APPLIED
PHYSICS
215

FLUCTUATION
PHENOMENA

APPLIED PHYSICS
2/5

APPLIED PHYSICS
215

Applied Physics 215

Fall Term 1960

Instructor: N. Bloembergen

Office Hour, Pierce 231
Tuesday at noon or by appointment

The course will emphasize the physical aspects of fluctuation phenomena. Mathematical concepts of probability theory will be used without any attempt at rigor. A general background in physics equivalent to Applied Physics 132 and 181 is prerequisite. A mathematical background equivalent to App. Math.105b is also required.

References

The mathematical theory of probability is treated in Math 109 and Statistics 150, and in the following textbooks:

E. Parzen, Modern Probability Theory and Its Applications, Wiley, 1960.

W. Feller, Probability Theory and Its Applications, Wiley, 1950.

Applications to physical problems are discussed in the following intermediate texts:

C. Kittel, Elements of Statistical Physics, Wiley.

R. L. Lindsay, Physical Statistics, Wiley.

A collection of a series of very important research review papers is obtainable in an inexpensive Dover publication,

Noise and Stochastic Processes, edited by N. Wax.

Noise Problems in Electrical and Communication Engineering are discussed in:

Lawson and Uhlenbeck, Threshold Signals, M.I.T. Rad. Lab. Series, Vol. 24, McGraw Hill, 1950.

Davenport and Root, Random Signals and Noise, Lincoln Laboratory Series, Vol. 1, McGraw Hill, 1959.

D. Middleton, Introduction to Statistical Communication Theory (comprehensive theoretical), McGraw Hill, 1960.

A. van der Ziel, Noise, (experimental circuits), Prentice Hall.

Fluctuations in the quantized electromagnetic field are discussed in: Smith, Jones and Chasmar, Detection and Measurement of Infrared Radiation, Oxford University Press.

Tentative Course Outline

Elementary Probability 5 lectures

Definitions, sample space, random event, joint and conditional probabilities, statistical independence, fluctuation in number of particles, linear random walk, Poisson distribution, Bernouilli trials.

Random Variables 4 lectures

Distribution functions, averages, moments, characteristic functions, multivariate Gaussian distribution, Maxwell-Boltzmann statistics.

Time-dependent Random Processes 4 lectures

Time average, stationary processes, auto- and cross-correlation, spectral density, Wiener-Khintchin theorem.

Shot Noise 3 lectures

The method of Rice; noise in electron tubes and photocells.

Hour Examination

Brownian Motion and Thermal Noise 6 lectures

Diffusion Equation. The method of Fokker-Planck. Brownian Motion of a free particle and in a field of force. Thermal noise in electrical circuits. Galvanometer with electrical and mechanical damping.

Noise figure and noise temperature, excess noise 2 lectures

Noise in non-linear circuits; the quadratic detector 2 lectures

Fluctuations in Thermodynamic Quantities 2 lectures

Fluctuations in the radiation field, waves and quanta, thermal detectors 5 lectures

APPLIED PHYSICS 215

FLUCTUATION PHENOMENA

NOTES

Professor: Nicolas Bloembergen
Room: Cruft 319, TTS at 11 AM

LECTURE I

Course Intent: *Physical Problems in Probability*

Background Reading List:

1. Parzan, E., *Modern Probability Theory & Applications*, Wiley 1960
2. Feller, W., " , " 1950
3. Ushensky, J.V., Introduction to Mathematical Probability, McGH 1957

Fields of Probability:

1. Games of chance
2. Genetics
3. Engineering (Quality Control)
4. Physics
5. Communications Engineering (signal to noise ratio)
6. Traffic (accidents)
7. Medical Science

Definition of Random or Chance Phenomenon:

This is characterized by the fact that there is not the same situation under corresponding circumstances. Situations not uniquely determined because knowledge of circumstances may not be complete.

Sample Space (s): Set of all possibilities

Example: Throw of die, S consists of 6 points, $k = 1, \ldots, 6$.

Throw of 2 dice, S consists of 36 points

Definition of Probability:

$$P(k) = \lim_{N\to\infty} \frac{\#\text{ of favorable outcomes}}{\#\text{ of trials } N}$$

An event is said to occur if outcome of random situation has its description contained in set A.

Mutually Exclusive Events (A independent of B):

$$P(A \text{ or } B) = P(A) + P(B)$$

Definition of Certainty: $P(S) = 1 = P(A \text{ or "not } A\text{"})$

Definition of Impossibility: $P = 0 = P(A \text{ and "not } A\text{"})$

An impossible event has probability zero, but an event whose probability is zero is not necessarily impossible. Example: probability of gas molecule(s) having a given velocity.

Joint Probabilities: $P(A_k) = \sum_{l=1}^{N} P(A_k, B_l)$

$$P(B_l) = \sum_{k=1}^{N} P(A_k, B_l)$$

LECTURE II 9/29/60

Mutually exclusive events have sets that do not overlap

Joint Probability: $P(A,B)$, probability that events A and B occur together.

Conditional Probability: $P(B/A) = \dfrac{P(A,B)}{P(A)}$, this is the probability that B occurs if one knows A is occurring.

Statistically Independent Events:

B is independent of A if $P(B/A) = P(B)$, in which case we have the product rule, viz,

$$P(A,B) = P(A)P(B)$$

Suppose the following series of events: $A_1, A_2, \cdots A_N$, that is, N events.

$P(A_m, A_n) = P(A_m)P(A_n)$ is a necessary but not sufficient condition for these N events to be statistically independent.

For example, take four events: $A_1, \cdots A_4$ and define three new events:

$B_1 \equiv [A_1 \text{ or } A_4]$ (means sum of A_1 and A_4 sample spaces)
$B_2 \equiv [A_2 \text{ or } A_4]$
$B_3 \equiv [A_3 \text{ or } A_4]$

Now $P(B_1) = 1/2$
$P(B_2) = 1/2$
$P(B_3) = 1/2$

and consider: $P(B_3 | B_1, B_2) = 1 \neq P(B_3)$ because one knows A_4 occurs, therefore these events are not statistically independent. We must have $P(A_1, A_2, \cdots A_N) = P(A_1)P(A_2) \cdots P(A_N)$ for the condition of being statistically independent.

Example 1: Given: 6 balls, identical size, shape, etc., with 4 white and 2 red

$P(W) = 2/3$ assuming $P(\text{any ball}) = 1/6$ indicating an "a priori" assumption must be made from a physical knowledge of the experiment.

Example 2:

Color:	R R R R R	W W	B
Hardness or Softness :	H H H H S	H S	H

$$P(H,R) = P(H|R)\,P(R) \; ; \quad P(H|R) = 4/5 \, , \quad P(R) = \frac{5}{8}$$

$$\therefore \; P(H,R) = \frac{4}{5} \times \frac{5}{8} = \frac{1}{2}$$

Physical Applications: Bernoulli's Distribution

Consider an experiment with two possible outcomes. In general, we will have two mutually exclusive events with probability p for event A and probability q for event B with $q = 1 - p$.

If we carry out the experiment N times to get n successes of event A, we will get for the total probability of these n successes over N trials for <u>a single arrangement</u> of the successes and failures, the value

(1) $P(n) = q^{N-n} p^{n}$ by applying the product rule for each success and failure events which are independent of each other.

However, there are N trials and the number of ways of arranging the results of these trials is $N!$. The number of ways of arranging the success and failures within these N trials is $\frac{N!}{n!(N-n)!}$. Thus the probability of n successes scattered in any manner throughout the trials is

(2) $$P_N(n) = p^{n} q^{N-n} \frac{N!}{n!(N-n)!}$$

Check: $\sum_{n=0}^{N} P_N(n) = (p+q)^N = 1$, the summation showing that $P_N(n)$ is a term of the binomial expansion.

Average number of successes n of event A:

(3) $$\bar{n} = \sum_{n=0}^{N} n \, P_N(n)$$

Mean Square:

(4) $\overline{n^2} = \sum_{n=0}^{N} n^2 P_N(n)$

A trick to calculate the moments of the Bernoulli Distribution:

(5) $(py+q)^N = \sum_{n=0}^{N} p^n q^{N-n} y^n \frac{N!}{n!\,(N-n)!}$

(6) Take $\frac{d}{dy}$: $Np(py+q)^{N-1} = \sum_{n=0}^{N} n p^n q^{N-m} y^{n-1} F_N(B)$

where $F_N(B) = \frac{N!}{n!\,(N-n)!}$

(7) Let $y = 1$: $Np = \sum_{n=0}^{N} n P_N(n) = \bar{n}$

(8) Take $\frac{d^2}{dy^2}$: $N(N-1)p^2(py+q)^{N-2} = \sum_{n=0}^{N} n(n-1) p^n q^{N-n} y^{n-2} F_N(B)$

(9) Let $y = 1$: $N(N-1)p^2 = \sum_{n=0}^{N} n(n-1) P_N(n) = \overline{n^2} - \bar{n}$

or: $(\bar{n})^2 - \bar{n}p = \overline{n^2} - \bar{n}$, $\overline{n^2} - (\bar{n})^2 = \bar{n}(1-p) = Npq$

(10) $\overline{(\Delta n)^2}$ is defined as the "mean square deviation" and is equal to $\overline{n^2} - (\bar{n})^2$

(11) $\therefore$ $(\overline{\Delta n})^2 = Npq$

(12) If $p \ll 1$, $q \sim 1$; $(\overline{\Delta n^2})^{1/2} \simeq (\bar{n})^{1/2}$

(13) Under these conditions, the relative fluctuation around the average may be equal to:

$$\frac{(\overline{\Delta n^2})^{1/2}}{\bar{n}} \simeq \sqrt{\frac{1}{\bar{n}}}$$

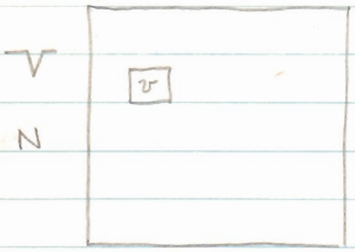
LECTURE III 10-1-60

Poisson Distribution:

Let N approach infinity and p approach zero such that

(1) $\lim\limits_{\substack{N\to\infty\\ p\to 0}} Np =$ finite constant

Consider the following gas:



V is the volume of the gas
N is total number of molecules
v is the volume of an element

Given: One box at one time with N independent, non-interacting molecules, each molecule representing a trial. However, interactions do occur, but we will postulate that these interactions do not affect the distribution in space of the independent molecules. What is the probability in N trials to find n molecules in volume v. We have to know the a priori probability of being in v which is assumed to be $\frac{v}{V}$. This may not be true as is the case in the atmosphere where a gravitational potential is present. We may now define along the lines of the Bernoulle distribution our distribution function in this case, which is the probability to find n specified molecules in volume v times probability of $N-n$ being outside, times the redistribution function:

(2) $\left(\frac{v}{V}\right)^{n}\left(1-\frac{v}{V}\right)^{N-n}\frac{N!}{(N-n)!\,n!}$

Inducing the anticipated constant to

take the Dirac limit keeping the density constant, that is, $\bar{n}$ = constant = Np, we take

(3) $\bar{n} = N\left(\frac{v}{V}\right)$, $\frac{v}{V} = \frac{\bar{n}}{N}$

Upon introducing this into (2) and taking $\lim_{N \to \infty}$, we also take $\lim_{p \to 0}$ and factor out the constant $\bar{n}$.

$$(4) \quad \lim_{N \to \infty} \frac{(\bar{n})^n}{n!} \frac{N(N-1)\cdots(N-n+1)\left(1-\frac{v}{V}\right)^{N-n}}{N^n}$$

$$= \lim_{N \to \infty} \left[\frac{(\bar{n})^n}{n!}\right] 1\left(1-\frac{1}{N}\right)\cdots\left(1-\frac{n-1}{N}\right)\left(1-\frac{v}{V}\right)^{N-n}$$

$$= \frac{(\bar{n})^n}{n!} \lim_{N \to \infty} \left(1-\frac{\bar{n}}{N}\right)^{N-n} = \frac{(\bar{n})^n}{n!} e^{-\bar{n}}$$

from $e^{-x} = \lim_{N \to \infty} \left(1-\frac{x}{N}\right)^N$

$\frac{(\bar{n})^n}{n!} e^{-\bar{n}}$ is called the Poisson distribution function.

Check: (5) $\bar{n} = \sum_{n=0,1}^{N=\infty} n \frac{(\bar{n})^n}{n!} e^{-\bar{n}} = \bar{n} \sum_{n=0}^{\infty} \frac{(\bar{n})^{n-1} e^{-\bar{n}}}{(n-1)!}$

Introduce: $n' = n-1$

(6) $\bar{n} = \bar{n} \sum_{n'=0}^{\infty} \frac{(\bar{n})^{n'}}{n'!} e^{-\bar{n}}$

but $\sum_{n'=0}^{\infty} \frac{(\bar{n})^{n'}}{n'!} \equiv e^{\bar{n}}$

(7) Therefore: $\bar{n} = \bar{n}$

The Poisson distribution is the limiting case of the Bernoulli distribution. Shows that the density is not constant, but has fluctuations which are very small.

At NTP: $\frac{N}{V} = 2.7 \cdot 10^{19}$ molecules/cc,

thus 10^{-3} cc contains $2.7 \cdot 10^{16}$ molecules $= \bar{n} = \frac{N}{V} v$
Under these conditions we find that

(8) $[\overline{(\Delta n)^2}]^{1/2} = [\bar{n^2} - (\bar{n})^2] = 1.65 \cdot 10^{8} = \sqrt{\bar{n}}$

(9) Relative Fluctuation: $\frac{[\overline{(\Delta n)^2}]^{1/2}}{\bar{n}} = \frac{1.65 \cdot 10^{8}}{\bar{n}}$
$= \frac{1}{1.65 \cdot 10^{8}}$

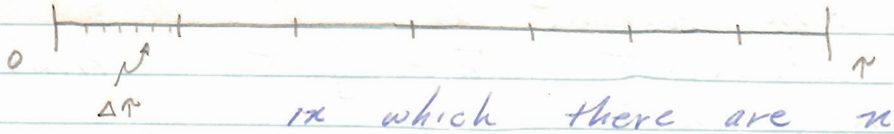
Is this always negligible?
Take a cube $v = 10^{-13}$ cc, about 5000 Å on a side. Now $\bar{n} = 2.7 \cdot 10^{+6}$ molecules.
Thus the relative fluctuation $= \frac{1}{\sqrt{\bar{n}}} \approx \frac{1}{1000}$.

For the index of refraction of air ($\mu = 1.00029$) the part due to the polarizibility of the air is .00029. This would mean a fluctuation in the index of refraction of .00000029, too small to be noticed except possibily for very short wavelengths. Near the critical point, it is extremely wrong to assume no effects from molecular interaction, therefore there will be large variations in μ. However, the above analysis applies to many problems:

① Radioactive Decay
② Electron emission

Assume a time interval (Δt) such that only one particle is emitted. This time interval will be small but finite. The a priori probability of electron emission is taken to be proportional to Δt; that is, $p = A \Delta t$ where A is a constant, and has the dimensions of reciprocal time.

Consider a long time τ:



in which there are n particles emitted. The number of trials N is equal to $\frac{\Delta\tau}{\tau}$. N is so large that only one particle can be emitted in $\Delta\tau$, either yes or no, thereby suggesting the Bernoulli distribution, viz.,

$$(10) \quad P_n(\tau) = (A\Delta\tau)^n (1-A\Delta\tau)^{N-n} \frac{N!}{(N-n)!\, n!}$$

Now take the limit as before as $N \to \infty$ and $\Delta\tau \to 0$ and get the Poisson distribution, that is,

$$(11) \quad P_n(\bar{n}) = \frac{(\bar{n})^n}{n!} e^{-\bar{n}}$$

with $\bar{n} = pN = A\Delta\tau N = A\tau$, sin $N = \frac{\Delta\tau}{\tau}$, so,

$$(12) \quad P_n(\tau) = \frac{(A\tau)^n}{n!} e^{-A\tau}$$

What is the probability to get no counts in time τ and the first count in between τ and $\tau + \Delta\tau$? Since the counts are statistically independent, we may use the product of each probability of the respective events.

$$(13) \quad P_0(\tau) = e^{-A\tau} \quad ; \quad P_1(\Delta\tau) = p = A\Delta\tau$$

$$(14) \quad \therefore \quad P = P_0(\tau) P_1(\Delta\tau) = A\Delta\tau\, e^{-A\tau}$$

This is also related to the probability of collisions: no collisions in τ with the first collision between τ and $\tau + \Delta\tau$.

Shot Effect:

Given a diode in the saturation region with saturation current i_0.
In time interval T: $\bar{n} = \frac{i_0}{e} T$ electrons on the average in T.

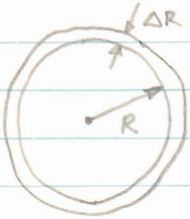
$$(15) \quad \overline{(\Delta n)^2} = \bar{n} = \frac{i_0 T}{e}$$

$$(16) \quad \sqrt{\frac{\overline{(\Delta n)^2}}{(\bar{n})^2}} = \sqrt{\frac{\overline{(\Delta i)^2}}{i_0^2}} = \sqrt{\frac{e}{i_0 T}}$$

because the number of electrons is proportional to i. This shows that a long time interval gives an accurate measure of current.

LECTURE IV 10-4-60

Consider the problem of the average distance between molecules at ordinary temperatures. If they are in a cubic lattice, the average distance is $(\frac{N}{V})^{1/3}$. However, we shall consider a gas. Take a sample of one molecule. How far to neighbor? Assume at distance R of a sphere about the sample molecule. We say that the neighbor is between R and $R+\Delta R$.



Apply reasoning that there is none in R and first in ΔR. The problem then reduces to the same problem as emission. From the result of this problem and the fact that the probability that a molecule will be in the sphere is $p = \frac{\frac{4}{3}\pi R^3}{V}$, we can write immediately:

$$(1) \quad w(R)_{\Delta R} = \left[e^{-\frac{\frac{4\pi}{3} R^3 N}{V}} \right] \left[N \frac{4\pi R^2}{V} \Delta R \right]$$

Average Distance:

$$(2) \quad D\,\Delta R = R\,w(R)\,\Delta R$$

$$(3) \quad D = \sum_{0}^{\infty} e^{-\frac{\frac{4\pi}{3}R^3 N}{V}} \cdot \frac{N}{V}\, 4\pi R^3\, \Delta R$$

We replace this by an integral for convenience of calculation:

$$(4) \quad D = \int_0^\infty e^{-\frac{\frac{4\pi}{3}R^3 N}{V}} \frac{N}{V}\, 4\pi R^3\, dR$$

$$= \frac{1}{\left(\frac{4\pi N}{3V}\right)^{1/3}} \underset{\overset{\|}{\Gamma\left(\frac{4}{3}\right)}}{\int_0^\infty e^{-x} x^{1/3}\, dx} = .55396 \left(\frac{V}{N}\right)^{1/3}$$

This is what one would expect from looking at the cubic lattice distance except for the constant which covers the random motion of the molecules.

Random Walk:

Make steps of length l either to right or left. Take N steps. How far do you get? The average is zero, but what is the mean squared displacement? This is the basic problem in diffusion and Brownian motion, that is, steps of length l with equal probability to right or left. What is the probability after N steps that one is at point ml. Could be anywhere between $-Nl$ and $+Nl$, that is:

$$(5) \quad -Nl,\ (-N+2)l,\ \ldots, ml,\ \ldots,\ (N-2)l,\ Nl$$

To be at ml, we must take $\frac{N+m}{2}$ steps to the right and $\frac{N-m}{2}$ steps to the left.

Now one sequence has probability $= \left(\frac{1}{2}\right)^N$

For all sequences:

$$(6) \quad P(m,N) = \frac{N!}{\left(\frac{N+m}{2}\right)! \left(\frac{N-m}{2}\right)!} \left(\frac{1}{2}\right)^N$$

or the Bernoulli distribution. However, it is clumsy to handle, thus make the approximation:

(7) Take $\lim_{N\to\infty}$ letting $\lim pN \to \infty$

Using Stirling's approximation: $N! \simeq \sqrt{2\pi N}\left(\frac{N}{e}\right)^N$

or $\ln N! = (N+\frac{1}{2})\ln N - N + \frac{1}{2}\ln(2\pi) + \ldots O_r(N^{-1})$

Thus:

$$(8) \quad \ln P(m,N) = (N+\tfrac{1}{2})\ln N - \tfrac{1}{2}(N+m)\ln\tfrac{1}{2}(N+m)$$
$$- \tfrac{1}{2}(N-m+1)\ln\tfrac{1}{2}(N-m) - \tfrac{1}{2}\ln(2\pi) - N\ln 2$$

We are interested in the region $\frac{m}{N} << 1$, because all steps in one direction be such a rare event with low probability so that it can be ignored. We will get an approximation that is good except in the tails of the distribution where $p \cong 0$ anyway. Now expand the ln terms thus:

$$(9) \quad \ln\tfrac{1}{2}(N+m) = \ln\tfrac{1}{2}N\left(1+\frac{m}{N}\right) = \ln\tfrac{1}{2}N + \ln\left(1+\frac{m}{N}\right)$$

$$= \ln\tfrac{1}{2}N + \frac{m}{N} - \frac{m^2}{2N^2} + \ldots$$

Do the same for other terms with the result:

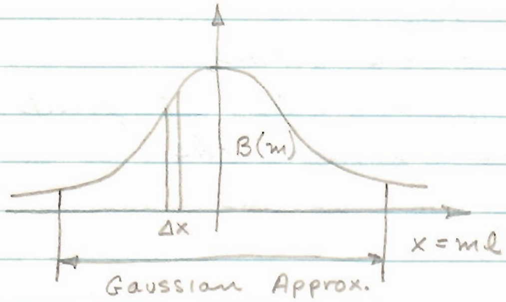
$$(10) \quad \ln P(m,N) = -\tfrac{1}{2}\ln N + \ln 2 - \tfrac{1}{2}\ln 2\pi - \frac{m^2}{2N}$$

+ terms in higher powers of $\frac{1}{N}$ which can be neglected.

