3},\ e^{i4\pi/3}$



(7) $\displaystyle R_1 = \lim_{z\to e^{i2\pi/3}} \left\{\frac{z^{a-1}}{z-e^{i4\pi/3}}\right\} = \frac{1}{i\sqrt{3}}\, e^{i2\pi/3(a-1)}$

(8) $\displaystyle R_2 = \lim_{z\to e^{i4\pi/3}} \left\{\frac{z^{a-1}}{z-e^{i2\pi/3}}\right\} = -\frac{1}{i\sqrt{3}}\, e^{i4\pi/3(a-1)}$

(9) $$\int_0^\infty \frac{x^{a-1}}{x^2+x+1}\,dx = \frac{-\pi}{i\sqrt{3}\sin\pi a}\left\{\exp\left[i\left(\frac{-\pi a-2\pi}{3}\right)\right] - \exp\left[i\underbrace{\left(\frac{\pi a-4\pi}{3}\right)}_{\frac{\pi a-4\pi+6\pi-6\pi}{3}}\right]\right\}$$

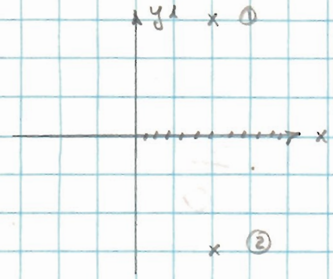
$$= \frac{2\pi}{\sqrt{3}}\,\frac{\sin\left[\frac{\pi a+2\pi}{3}\right]}{\sin\pi a} = \frac{2\pi}{\sqrt{3}}\,\frac{\cos\left[\frac{\pi}{2}-\frac{\pi a}{3}-\frac{2\pi}{3}\right]}{\sin\pi a}$$

$$= \frac{2\pi}{\sqrt{3}}\,\frac{\cos\left[\frac{\pi+2\pi a}{6}\right]}{\sin\pi a}, \quad 0<a<2$$

43. ii. (1) show $\displaystyle\int_0^\infty \frac{x^{a-1}\,dx}{x^2-x+1} = \frac{2\pi}{\sqrt{3}} \frac{\sin\left[\frac{\pi+2\pi a}{3}\right]}{\sin \pi a}$; $0<a<2$

(2) Consider : $\displaystyle\int_C \frac{z^{a-1}\,dz}{z^2-z+1}$; where $z^a I(z) \to 0$ as $|z|\to\infty$, $a<2$
$z^a I(z) \to 0$ as $|z| \to 0$, $a>0$

and $z^2-z+1=0$: $z = \frac{1}{2} \pm i\frac{\sqrt{3}}{2} = e^{i\pi/3}, e^{i5\pi/3}$



Thus we satisfy the conditions of the previous problem and proceed in the exact same manner:

$$\int_0^\infty x^{a-1} I(x)\,dx = \frac{-\pi e^{-i\pi a}}{\sin \pi a} \sum R\left[z^{a-1} I(z)\right]$$

(3) $\displaystyle R_1 = \lim_{z\to e^{i\pi/3}} \left\{\frac{z^{a-1}}{z-e^{i5\pi/3}}\right\} = \frac{e^{i\pi/3(a-1)}}{e^{i\pi/3}-e^{i5\pi/3}} = \frac{1}{i\sqrt{3}} e^{i\pi/3(a-1)}$

(4) $\displaystyle R_2 = \lim_{z\to e^{i5\pi/3}} \left\{\frac{z^{a-1}}{z-e^{i\pi/3}}\right\} = \frac{e^{i5\pi/3(a-1)}}{e^{i5\pi/3}-e^{i\pi/3}} = -\frac{1}{i\sqrt{3}} e^{i5\pi/3(a-1)}$

(6) $\displaystyle\int_0^\infty \frac{x^{a-1}\,dx}{x^2-x+1} = \frac{-\pi}{i\sqrt{3}\sin\pi a}\left[\exp\left\{i\left(\frac{-2\pi a-\pi}{3}\right)\right\} - \exp\left\{i\underbrace{\left(\frac{2\pi a-5\pi}{3}\right)}_{\frac{2\pi a-5\pi+6\pi-6\pi}{3}}\right\}\right]$

$\displaystyle = \frac{2\pi}{\sqrt{3}} \frac{\sin\left[\frac{2\pi a+\pi}{3}\right]}{\sin\pi a}$