Simplifying,

$$(11)\quad P(m,N) = \left(\frac{2}{\pi N}\right)^{1/2} e^{-\frac{m^2}{2N}}$$

which is called a Gaussian distribution, and with which we can replace the Bernoulli distribution; however, it is only defined for integral values of m. Consider the following diagram:



Take $x = ml$ as a continuous variable. An interval of Δm has $\frac{\Delta x}{2l}$ points because m can be either even or odd. Take Δm so small such that $P(m,N)$ is constant, so we have:

$$(12)\quad P(m,N)\,\Delta m = \left(\frac{2}{\pi N}\right)^{1/2} e^{-\frac{m^2}{2N}}\,\Delta m$$

with the m's two apart, then,

$$(13)\quad P(x)\,\Delta x = \left(\frac{1}{2\pi N l^2}\right)^{1/2} e^{-\frac{x^2}{2Nl^2}}\,\Delta x$$

To apply this to diffusion: Assume N' steps in each time interval is constant, $N = N'T$. Now let $N' \to \infty$, $t \to 0$, such that

$$(14)\quad \lim N' l^2 = \text{constant} = 2D$$

Then $2\pi N l^2 = 2\pi N' l^2 T = 4\pi D T$

$$(15)\quad \therefore\ P(x,t)\,\Delta x = \frac{1}{(4\pi D t)^{1/2}}\, e^{-\frac{x^2}{4Dt}}\,\Delta x$$

which is also the solution of the differential equation of motion for diffusion with δ function at origin at $t = 0$.

LECTURE I   10-6-60

REFERENCES: CHAPTERS 3,4,5,6 ; Davenport and Routh
Part I in Middleton

Random Variable: x is able to take on discreet or continuous range. ASSIGN a probability to each point of sample space so that it lies between 0 and 1.

Define P(E) as the probabity $x \leq E$, this overcomes the difficulty of zero probability at points on the continuous distribution curve. Define the probability density or frequency function as

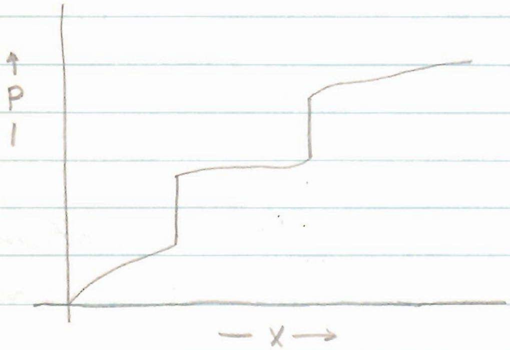
$$(1) \quad f(x) = \frac{\partial P}{\partial x}$$

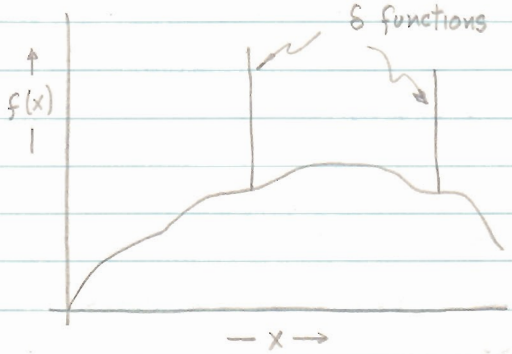
with $f(x)\,\Delta x$ denoting probability to find x in the interval $x, x+\Delta x$. Also

$$(2) \quad \int_{-\infty}^{\infty} f(x)\,dx = 1$$

which means that x must be found somewhere in sample space.

However, certain distributions may be discontinuous at certain points. Therefore, a set of δ functions may be added to the integral.





Joint Probability:

$f(x,y)\,\Delta x\,\Delta y$ denotes joint probability that x is found in $x, x+\Delta x$ and y in $y, y+\Delta y$.

Complex Random Variable: $z = x+iy$ can be introduced through joint probability definition.

Conditional Probability:

What is the probability that y is in $y, y+\Delta y$ when we know that x is in $x, x+\Delta x$?

Answer: $\dfrac{f(x,y)\,\Delta x\,\Delta y}{f(x)\,\Delta x}$

Statistical Independence:

The probability of finding x in $x, x+\Delta x$ is independent of the probability of finding y in $y, y+\Delta y$. Statistical Independence occurs if and only if $f(x,y) = f_1(x)\,f_2(y)$.

The probability, under statistical independence, for x to in $x, x+\Delta x$ if one knows y is in $y, y+\Delta y$ is:

$$(3) \quad \int_x^{x+\Delta x} f_1(x)\,dx = f_1(x)\,\Delta x$$

Problem of a Function of a Random Variable:

A function of a random variable is also a random variable, for example, $\mu = x^2$. The probability that μ takes on x^2 is the probability that x takes x. In general, $\mu = \mu(x)$, $\mu(x)$ given, then what is $F(\mu)$. Assume inversion is possible: $x = x(\mu)$, then,

$$(4) \quad \int_x^{x+\Delta x} f(x)\,dx = \int_\mu^{\mu+\Delta\mu} \underbrace{f(x)\left|\frac{dx}{d\mu}\right|}_{F(\mu)} d\mu \; ; \quad \left|\frac{dx}{d\mu}\right| \text{ because interval in sample space is positive.}$$

Function of Two Random Variables:

Given: $u = u(x,y)$, $v = v(x,y)$, $x = x(u,v)$, $y = y(u,v)$

Consider:

$$f(x,y)\,dx\,dy = \underbrace{f(x,y)\left|\frac{\partial(x,y)}{\partial(u,v)}\right|}_{F(u,v)} du\,dv \leftarrow \text{Jacobian}$$

with $$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Problem of Averages:

The statistical average of a random function $g(x)$ is:

$$(5) \quad \overline{g(x)} = \underbrace{\sum_n g(x_n)\,p(x)}_{\text{discreet}} \Rightarrow \int_{-\infty}^{\infty} \underbrace{g(x)\,f(x)}_{\text{continuous}}\,dx$$

In particular: $\bar{x} = \int_{-\infty}^{\infty} x\,f(x)\,dx$,

and $\overline{x^n} = \int_{-\infty}^{\infty} x^n f(x)\,dx$ is the nth moment of $f(x)$.

Joint Moment.

$$(6) \quad \overline{x^m y^n} = \iint_{-\infty}^{\infty} x^m y^n f(x,y)\,dx\,dy$$

Central Moments are moments with respect to $\bar{x}$ as origin. First central moment $= 0$, second central moment $=$ variance $= \overline{(x-\bar{x})^2}$

Co-variance:

$$(7) \quad \overline{(x-\bar{x})(y-\bar{y})} = \iint_{-\infty}^{\infty} (x-\bar{x})(y-\bar{y})\,f(x,y)\,dx\,dy$$

where $\bar{x} = \iint_{-\infty}^{\infty} x\,f(x,y)\,dx\,dy$

Characteristic Functions:

(8) $$M_x(iu) \equiv \overline{e^{iux}} = \int_{-\infty}^{\infty} e^{iux} f(x)\,dx$$

= Fourier Transform of probability density function $f(x)$, then $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iux} M_x(iu)\,du$.

Moment Generation:

(9) $$\frac{dM_x(iu)}{du} = i\int_{-\infty}^{\infty} x e^{iux} f(x)\,dx$$

take $u=0$; find $\bar{x} = -i \left.\frac{dM_x(iu)}{du}\right]_{u=0}$

In general:

(10) $$\overline{x^n} = (-i)^n \left.\frac{d^n M_x}{du^n}\right]_{u=0}$$

If characteristic function is known, moments are known and vice-versa.
Consider the Taylor expansion for the characteristic function:

(11) $$M_x(iu) = \sum_0^{\infty} \overline{x^n}\,\frac{(iu)^n}{n!}$$

Joint Characteristic Function:

$$M_{xy}(iu, iv) = \iint_{-\infty}^{\infty} e^{iux+ivy} f(x,y)\,dx\,dy = \overline{e^{iux+ivy}}$$

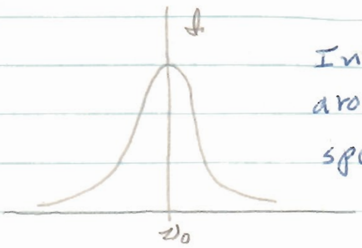
Statistical Independence of Two Variables in Terms of Moments:

$\overline{x^m y^n} = \overline{x^m}\;\overline{y^n}$ is a necessary condition related to the separation of the integrand. For statistical independence, $M_{x,y}(iu, iv) = M_x(iu)\,M_y(iv)$. Two variables are linearly independent $\overline{xy} = \bar{x}\,\bar{y}$, then will find the co-variance zero.

LECTURE VI 10-8-60

Example of Distribution Where No Moments Exist: Cauchy-Lorentz Distribution:

(1) $f(\nu) = \frac{1}{\pi} \frac{u}{1+(\nu-\nu_0)^2}$



Intensity around a spectral line

A very famous distribution is the Gaussian Distribution. For example, this is the distribution of momentum of the molecules of an ordinary gas. The probability of p_x to be in $p_x, p_x+\Delta p_x$ is:

(2) $\frac{1}{(2\pi m kT)^{1/2}} e^{-\frac{p_x^2}{2mkT}} dp_x = f(p_x)\,dp_x$

In Three Dimensions we have the Maxwell-Boltzmann Distribution which is trivariant.

(3) $f(p_x, p_y, p_z)\,dp_x\,dp_y\,dp_z = \frac{1}{(2\pi mkT)^{3/2}} e^{-\frac{p_x^2+p_y^2+p_z^2}{2mkT}}$

$dp_x\ dp_y\ dp_z$

It is seen that p_x, p_y, p_z are statistically independent.

What is the probability for the kinetic energy of the molecules to lie between $E, E+dE$, where $E = \frac{p_x^2+p_y^2+p_z^2}{2m}$?

Transform the momentum to spherical co-ordinates in p, ϑ, φ:

(4) $$\frac{1}{(2m\pi kT)^{3/2}} e^{-\frac{p^2}{2mkT}} \underbrace{p^2 \sin\vartheta}_{\substack{\text{Jacobian} \\ \frac{\partial(p_x\,p_y\,p_z)}{\partial(p,\,\vartheta,\,\varphi)}}}\, dp\, d\vartheta\, d\varphi$$

Now we have:

$$(5)\quad f(p) = \int_0^{\pi}\int_0^{2\pi} f(p,\vartheta,\varphi)\sin\vartheta\, d\vartheta\, d\varphi = \frac{4\pi p^2}{(2\pi m kT)^{3/2}}\, e^{-\frac{p^2}{2mkT}}$$

Now: $p^2 dp = 1/2\, p\, d(p^2) = m\,(2mE)^{1/2} dE$

Then:

$$(6)\quad f(E)\,dE = \frac{2\pi}{(\pi kT)^{3/2}}\, E^{1/2}\, e^{-E/kT}\, dE$$

However, it would be easier to calculate $\bar{E}$ by the following method:

$$(7)\quad \bar{E} = \frac{\overline{p_x^2} + \overline{p_y^2} + \overline{p_z^2}}{2m} = 3/2\, kT$$

calculating the average value of each component of the momentum squared.

For the joint moment, we have, by statistical independence:

$$(8)\quad \overline{p_x^n p_y^m} = \overline{p_x^n}\;\overline{p_y^m}$$

Important: p_x and p_x^2 are linearly independent because the first moment vanishes. This is an example that linear independence is weaker than statistical independence.

The Maxwell-Boltzmann distribution can be generalized to take into account other effects such as the gravitational potential.

$$(9)\quad f(x,y,z,p_x,p_y,p_z) = C\, \underbrace{e^{-\frac{V(x,y,z)}{kT}}}_{\text{Not Gaussian}}\; e^{-\frac{p_x^2+p_y^2+p_z^2}{2mkT}}$$

For gravity: $V = mgz$

If $V = 1/2\,\alpha x^2 + 1/2\,\beta y^2 + 1/2\,\gamma z^2$ (harmonic oscillator) then we get a Gaussian distribution in both position and momentum.

Properties of the One Dimensional Gaussian Distribution:

(10) $$w(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$\overline{x^n} = 0 \quad \text{for } n \text{ odd}$$
$$\overline{x^n} = 1.3.5. \dots (n-1) \quad \text{for } n \text{ even}$$

which is found by integrating the following integral by parts and obtaining a recursion equation:

(11) $$\overline{x^n} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^n e^{-\frac{x^2}{2}} dx$$

Consider the new variable; $y = \sigma x + m$. Then:

(12) $$w(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{(y-m)^2}{2\sigma^2}} dy$$

and $$\overline{(y-m)^n} = 1 \cdot 3 \cdot 5 \cdot 7 \cdot \dots (n-1)\, \sigma^n \quad \text{for } n \text{ even}$$

Characteristic Function:

(13) $$M(t) = 1 + \frac{(jt)^2 \sigma^2}{2} + \dots = 1 + (j\sigma)^2 \frac{t^2}{2} + 1 \cdot 3\, (j\sigma)^4 \frac{t^4}{4!}$$
$$+ \dots + \underbrace{1 \cdot 3 \cdot 5 \cdot \dots (2n-1)\, (j\sigma)^{2n} \frac{t^{2n}}{(2n)!}}_{\left(\frac{j^2\sigma^2}{2}\right)^n \frac{t^{2n}}{n!}}$$
$$= e^{-\frac{\sigma^2 t^2}{2}}$$

We take the Fourier transform of $w(y)$ with the above characteristic:

(14) $$\int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{\sigma^2 t^2}{2}} e^{j(y-m)t} dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\left[\frac{\sigma t}{\sqrt{2}} + \frac{y-m}{j\sqrt{2}\sigma}\right]^2} dt$$
$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-m)^2}{2\sigma^2}}$$

The characteristic of a Gaussian is a Gaussian.

LECTURE VII 10-11-60

Let us start with two statistically independent variables, x, y. The density function is:

(1) $f_1(x) f_2(y)$

Consider $z = x + y$ which is also a random variable, with mean $\bar{z} = \bar{x} + \bar{y}$.
Also, $\overline{z^2} - (\bar{z})^2 = \overline{x^2} - (\bar{x})^2 + \overline{y^2} - (\bar{y})^2$

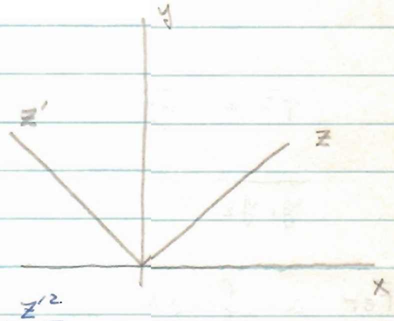
Consider the Gaussian:

(2) $\frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}} dx\, dy$, with σ the same

Define:

(3) $z = \frac{1}{\sqrt{2}}(x+y)$

Geometrically:



$z' = \frac{1}{\sqrt{2}}(x-y)$

(4) Then: $x^2 + y^2 = z^2 + z'^2$

and we have: $\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{z^2}{2\sigma^2}} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{z'^2}{2\sigma^2}}$

Integrate over z' and get $\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{z^2}{2\sigma^2}}$

Consider a multi-variate Gaussian Distribution:

(5) $w(x_1, x_2, \ldots x_n) = \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{x_1^2}{2\sigma_1^2}} \frac{1}{\sqrt{2\pi\sigma_2^2}} e^{-\frac{x_2^2}{2\sigma_2^2}} \ldots \frac{1}{\sqrt{2\pi\sigma_n^2}} e^{-\frac{x_n^2}{2\sigma_n^2}}$

Make a linear transformation for two random variables.

(6) $y_1 = a_{11} x_1 + a_{12} x_2$
$y_2 = a_{21} x_1 + a_{22} x_2$

Find $x_1(y_1, y_2)$ and $x_2(y_1, y_2)$ and substitute in (5).

Multiply by the Jacobian and get the bivariate Gaussian distribution.

$$(7) \quad \frac{1}{2\pi\sigma\tau(1-\rho^2)^{1/2}} \exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{y_1^2}{\sigma^2}+\frac{y_2^2}{\tau^2}-\frac{2\rho}{\sigma\tau}y_1y_2\right)\right] = w(y_1 y_2)$$

We assert that:

$$(8) \quad \overline{y_1^2} = \sigma^2, \quad \overline{y_2^2} = \tau^2, \quad \overline{y_1 y_2} = \rho\sigma\tau$$

ρ is the correlation coefficient, a sort of normalized joint linear moment, with the restriction that: $-1 \leq \rho \leq 1$. If $\rho = 0$, then we have two independent Gaussian distributions.

$$(9) \quad \sigma^2 = \overline{y_1^2} = a_{11}^2 \underset{\sigma_1^2}{\underset{\shortparallel}{\overline{X_1^2}}} + a_{12}^2 \underset{\sigma_2^2}{\underset{\shortparallel}{\overline{X_2^2}}}$$

$$\tau^2 = \overline{y_2^2} = a_{21}^2\sigma_1^2 + a_{22}\sigma_2^2$$

$$\overline{y_1 y_2} = \rho\sigma\tau = a_{11}a_{21}\sigma_1^2 + a_{12}a_{22}\sigma_2^2$$

For a fixed y_2 (conditional probability), we get a Gaussian distribution for y_1.

Consider the General Case: A joint Gaussian distribution function for n independent variables, viz., equation (5):

$$(10) \quad w(X_1 \ldots X_n) = \prod_{\lambda=1}^{n} \frac{1}{\sqrt{2\pi\sigma_\lambda^2}} e^{-\frac{X_\lambda^2}{2\sigma_\lambda^2}}$$

Consider $y_1, \ldots, y_s$. s may be less than n because the integration over $n-s$ will give unity. We define the transformation:

$$(11) \quad y_k = \sum_{\lambda=1}^{n} a_{k\lambda} X_\lambda$$

which will be distributed according to an s variate Gaussian distribution.

Then:

$$(12)\quad w(y_1, \ldots, y_s) = \frac{1}{(2\pi)^{3/2}\, B^{1/2}} \exp\left\{ -\frac{1}{i B} \sum_{\substack{l=1\\k=1}}^{s} B_{kl}\, y_k\, y_l \right\}$$

where $B = |B_{kl}|$, and B_{kl} = cofactor of element b_{kl} in the b matrix defined by:

$$(13)\quad b_{kl} = \sum_{i=1}^{n} a_{ki}\, a_{li}\, \sigma_i^2 = \langle y_k\, y_l \rangle$$

Proof: Introduce a δ function such that:

$$(14)\quad \delta(x-x') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{it(x-x')}\, dt$$

$$(15)\quad w(y_1, \ldots, y_s) = \frac{1}{(2\pi)^{\frac{n}{2}+s}\, \sigma_1 \cdots \sigma_n} \int e^{-\frac{1}{2} \sum_1^n \frac{x_i^2}{\sigma_i^2}}\, dx_1 \ldots dx_n$$

$$\cdot \int \prod_{k=1}^{s} \exp\left[i\, t_k \left(y_k - \sum_{i=1}^{n} a_{ki}\, x_i \right) \right] dt_1 \ldots dt_s$$

We have taken the old distribution and multiplied by the transformation expressed by the δ function. We now can make perfect squares out of the x's. Thus, integrating over the x's and keeping y and t as parameters:

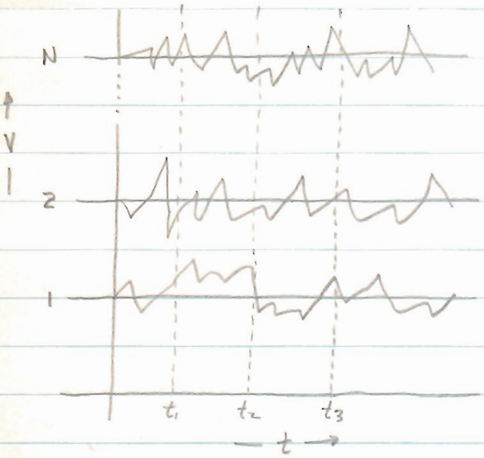
$$(16)\quad w(y_1, \ldots, y_s) = \frac{1}{(2\pi)^s} \int_{-\infty}^{\infty} dt_1 \ldots dt_s\, \exp\left[i \sum_{k=1}^{s} t_k\, y_k - \frac{1}{2} \sum_{\substack{k=1\\l=1}}^{s} b_{kl}\, t_k\, t_l \right]$$

Now we have the characteristic function of the y distribution, so the problem is essentially solved. What is left to do is perform a Fourier transformation. Make perfect squares in t's. Introduce the transformation $u_m = \sum_k C_{mk}\, t_k$ to make perfect squares in u. Make another transformation involving a linear combination of the y's: $\sum_k y_k t_k = \sum_k \eta_k u_k$

Substitute, integrate, and get a distribution in η. Then reverse and finally get distribution in y.

Random Processes:

Consider an ensemble of N diodes with noise signals present.



Considering voltages at different times in the ensemble of diodes, we get in general, the diagram on the left. This is mathematically described by a series of joint distribution functions.

$$\left.\begin{array}{l} w_1(V_1 t_1) \\ w_2(V_1 t_1, V_2 t_2) \\ \vdots \end{array}\right\}$$

An infinite number will describe a random process, and V and t will become continuous.

LECTURE VIII 10-13-60

Correlation Coefficient:

(1) $\overline{y_1^2} = \sigma^2$, $\overline{y_2^2} = \tau^2$, $\overline{y_1 y_2} = \rho\sigma\tau$

(2) $\overline{\left(\frac{y_1}{\sigma} + \frac{y_2}{\tau}\right)^2} = 1 + 1 + \frac{2\,\overline{y_1 y_2}}{\sigma\tau} \geq 0$

then $2(1+\rho) \geq 0$

Start again with

(3) $\overline{\left(\frac{y_1}{\sigma} - \frac{y_2}{\tau}\right)^2} = 1 + 1 - \frac{2\,\overline{y_1 y_2}}{\sigma\tau} \geq 0$

then $2(1-\rho) \geq 0$

Therefore: $\rho \leq 1$

If $\rho = 0$, $\overline{y_1^n y_2^m} = \overline{y_1^n}\ \overline{y_2^m}$ for a Gaussian distribution whose random variables are linearly and statistically independent.

Gaussian Processes:

We have $w_1(x_1 t_1)$ which is the probability that we are between x_1, $x_1 + \Delta x_1$; t_1, $t_1 + \Delta t_1$, and we can form the joint probabilities $w_2(x_1 t_1, x_2 t_2)$; $\omega_3(x_1 t_1, x_2 t_2, x_3 t_3)$. Consider the noise voltages on a large number of diodes, as per last lecture. A purely random process is defined as $w_2(x_1 t_1, x_2 t_2) = w_1(x_1 t_1)\, w_2(x_2 t_2)$.

Markoff Process:

The probability depends on past history, but if known at one previous time, knowledge of all others is not needed, that is, for example:

(4) $P(x_3 t_3 | x_1 t_1, x_2 t_2) = P(x_3 t_3 | x_2 t_2)$ does not depend on $P(x_1 t_1)$

Now through a combination of a pure random process and an Markovian process, we should be able to find the joint probability. Consider $w_1(x_2 t_2)$, then:

(5) $P_2(x_1 t_1 | x_2 t_2)\, w_1(x_2 t_2) = w_2(x_1 t_1, x_2 t_2)$

$P_3(x_3 t_3 | x_2 t_2)\, w_1(x_2 t_2) = w_2(x_3 t_3, x_2 t_2)$

Now: $P(x_3 t_3 | x_2 t_2)\, P(x_1 t_1 | x_2 t_2)\, w_1(x_2 t_2) = \omega_3(x_1 t_1, x_2 t_2, x_3 t_3)$

$$= \frac{w_2(x_1 t_1, x_2 t_2)\, w_2(x_3 t_3, x_2 t_2)}{w_1(x_2 t_2)}$$

P_2 must satisfy the Smolochowsky Equation. We proceed as follows:

(6) $\omega_3(x_1 t_1, x_2 t_2, x_3 t_3) = P(x_3 t_3 | x_2 t_2, \underset{\substack{\downarrow \\ \text{Not needed (Markoff Process)}}}{x_1 t_1})\, P(x_2 t_2 | x_1 t_1)\, w_1(x_1 t_1)$

We now integrate over x_2 space and get:

$$(7) \quad w_2(x_3 t_3, x_1 t_1) = P(x_3 t_3 | x_1 t_1)\, w_1(x_1 t_1)$$

$$= w_1(x_1 t_1) \int P(x_3 t_3 | x_2 t_2)\, P(x_1 t_1 | x_2 t_2)\, dx_2$$

Upon simplifying, we have the Smolchowsky Equation.

$$(8) \quad P(x_3 t_3 | x_1 t_1) = \int P(x_3 t_3 | x_2 t_2)\, P(x_1 t_1 | x_2 t_2)\, dx_2$$

Example of a Markoff Process: Random Walk.

The position after n steps depends not on the probability of the previous step but only on the position of the previous step and not those before. If one step position away is not known, knowledge of two steps away helps.

Stationary Random Process:

This is a process where the choice of time co-ordinates does not matter. All co-ordinate probability functions are dependent only on a time interval.

Example: noise voltage across resistors or diodes

Transient processes are not stationary random processes.

Consider Voltage and Current:

$$(9) \quad \omega_3(V_1 \varphi_1 t_1, V_2 \varphi_2 t_2, V_3 \varphi_3 t_3) = \omega_{3V}(V_1 t_1, V_2 t_2, V_3 t_3)\, \omega_{3\varphi}(\varphi_1 t_1, \varphi_2 t_2, \varphi_3 t_3)$$

$V_1 t_1$, etc., is now redundant symbology as there are no longer samples involved.

Correlation Functions: $R_x(t_1 t_2) = \overline{x(t_1)\, x(t_2)} = \overline{x_1 x_2}$

For two fixed times, R_x becomes the joint linear moment. For a stationary process, $R_x(t_1 t_2) = R_x(\tau) = \overline{x(0)\, x(\tau)}$

Joint Linear Moment around the Mean Value; Normalized Auto-Correlation Function:

$$(10)\quad \rho_x(t_1 t_2) = \frac{\overline{(x_1-\bar{x}_1)(x_2-\bar{x}_2)}}{\sigma_1\sigma_2} = \frac{\overline{x_1 x_2} - \bar{x}_1\bar{x}_2}{\sigma_1\sigma_2}$$

$$\sigma_1 = \left[\overline{(x_1-\bar{x}_1)^2}\right]^{1/2}, \quad \sigma_2 = \left[\overline{(x_2-\bar{x}_2)^2}\right]^{1/2}$$

For a stationary process:

$$(11)\quad \rho_x(\tau) = \frac{R_x(\tau) - (\bar{x})^2}{\sigma^2}$$

LECTURE IX 10-15-60

Two Simultaneous Random Processes:

Cross-Correlation Function:

$$(1)\quad R_{xy}(t_1 t_2) = \overline{x_1 y_2^*} \longleftarrow \text{if } x \text{ and } y \text{ are complex}$$

x_1 taken at t_1

y_2 taken at t_2

Auto-correlation for a complex variable:

$$(2)\quad R_x(t_1 t_2) = \overline{x_1 x_2^*}$$

For a stationary process:

$$(3)\quad R_{xy}(\tau) = \overline{x(0)\,y^*(\tau)}$$

We can show: $R_{xy}(\tau) = R_{yx}^*(-\tau)$

$R_x(\tau) = R_x^*(-\tau)$

If the random variable is real, R_x is an even function.

Proof: $R_x(t, t+\tau) = R_x(t-\tau, t) = \overline{x(t-\tau)\,x^*(t)}$

$$= \left[\overline{\{x^*(t-\tau)\,x(t)\}}\right]^* = R_x^*(-\tau)$$

From the foregoing, taking the complex conjugate and the statistical average can be interchanged. If stationary, Rx is a decreasing function of time. That is, take

$$(4) \quad \overline{\left(\frac{x(0)}{\sigma} \pm \frac{x(\tau)}{\sigma}\right)^2} \geq 0$$

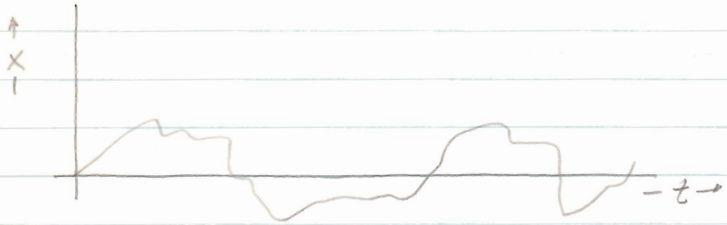
$$\underbrace{\frac{\overline{x^2(0)}}{\sigma^2} + \frac{\overline{x^2(\tau)}}{\sigma^2}} \pm 2\,\frac{\overline{x(0)\,x(\tau)}}{\sigma^2} \geq 0$$

$$2\,\frac{\overline{x^2(0)}}{\sigma^2} \pm 2\,\frac{\overline{x(0)\,x(\tau)}}{\sigma^2}$$

and $\left|\overline{x(0)\,x(\tau)}\right| \leq \overline{x^2(0)}$

Thus the correlation function decreases as time progresses for a stationary process.

Integrals Along the Process:



$$(5) \quad y(s) = \int_a^b h(t)\, x(s,t)\, dt$$

limit of sum of random variables $x(s_n, t_n)$

depends on circuit configuration

random variable such as position of electrons in device.

Take h(t) constant:

$$(6) \quad y = \frac{1}{b-a}\int_a^b x(s,t)\, dt$$ = time average of the random process over a finite time interval.

Now;

$$(7)\quad \langle x \rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(s,t)\,dt \; = \text{time average of } x.$$

If the process is stationary, the limit exists and has physical significance only in this type of process as a rule.

Time Auto and Cross Correlation:

$$(8)\quad R_x = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, x(t+\tau)\, dt \quad ; \text{ time auto-correlation}$$

$$(9)\quad R_{xy}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, y^*(t+\tau)\, dt \quad ; \text{ time cross-correlation.}$$

Ergodicity: The time average is the same as the statistical average. This means that the time average over a large number of samples at one time is the same as the time average of one sample over a large time interval.

General Problem of Random Flights:

Consider step j in a series of N steps. The distribution function is $\tau_j(\bar{r}_j)\,d\bar{r}_j$, the probability density for the j th step.

What is the probability to arrive in $\bar{R}$, $\bar{R}+d\bar{R}$ after N steps?

$$(10)\quad W_N(\bar{R}) = \underset{\substack{3N \\ \uparrow \\ \text{subject to the} \\ \text{restriction} \\ \sum \bar{r}_j = \bar{R}}}{\int\int\int\int\int\int\int\cdots} \left[\frac{1}{(2\pi)^3} \int\int\int e^{i\left(\sum_{j=1}^{N} \bar{r}_j - \bar{R}\right)\cdot \bar{p}}\, d\bar{p} \right] \prod_{j=1}^{N} \tau_j(\bar{r}_j)\, d\bar{r}_1 \cdots d\bar{r}_j \cdots d\bar{r}_N$$

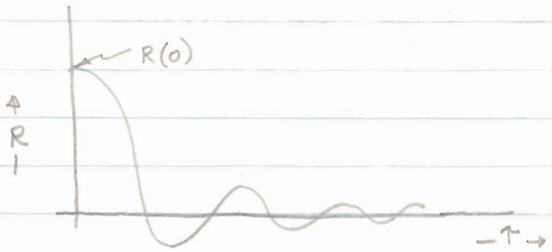
Because of the awkward boundary conditions, we multiply by the 3 dimensional δ function, $\delta\left(\sum_{j=1}^{N} \bar{r}_j - \bar{R}\right)$.

LECTURE X 10-20-60

Errata: $\frac{2\overline{x^2}}{\sigma^2} \pm 2\,\frac{\overline{x(0)\,x(\tau)}}{\sigma^2} \geq 0$

then $R(0) \geq |R(\tau)|$, which means that $R(\tau)$ has a limit which is $R(0)$.

Example:



Random Flights: take a step in an arbitrary direction of arbitrary length. Let $\tau(\bar{r})\,d\bar{r}$ be the probability of a step in the direction $\bar{r}$, $\bar{r}+d\bar{r}$. After N steps what is the position $\bar{R}$ which is the total displacement $\bar{R} = \sum_j \bar{r}_j$ where $\bar{r}_j$ is the direction and length of the jth step?

$$(1)\quad W_N(R)\,dR = \underset{3N}{\iiint} \prod_{j=1}^{N} \tau_j(\bar{r}_j)\,d\bar{r}_j$$

with BC of $\sum \bar{r}$, lies between $\bar{R}$, $\bar{R}+d\bar{R}$

We can get rid of the cumbersome boundary conditions with a δ function:

$$(2)\quad \delta\left(\sum_j \bar{r}_j - \bar{R}\right) = \delta\left(\sum x_j - X\right)\delta\left(\sum y_j - Y\right)\delta\left(\sum_j z_j - Z\right)$$

which can be written in an integral form as:

$$(3)\quad \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\left(\sum x_j - X\right)\rho_x}\,d\rho_x \quad \text{, etc.}$$

Now:

$$(4)\quad W_N(\bar{R})\, d\bar{R} = \frac{1}{(2\pi)^3}\, d\bar{R} \iiint_{\rho_x \rho_y \rho_z} e^{-i\bar{\rho}\cdot\bar{R}}\, A_N(\bar{\rho})\, d\bar{\rho}$$

$$\text{where } A_N(\bar{\rho}) = \prod_{j=1}^{N} \iiint_{3N} e^{i\bar{\rho}\cdot\bar{r}_j}\, \tau_j(\bar{r}_j)\, d\bar{r}_j$$

$$= \left[\iiint e^{i\bar{\rho}\cdot\bar{r}}\, \tau(\bar{r})\, d\bar{r} \right]^N \text{ for identical } \tau_j$$

Now assume each τ_j is a Gaussian Distribution, although not necessarily the same in each step.

$$(5)\quad \tau_j = \frac{1}{(2\pi l_j^2/3)^{3/2}}\, e^{-3|\bar{r}_j|^2/2 l_j^2}$$

where l_j is the mean displacement in the j th step. We get for $A_N(\bar{\rho})$:

$$(6)\quad A_N(\bar{\rho}) = \exp\left[-|\bar{\rho}|^2 \sum_{j=1}^{N} \frac{l_j^2}{6} \right] = \exp\left\{ \frac{-|\bar{\rho}|^2\, N\, \overline{l^2}}{6} \right\}$$

and:

$$(7)\quad W_N(\bar{R})\, d\bar{R} = \frac{1}{(2\pi N \overline{l^2}/3)^{3/2}} \exp\left\{ \frac{-3|\bar{R}|^2}{2N\overline{l^2}} \right\}$$

$\overline{l^2}$ is the mean squared distance of each step.

Regardless of the special nature of τ, $W(\bar{R})$ tends in large N to be a Gaussian. This is called the Central Limit Theorem.

$$(8)\quad A_N(\rho) = \left[\int e^{i\rho x}\, \tau(x)\, dx \right]^N = \left[1 + i\rho\langle x\rangle - \tfrac{1}{2}\rho^2\langle x^2\rangle + \cdots \right]^N$$

If $N \gg 1$, then:

$$(9)\quad A_N(\rho) = e^{iN\rho\langle x\rangle - \frac{1}{2}N\rho^2\langle x^2\rangle}$$

and:

$$(10)\quad W(X) = \frac{1}{2\pi}\int e^{-\frac{1}{2}N\rho^2\langle x^2\rangle - i\rho(X - N\langle x\rangle)}\,d\rho$$

$$= \frac{1}{\left[2\pi N\langle x^2\rangle\right]^{1/2}}\, e^{-\frac{(X-N\langle x\rangle)^2}{2N\langle x^2\rangle}}$$

which is the central limit.

This means that we can sample an arbitrary distribution function many times and get a Gaussian. We can therefore say that the distribution of the sample mean is a Gaussian.

$$(11)\quad W(\underset{\substack{\text{over}\\ N\\ \text{samples}}}{X_{\text{mean}}}) = \frac{1}{\left[2\pi\,\frac{\langle x^2\rangle}{N}\right]^{1/2}}\, e^{-\frac{(X_m - \langle x\rangle)^2}{2\,\frac{\langle x^2\rangle}{N}}}$$

LECTURE XI 10-22-60

Sample Means: $S_N = \dfrac{A_1 + A_2 + \cdots + A_N}{N}$ which tends to a Gaussian at high N.

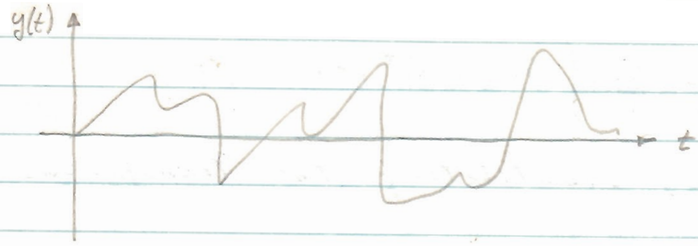
Mean Square Deviation:

$$\overline{(S_N - \underset{\substack{\uparrow\\ \text{true}\\ \text{Mean}}}{S_m})^2} = \frac{\overline{(A_j - S_m)}^2}{N}$$

Each measurement is statistically independent of the others. Only errors that are averaged out are statistical errors, not systematic errors.

Wiener - Khintchine Theorem:

Power Spectral Density:



(1) $\mathcal{R}_y(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_0^T y(t)\,y(t+\tau)\,dt$

Consider $y(t)$ to be bounded in a finite time interval.

Introduce $Y_T = y(t) \quad 0 \leq t \leq T$
$= 0 \quad$ elsewhere

T can be taken so large that times beyond no longer are important. One could also define $Y_T(t)$ to be periodic in T. However, we will just say that $\int_{-\infty}^{\infty} Y_T(t)\,dt$ exists.

Expanding and defining in Fourier integrals:

(2) $Y_T(t) = \int_{-\infty}^{\infty} S_T(f)\,e^{-2\pi i f t}\,df$

$S_T(f) = \int_{-\infty}^{\infty} Y_T(t)\,e^{2\pi i f t}\,dt$

} no 2π factor because f is used and not ω.

If $y(t)$ represents the current in a 1Ω resistor, then $\frac{1}{T}\int_0^T y^2(t)\,dt$ is the average power dissipated, and:

$\frac{S_T^2(f)}{T}$ represents the power dissipated in a unit frequency interval around f.

If $Y_T(t)$ is real, then $S_T(f) = S_T^*(-f)$

Returning to the correlation factor:

$$(3) \quad \mathcal{R}_y(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_0^T y(t)\, y^*(t+\tau)\, dt$$

$$\Rightarrow \quad \mathcal{R}_Y(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_0^T Y_T(t)\, Y_T^*(t+\tau)\, dt$$

when $T >> \tau$. Now, substituting the transforms for the Y_T's:

$$(4) \quad \mathcal{R}_Y(\tau) = \lim_{T\to\infty} \iint_{-\infty}^{\infty} \frac{S_T(f)\, S_T(f')}{T}\, e^{2\pi i f'\tau}\, df\, df' \underbrace{\int_{-\infty}^{\infty} e^{-2\pi i f t}\, e^{2\pi i f' t}\, dt}_{\delta(f-f')}$$

$$= \lim_{T\to\infty} \int_{-\infty}^{\infty} \frac{S_T(f)\, S_T^*(f)}{T}\, e^{2\pi i f\tau}\, df$$

$$= \lim_{T\to\infty} \int_0^{\infty} \frac{2\,|S_T(f)|^2}{T} \cos 2\pi f t\, df \quad , \text{ if } Y_T \text{ is real.}$$

Now introduce $G(f) = \lim_{T\to\infty} \frac{2\,|S_T(f)|^2}{T}$ for positive frequencies and is called the power spectral density. Therefore:

$$(5) \quad \mathcal{R}_Y(\tau) = \int_0^{\infty} G(f) \cos 2\pi f t\, df$$

The time auto-correlation factor is thus the Fourier transform of the Power Spectral Density. We could have defined:

$$(6) \quad G'(f) = \lim_{T\to\infty} \frac{S_T(f)\, S_T^*(f)}{T}$$

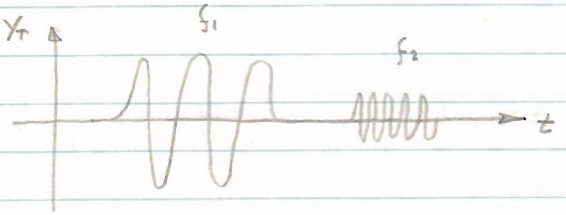
which is half the previous $G(f)$ because $G'(f)$ is over both positive and negative frequencies with $G'(f) = G'(-f)$ for real processes.

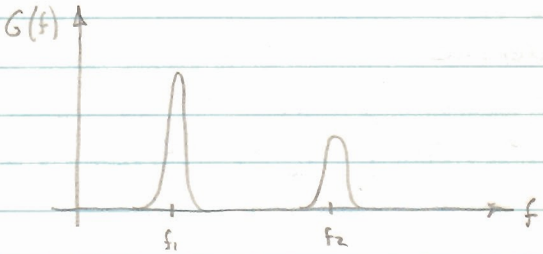
$$(7) \quad G'(f) = \tfrac{1}{2}\, G(f)$$

The inverse transform is:

(8) $$4\int_0^\infty \mathcal{R}_y(\tau)\cos 2\pi f\tau\, d\tau = G(f)$$

Examples:





LECTURE XII 10-25-60

Power Spectral Density:

(1) $$\mathcal{R}_y(\tau) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} y(t)\,y^*(t+\tau)\,dt$$

(2) $$G(f) = \int_{-\infty}^{\infty}\mathcal{R}_y(\tau)\,e^{2\pi i f\tau}\,d\tau$$

(3) $$G(f) = \lim_{T\to\infty}\frac{S_T(f)\,S_T^*(f)}{T}$$

For an ergotic process, we can replace $\mathcal{R}$ by R, because the time average is equal to the statistical average. For ergotic stationary processes:

(4) $$G(f) = \int_{-\infty}^{\infty} R(\tau)\,e^{2\pi i f\tau}\,d\tau$$

$$R(\tau) = \overline{y(t)\,y(t+\tau)}$$

We can extend this reasoning to non-ergotic processes if $R(\tau)$ exists and is independent of the time. This is called a wide-sense stationary process. Then the existence of $R(\tau)$ implies $G(f)$ exists. $G(f)$ is now the statistical power density and is non-negative because of upper bound on $R(\tau)$.

$$\overline{\left|\int_0^T y(t)e^{2\pi i f t}\,dt\right|^2} \geq 0$$

We may now construct:

$$(5)\quad \frac{1}{T}\int_0^T y(t)\, e^{2\pi i f t}\, dt \int_0^T y^*(s)\, e^{2\pi i f s}\, ds$$

$$= \frac{1}{T}\int_0^T\int_0^T R(t-s)\, e^{2\pi i f(t-s)}\, ds\, dt$$

We define $\tau = t-s$

$$(6)\quad (5) = \frac{1}{T}\int_0^T\int_\tau^T R(\tau)\, e^{2\pi i f \tau}\, d\tau\, dt + \frac{1}{T}\int_{-T}^{0}\int_0^{T+\tau} R(\tau)\, e^{-2\pi i f \tau}\, d\tau\, dt \geq 0$$

$$= \int_{-T}^{+T}\left(1-\frac{|\tau|}{T}\right) R(\tau)\, e^{-2\pi i f \tau}\, d\tau$$

We now define $R_T(\tau) = \left(1-\frac{|\tau|}{T}\right) \quad |\tau| \leq T$; then:

$= 0 \quad |\tau| > T$

(7) $\lim_{T\to\infty} R_T(\tau) \Rightarrow R(\tau)$ for every τ, that is, $\lim_{T\to\infty} R_T(\tau)\, R(\tau) = R(\tau)$

Therefore:

$$(8)\quad \lim_{T\to\infty}\int_{-T}^{T}\left(1-\frac{|\tau|}{T}\right) R(\tau)\, e^{-2\pi i f \tau}\, d\tau = \int_{-\infty}^{\infty} R(\tau)\, e^{-2\pi i f \tau}\, d\tau \geq 0$$

which shows that the statistical spectral density is positive.