NB: In neither one of these problems does the value $a=1$ cause difficulty, because:

i. $\displaystyle\lim_{a\to 1} \frac{2\pi}{\sqrt{3}} \frac{\cos\left[\frac{\pi+2\pi a}{6}\right]}{\sin\pi a} = \frac{2\pi}{\sqrt{3}} \lim_{a\to 1} \frac{-\frac{2\pi}{6}\sin\left[\frac{\pi+2\pi a}{6}\right]}{\pi\cos\pi a}$

$\displaystyle = \frac{2\pi}{3\sqrt{3}}$

ii. $\displaystyle\lim_{a\to 1} \frac{2\pi}{\sqrt{3}} \frac{\sin\left[\frac{2\pi a+\pi}{3}\right]}{\sin\pi a} = \frac{2\pi}{\sqrt{3}} \lim_{a\to 1} \frac{\cos\left[\frac{2\pi a+\pi}{3}\right]\cdot\frac{2\pi}{3}}{\pi\cos\pi a}$

$\displaystyle = \frac{2\pi}{3\sqrt{3}}$

Problems 43, 44, 45
Continued

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44. (9)

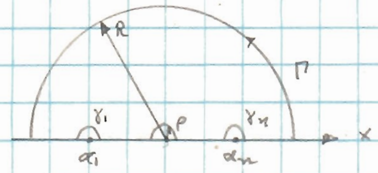
(1) show $P\int_0^\infty \frac{x^4}{x^6-1}\,dx = \frac{\sqrt{3}\,\pi}{6}$

(2) Consider $\int_C \frac{z^4}{z^6-1}\,dz$ where $z\,I(z) \to 0$ as $|z| \to \infty$

$z^6 = 1$; $z = e^{i\left(\frac{2n\pi}{6}\right)} = 1, e^{i\pi/3}, e^{i2\pi/3}, -1, e^{i4\pi/3}, e^{i5\pi/3}$
so that there are two simple poles on the real axis.

(3) Consider : $\int_C I(z)\,dz$ where $z\,I(z) \to 0$ as $|z| \to \infty$ and $I(z)$ is a rational function with simple poles on the real axis.

C:



As $R \to \infty$, $\int_\Gamma \to 0$

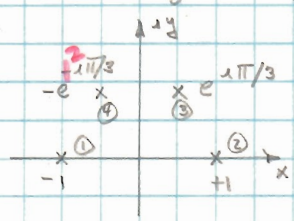
$$\therefore \int_{-\infty}^{\alpha_1-\rho} + \cdots + \int_{\alpha_n-\rho}^{\infty} I(x)\,dx + \int_{\gamma_1} + \cdots + \int_{\gamma_n} I(z)\,dz = 2\pi i \sum R_+$$

(4) From Cauchy's Residue theorem II, Phillips, p. 117: $\int_{\gamma_n} I(z)\,dz = \pi i\, R(\alpha_n)$

(5) Define: $P\int_{-\infty}^{\infty} I(x)\,dx \equiv \lim_{\rho \to 0}\left\{\int_{-\infty}^{\alpha_1-\rho} + \cdots + \int_{\alpha_n-\rho}^{\infty} I(x)\,dx\right\}$

(6) $\therefore\ P\int_{-\infty}^{\infty} I(x)\,dx = 2\pi i \sum R_+ + \pi i \sum R_0$

(7) The integral of the problem is exactly this case:



Let α be any pole of $I(z)$ or root of $D(z)$:

$$\alpha^6 = +1$$

$$\therefore\ R = \lim_{z\to\alpha}\left\{\frac{z^4(z-\alpha)}{z^6-1}\right\} = \lim_{z\to\alpha}\left\{\frac{5z^4-4\alpha z^3}{6z^5}\right\}$$

$$= \frac{\alpha^4}{6\alpha^5} = \frac{1}{6\alpha} = \frac{\alpha^5}{6}$$

(8) $\pi i \sum R_0 = \frac{\pi i}{6}(1-1) = 0$; $2\pi i \sum R_+ = \frac{\pi i}{3}\left[e^{-i\pi/3} - e^{i\pi/3}\right]$

$$= \frac{2\pi}{3}\sin \pi/3 = \frac{\sqrt{3}\,\pi}{3}$$

(9) $\int_0^\infty \frac{x^4\,dx}{x^6-1} = \frac{1}{2}\int_{-\infty}^{\infty}\frac{x^4\,dx}{x^6-1} = \frac{\sqrt{3}\,\pi}{6}$

45. (1) Show $\displaystyle\int_0^\infty \frac{\sinh ax}{\sinh \pi x}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sinh ax}{\sinh \pi x}\,dx = \frac{1}{2}\tan\frac{1}{2}a \,;\ -\pi < a < \pi$

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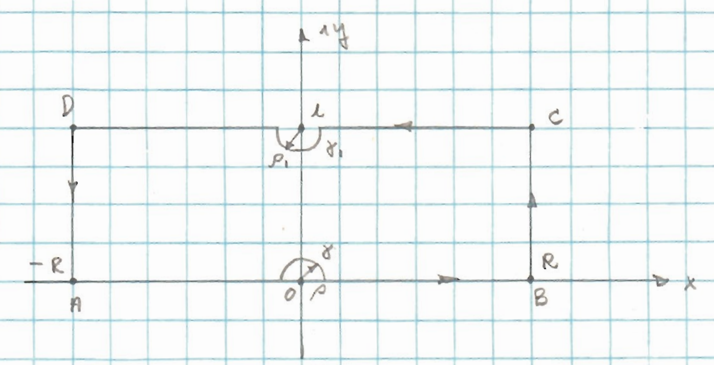
(2) Consider $\displaystyle\int_C \frac{e^{az}}{\sinh \pi z}\,dz$

$\sinh \pi z$ has simple zeros at $z = ni \ (n = 0, \pm 1, \pm 2, \cdots)$ thus the integrand has simple poles at these points. Consider the following contour C:



(3) $$\int_C = \int_{-R}^{-\rho} \frac{e^{ax}}{\sinh \pi x}\,dx + \int_\rho^R \frac{e^{ax}}{\sinh \pi x}\,dx + \int_0^1 \frac{e^{a(R+iy)}}{\sinh \pi (R+iy)}\,dy$$

$$+ \int_1^0 \frac{e^{a(-R+iy)}}{\sinh \pi(-R+iy)}\,dy + \int_\gamma + \int_{\gamma'} \frac{e^{az}}{\sinh \pi z}\,dz$$

$\dfrac{z e^{az}}{\sinh \pi z} =$

$$+ \int_R^{\rho_1} \frac{e^{a(x+i)}}{\sinh \pi (x+i)}\,dx + \int_{-\rho_1}^{-R} \frac{e^{a(x+i)}}{\sinh \pi (x+i)}\,dx = 0$$

(4) We take residues at $z = 0$ and $z = i$ in accordance with Phillip's Theorem on semicircles around simple poles; # 43.

$$\lim_{z\to 0}\left\{\frac{z e^{az}}{\sinh \pi z}\right\} = \lim_{z \to 0}\left\{\frac{a z e^{az} + e^{az}}{\pi \cosh \pi z}\right\} = \frac{1}{\pi} \quad ; \ \pi i R = i$$

$$\lim_{z\to i}\left\{\frac{(z-i) e^{az}}{\sinh \pi z}\right\} = \lim_{z \to i}\left\{\frac{a(z-i) e^{az} + e^{az}}{\pi \cosh \pi z}\right\} = -\frac{e^{ia}}{\pi} \ ; \ \pi i R = -i e^{ia}$$

(5) Also note that $\sinh \pi (x+i) = -\sinh \pi x$

(6) $\displaystyle\int_{CB} - \int_{DA} \to 0$ as $R \to \infty$: As $R \to \infty$ and $\rho_1, \rho \to 0$ independently, for $-\pi < a < \pi$

$$(1 + e^{ia}) \int_{-\infty}^{\infty} \frac{e^{ax}}{\sinh \pi x}\,dx = +i\,(1 - e^{ia})$$

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{\sinh \pi x}\,dx = \tan\frac{a}{2} \quad ; \quad \int_{-\infty}^{\infty} \frac{e^{-ax}}{\sinh \pi x}\,dx = -\tan\frac{a}{2}$$