Consequences: $R(\tau) = \int_{-\infty}^{\infty} G(f)\, e^{-2\pi i f \tau}\, df$

$R(0) = \int_{-\infty}^{\infty} G(f)\, df \rightarrow \int_{-\infty}^{\infty} \frac{y(t)\, y^*(t)}{T}\, dt = \int_{-\infty}^{\infty} \frac{S(f)\, S^*(f)}{T}\, df$

For real, stationary processes, $G(f) = G(-f)$.

Reference on W-K Theorem: See Davenport and Root, Ch. 6.

Examples:

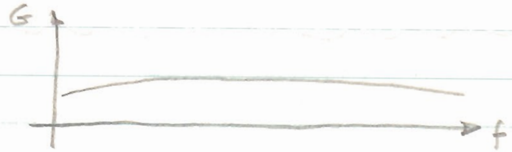
I: Take a stationary random process with a Gaussian correlation function and with a normalized auto-correlation function, $R(0) = 1$.

$$R(\tau) = e^{-\frac{\tau}{2\epsilon^2}}$$

$$G(f) = \int_{-\infty}^{\infty} e^{-\frac{\tau^2}{2\epsilon^2}} e^{2\pi i f \tau} d\tau = \epsilon \sqrt{2\pi}\, e^{-\frac{(2\pi f \epsilon)^2}{2}}$$

Note that: $\int_{-\infty}^{\infty} G(f)\, df = 1$

Point of interest: if $\epsilon \to 0$, $R(\tau) \to \delta(\tau)$, a purely random process, and the power spectral density → constant = 0. However, physically, purely random processes do not occur. As ϵ gets small, $G(f) \to$ constant, $G(f) \to$ constant $\neq 0$, or a wide spectrum.



II: Take the random process: $y(t) = A \cos(\omega_0 t + \varphi)$. A and φ are independent random variables. φ is distributed uniformly over $0 - 2\pi$, $f(\varphi) = \text{constant} = 2\pi$.

$$\overline{y(t)\, y(t+\tau)} = \tfrac{1}{2} \overline{A^2} \cos \omega_0 \tau + \tfrac{1}{2} \overline{A^2}\, \overline{\cos(2\omega_0 t + 2\varphi + \omega_0 \tau)}$$

$$\overline{\cos(2\omega_0 t + 2\varphi + \omega_0 \tau)} = \int_{-\infty}^{\infty} f(\varphi) \cos(2\omega_0 t + 2\varphi + \omega_0 \tau)\, d\varphi$$

$$= 2\pi \int_0^{2\pi} \cos(2\omega_0 t + 2\varphi + \omega_0 \tau)\, d\varphi = 0$$ since the cos is even over the interval.

$\overline{y(t)\, y(t+\tau)} = \tfrac{1}{2} \overline{A^2} \cos \omega_0 \tau$, thus we have a stationary process in the wide sense.

Now;

$$G^+(t) = 2\bar{A}^2 \int_0^\infty \cos \omega_0 \tau \; \cos 2\pi f \tau \, d\tau = \tfrac{1}{2} \bar{A}^2 \, \delta(f-f_0)$$

Because:

$$\int_0^\infty \cos 2\pi f \tau \, d\tau = \tfrac{1}{2} \int_0^\infty (e^{2\pi i f \tau} + e^{-2\pi i f \tau}) \, d\tau$$

$$= \tfrac{1}{2} \int_{-\infty}^\infty e^{2\pi i f \tau} \, d\tau = \tfrac{1}{2} \delta(f)$$

$$\therefore \int_0^\infty \cos \omega_0 \tau \; \cos 2\pi f \tau \, d\tau = \int_0^\infty \left[\tfrac{1}{2} \cos \{2\pi (f+f_0)\tau\} + \tfrac{1}{2} \cos \{2\pi (f-f_0)\tau\} \right] d\tau$$

$$= \tfrac{1}{4} \delta(f-f_0)$$

We see that this is a non-ergotic process.

LECTURE XIII 10-27-60

Non-Stationary Random Processes:

Take $y(t) = A \cos \omega_0 t$

$$\underset{\text{random process}}{\underset{\uparrow}{y(t)}} = \underset{\text{random variable}}{\underset{\uparrow}{A}} \cos \omega_0 t$$

(1) $y(t)\, y(t+\tau) = \tfrac{1}{2} A^2 \cos \omega_0 \tau + \tfrac{1}{2} A^2 \cos (2\omega_0 t + \omega_0 \tau)$

This depends on t explicitly and the process is therefore not stationary and cannot have a Fourier transform to the PSD. However, we can construct $R_y(\tau)$ for each individual sample as follows:

(2) $$R_y(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_0^T y(t)\, y(t+\tau)\, dt$$

$$= \tfrac{1}{2} A^2 \cos \omega_0 \tau + \lim_{T\to\infty} \frac{\tfrac{1}{2} A^2}{T} \int_0^T \cos (2\omega_0 t + \omega_0 \tau)\, dt$$

$$= \tfrac{1}{2} A^2 \cos \omega_0 \tau$$

We can still define the PSD for each sample even though it is statistically not stationary. From previous results, the spectral density is obviously a δ function.

Example: Random Telegraph Wave

One Sample:

We will make the Fourier transform of the auto-correlation function for each sample function. The important thing about the process is to indicate the number of zero crossings in a given time interval. We say that the average number of zero crossings is a per second. We assume that the number of zero crossings k in a time interval T will be Poisson distributed.

$$(3) \quad P_T(k) = \frac{(aT)^k}{k!} e^{-aT}$$

We now construct the auto-correlation function $R_Y(\tau)$, saying that $y(t)\,y(t+\tau) = +1$ if number of 0 crossings is even
$= -1$ " " " " " " odd

The probability for an even number can be found from the Poisson distribution.

$$(4) \quad R_Y(\tau) = +1 \sum_{\substack{\text{even}\\ k=0}}^{\infty} \frac{(a\tau)^k}{k!} e^{-a\tau} - 1 \sum_{\substack{\text{odd}\\ k=1}}^{\infty} \frac{(a\tau)^k}{k!} e^{-a\tau}$$

$$= e^{-a\tau} \sum_{\substack{\text{all values}\\ k=0}}^{\infty} \frac{(-a\tau)^k}{k!} = e^{-2a|\tau|}$$

Taking the absolute value of τ prevents a catastrophe because the interval is assumed positive in the Poisson distribution.

Spectral Density:

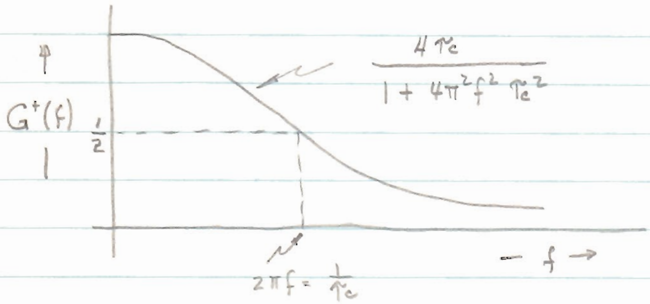
$$(5)\quad G(f) = \int_{-\infty}^{\infty} R(\tau)\, e^{i\omega\tau}\, d\tau = \int_0^{\infty} e^{(i\omega-2a)\tau}\, d\tau + \int_{-\infty}^{0} e^{(i\omega+2a)\tau}\, d\tau$$

$$\overset{\shortparallel}{\int_0^{\infty} e^{-(i\omega+2a)\tau'}\, d\tau'}$$

$$= \frac{-1}{i\omega - 2a} + \frac{1}{i\omega + 2a} = \frac{4a}{(2a)^2 + \omega^2} = \frac{1/a}{1 + \left(\frac{2\pi f}{2a}\right)^2}$$

$$= \frac{2\tau_c}{1+\omega^2\tau_c^2}$$, defining $\tau_c = \frac{1}{2a}$ with $R(\tau) = e^{-\tau/\tau_a}$

This is called the Lorentz form of the spectrum:



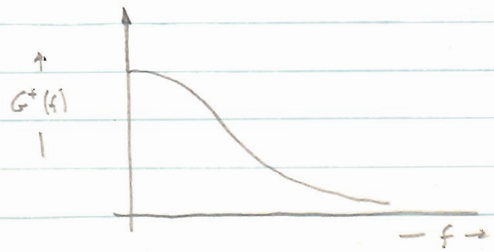
This happens to be normalized because $\int_0^{\infty} G^+(f)\, df$

$$= \frac{4\tau_c}{2\pi\tau_c}\int_0^{\infty} \frac{1}{1+x^2}\, dx = \frac{2}{\pi}\tan^{-1} x \Big]_0^{\infty} = 1$$

If the telegraph problem is binary in nature:

We get $R_Y(\tau) = \frac{1}{4} + \frac{1}{4}e^{-2a|\tau|}$

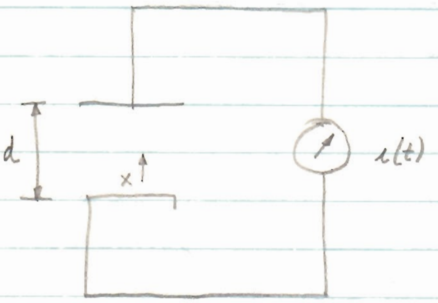
We get for the PSD a δ function for the constant term and then the regular shape of the Lorentz curve.



Shot Noise:

References: Ch. 7, Davenport and Root
Wax: S. O. Rice

Consider a Diode:



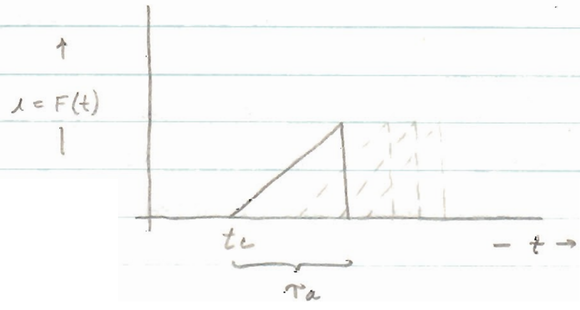
Assume that one electron is emitted from the cathode. As the electron comes closer to the anode, the charge on the anode changes and the current is $\frac{dq}{dt} = i$

Now the charge induced on the anode is postulated to be $\frac{ex}{d}$ and then:

(6) $i = \frac{e\dot{x}}{d}$; $\dot{x} = \frac{eF}{m}(t - \underset{\uparrow}{t_c}) = \frac{eV_a}{md}(t - t_c)$

time of departure from cathode

(7) $i = \left(\frac{e}{d}\right)^2 \frac{V_a}{m}(t - t_c)$, which plots as:

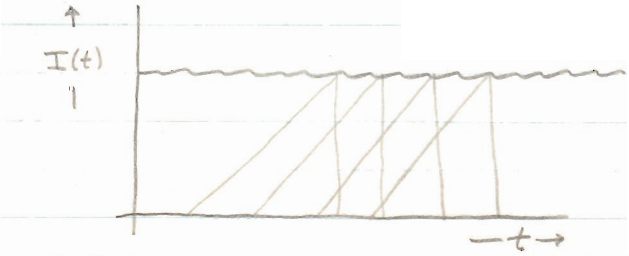


(8) $d = \frac{1}{2}\frac{eV_a}{md}\tau_a^2$

Now the total current is $I(t) = \sum_{k\ \text{electrons}}^{K} F(t - t_c)$

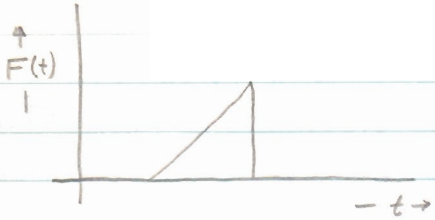
K ← random variable

t_c ← random variable, distributed uniformly



LECTURE XIV   10-29-60

Shot Noise:



Assuming no space charge, $F(t-t_k)$ is the current pulse for the k th election

The total corrent $I_K(t) = \sum_k^K F(t-t_k)$ is a random variable. What is the mean, variance, correlation, PSD, etc. ?

(1) $$\overline{I_K(t)} = \int_0^T \frac{dt_1}{T} \cdots \frac{dt_k}{T} \cdots \frac{dt_K}{T} \sum_{k=1}^{K} F(t-t_k)$$

For the k th electron:

(2) $$I_k = \frac{1}{T}\int_0^T F(t-t_k)\,dt_k$$

Then for all, since the average of a sum is the sum of the averages:

(3) $$\overline{I_K(t)} = \sum_{k=1}^{K} \frac{1}{T}\int_0^T F(t-t_k)\,dt_k = \frac{K}{T}\int_0^T F(t-t_k)\,dt_k$$ since

$F(t-t_k)$ is the same for all electrons. This is the average current of K electrons in a time T arriving at the anode.

(4) Now the probability of having K electrons in a time T is

$$\overline{I(t)} = \sum_{K=0}^{\infty} \frac{P(K)\,K}{T}\int_0^{\infty} F(t-t_k)\,dt_k = Ne$$ = average number of electrons per second.

Therefore, the total mean square current is:

(5) $$\overline{I^2(t)} = \sum_{K=0}^{\infty} P(K)\int_0^T \cdots \int_0^T \frac{dt_1}{T} \cdots \frac{dt_K}{T} \sum_{k=0}^{K}\sum_{m=0}^{K} F(t-t_k)\,F(t-t_m)$$

$$= \sum_{K=0}^{\infty} \frac{P(K)\,K}{T} \int_0^T F^2(t-t_k)\,dt_k \;+\; \sum_{K=0}^{\infty} \frac{P(K)\,K(K-1)}{T^2} \int_0^T\!\!\int_0^T F(t-t_k)\,F(t-t_m)\,dt_k\,dt_m$$

$\downarrow$ perfect squares term $\downarrow$

$\downarrow$ cross product term $\downarrow$

$$= N \int_0^T F^2(t-t_k)\,dt_k \;+\; N^2 \left[\int_0^T F(t-t_k)\,dt_k \right]^2$$

$$\shortparallel$$

$$N^2 e^2$$

$$\shortparallel$$

square of average current.

Now let $T \rightarrow \infty$, then we can write:

$$(6) \quad \overline{I^2(t)} - \left(\overline{I(t)}\right)^2 = N \int_0^{\infty} F^2(t-t_k)\,dt_k$$

Correlation Function:

The same procedure can be used as was used in finding the variance; we get:

$$(7) \quad \overline{I(t)\,I(t+\tau)} = N^2 e^2 + N \int_{-\infty}^{\infty} F(t-t_k)\,F(t-t_k+\tau)\,dt_k$$

We see that $\overline{\left(I(t) - \overline{I(t)}\right)\left(I(t+\tau) - \overline{I(t)}\right)}$ $\Uparrow$

The PSD of N independent events is N times the PSD of one event. From the Fourier transform of the correlation function:

$$(8) \quad G_+(f) = 2N \int_{-\infty}^{\infty} F(t)\,F(t+\tau)\,e^{2\pi i f \tau}\,d\tau + 2N^2e^2\,\delta_+(f)$$

which is the PSD of the deviation from the mean.

Now $F(t)\,F(t+\tau)$ vanishes for $\tau > \tau_a$.

For $f << \tau_a^{-1}$, the exponent almost vanishes and we have;

$$(9) \quad G^+(f) = 2N \iint_{-\infty}^{+\infty} F(t)\, F(t+\tau)\, d\tau\, dt = 2N \left[\int_{-\infty}^{\infty} F(t)\, dt \right]^2 = 2Ne^2$$

$$= 2\, \overline{I(t)}\, e$$

This then is a wide spectrum, holding rigourously only for δ functions of current and will tail off as $f \rightarrow \tau_a^{-1}$. If it did not, the power would be infinite. If we know $F(t)$ to be of the form Δ, then we can evaluate for $G^+(f)$. However, $G^+(f=0) = 2\, \overline{I(t)}\, e$ is always true regardless of τ_a.

What is the statistical average over a finite time period? Consider a large sample of diodes in time 0 to T_0.

$$(9) \quad \langle \Delta I \rangle_{T_0} = \frac{1}{T_0} \int_0^{T_0} \underbrace{\left(I(t) - \overline{I(t)} \right)}_{\Delta I(t)} dt$$

$$(10) \quad \overline{\left\{ \langle \Delta I \rangle_{T_0} \right\}^2} = \overline{\frac{1}{T_0^2} \int_0^{T_0} \int_0^{T_0} \Delta I(t)\, \Delta I(t')\, dt\, dt'}$$

LECTURE XV 11-1-60

· Continuation of Shot Noise:

(1) $$\langle \Delta I \rangle_{T_0} = \frac{1}{T_0}\int_0^{T_0} \Delta I(t)\, dt \quad , \quad \Delta I(t) = I(t) - \bar{I}$$

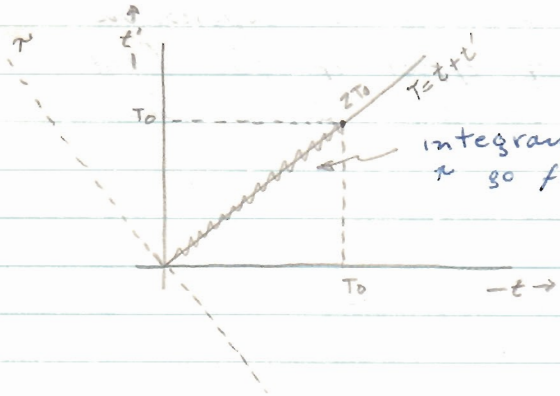
$$\left(\langle \Delta I \rangle_{T_0}\right)^2 = \frac{1}{T_0^2}\int_0^{T_0}\int_0^{T_0} \overline{\Delta I(t)\, \Delta I(t')}\, dt\, dt'$$

If we had an instantaneous ammeter, we would measure $\overline{\Delta I(t)^2}$. However, in reality, we see $\left(\langle \Delta I \rangle_{T_0}\right)^2$.

Let $t' \rightarrow t + \tau$, $T_0 >> \tau_a$. Then

(2) $$\left(\langle \Delta I \rangle_{T_0}\right)^2 = \frac{1}{T_0^2}\int_0^{T_0}\int_{-t}^{T_0 - t} \overline{\Delta I(t)\, \Delta I(t+\tau)}\, dt\, d\tau$$

For a stationary process, the integral is independent of t.



integrand exists only around this axis. We can then let τ go from $-\infty$ to $+\infty$.

(3) $$\frac{1}{T_0^2}\int_0^{T_0} dt \int_{-\infty}^{\infty} \overline{\Delta I(t)\, \Delta I(t+\tau)}\, d\tau$$

$$= \frac{1}{T_0}\int_{-\infty}^{\infty} \overline{\Delta I(t)\, \Delta I(t+\tau)}\, d\tau$$

We could have defined a new axis T. Would then get:

(4) $$\frac{1}{T_0^2}\int_0^{2T_0}\int_{-\infty}^{\infty} \overline{\Delta I(t)\, \Delta I(t+\tau)}\, dT\, d\tau$$

which is twice what we had before: The error is in the Jacobian.

First:

$$\left.\begin{array}{l} t = t \\ t = t + \tau \end{array}\right\} |J| = 1$$

Second:

$$|J| = \tfrac{1}{2}$$

Continuing:

(5) $$\int_{-\infty}^{\infty} \overline{\Delta I(t)\,\Delta I(t+\tau)}\,d\tau = \frac{1}{T_0}\,G\,(f=0) = \frac{1}{2T_0}\,G^{+}(f=0)$$

$$= \frac{e\bar{I}}{T_0}$$ which decreases as time increases.

The spectral density at zero frequency is the long time average of the statistical average. We can derive physically that for N electrons arriving on the average in time T_0, $\Delta N^2 = N$, $\bar{I} = \frac{Ne}{T_0}$, then:

(6) $$\left(\langle \Delta I(t) \rangle_{T_0}\right)^2 = \frac{e^2}{T_0^2}\,\Delta N^2 = \frac{e^2 N}{T_0^2} = \frac{e\bar{I}}{T_0}$$

which gives the same result as (5).