(7) $$\int_{-\infty}^{\infty} \frac{e^{xa}}{\sinh \pi x}\,dx - \int_{-\infty}^{\infty} \frac{e^{-ax}}{\sinh \pi x}\,dx = 2\int_{-\infty}^{0} \frac{\sinh ax}{\sinh \pi x}\,dx = 2\tan\frac{a}{2}, \text{ or, } \int_0^\infty \frac{\sinh ax}{\sinh \pi x}\,dx = \frac{1}{2}\tan\frac{a}{2}$$

$-\pi < a < \pi$

1960-61

HARVARD UNIVERSITY

APPLIED MATHEMATICS 201

1. (i) Sketch briefly a proof of Laurent's theorem, that if C_1 and C_2 are two circles, each with its center at the origin, and $\underline{f(z)}$ is regular on and between C_1 and C_2, then at any point $\underline{z}$ in the annulus between the circles

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$$

where

$$a_n = \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(z)}{z^{n+1}}\, dz,$$

and Γ is any closed contour enclosing the origin and lying within the annulus.

Prove that when $\underline{z}$ is not zero or infinite

$$\exp\left(z + \frac{1}{2z^2}\right) = \sum_{n=-\infty}^{\infty} a_n z^n$$

where

$$a_n = \frac{1}{\pi e^{1/2}} \int_0^{\pi} \exp\left\{(1 + \cos\theta)\cos\theta\right\} \cos\left\{n\theta - (1 - \cos\theta)\sin\theta\right\} d\theta.$$

(ii) (a) Which of the functions listed below is single-valued, and how many values (or branches) have each of the other functions?

$$\frac{e^{i\theta}+e^{-i\theta}}{2} + \frac{e^{2i\theta}+2+e^{-2i\theta}}{4}$$

(continued)

$$= \frac{2e^{i\theta}+2e^{-i\theta}+e^{2i\theta}+2+e^{-2i\theta}}{4}$$

(b) What is the position and nature of their singularities, including branch points, in the z-plane?

(c) What is the nature of the point at infinity for each function?

(d) Are there any singularities of any one branch of any of the many-valued functions?

Sketch diagrams to show cuts in the z-plane such each branch of a many-valued function is a regular function in the cut z-plane (apart from any singularities under (d) above).

Which functions, or branches of functions, admit of expansions of the form (i) $\sum_{n=0}^{\infty} a_n z^n$, (ii) $\sum_{n=0}^{\infty} b_n z^{-n}$ in any region of the z-plane.

In which annular region (if any) centered on the origin can a branch or branches of $f_4(z)$ and $f_5(z)$ be expanded in a Laurent series, $\sum_{n=-\infty}^{\infty} a_n z^n$.

(Answers in tabular form will be accepted. Proofs of statements are not here required.)

$$f_1(z) = \cosh z^{1/2}; \quad f_2(z) = (1 + z^2)^{1/3}; \quad f_3(z) = (1 + z^3)^{1/3};$$

no no

$$f_4(z) = [z(1 + z^2)]^{1/2}; \quad f_5(z) = [z/(z - i)(z - 2)]^{1/2};$$

no no

$$f_6(z) = \frac{1}{\mathrm{Log}\left\{z + (z^2 + 1)^{1/2}\right\}}.$$

no

2. Find a function $\underline{f}(\underline{z})$ such that $\zeta = \underline{f}(\underline{z})$ transforms the two circles $|\underline{z} - 17| = 15$ and $|\underline{z} - 10| = 6$ into the circle $|\zeta| = 1$ and a concentric circle $|\zeta| = \underline{k}$ with $\underline{k} > 1$, and find the value of $\underline{k}$.

If $\underline{w} = \underline{U} + \underline{iV}$ is a regular function of $\underline{z}$ between and on the two circles in the $\underline{z}$-plane, and $\underline{U} = 0$ on the first circle and $\underline{U} = 1$ on the second circle, find an expression for $\underline{w}$ in terms of $\underline{z}$ between the two circles.

[Hint. If the two circles C_1 and C_2,

$$|z - a_1| = r_1, \quad |z - a_2| = r_2$$

($\underline{a}_1$, $\underline{a}_2$, $\underline{r}_1$, $\underline{r}_2$ real), with C_1 inside C_2, are considered as two of a system of coaxal circles of which the limiting points are at $(x_1, 0)$ and $(x_2, 0)$, show that

$$-x_1x_2 + a_1(x_1 + x_2) = a_1^{\,2} - r_1^{\,2}$$

$$-x_1x_2 + a_2(x_1 + x_2) = a_2^{\,2} - r_2^{\,2}.$$

It may be assumed that the limiting points are inverse points with respect to each circle of a system of coaxal circles.]

3. Use contour integration and the method of residues to evaluate

(i) $\displaystyle\int_0^{\infty} \frac{x^2}{x^8 + 1}\,dx$ (ii) $\displaystyle\int_0^{\infty} \frac{dx}{1 + x^5}$ (iii) $\displaystyle\int_0^{\infty} \frac{x^{1/2}}{x^2 - 4x + 5}\,dx.$

4. Explain briefly how to calculate the number of complex zeros of a polynomial, $\underline{f}(\underline{z})$, in $\underline{z}$ which are represented by points in the first quadrant of the $\underline{z}$-plane, when $\underline{f}(\underline{z})$ has no zeros on the positive real axis, on the positive imaginary axis, or at the origin.

Show that for all positive values of $\underline{a}$

$$z^7 + z + a = 0$$

has two complex roots in each of the first and fourth quadrants, and one in each of the second and third quadrants.

Show that there are no roots in the first quadrant inside the circle $|z| = 1$ for any positive value of a.

5. (a) If

$$w(\zeta) = U + iV = A \log(\zeta - a) + B \log(\zeta - b),$$

where A, B, a, and b are real, $a > b$, and $\log(\zeta - a)$, $\log(\zeta - b)$ are real when ζ is real and greater than a and b, what are the values of V at the points on the real axis other than a and b?

(b) By considering the transformation $\zeta = \cosh kz$, where k is a constant to be determined, show how to map the open rectangle $0 < y < h$, $x > 0$, in the z-plane on the upper half-plane of ζ.

(c) Given that $w = U + iV$ is a regular function of z in the open rectangle in (b), that V tends to zero as x tends to infinity, that $V = 0$ on $y = 0$ and on $y = h$, and that $V = 1$ on $x = 0$, show that

$$w = \frac{2}{\pi} \log \tanh \frac{\pi z}{2h}$$

and that

$$V = \frac{2}{\pi} \arctan \left[\frac{\sin \pi y/h}{\sinh \pi x/h} \right].$$

6. Prove (for $x > 0$, $a > 0$, $t > 0$) that

$$\int_0^\infty \exp(-x^2u^2) \cos au \, du = \frac{\sqrt{\pi}}{2x} \exp(-a^2/4x^2)$$

and

$$\int_0^\infty \exp(-tu^2) \sin au \frac{du}{u} = \sqrt{\pi}\int_0^y \exp(-w^2) dw,$$

where $y = a/(2\underline{t}^{1/2})$.

If $\phi(\underline{p})$ is the $\underline{p}$-multiplied Laplace transform (or Heaviside operational representation) of $\underline{f}(\underline{t})$, it may be assumed that

$$\phi(u) = u \int_0^\infty e^{-ut} f(t) dt, \qquad f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{\lambda t} \frac{\phi(\lambda)}{\lambda} d\lambda,$$

with a suitable restriction on $\underline{c}$. Use the result above to prove that if

$$\phi(p) = \exp(-ap^{1/2}) \quad (a > 0)$$

then

$$f(t) = \operatorname{erfc}(a/2t^{1/2})$$

where

$$\operatorname{erfc} x = \frac{2}{\sqrt{\pi}} \int_x^\infty \exp(-w^2) \, dw.$$

Also show that if sech $\underline{p}^{1/2}$ is the transform of $\underline{F}(\underline{t})$, then $\underline{F}(\underline{t})$ may be expressed in the form

$$2 \sum_{n=0}^\infty (-1^n \operatorname{erfc} \left(\frac{2n+1}{2t^{1/2}}\right),$$

and explain why this form is suitable for calculation for small values of $\underline{t}$.

$$\cancel{\cos 2\theta = 2 \sin\theta \cos\theta}$$

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$

$$\sin 2\theta = 2 \sin\theta \cos\theta$$

$$\cos(x+y) = \sin x \sin y - \cos x \cos y$$

$$\cos 2\theta = -\sin^2\theta + \cos^2\theta = -1 + 2\cos^2\theta$$