Space Charge Limited Case:

The presence of space charge leads to the suppression of fluctuations because of the damping effect of the field of the other electrons. Reference: van der Ziel, Ch. 5. The result is that:

(7) $$G^{+}(f) = 2e\bar{I}\,\Gamma^2 \quad , \quad 0 < \Gamma < 1$$

cathode temperature

where $$\Gamma^2 \simeq 3\left(1 - \pi/4\right)\frac{2kT_c\,g_d}{e\bar{I}}$$

noise suppression factor

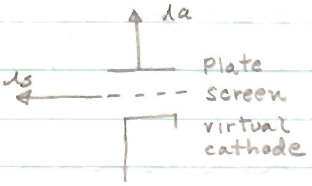
(8) $$g_d = \frac{\partial I}{\partial V_a} = \frac{3}{2}\,\frac{\bar{I}}{V_a - V_m}$$

The same formula can be used for the shot noise in a triode.

However, a different treatment is required for pentodes.

Pentodes:

A new factor enters because the current flow is distributed between the screen and anode. We use the following diagram:



We say that the chance for an electron to hit the anode is;

$$(9) \quad p = \frac{i_a}{i_a + i_s} \quad \text{with} \quad q = 1 - p = \frac{i_s}{i_a + i_s}$$

We examine the partition noise in the case $\Gamma = 0$. We suppose that there are N electrons before the screen in time T_0, $\overline{n_a} = pN$, and $\overline{\Delta n_a^2} = pqN$ (Bernoulli distributed). From previous arguments;

$$(10) \quad \overline{\Delta I_a^2} = \frac{e^2 pqN}{T_0^2} = \frac{\overline{i_a}\,\overline{i_s}\,e}{T_0\,(\overline{i_a + i_s})}$$

$$(11) \quad \therefore \quad G^+_{part}(f_0) = \frac{2\,\overline{i_a}\,\overline{i_s}\,e}{i_a + i_s}$$

Now consider $\Gamma \neq 0$. There is no relation between the time of emission and the probability of hitting the screen or the plate. Partition and emission time are statistically independent, and the mean squared deviations can be added, that is,

$$(12) \quad \overline{\Delta n_a^2} = \underbrace{pqN}_{\substack{\text{random}\\ \text{partition}}} + \underbrace{p^2 N \Gamma^2}_{\substack{\text{because } (\Delta n_a)^2 = p^2 \Delta N^2 \text{ and from (6)}\\ \text{random}\\ \text{emission term}}}$$

Total Fluctuation:

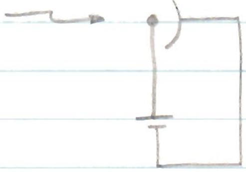
$$(13) \quad \overline{(\langle \Delta i_a \rangle)^2} = \frac{pqNe^2}{T_0^2} + \frac{p^2 N e^2 \Gamma^2}{T_0^2}$$

$$(14) \quad G^+_{i_a}(f=0) = 2e\,\overline{i_a}\left[\frac{\overline{i_s} + \overline{i_a}\,\Gamma^2}{i_s + i_a}\right]$$

If the screen is wide we do not have random partition noise but have essentially two triodes in parallel. If $\Gamma = 1$, equation (14) reduces to equation XIV (9). The effect of partition does not appear because the purely random process of emmission is not affected by the purely random process of partition.

LECTURE XVI   11-3-60

The Phototube:



Suppose $\mathcal{N}$ photons/sec are emitted from the light source. η is the efficiency of the phototube surface. No interactions between photons in the light beam.

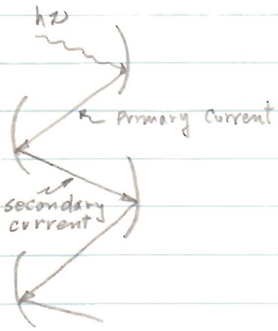
We can immediately write for the current:

$$(1) \quad \overline{n_e} = \overline{\Delta n_e^2} = \eta \mathcal{N} T_0 \quad \text{and}$$

$$(2) \quad \overline{\langle \Delta i_{ph}^2 \rangle} = \frac{\eta \mathcal{N} e^2}{T_0} = \frac{e \bar{I}_{ph}}{T_0}$$

Now $G_{ph}^{+}(f=0) = 2e\bar{I}_{ph}$ which is the same as the shot noise in the space-charge free diode.

Photomultipliers:



all anode emissions are independent of each other. The number of electrons emitted per incident electron = $\bar{p}$. What are the fluctuations in the current and the PSD?

Now: $\bar{I}_2 = \bar{I}_{ph}\,\bar{p}$, $G_2^{+}(f=0) = 2e\bar{I}_{ph}\,\bar{p}^2$

The first $\bar{p}$ is because of the fraction of primary current. The second $\bar{p}$ is because of the different effective "charge" of the secondary electrons.

Let us take the more general case when p is a random process. We have that the number of secondaries emitted per primary is p with probability β_p where:

$$(3) \quad \sum_{p=0}^{\infty} \beta_p = 1 , \quad \sum_{p=0}^{\infty} \beta_p \, p = \bar{p}$$

Now: N_{sec} (for N_{pri} primaries) $= \sum_{i=1}^{N} p_i = \sum_{i=1}^{N} (\bar{p} + \Delta p_i)$

and $$\overline{N^2_{sec}} = \overline{\left\{ \sum_{i=1}^{N} (\bar{p} + \Delta p_i) \right\}^2} = N^2 \bar{p}^2 + N \overline{(\Delta p_i^2)}$$

$$\parallel$$

$$\left\{ N \bar{p} + \sum_{i=1}^{N} \Delta p_i \right\}^2 \quad \left(\text{because } \overline{\Delta p_i \Delta p_j} = 0 , \ \overline{\Delta p_i} \, \bar{p} = 0 \right)$$

for fixed N_{pri} incident. We now want to average over N_{pri}: we can replace $N_{pri} \to \overline{N_{pri}}$, $N^2_{pri} \to \overline{N^2_{pri}}$ because N_{pri} and N_{sec} are statistically independent. Therefore:

$$(4) \quad \overline{N^2_{sec}} = \overline{N^2} \, \bar{p}^2 + \bar{N} \, \overline{(\Delta p_i^2)}$$

$$\text{and} \quad \overline{\Delta N^2_{sec}} = \overline{\Delta N^2_{pri}} \, \bar{p}^2 + \bar{N}_{pri} \overline{\Delta p^2}$$

subtract; $\bar{N}^2 \bar{p}^2 = \bar{N}^2_{sec}$,

$$\overline{N^2_{pri}} \, \bar{p}^2 - \bar{N}^2 \bar{p}^2 = \overline{\Delta N^2_{pri}}$$

$$\overline{N^2_{sec}} - \bar{N}^2_{pri} \bar{p}^2 = \overline{N^2_{sec}} - \bar{N}_{sec}^2$$

$$= \overline{\Delta N^2_{sec}}$$

We may continue this and arrive at results for any number of electrodes.

Further Fluctuations in Vacuum Tubes:

1) Secondary Emission
2) Photo-effect of light from cathode
3) Fluctuations in grid current.

Case of Floating Grid:

$$i_g = i_{eg} + i_{ions} + i_{ph} = 0$$

$$G_g^+(f=0) = 2e|i_{eg}| + 2e|i_{ions}| + 2e|i_{ph}|$$

Proof That Shot Noise has Gaussian Distribution and $i(t)$ is a Gaussian process. Consider current due to K electrons.

$$(5)\quad I_K(t) = \sum_{k=1}^{K} F(t-t_k)$$

We now write the probability of I if we know K electrons are arriving:

$$(6)\quad p(I_K) = \frac{1}{T_0}\int_0^{T_0}\cdots\int \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iI_K u}\, e^{iu\sum_{k=1}^{K} F(t-t_k)}\, du \cdot dt_1 \cdots dt_k$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iuI_K}\left[\frac{1}{T_0}\int_0^{T_0} e^{iuF(t+\tau)}\,d\tau\right]^K du$$

Now K is Poisson distributed: $f(K) = \frac{(\bar{n}T_0)^K}{K!}e^{-\bar{n}T_0}$
Then:

$$(7)\quad p(I) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iuI}\, e^{-\bar{n}T_0} \sum_{K=0}^{\infty} \frac{\left[\bar{n}\int_0^{T_0} e^{iuF(t-\tau)}\,d\tau\right]^K}{K!}\, du$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iuI}\, \exp\left[\bar{n}\int_0^{T_0}\left\{e^{iuF(t+\tau)} - 1\right\} d\tau\right] du$$

Now $\bar{i} \sim \bar{n}$. Introduce normalized random variables:
$\sigma^2 = \bar{n}\int_{-\infty}^{\infty} F^2(\tau)\,d\tau$, $x = \frac{I-\bar{I}}{\sigma}$, $u' = \sigma u$

Then:

replace with $-\infty$ to $+\infty$

$$(8)\quad p(x) = \frac{1}{2\pi\sigma}\int_{-\infty}^{\infty} e^{iu'x}\, \underbrace{e^{-iu'\frac{\bar{I}}{\sigma}}\, \exp\left\{\bar{n}\int_0^{T_0}\left[e^{iu'F(t+\tau)} - 1\right] d\tau\right\}}_{\text{expand in power series}}\, du'$$

$$e^{-iu'\bar{I}/\sigma}\, e^{iu'\bar{n}/\sigma \int_{\infty}^{\infty} F(\tau)\,d\tau} \qquad = e\ (\text{charge})$$
$$+\ e^{-\frac{u'^2}{\sigma^2}\bar{n}\int_{-\infty}^{\infty} F^2(\tau)\,d\tau} + \text{higher terms}$$

As $\bar{n}$ goes to ∞, higher terms vanish
and $p(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$

LECTURE XVII 11-5-60

Joint Gaussian Shot Noise:

Shot noise may be distributed according to a multivariate Gaussian.

Fourier Representation of the Shot Noise Current

(1) $$i(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{2\pi n t}{T} + b_n \sin\frac{2\pi n t}{T}\right)$$

where $$a_n = \frac{2}{T}\int_0^T i(t)\cos\frac{2\pi n t}{T}\,dt$$

$$b_n = \frac{2}{T}\int_0^T i(t)\sin\frac{2\pi n t}{T}\,dt$$

If $i(t)$ is a random process a_n and b_n are random variables. If $i(t)$ gaussian, a_n, b_n are gaussian and also their sum. Consider:

(2) $$\left.\begin{matrix} i(t) \\ i(t') \\ i(t'') \end{matrix}\right\}$$ may be summed to form gaussian

We have for the coefficients:

(3) $$\left.\begin{matrix} \overline{a_n} = \overline{b_n} = 0 \\ \overline{a_n^2} = \overline{b_n^2} \\ \overline{b_n b_m} = \overline{a_n a_m} = 0 \;,\; \overline{a_n b_m} = 0 \end{matrix}\right\}$$ That is, the coefficients are statistically independent.

We must prove this:

(4) $$\overline{a_n a_m} = \frac{4}{T^2}\int_0^T\int_0^T \overline{i(t)\, i(t+\tau)}\cos\frac{2\pi n t}{T}\cos\frac{2\pi m(t+\tau)}{T}\,dt\,d\tau$$

writing $t' = t+\tau$. Integrand is stationary process so we can change limits on t to $\pm\infty$.

Expanding the trig terms:

$$(5) \quad \overline{a_n a_m} = \frac{4}{T^2} \int_0^T \int_{-\infty}^{\infty} R_y(\tau) \Big(\cos\frac{2\pi n t}{T} \cos\frac{2\pi m t}{T} \cos\frac{2\pi m \tau}{T}$$

$$- \cos\frac{2\pi n t}{T} \sin\frac{2\pi m t}{T} \sin\frac{2\pi m \tau}{T} \Big) dt\, d\tau$$

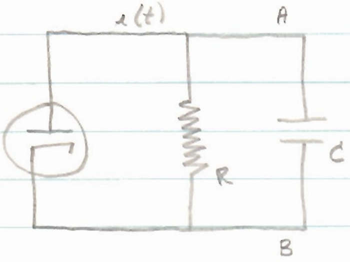
$$= \frac{2}{T} \int_{-\infty}^{\infty} R_y(\tau) \cos\frac{2\pi n \tau}{T}\, d\tau\, \delta_{nm} = \frac{1}{T} G^+\left(f = \frac{n}{T}\right) \delta_{nm}$$

Therefore: $\overline{a_n^2} = \overline{b_n^2} = \frac{1}{T} G^+\left(f = \frac{n}{T}\right)$.

We note that as T is larger, Fourier components become more dense. Inverting we have:

$$(6) \quad G^+(f)\, df = \sum_{n=fT}^{n=T(f+df)} \tfrac{1}{2}\left(\overline{a_n^2} + \overline{b_n^2}\right) = \frac{T}{2}\left(\overline{a_n^2} + \overline{b_n^2}\right) df$$

This representation is helpful in solving specific problems.
Example:



what is V_{AB}?

Circuit equation:

$$C \frac{dV}{dt} + \frac{V}{R} = i(t)$$

Since $i(t)$ is sum of periodic signals, so will $V(t)$:
Substituting; $i(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} C_n \cos\left(\frac{2\pi n t}{T} + \varphi_n\right)$

$$V(t) = \frac{a_0}{2} R + \mathrm{Re} \sum_{n=1}^{\infty} \frac{C_n \exp\left(\frac{2\pi i n t}{T} + i\varphi_n\right)}{\frac{1}{R} + i\frac{2\pi n}{T} C}$$

where $C_n^2 = a_n^2 + b_n^2$, with distribution $e^{-\frac{a_n^2 + b_n^2}{2\sigma^2}}\, da_n\, db_n$

$= e^{-\frac{C_n^2}{2\sigma^2}} C_n\, dC_n\, d\varphi_n$ which is not gaussian but is the radial distribution.

We may write for the sum term in the current,

$$\sum_{n=1}^{\infty} C_n \cos\left(\frac{2\pi n t}{T} + \varphi_n\right) = \mathrm{Re} \sum_{n=1}^{\infty} C_n \exp\left(\frac{2\pi i n t}{T} + i\varphi_n\right)$$

If c_n and φ_n are random variables, $v(t)$ is random process. We see that $\overline{V(t)} = \frac{a_0}{2} R$. What is spectral density?

$$(7) \quad G_v^+(f) = \frac{a_0^2}{4} R^2 \, 2\, \delta^+(f) + \frac{\frac{T}{2} \overline{c_n^2}}{\left| \frac{1}{R} + i \frac{2\pi n}{T} C \right|^2}$$

$$= \frac{a_0^2}{4} R^2 \, 2\, \delta^+(f) + \frac{G_i^+(f)}{\left| \frac{1}{R} + i\, 2\pi f C \right|^2}$$

$$= \frac{a_0^2 R^2}{2} \delta^+(f) + |Z|^2 G_i^+(f)$$

$$= |Z|^2 G_i^+(f)$$, incorporating the δ function at the origin in G_i^+.

We return to examine the properties of the C_n's in the current representation:

$$(8) \quad \overline{i(t)\, i(t+\tau)} = \sum_{n=1}^{\infty} \overline{C_n^2} \, \overline{\cos\left(\frac{2\pi n t}{T} + \varphi_n\right) \cos\left(\frac{2\pi n \{t+\tau\}}{T} + \varphi_n\right)}$$

cross product terms vanish since statistical average vanishes

$$= \sum_{n=1}^{\infty} \overline{C_n^2} \, \overline{\cos^2\left(\frac{2\pi n t}{T} + \varphi_n\right)} \cos \frac{2\pi n \tau}{T} + \overline{\cos\left(\frac{2\pi n t}{T} + \varphi_n\right) \sin\left(\frac{2\pi n t}{T} + \varphi_n\right)} \sin \frac{2\pi n \tau}{T}$$

$$= \frac{1}{2} \sum_{n=1}^{\infty} \overline{C_n^2} \cos \frac{2\pi n \tau}{T} = \frac{1}{2} \int_0^{\infty} T \, \overline{C^2_{n=Tf}} \cos 2\pi f \tau \, df$$

The whole point is that the output spectral density is equal to input spectral density times the system function either an impedance or a transfer function.

LECTURE XVIII 11-8-60

Linear Fixed Parameter System:

Input $x(t)$ gives output $y(t)$. Meaning of linear:
$a_1 X_1 + a_2 X_2 \Rightarrow a_2 y_2 + a_1 y_1$

Fixed Parameter: R, C, L not functions of time.
Linear amplifiers belong to this class.
The result is:

$$(1) \quad G_{out}(f) = |A(\omega)|^2 \, G_{in}(f)$$

where $A(\omega)$ is the system function, be it impedance or transfer function.

Sometimes more convenient to use impulse response function $h(t)$ where relation of output to input is

$$(2) \quad y(t) = \int_{-\infty}^{\infty} h(t') \, x(t-t') \, dt'$$

To see meaning of $h(t')$ let $x(t-t')$ be δ function. Then output is $h(t)$ or $y(t) = h(t)$. To be physically possible, must have $h(t) = 0$ for $t < 0$ so that lower limit on integral is zero. To show relation between $h(t)$ and $A(\omega)$, we Fourier transform:

$$(3) \quad x(t-t') = \delta(t-t') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega(t-t')} \, d\omega \text{. Then}$$

$$(4) \quad y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(t') \, e^{-i\omega t'} \, e^{i\omega t} \, dt' \, d\omega$$

$$= \int_{-\infty}^{\infty} A(\omega) \, e^{i\omega t} \, d\omega$$

$$(5) \quad A(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(t') \, e^{-i\omega t'} \, dt'$$

We have shown that $h(t)$ is just the Fourier transform of $A(\omega)$.

What happens when random process passes through the system?

$$(6) \quad \overline{y(t)} = \int_{-\infty}^{\infty} h(t') \, \overline{x(t-t')} \, dt' = \overline{x(t)} \int_{-\infty}^{\infty} h(t') \, dt'$$

If $x(t-t')$ is bounded, $y(t)$ exists if

$$(7) \quad \int_{-\infty}^{\infty} |h(t')| \, dt$$ exists and it does for stable systems, or

$$|y(t)| \leq \int_{-\infty}^{\infty} |h(t')| |x(t-t')| \, dt \leq A \int_{-\infty}^{\infty} |h(t')| \, dt$$

What is the correlation function of the output as related to the input?

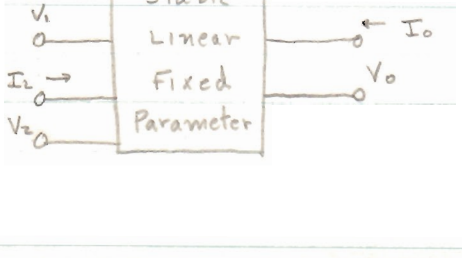
$$(8) \quad \overline{y(t) \, y(t+\tau)} = R_y(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\alpha) \, h(\beta) \, \underbrace{\overline{x(t-\alpha) \, x(t+\tau-\beta)}}_{\overset{\shortparallel}{R_x(\tau+\alpha-\beta)}} \, d\alpha \, d\beta$$

If the input is stationary, the output is stationary.

$$(9) \quad G_{out}(f) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\alpha) \, h(\beta) \, e^{-2\pi i f \alpha} \, e^{2\pi i f \beta} \int_{-\infty}^{\infty} e^{2\pi i f (\tau+\alpha-\beta)} \, R_x(\tau+\alpha-\beta) \, dt \, d\alpha \, d\beta$$

$$= \underbrace{\int_{-\infty}^{\infty} h(\alpha) \, e^{-2\pi i f \alpha} \, d\alpha \int_{-\infty}^{\infty} h(\beta) \, e^{2\pi i f \beta} \, d\beta}_{|A(\omega)|^2} \quad G_{in}(f)$$

Extension to n-pair Terminal Networks:



This network can be described as the matrix of the equations:

(10) $$I_1 = y_{01} V_0 + \dots + y_{0N} V_N$$

$$\vdots$$

$$I_N = y_{N0} V_0 + \dots + y_{NN} V_N$$

y's are the short circuit transfer admittances.

Given V_N input voltages, what is open circuit output voltages?

(11) $$V_0 \text{ (open circuit)} = \sum_{k=1}^{N} -\frac{y_{0k}}{y_{00}} V_k = \sum_{k=1}^{N} A_k(\omega) V_k$$

This is of interest to us when V's are noise voltages. Must examine if V's are correlated or not. We now write the result for $G_{out}(f)$ for the same procedure as for the two pair terminal network.

(12) $$G_{out}(f) = \sum_{k=1}^{N} \sum_{k'=1}^{N} A_k(\omega) A^*_{k'}(\omega) \, G_{in_{kk'}}(f)$$

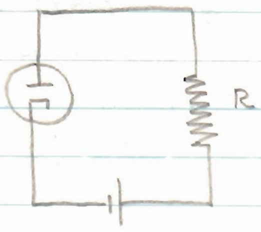
where $G_{in_{kk'}}(f)$ is the Fourier transform of the cross-correlation function $\overline{V_k(t) V_{k'}(t+\tau)}$

For uncorrelated noise inputs:

(13) $$G_{out}(f) = \sum_{k=1}^{N} |A_k(\omega)|^2 G_k(f)$$

LECTURE XIX 11-10-60

Thermal Noise:

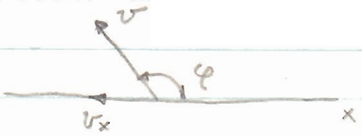


Other random processes present other than random emission. For example, thermal noise in resistor. Because of random motion of electrons, can get momentary unequal charge distribution in resistor. Also called Johnson noise.

Drude-Lorentz electron model:
Momentum will be distributed according to:

$$(1) \quad \exp\left[-\frac{p^2}{2mkT}\right] p^2\,dp$$

Consider direction as random variable:

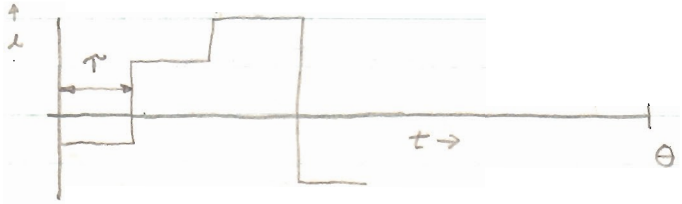


Consider strip of length L bent into ring with direction x along the ring. The current due to one electron is:

$$(2) \quad i_e = \frac{e v_x}{L}$$

v_x/L is the number of times per second that electron goes thru given cross-section of ring.

Introduce resistance: what is collision time? In simple model, take this time to be always τ. Electron only changes direction on collision, or φ changes. Current looks like:



θ is total time interval. We will analyse current in terms of Fourier components, which are:

$$(3)\quad a_k = \frac{2}{\theta}\int_0^\theta i_e(t)\cos 2\pi f_k t\,dt \qquad \Bigg\} \quad f_k = \frac{k}{\theta} = \text{frequency of } k \text{ th harmonic.}$$

$$b_k = \frac{2}{\theta}\int_0^\theta i_e(t)\sin 2\pi f_k t\,dt$$

or:

$$(4)\quad a_k = \frac{2e}{\theta L}\sum_{i=1}^{\theta/\tau}\int_{t_i}^{t_i+\tau} v_{x_i}\cos 2\pi f_k t\,dt$$

$$= \frac{2e}{\theta L}\sum_{i=1}^{\theta/\tau}\int_0^{\tau} v_{x_i}\cos 2\pi f_k (t_i + t')\,dt'$$

since there are θ/τ number of periods. Also:

$$(5)\quad a_k = \frac{2e\tau}{\theta L}\sum_{i=1}^{\theta/\tau} v_{x_i}\cos 2\pi f_k t_i$$

We take a uniform distribution for φ such that:

$$(6)\quad \overline{a_k} = 0$$

$$(7)\quad \overline{a_k^2} = \frac{4e^2\tau^2}{\theta^2 L^2}\sum_i\sum_j \overline{v_{x_i} v_{x_j}}\cos 2\pi f_k t_i \cos 2\pi f_k t_j$$

$$= \frac{4e^2\tau^2}{\theta^2 L^2}\sum_i^{\theta/\tau} \overline{v_{x_i}^2}\cos^2 2\pi f_k t_i$$

since collisions are so strong that no memory is retained and cross-products vanish.

Now, $\frac{\overline{p_x^2}}{2m} = \frac{kT}{2}$, $\overline{v_x^2} = \frac{kT}{m}$, then:

$$(8)\quad \overline{a_k^2} = \frac{4e^2\tau^2}{\theta^2 L^2}\cdot\frac{kT}{m}\cdot\frac{1}{2}\frac{\theta}{\tau} = \frac{2e^2\tau kT}{\theta m L^2} = \overline{b_k^2}$$

replacing Σ by $\int$ because $\tau \ll \theta$.

Therefore:

$$(9) \quad G^+(f_k) = \lim_{\theta \to \infty} \left[\theta \, \frac{\overline{a_k^2} + \overline{b_k^2}}{2} \right] = \frac{2 e^2 \tau k T}{m L^2}$$

We now wish to get an expression for the resistance. For N electrons, multiply (9) by N, because the electrons are independent. The number of electrons per unit volume $= \frac{N}{LA} = n$. Now the relation between conductivity, n, and τ is:

$$(10) \quad \sigma = \frac{n e^2 \tau}{2m}$$

because average drift velocity of electrons in a field is $\overline{v}_{x\,drift} = \frac{eE}{2m}\tau$. Thus:

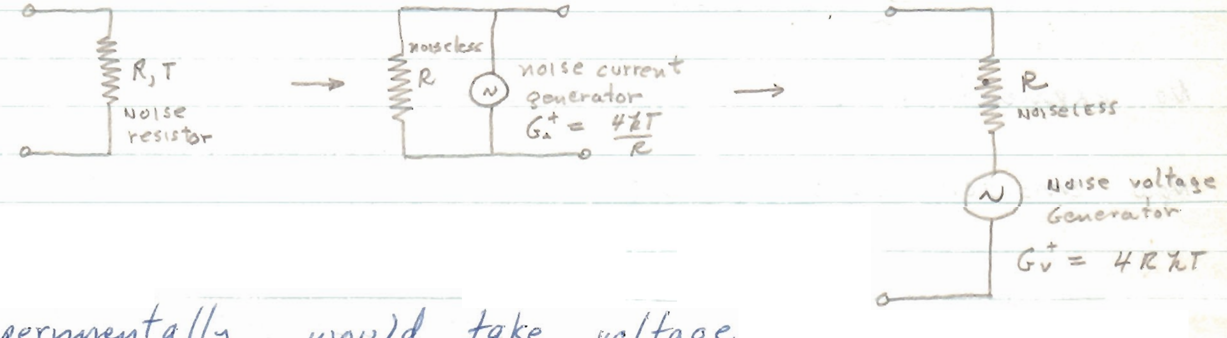
$$(11) \quad G^+(f_k) = \frac{4 \sigma k T A}{L}$$

Now $R = \frac{L}{\sigma A}$ or:

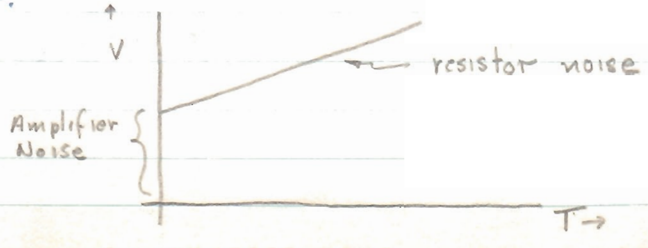
$$(12) \quad G^+(f_k) = \frac{4kT}{R}$$

which is PSD for current through the resistor as measured by an infinitely fast ammeter which shorts out the resistor.

Equivalent Circuits:



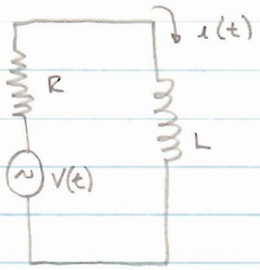
Experimentally, would take voltage output, amplify it, and read with quadratic detector.



LECTURE XX   11-12-60

General Formulation of Johnson Noise:

Consider:



We take for $\overline{i^2}$:

(1) $\frac{1}{2} L \overline{i^2} = \frac{kT}{2}$

from Brownian motion of electrons and equipartion of energy, and current flow as linear combination of electron momentum in one direction. Then kinetic energy is associated with one degree of freedom. Assume $G_V(f)$ as wide spectral density,

(2) $G_V(f) = 2\sigma^2$

Good assumption because L will damp out high frequency components. For current PSD:

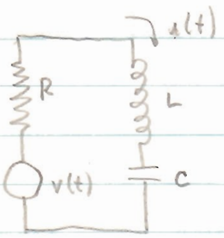
(3) $G_i(f) = \dfrac{2\sigma^2}{R^2 + \omega^2 L}$, and,

(4) $$\overline{i^2} = \int_0^\infty \frac{2\sigma^2 \, df}{R^2 + 4\pi^2 f^2 L^2} = \frac{2\sigma^2}{2\pi RL} \arctan \frac{2\pi f L}{R}\Big]_0^\infty = \frac{\sigma^2}{2RL}$$

$= \dfrac{kT}{L}$ from $G_V(f) = 2\sigma^2 = 4RkT$

No exponential $e^{-i2\pi f\tau}$ since $\tau = 0$.

Now consider RLC circuit:



(5) $G_i(f) = \dfrac{2\sigma^2}{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$

(6) $$\overline{i^2} = \int_0^\infty \frac{2\sigma^2 \, df}{R^2 + \left(2\pi f L - \frac{1}{2\pi f C}\right)^2}$$

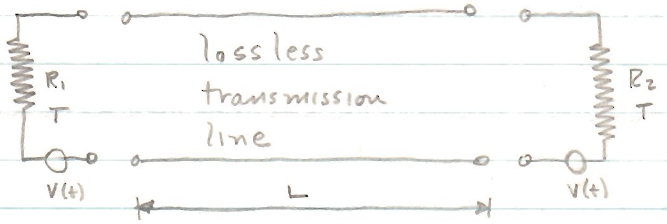
Evaluate by contour integration and get: $\overline{i^2} = \frac{kT}{L}$.
Now:

(7) $$G_{V_c}(f) = \frac{4RkT}{\omega^2 C^2 R^2 + (\omega^2 LC - 1)^2}$$

(8) $$\frac{1}{2} C \overline{V_c^2} = 2RkTC \int_0^\infty \frac{df}{\omega^2 C^2 R^2 + (\omega^2 LC - 1)^2} = \frac{kT}{2}$$

by contour integration, which is the potential energy of the capacitor. This is because circuit represents the harmonic oscillator in terms of V and i, and one gets the usual equipartition of energy in terms of ½ kT.

Nyquist Theorem:



Assume $R_1 = R_2 = R$

Using thermodynamic equilibrium, power delivered by each resistor must balance.

How many modes are there? $N\lambda = L$ or $\frac{N}{L} = \frac{1}{\lambda} = \frac{f}{c}$

If $\frac{L\,\Delta f}{c} = 1$, there is one mode in Δf

or number of modes in Δf = $\frac{L\,\Delta f}{c}$

The number of modes passing per unit time through a cross-section, is Δf. Each mode carries an energy kT since the waves are harmonic oscillators with H carrying kinetic energy and E potential energy. Wave is at temperature T because it only see the resistor reservoirs at each end. Therefore the power transported by the modes is $kT\,\Delta f$.

Calculate power one resistor delivers to the other as load:

(9) Power delivered by noise voltage generator in R_1 to Load R_2

$$= \frac{\overbrace{4 R_1 k T\, df}^{G(f)}}{(R_1+R_2)^2} R_2$$

$$= k T\, df \quad \text{for } R_1 = R_2$$

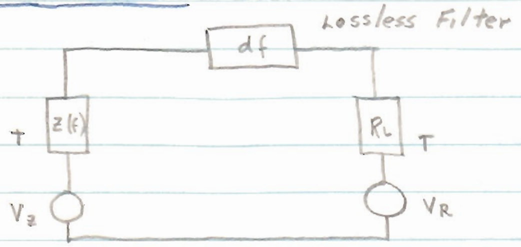
By working backwards, can find power generated by $R_1 = 4 R_1 k T$. This now shows the wideness of the spectrum of the Johnson noise.

Even without transmission line:

$$(10) \qquad \frac{G_{R_1}(f)\, R_2\, df}{(R_1+R_2)^2} = \frac{G_{R_2}(f)\, R_1\, df}{(R_1+R_2)^2}$$

for thermodynamic equilibrium.

LECTURE XXI 11-15-60



(1) $Z(f) = R(f) + i\, Y(f)$

What is power delivered to load resistor?

$$(2.) \qquad \frac{G_{V_Z}(f)\, R_L}{|Z + R_L|^2} = \frac{G_R(f)\, R(f)}{|Z + R_L|^2}$$

Thus power delivered to load resistor must equal power delivered by load resistor for thermodynamic equilibrium.

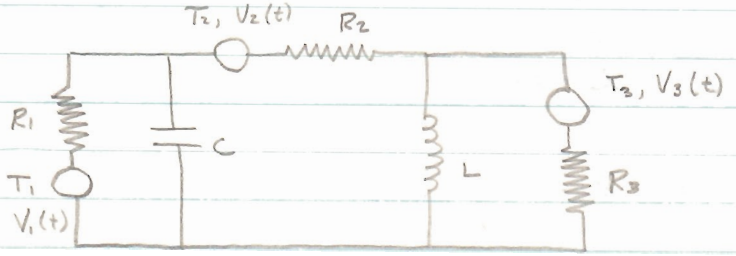
And, $G_{V_Z} = \mathrm{Re}\, Z \;\; F(T,f)$

$$G_{R_L} = R_L \;\; F(T,f)$$

Nyquist has shown that the universal function $F(T,f)$ is not dependent on f and is equal to $4kT$.

When the temperatures are not equal, thermodynamic equilibrium does not exist and a net flow of power occurs until power dissipation heats up the cooler resistor until equilibrium is restored.

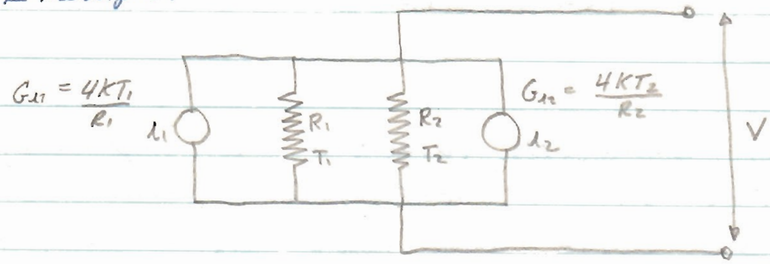
In general, one sees physically that only dissipative elements can generate noise voltages.



$G_1(f) = 4R_1 KT$
$G_2(f) = 4R_2 KT$
$G_3(f) = 4R_3 KT$

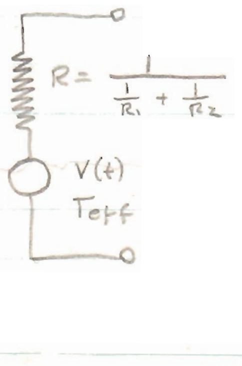
All noise voltages are uncorrelated, that is, $\overline{V_1(t)\,V_2(t')} = 0$

Example:



$$\left[\frac{4kT_2}{R_2} + \frac{4kT_1}{R_1}\right]\frac{1}{\left(\frac{1}{R_1}+\frac{1}{R_2}\right)^2} = G_V(f)$$

Thèvenin Equivalent Circuit:

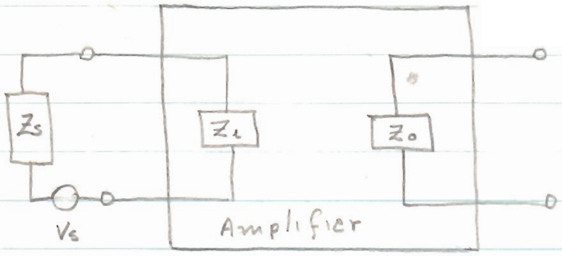


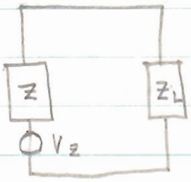
$$4Rk\,T_{eff} = \left[\frac{4kT_2}{R_2} + \frac{4kT_1}{R_1}\right]R^2$$

$$T_{eff} = T_2\frac{R}{R_2} + T_1\frac{R}{R_1}$$

If $T_1 = T_2$; $T_{eff} = T_1 = T_2$

Can measure noise in circuits expermentally.





Maximum incremental noise power delivered to load is

$$\frac{G_V(f)\,\text{Re}\,Z_L}{|Z+Z_L|^2} = \frac{G_V(f)}{4R} = kT$$

Define: Available power gain

$$= \frac{\text{Power available at output}}{\text{Power available from source}}$$

$$= \frac{V_{out}^2/4R_o}{V_s^2/4R_s} = \frac{R_s}{R_o}\frac{V_{out}^2}{V_{in}^2}\left|\frac{Z_s}{Z_s+Z_i}\right|^2$$

$$= \frac{R_s}{R_o}\left|\frac{A(\omega)\,Z_s}{Z_i+Z_s}\right|^2 = P(f)$$

Noise power available at output $= \int_0^\infty kT\,P(f)\,df$

$$= \int_{f_0-\frac{BW_{eff}}{2}}^{f_0+\frac{BW_{eff}}{2}} kT\,P(f_0)\,df = kT\,(BW_{eff})\,P(f_0)$$

where BW_{eff} is the bandwidth obtained by replacing the frequency distribution with a rectangular figure with the same area as under curve and same hieght as $P(f_0)$.

Calibrate system by placing diode across source and increase cathode temperature until noise output is doubled according to:

$$(2eI_a)_{\substack{\text{doubling}\\ \text{noise output}}} = \frac{4kT}{R}$$

LECTURE XXII 11-19-60

Johnson Noise Spectrum: $G_i^+(f) = \frac{4kT}{R}$

1) Nyquist argument: no cutoff (must involve QM)

2) Resistor: HF cutoff at collision frequencies, $f > \frac{1}{\tau_a} = 10^{13}$ cps

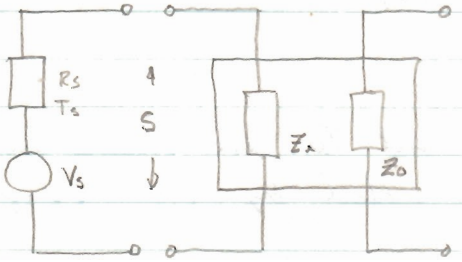
average energy $= \frac{h\nu}{e^{h\nu/kT}-1} \approx h\nu$ if $h\nu << kT$

QM cutoff $= f > \frac{kT}{h}$

At room temperature $f \approx 3 \cdot 10^{14}$ cps

3) Ultimately, capacitive and inductive effects bring about cutoff much lower than those above.

Noise in Circuits:

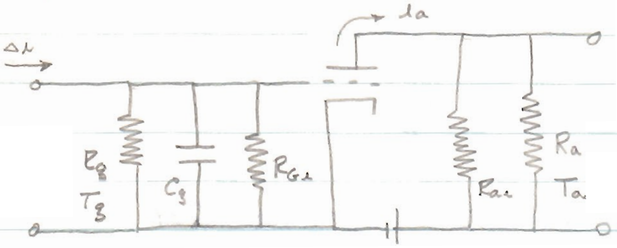


Noise Figure:

$$F = \frac{\left(\frac{\text{signal power}}{\text{noise power}}\right)\text{available in}}{\left(\frac{\text{signal power}}{\text{noise power}}\right)\text{available out}}$$

Amplifier always adds noise from its own resistors at finite temperature and shot noise.

Example: Actual Triode Amplifier



We refer all noise to an input grid current. Thus,

$$G_{\substack{\text{effective PSD}\\ \text{input}\\ \text{grid}\\ \text{current}}}(f) = \frac{4kT_s}{R_s} + \frac{4kT_g}{R_g} + 2e\underset{\text{shot noise in grid}}{\left\{|i_g^+| + |i_g^-|\right\}}$$

$$+ \underbrace{\frac{2e\,i_a}{g^2|Z|^2}}_{\substack{\text{plate shot}\\ \text{noise referred}\\ \text{to grid}}} + \underset{\text{noise in output resistor referred to grid.}}{\frac{4kT_a}{R_a}\left(\frac{R_a}{R_a + R_{aL}}\right)^2 \frac{1}{g^2|Z|^2}}$$

$$Z = \frac{1}{\frac{1}{R_g} + \frac{1}{R_{gL}} + j\omega C_g}$$

Noise figure is obtained by dividing above expression by PSD of noiseless amplifier or $\frac{4kT_s}{R_s}$.

Also, we can define an effective noise temperature which may be more useful than a noise figure, writing

$$G_{\substack{\text{effective PSD}\\ \text{input}\\ \text{grid}\\ \text{current}}}(f) = \frac{4k}{R_s}(T_s + T_N)$$

with $kT_N = (F-1)kT_s$. F and T_N are related in that they measure the noise power added by the amplifier.

Refer now to original diagram: Assume signal power S with signal to noise ratio: S/kT_sB, B = bandwidth

what is F?

$$F = \frac{S/kT_sB}{GS/(GkT_sB + GP_N)} = 1 + \frac{P_N}{kT_sB} = 1 + \frac{T_N}{T_s}$$

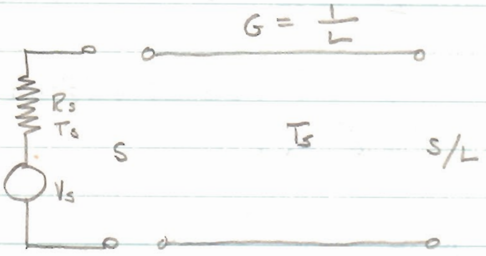
Cascaded Amplifier Stages:

Consider two stages:

$$F_{1+2} = \frac{S / k T_s B}{G_1 G_2 S / (G_1 G_2 k T_s B + G_1 G_2 k T_{N_1} B + G_2 k T_{N_2} B)}$$

$$= 1 + \frac{T_{N_1}}{T_s} + \frac{T_{N2}}{G_1 T_s} = F_1 + \frac{F_2 - 1}{G_1}$$

Noise generated in later stages does not contribute as much to overall noise figure as gain factor appears in the denominator. If we have attenuator, reverse occurs, and noise in later stages contributes more. For example, consider a transmission line.



If matched to source and at same temperature,

$$F = L = \frac{1}{G}$$

LECTURE XXIII 11-22-60

Continuation of Circuit Noise:

Johnson noise in an RL circuit is mathematically equal to the Brownian motion of a free particle. The Langevin equations for each are:

Johnson Noise	Brownian Motion
$L \frac{di}{dt} + Ri = V(t)$	$m \frac{dv}{dt} + \beta v = F(t)$

F(t) is force due to the collisions of molecules on the particle.

Johnson Noise

$$\overline{i^2_{t\to\infty}} = \frac{kT}{L}$$

Brownian Motion

$$\overline{v^2_{t\to\infty}} = \frac{kT}{m}$$

Thus we see that equipartition occurs after a long enough time. General Solution of circuit Langevin equation:

(1) $$i = i_0 e^{-\frac{R}{L}t} + i_0' e^{-\frac{R}{L}t} \; ; \quad i_0' = \frac{1}{L}\int_0^t V(\xi)\, e^{\frac{R}{L}\xi}\, d\xi$$

(2) $$\overline{i} = i_0 e^{-\frac{R}{L}t} \, , \quad \overline{V(\xi)} = 0$$

(3) Now $$f(i_{t\to\infty}) = \frac{1}{\sqrt{2\pi \frac{kT}{L}}}\, e^{-\frac{Li^2}{2kT}}$$

(4) Then $$\overline{i^2} = i_0^2 e^{-2\frac{R}{L}t} + e^{-2\frac{R}{L}t}\frac{1}{L^2}\int_0^t\!\!\int_0^t \underbrace{\overline{V(\xi)V(\eta)}}_{2RkT\,\delta(\xi-\eta)}\, e^{\frac{R}{L}(\xi+\eta)}\, d\xi\, d\eta$$

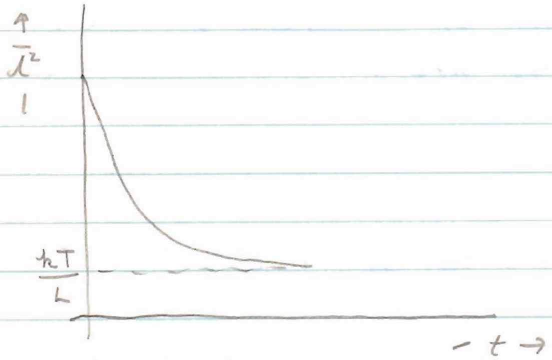
Make transformation $u = \xi - \eta$, $v = \xi + \eta$,
Then double integral is:

(5) $$e^{-2\frac{R}{L}t}\frac{1}{L^2}\int_0^{2t}\!\!\int_{-\infty}^{\infty} 2RkT\,\delta(u)\, e^{\frac{R}{L}v}\,\frac{1}{2}\, du\, dv$$

We get:

(6) $$\overline{i^2} = i_0^2 e^{-2\frac{R}{L}t} + \frac{2kRT}{L^2} e^{-2\frac{R}{L}t}\int_0^{2t} e^{\frac{R}{L}v}\cdot\frac{1}{2}\, dv$$

$$= \frac{kT}{L} + e^{-\frac{2R}{L}t}\left(i_0^2 - \frac{kT}{L}\right)$$



We see that $\overline{i^2}$ decays to its equipartition value after a long time. Is all right to take V as δ function as long as fluctuation time is less than circuit time constant.

The analogous result for Brownian motion is:

$$(7) \quad \overline{v^2} = \frac{kT}{m} + e^{-\frac{2B}{M}t}\left(v_0^2 - \frac{kT}{m}\right)$$

For higher moments, we take

$$\overline{V(t_1) \cdots V(t_{2n})} = \sum_{\substack{\text{all} \\ \text{pairs}}} \overline{V(t_x) V(t_y)} \; ; \quad = 0 \quad \text{for } 2n+1$$

or odd.

How many ways can we divide up $2n$ elements in n pairs?

$$\frac{(2n)!}{2^n \; n!}$$

We can prove that given above conditions, the distribution is a gaussian. Expand $V(t)$ in a Fourier series ($V(t)$ periodic with period T).

$$(8) \quad V(t) = \sum_k \left(a_k \cos 2\pi f_k t + b_k \sin 2\pi f_k t\right) \; ; \quad f_k = \frac{k}{T}$$

$$\text{with} \quad a_k = \frac{2}{T} \int_0^T V(t) \cos 2\pi f_k t \, dt$$

Take $2n$ th moment of a_k

$$(9) \quad \overline{a_k^{2n}} = \left(\frac{2}{T}\right)^{2n} \iiint \cdots \int \cos 2\pi f_k t_1 \cdots \cos 2\pi f_k t_{2n}$$

$$\cdot \overline{V(t_1) \cdots V(t_{2n})} \, dt_1 \cdots dt_{2n}$$

$$= \frac{(2n)!}{2^n \, n!} \left[\left(\frac{2}{T}\right)^2 \int_0^T \int_0^T \cos 2\pi f_k t_x \cos 2\pi f_k t_y \, \overline{V(t_x) V(t_y)} \, dt_x \, dt_y\right]^n$$

$$= 1 \cdot 3 \cdot 5 \cdots (2n-1)\left(\overline{a_k^2}\right)^n$$ which is the $2n$ th moment of a gaussian.

We can also show that a_k and b_k are independent gaussian distributed. We can plug in differential equation the series $V(t)$ and get $i(t)$ as a linear function of a_k and b_k. Thus $i(t)$ is gaussian distributed. Thus Johnson Noise and Brownian motion are gaussian random processes.

In Brownian motion, one is usually not interested in velocity but in displacement. We must integrate solution.

Note that $\overline{i(t)\,\underset{\substack{\shortparallel \\ i_0}}{i(t_0)}} = i_0^2\, e^{-\frac{R}{L}t}$ and this is

The fourier transform of the Lorentzian PSD of this circuit $G_i(f) \cong \dfrac{1}{1+\frac{\omega^2 L^2}{R^2}}$

LECTURE XXIV 11-26-60

Langevin Equation:

$$(1) \quad m\frac{dv}{dt} + \beta v = F(t)$$

whose equation is: $v = v_0 e^{-\beta/m\, t} + \frac{1}{m} e^{-\beta/m\, t} \int_0^t F(\xi) e^{\frac{\beta}{m}\xi}\, d\xi$

with $\bar{v} = v_0 e^{-\beta/m\, t}$ and $\overline{v v_0} = v_0^2\, e^{-\beta/m\, t}$ $\left(e^{-\frac{\beta}{m}t} = \rho\right)$

and $\overline{v^2} = \frac{kT}{m} + \left(v_0^2 - \frac{kT}{m}\right) e^{-\frac{2\beta}{m}t}$

$v(t)$ is a gaussian process and $v(t_1)$, $v(t_2)$ has a joint gaussian distribution. Thus:

$$(2) \quad W(v, v_0, t) = \left[\frac{m}{2\pi kT\left(1-e^{-\frac{2\beta}{m}t}\right)}\right]^{1/2} \exp\left[-\frac{m}{2kT}\, \frac{\left(v - v_0 e^{-\frac{\beta}{m}t}\right)^2}{1-e^{-2\beta/m\, t}}\right]$$

Note that in limit of $t \to \infty$, W is the Maxwell Boltzmann distribution independent of v_0. In observing Brownian motion, one observes the mean displacement over .1 second instead of the velocity. Therefore, integrating (2):

$$(3) \quad x = x_0 + \frac{mv_0}{\beta}\left(1 - e^{-\beta/m\, t}\right)$$

$$+ \frac{1}{m}\int_0^t e^{-\beta/m\, t'}\, dt' \int_0^{t'} e^{\frac{\beta}{m} t''} F(t'')\, dt''$$

$$= x_0 + \frac{mv_0}{\beta}\left(1 - e^{-\beta/m t}\right) - \frac{1}{\beta} e^{-\beta/m t} \int_0^t e^{-\beta/m\, t''} F(t'')\, dt''$$

$$+ \frac{1}{\beta}\int_0^t F(t'')\, dt''$$ by partial integration.

The mean displacement:

$$(4) \quad \overline{x - x_0} = \frac{mv_0}{\beta}\left(1 - e^{-\beta/m\, t}\right)$$

For the mean square displacement, using $\overline{F(t'')F(t''')} = 2\beta kT\, \delta(t'' - t''')$:

$$(5) \quad \overline{(x - x_0)^2} = \frac{2\beta kT}{\beta^2} + \frac{m^2 v_0^2}{\beta^2}\left(1 - e^{-\frac{\beta}{m}t}\right)^2 + \frac{2\beta kT}{\beta^2}\left(-3 + 4e^{-\frac{\beta}{m}t} - e^{-\frac{2\beta}{m}t}\right)$$

The displacement will be distributed as a gaussian:

$$(6) \quad W_{v_0}(x, x_0, t) = \left[\frac{\beta^2/m}{2\pi kT\left(2\frac{\beta}{m}t - 3 + 4e^{-\beta/mt} - e^{-2\beta/mt}\right)}\right]^{1/2}$$

$$\cdot \exp\left[\frac{\beta^2}{2mkT} \frac{\left\{x - x_0 - \frac{mv_0}{\beta}\left(1 - e^{-\frac{\beta}{m}t}\right)\right\}^2}{2\frac{\beta}{m}t - 3 + 4e^{-\beta/mt} - e^{-2\beta/mt}}\right]$$

We will be more interested in $\overline{(x - x_0)^2}$ than in distribution. We now averge $\overline{(x - x_0)^2}$ over the velocity which we take to be distributed Maxwellian.

$$(7) \quad \overline{(x-x_0)^2} = \frac{2mkT}{\beta^2}\left(\frac{\beta}{m}t - 1 + e^{-\frac{\beta}{m}t}\right)$$

$$= \frac{2kT}{\beta} t \qquad \text{for} \quad t \gg \frac{m}{\beta}$$

$$= \underset{\underset{v_0^2}{\parallel}}{\frac{kT}{m}} t^2 \qquad \text{for} \quad t \ll \frac{m}{\beta}$$

In Brownian motion we are interested in $t \gg \frac{m}{\beta}$. For heavy objects it would be $t \ll \frac{m}{\beta}$.

We now calculate $\overline{(x-x_0)^2}$ for .1 second:

$$m = \frac{4\pi}{3} a^3 \underset{\downarrow}{(\rho - \rho_0)} \; ; \quad \beta = 6\pi\eta a \quad \text{with } a \sim 10^{-3} \text{ cm}$$

do not consider because particles are assumed to be free.

If particle is too small, displacement is too large for eye to follow properly. In H_2O and in .1 sec, $\overline{(x-x_0)^2} \cong a^2$. This is the optimum smallest size of particle. We could observe for long time and find $\overline{(x-x_0)^2}$ indeed becomes proportional to time.

Note that as $t \to \infty$, (6) becomes:

$$(8) \quad \frac{1}{(4\pi D t)^{1/2}} e^{-\frac{(x-x_0)^2}{4Dt}} \; ; \quad D = \frac{kT}{\beta}$$

Large times means that particle has forgotten its initial velocity.

This discussion shows that the displacement is large enough that it may be observed, yet $\overline{(x-x_0)^2}$ is of the order of the resolving power of the eye so that diffusion may be observed directly.

LECTURE XXV   11-29-60

Errata: $\overline{(x-x_0)^2} = \frac{m^2 v_0^2}{\beta^2}\left(1-e^{-\beta/m t}\right)^2 + \frac{2\beta kT}{\beta^2} t + \frac{m 2\beta kT}{\beta^3}\left(-3+4e^{-\beta/m t} - e^{-\frac{2\beta}{m}t}\right)$

Fokker Planck Method:

Master Equation:

What is the probability to jump from x to $x', x'+dx'$ in time $t, t+dt$?

(1) $P(x', t'=t+\Delta t \mid x\, t) = P(x'|x)\,\Delta t\,\Delta x'$

Number of jumps out of interval dx in time dt is $W(x,t)\,\cancel{dx} \int P(x'|x)\,dx'\,dt$

Number of jumps into: $\int P(x|x')\,W(x',t)\,dx'\,dt$

(2) Net flow: $\frac{\partial W(x,t)}{\partial t} = \int \left\{ W(x',t)\,P(x|x') - W(x,t)\,P(x'|x) \right\} dx'$

which is the master equation. The process is Markovian since past history is irrelevant and also the assumption is made that probability is proportional to time interval. Now derive Fokker-Planck equation from it. Assume $P(x'/x)$ is a slowly varying function of x. If $x'-x=-y$ is the jump distance, $P(x'/x)$ rapidly $\to 0$ for large y. Let $P(x'/x) \equiv P(x,y)$ in order to bring in y explicitly. Then, from (2):

(3) $\frac{\partial W(x,t)}{\partial t} = \int W(x-y,t)\,P(x-y,y)\,dy - W(x,t)\underbrace{\int P(x,-y)\,dy}_{\text{probability per unit time to jump at all.}}$

Enacting the assumption of slowly varying $P(x'/x)$ we may expand in a Taylor series:

$$(4)\quad W(x-y,t)\,P(x-y,y) = W(x,t)\,P(x,y) - y\left\{\frac{\partial}{\partial x} W(x,t)\,P(x,y)\right\}_x + \frac{1}{2}y^2\left\{\frac{\partial^2}{\partial x^2} w(x,t)\,P(x,y)\right\}_x + \cdots$$

Then master equation becomes:

$$(5)\quad \frac{\partial W(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left[W(x,t)\underbrace{\int y\,P(x,y)\,dy}_{\substack{\bar{y}\ :\ \text{mean}\\ \text{displacement}\\ \text{per unit time due}\\ \text{to jumps}}}\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[W(x,t)\underbrace{\int y^2\,P(x,y)\,dy}_{\substack{\overline{y^2}\ :\ \text{mean square}\\ \text{displacement per}\\ \text{unit time due to}\\ \text{jumps}}}\right]$$

Define: $A(x) = \lim\limits_{\Delta t\to 0}\dfrac{\bar{y}}{\Delta t}$, $\quad B(x) = \lim\limits_{\Delta t\to 0}\dfrac{\overline{y^2}}{\Delta t}$

Now the Fokker-Planck equation is:

$$(6)\quad \frac{\partial W(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left\{A(x)\,W(x,t)\right\} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left\{B(x)\,W(x,t)\right\}$$

Apply to random walk: $A(x)$, $B(x)$ are constants since independent of $P(x,y)$, $w(x,t)$. From right-left symmetry, $\overline{\Delta x} = 0$ and $\overline{\Delta x^2} = a^2 N\,\Delta t$, therefore $B = a^2 N$. Then, for FP equation:

$$(7)\quad \frac{\partial W}{\partial t} = \underset{\overset{\parallel}{D}}{\frac{1}{2}a^2 N}\,\frac{\partial^2 W}{\partial x^2} \qquad \text{which is diffusion equation.}$$

$$D \Rightarrow \overline{\Delta x^2} = 2D\,\Delta t$$

Fundamental solution: Boundary conditions at $t=0$; $W = \delta(x-x_0)$ and we know that solution is a gaussian, thus we write:

$$(8)\quad W(x,t) = \frac{1}{\sqrt{4\pi D t}}\, e^{-\frac{(x-x_0)^2}{4Dt}}$$

from previous knowledge that random walk in limit of large number of steps is gaussian.

Assumption of large number of steps is implicit in derivation of FP equation.

Now apply to Brownian Motion:
Use FP equation on velocity or momentum of particles, rather than on position.

$$(9) \quad A(v) = \lim_{\Delta t \to 0} \frac{\langle \Delta v \rangle}{\Delta t} = -\frac{\beta}{m} v$$

from $m \frac{dv}{dt} + \beta(v) = F(t)$; $\overline{F(t)} = 0$

We obtain $B(v)$ from Langevin equation:

$$\overline{v} = V_0 \, e^{-\frac{\beta}{m} \Delta t}$$

$$\overline{v^2} = \frac{kT}{m} + \left(v_0^2 - \frac{kT}{m} \right) e^{-\frac{2\beta}{m} \Delta t}$$

$$\lim_{\Delta t \to 0} \frac{\overline{(v - v_0)^2}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\overline{v^2} - 2 v_0 \overline{v} + v_0^2}{\Delta t} = \frac{2 \beta k T}{m^2} = B(v)$$

Then the FP equation for Brownian motion:

$$(10) \quad \frac{\partial W(v,t)}{\partial t} = -\frac{\beta}{m} \frac{\partial}{\partial v} \left\{ v \, W(v,t) \right\} + \frac{\beta k T}{m^2} \frac{\partial^2 W}{\partial v^2}$$

whose solution we already know:

$$(11) \quad W(v, v_0, t) = \left[\frac{m}{2\pi k T \left(1 - e^{-\beta/m\, t} \right)} \right]^{1/2} \exp \left[-\frac{m}{2kT} \frac{\left(v - v_0 e^{-\beta/m\, t} \right)^2}{1 - e^{-\beta/m\, t}} \right]$$

This method is seen to lead to difficult differential equations. However, in external fields of force, there is usually no other way.

* * *

LECTURE XXVI 12-1-60

Brownian Motion:

The displacements are not strictly Markovian.

(1) $$\overline{x - x_0} = \frac{m v_0}{\beta}\left(1 - e^{-\beta/m t}\right)$$

(2) $$\overline{(x-x_0)^2} = \frac{m^2 v_0^2}{\beta^2}\left(1 - e^{-\frac{\beta}{m}t}\right)^2 + \frac{2\beta k T}{\beta^2} t + \frac{m 2 \beta k T}{\beta^3}\left(-3 + 4e^{-\frac{\beta}{m}t} - e^{-\frac{2\beta}{m}t}\right)$$

Now:

(3) $$\overline{\overline{x - x_0}} = 0$$

(4) $$\overline{\overline{(x-x_0)^2}} = \frac{2 m k T}{\beta^2}\left[\frac{\beta}{m} t - 1 + e^{-\frac{\beta}{m}t}\right]$$

$x(t)$ may be considered as a Markoff process, if only times $t \gg \frac{m}{\beta}$ are considered.

In the FP equation: $A(x) = 0$

(5) $B(x) = \frac{2kT}{\beta}$, letting t become small but greater than β/m. Thus,

(6) $$\frac{\partial W}{\partial t} = \frac{1}{2}\,\frac{2kT}{\beta}\,\frac{\partial^2 W}{\partial x^2} = D\,\frac{\partial^2 W}{\partial x^2} \quad ; \quad \boxed{D = \frac{kT}{\beta}} \text{ (Einstein)}$$

Thus process is Markovian in $t \gg \frac{\beta}{m}$ for free particle in a fluid.

Particle in a Field of Force:

Gravitational Force: $K = \frac{4\pi a^3}{3}(\rho - \rho_0) g$

Now the drift velocity is given by the balance between the force and the damping.

(7) $\frac{d\langle x\rangle}{dt} = \frac{K}{\beta}$, of which we only observe the average damping $\beta\langle v\rangle$

We can now write the Smoluchowske Equation:

(8) $\frac{\partial W}{\partial t} = -\frac{K}{\beta}\frac{\partial W}{\partial x} + D\frac{\partial^2 W}{\partial x^2}$; $A = \frac{K}{\beta}$

We find the stationary solution:

(9) $D\frac{\partial W}{\partial x} = \frac{K}{\beta}W$; or $W =$ constant $\times$ $e^{\frac{K}{D\beta}x}$

We can see how this fits with the thermodynamic argument that we should get Maxwell-Boltzmann distribution:

(10) $W =$ constant $\times$ $e^{-\frac{V}{kT}}$, $V = -Kx$, $W =$ constant $\times$ $e^{\frac{Kx}{kT}}$

but $D = \frac{kT}{\beta}$ so the two results are equal.

If we multiply (9) by N_0, we have + diffusion current on the left and the drift current on the right.

In any field of force we have the general Smoluchowski equation:

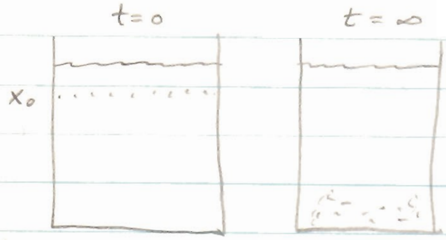
(11) $\frac{\partial W}{\partial t} = -\frac{1}{\beta}\frac{\partial (KW)}{\partial x} + D\frac{\partial^2 W}{\partial x^2}$

For harmonic oscillator force, $K = \alpha x$:

(12) $\frac{\partial W}{\partial t} = -\frac{\alpha}{\beta}\frac{\partial (xW)}{\partial x} + D\frac{\partial^2 W}{\partial x^2}$

which is the FP equation that was obtained previously. This is good only for highly damped harmonic oscillator.

Sedimentation Problem:



Particles in fluid in constant field of force.

The Smoluchowski equation is:

$$(13)\quad \frac{\partial W(x,y,z,t)}{\partial t} = -\frac{\frac{4\pi a^3}{3}(\rho-\rho_0)g}{\beta}\frac{\partial W}{\partial z} + D\left[\frac{\partial^2 W}{\partial z^2} + \frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial x^2}\right]$$

Consider only z direction and that homogeniety exists in the other directions.

$$(14)\quad \frac{\partial W}{\partial t} = c\frac{\partial W}{\partial z} + D\frac{\partial^2 W}{\partial z}$$

Substitute: $W = U \exp\left\{-\frac{c}{2D}(z-z_0) - \frac{c^2}{4D}t\right\}$

Thus $\frac{\partial U}{\partial t} = D\frac{\partial^2 U}{\partial z^2}$

Boundary Conditions: $t=0$: $W = \delta(z-z_0)$

$t \geq 0$: $D\frac{\partial W}{\partial z} + cW = 0$ for $z=0$

For U: $U = \delta(z-z_0)$ at $t=0$

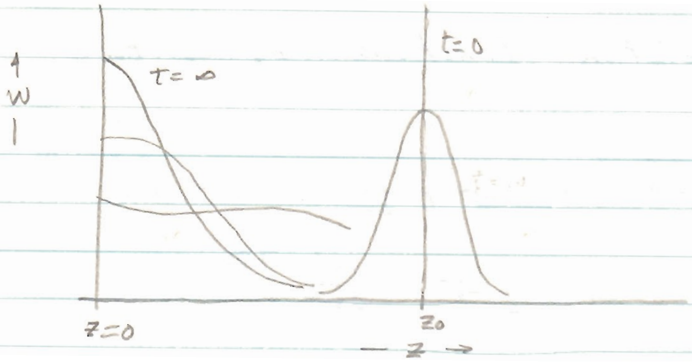
$D\frac{\partial U}{\partial z} + \frac{1}{2}cU = 0, \quad t>0$

We now solve and write the solution:

$$(15)\quad W(z,z_0,t) = \frac{1}{2(\pi Dt)^{1/2}}\left[\exp\left\{\frac{-(z-z_0)^2}{4Dt}\right\} + \exp\left\{\frac{+(z-z_0)^2}{4Dt}\right\}\right]$$

$$\cdot \exp\left[-\frac{c}{2D}(z-z_0) - \frac{c^2}{4D}t\right] + \frac{c}{D\pi^{1/2}}e^{-\frac{cz}{D}}\int_{\frac{z+z_0-ct}{2(Dt)^{1/2}}}^{\infty} e^{-x^2}dx$$

Plotting for different times "t":



For discussion of this problem, see Wax, p. 59 Chandrasekhar.

LECTURE XXVI 12-3-60

Harmonic Oscillator:

The equation of motion is:

$$(1) \quad m\ddot{x} + \beta\dot{x} + \alpha x = F(t)$$

Not Markovian, as it involves a second time derivative, hence, it is difficult to solve for the displacement even though given the distribution for F: However, we can write two coupled linear first order differential equations:

$$(2) \quad \dot{x} = \frac{p}{m}$$

$$(3) \quad \dot{p} = -\beta\frac{p}{m} - \alpha x + F(t)$$

For the stationary state; we can take the spectral density:

$$(4) \quad G_x(f) = \frac{2\beta kT}{|-m\omega^2 + i\beta\omega + \alpha|^2}$$

$$(5) \quad G_p(f) = \frac{2\omega^2\beta kT m^2}{(\alpha - m\omega^2)^2 + \omega^2\beta^2}$$

(6) $$G_{px}(f) = \frac{2i\omega\beta kTm}{(\alpha - m\omega^2)^2 + \omega^2\beta^2}$$

We can find the correlation functions by contour integration:

(7) $$\overline{x(t)\,x(t+\tau)} = \frac{kT}{m\omega_0^2}\, e^{-\frac{\beta\tau}{2m}} \left(\cos\omega_1\tau + \frac{\beta}{2\omega_1} \sin\omega_1\tau \right)$$

where $\omega_0^2 = \frac{\alpha}{m}$; $\omega_1 = \left[\omega_0^2 - \frac{\beta^2}{4m^2}\right]^{1/2}$

(8) $$\overline{p(t)\,p(t+\tau)} = m\,kT\, e^{-\frac{\beta\tau}{2m}} \left(\cos\omega_1\tau - \frac{\beta}{2\omega_1} \sin\omega_1\tau \right)$$

(9) $$\overline{x(t)\,p(t+\tau)} = \int_{-\infty}^{\infty} G_{px}(f)\, df\, e^{i\omega\tau} = \frac{4\beta kT}{m} \int_0^{\infty} \frac{\omega \sin\omega\tau}{(\alpha - m\omega^2)^2 + \omega^2\beta^2}\, df$$

$$= \frac{kT}{\omega_1}\, e^{-\frac{\beta\tau}{2m}} \sin\omega_1\tau$$

We see that the cross correlation vanishes for $\tau = 0$ as expected because at $\tau = 0$, we have Maxwell - Boltzmann distribution. We have at $\tau \neq 0$ a multivariate gaussian distribution and we could write out this distribution hence we have solved completely for the stationary state, that is, we have $W(x\,p,\, x_0\,p_0,\, \tau)$

We could do same procedure as for free particle to find a time - dependent position distribution by solving the second order equation. See result in Uhlenbeck and Ornstein in Wax.

Setting up FP equation in two variables, p and x. Must find:

$$A(x) = \lim \frac{\overline{\Delta x}}{\Delta t} = \frac{p}{m}$$

$$A(p) = \lim \frac{\overline{\Delta p}}{\Delta t} = -\frac{\beta p}{m} - \alpha x$$

$$B(x) = \lim \frac{\overline{\Delta x^2}}{\Delta t} = 0$$

$$B(x\,p) = \lim \frac{\overline{\Delta x \,\Delta p}}{\Delta t} = 0$$

$$B(p) = \lim \frac{\overline{\Delta p^2}}{\Delta t} = \frac{2\beta kT}{m^2}$$

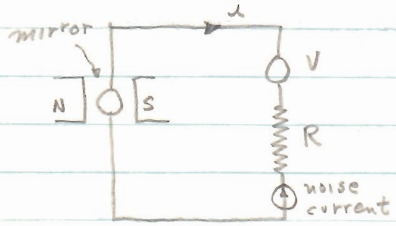
We can now write FP equation:

$$(10) \quad \frac{\partial W(x,p,t)}{\partial t} = -\frac{p}{m}\frac{\partial W}{\partial x} + \frac{\partial}{\partial p}\left[\left(\frac{\beta p}{m} + \alpha x\right)W\right] + \frac{\beta kT}{m^2}\frac{\partial^2 W}{\partial p^2}$$

Can find result in Uhlenbeck and Wang in Wax.

The point of this example is to indicate the method of forming FP equation by splitting up second order differential equation.

Example: Galvanometer Damping:



Introduce notation:

Moment of Inertia : $\mathcal{I}$
Torsion constant : α
Deflection ; ϑ

Assume square loop of dimensions a, b in a constant magnetic field Ho.

Lorentz Torque: $K = \frac{nHiab}{c} \cos\vartheta = \gamma i$, $\vartheta << 1$

The mechanical equation of motion:

$$(11) \quad \mathcal{I}\ddot{\vartheta} + \beta\dot{\vartheta} + \alpha\vartheta = \gamma i + F(t)_{mech}$$

Electrical equation:

$$(12) \quad \underset{\overset{\shortparallel}{\text{neglect}}}{-L\frac{di}{dt}} + iR = \underset{\overset{\downarrow}{\text{signal voltage}}}{v_s} + \underset{\overset{\downarrow}{-\gamma\dot{\vartheta}}}{v_{induced}} + F_{el}(t)$$

from $v_{\text{induced}} = -\frac{1}{c}\frac{d}{dt}\int \bar{H}\cdot d\bar{A} = -\frac{Hab\,n}{c}\cos\vartheta\,\dot{\vartheta} = -\gamma\dot{\vartheta}$

Solve simultaneously:

$$(13)\quad \mathcal{I}\ddot{\vartheta} + \Big(\beta + \underbrace{\frac{\gamma^2}{R}}_{\substack{\text{electrical}\\ \text{damping}}}\Big)\dot{\vartheta} + \alpha\vartheta = \frac{\gamma v_s}{R} + F_{mech}(t) + \frac{\gamma}{R}F_{el}(t)$$

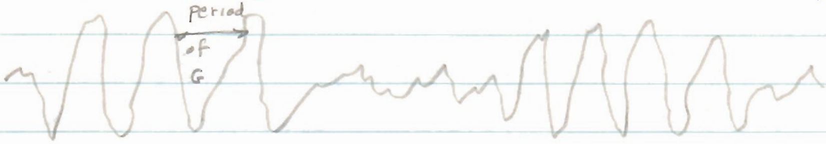
$$(14)\quad G^+_{\vartheta}(f) = \frac{4\beta k T_{air} + \frac{\gamma}{R}4RkT_R}{\left|-\omega^2\mathcal{I} + i\omega\left(\beta+\frac{\gamma^2}{R}\right)+\alpha\right|^2}$$

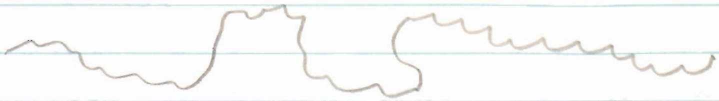
$$(15)\quad \overline{\vartheta^2} = \int_0^\infty G^+\,df = \frac{\beta T_{air} + \frac{\gamma^2}{R}T_R}{\beta + \frac{\gamma^2}{R}}\;\frac{k}{\alpha}$$

$$= \frac{kT}{\alpha} \quad \text{where} \quad T_{air} = T_R = T$$

so that under equilibrium, equipartition is observed.

Some periodicity observed in underdamped noise:



For overdamped: 

However, $\overline{\vartheta^2}$ is same for both cases.

LECTURE XXVII 12-6-60

Steady State Solution of FP Equation:

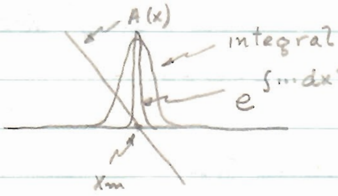
(1) $$\frac{\partial W}{\partial t} = -\frac{\partial}{\partial x}(AW) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(BW)$$

(2) $$\frac{\partial W}{\partial t} = 0 : \quad AW = \frac{1}{2}\frac{\partial}{\partial x}(BW)$$

$$\frac{2A(x)}{B(x)} = \frac{1}{BW}\frac{\partial}{\partial x}(BW)$$

or $$W(x) = \frac{\text{constant}}{B(x)}\, e^{\int_{-\infty}^{x}\frac{2A(x')}{B(x')}dx'}$$

We argue that most of the contribution to the integral comes from regions where $A(x)$ changes sign or $A(x)=0$. Method of steepest descent:



Therefore we expand $A(x)$ about x_m:

$$A(x') = 0 + \left(\frac{\partial A}{\partial x}\right)_{x_m}(x-x_m) + \cdots$$

Now $B(x_m)$ is positive and we hope that $A(x_m)$ is negative. We have in the exponent instead of the integral:

$$\frac{2\left(\frac{\partial A}{\partial x}\right)_{x_m}\frac{(x-x_m)^2}{2}}{B(x_m)}$$ with the result that:

(3) $$W_{ss}(x) = \frac{\text{constant}}{B(x)_m}\, e^{-\frac{(x-x_m)^2}{2\sigma^2}} \; ; \quad \sigma^2 = \frac{-B(x_m)}{2\left(\frac{\partial A}{\partial x}\right)_{x_m}}$$

Thus, we get a gaussian under this approximation which is generally valid as the probability decreases rapidly as one moves away from x_m. This whole argument rather follows from the central limit theorem. The FP equation will usually tend to a gaussian. This same result could also be had directly from the master equation.

what we have said is that the mean is the most probable value and large variations from the mean do not occur.

Example: Fluctuations in Semi-conductor Carrier Density.

(4) Probability of n carriers $= P(n)$. Setting up the master equation with probability of absorption α; generation β:

(5) $$\frac{dP(n)}{dt} = \alpha(n+1)\,P(n+1) + \beta(n-1)\,P(n-1) - \left[\alpha(n) + \beta(n)\right]P(n)$$

For an intrinsic material:

$$\alpha(n) = an^2 \quad , \quad \beta(n) = \text{constant}$$

For n type:

$$\alpha(n) = a'n^2 \quad , \quad \beta(n) = b\,(Nd-n)$$

Radioactive Decay:

$\alpha(n) = a'n \quad , \quad \beta(n) = 0$ since only decay occurs.

Fluctuations in state of gas under EM radiation:
n is number in excited state.

(absorbtion of photons) $\alpha(n) = \{A + B\rho\}\,n$, ρ = density of electrons

stimulated emmision spontaneous emmision

$\beta(n) = B\rho\,(N-n)$ (emmision of photons)

number in excited state

Random Walk:

$$\alpha(n) = \beta(n) = \text{constant}$$

Now calculate fluctuations:

(6) $$\frac{d\bar{x}}{dt} = \frac{d}{dt}\int x\, P(x)\, dx = -\langle \alpha(n) \rangle + \langle \beta(n) \rangle$$

from equation (5). Now make linear assumption; Expand $\alpha(n)$, $\beta(n)$ around steady state value (corresponding to x_m):

$$\alpha(n) = \alpha(n_m) + \alpha'(x - n_m) + \cdots$$

$$\beta(n) = \beta(n_m) + \beta'(n - n_m) + \cdots$$

Thus, since $\frac{d x_m}{dt} = 0$:

(7) $$\frac{d(\bar{n} - n_m)}{dt} = -\alpha(n_m) + \beta(n_m) + (-\alpha' + \beta')(\bar{n} - n_m)$$

Note that the rate of decay depends on the time which one would expect. The system decays exponentially to its equilibrium at a characteristic time $\frac{1}{\tau} = (-\alpha' + \beta')$.

We can do the same thing for the mean square deviation;

(8) $$\frac{\partial}{\partial t}\int (n - n_m)^2 P(n)\, dn$$, method of solution is in M. Lax, Rev. Mod. Phys. 32, 25, (1960)

The result is:

(9) $$\frac{d\,\overline{(\bar{n} - n_m)^2}}{dt} = -\frac{2}{\tau}\left\{\overline{(n - n_m)^2} + \text{constant}\right\}$$ where τ is as above.

The constant would occur alone in the steady state;

$$\text{constant} = -\left[\frac{\alpha(n_m) + \beta(n_m)}{2\left\{-\alpha'(n_m) + \beta'(n_m)\right\}}\right]$$

This method of derivation does not assume any Gaussian properties.

LECTURE XXVIII 12-8-60

Fluctuation in Semi-conductor Carrier Concentration:

$$(1)\quad A(n) = \lim \frac{\overline{\Delta n}}{\Delta t} = (-1)\,\alpha(n) + (+1)\,\beta(n) = -\alpha + \beta$$

$$(2)\quad B(n) = \lim \frac{\overline{(\Delta n)^2}}{\Delta t} = 1\,\alpha(n) + 1\,\beta(n) = \alpha + \beta$$

Then the FP equation is:

$$\frac{\partial}{\partial t}\int n\,P(n)\,dn = -\int n \frac{\partial}{\partial n}\Big[(-\alpha+\beta)P\Big]\,dn + \frac{1}{2}\int n \frac{\partial^2}{\partial n^2}\Big\{(\alpha+\beta)P\Big]\,dn$$

$$\text{or}\quad \frac{\partial \bar{n}}{\partial t} = +\int(-\alpha+\beta)P\,dn - \frac{1}{2}\underset{\substack{\shortparallel \\ 0}}{\int_0^\infty (\alpha+\beta)P\,dn}$$

from boundary conditions

$$= \overline{-\alpha(n)+\beta(n)}$$

In the steady state: $n = n_0$, $\alpha(n_0) = \beta(n_0)$

We then expand linearly around the steady state:

$$(3)\quad \frac{\partial \bar{n}}{\partial t} \simeq \overline{-\alpha(n_0) + \beta(n_0)} \simeq -\alpha(n_0) + \beta(n_0) + (-\alpha' + \beta')_{n_0}\,(\bar{n} - n_0)$$

$$= \frac{\partial(\bar{n} - n_0)}{\partial t}$$

which is good for small fluctuations from $\bar{n}$.

$$\tau_{decay} = (-\alpha' + \beta')^{-1}_{n_0}$$

For the mean square deviation:

$$\frac{\partial}{\partial t}\int (n-n_0)^2 P\,dn = -\int (n-n_0)^2 \frac{\partial}{\partial n}\Big[(-\alpha+\beta)P\,dn$$

$$+ \frac{1}{2}\int (n-n_0)^2 \frac{\partial^2}{\partial n^2}\Big[(\alpha+\beta)P\Big]\,dn$$

$$= 2\int (n-n_0)(-\alpha+\beta)P\,dn = 2\,\overline{(n-n_0)(-\alpha+\beta)}$$

by partial integration

or, in the linear approximation:

$$(4) \qquad \frac{\partial\, \overline{(n-n_0)^2}}{\partial t} = 2\, \overline{(n-n_0)^2}\, (-\alpha' + \beta')_{n_0}$$

The second term gives by partial integration,

$$(5) \qquad \overline{\alpha + \beta}$$

or, in the linear approximation:

$$(6) \qquad \frac{\partial\, \overline{(n-n_0)^2}}{\partial t} = 2\, \overline{(n-n_0)^2}\, (-\alpha' + \beta')_{n_0} + \alpha(n_0) + \beta(n_0)$$

and in the steady state $\dfrac{\partial\, \overline{(n-n_0)^2}}{\partial t} = 0$;

$$(7) \qquad \overline{(n-n_0)^2}_{ss} = - \frac{\alpha(n_0) + \beta(n_0)}{2\left\{-\alpha'(n_0) + \beta'(n_0)\right\}}$$

The previous solution to the FP equation gave:

$$(8) \qquad P(n)_{ss} = \text{constant } e^{-\frac{(n-n_0)^2}{2\sigma^2}}$$

where $\sigma^2 = - \dfrac{B(n_0)}{2\left(\frac{\partial A}{\partial n}\right)_{n_0}}$ which checks with (7).

Upon solution of equations (3) and (6) we have ;

$$(9) \qquad \overline{[n(0) - n_0]\, \overline{[n(t) - n_0]}} = \overline{[n(0) - n_0]^2}\, e^{-t/\tau_{decay}}$$

$$\text{or} \qquad \overline{\Delta n(0)\, \Delta n(t)} = \overline{\Delta n^2(0)}\, e^{-t/\tau_{decay}}$$

and :

$$(10) \qquad G_{\Delta n}(f) = \overline{\Delta n^2(0)}\, \frac{2\, \tau_{decay}}{1 + \omega^2 \tau_{decay}^2}$$

or a Lorentzian spectral density.

Example: n-type Semi-conductor:

$$\alpha(n) = a n^2 \quad , \quad \beta(n) = b\,(N_d - n)$$

which in the steady state:

$$a n_0^2 = b\,(N_d - n_0)$$

or $$n_0 = -\frac{b}{2a} + \left[\frac{b^2}{4a^2} + \frac{b}{a} N_d\right]^{1/2}$$

and $$\overline{(n-n_0)^2_{ss}} = \frac{a n_0^2}{2 a n_0 + b} = \frac{n_0 (N_d - n_0)}{2 N_d - n_0}$$

At low temperature, only small fraction of donors ionized so that $N_d >> n_0$ and

$$\overline{(n-n_0)^2_{ss}} = \frac{n_0}{2}$$

At high temperatures: $N_d \approx n_0$, $N_d - n_0 \approx 0$ so very small deviations.

This is not purely random creation process as we would expect, $\Delta n^2 = n_0$, as there is not an infinite supply of donors. The $\frac{1}{2}$ comes from the creation relation $\alpha(n) = a n^2$.

We now consider shot noise in a semi-conductor when an electric field is applied. Recall that Johnson noise is related not to fluctuations in carrier density but to carrier velocity. Semi-conductors become noisier when more current is passed; this does not happen in a metal.

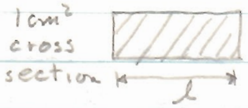
LECTURE XXIX 12-10-60

Noise in Semiconductors:

(1) $\overline{(n - n_0)^2} = \dfrac{n_0 (N_d - n_0)}{2 N_d - n_0}$

(2) μ is mobility : $v_{drift} = \mu E$, $\sigma = n e \mu$

(3) Lifetime $\tau = (\alpha' - \beta')^{-1}$



The average current produced by one electron during its lifetime

(4) $e \dfrac{x}{l} \cdot \dfrac{1}{\tau} = e \dfrac{\mu E \tau}{l} \dfrac{1}{\tau} = \dfrac{e}{\tau_2}$

(5) or $\bar{I} = n_0 \dfrac{e}{\tau_2}$; $\overline{\Delta I^2} = \dfrac{e^2}{\tau_2^2} \overline{\Delta n^2}$, τ_2 is transit time $= \dfrac{1}{\tau_2} = \dfrac{\mu E}{l}$

The correlation functions is:

(6) $\overline{\Delta I(t) \Delta I(t+\tau)} = \dfrac{e^2}{\tau_2^2} \overline{\Delta n(t) \Delta n(t+\tau)} = \dfrac{e^2}{\tau_2^2} \overline{\Delta n^2} e^{-t/\tau}$

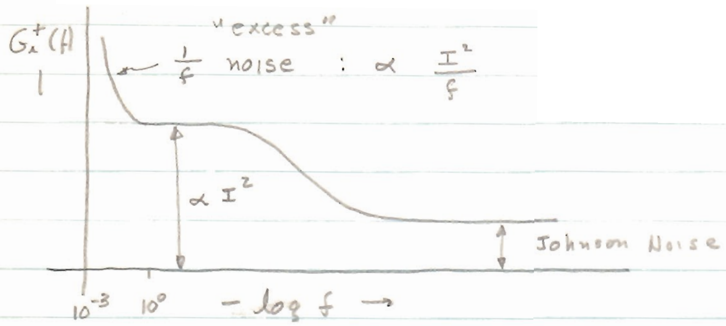
(7) $G_I^+(f) = \dfrac{e^2}{\tau_2^2} \dfrac{n_0 (N_d - n_0)}{2 N_d - n_0} \dfrac{4\tau}{1 + \omega^2 \tau^2}$, vacuum diode, PSD = $2 e I_0$

In diode we had high cutoff and constant PSD because of short transit time, however, now transit time is long enough to be of interest.

Now:

(8) $\dfrac{1}{\tau_2^2} = \dfrac{\mu^2 E^2}{l^2} = \dfrac{\mu^2}{\sigma^2} \dfrac{I^2}{l^2} = \dfrac{I^2}{n_0^2 e^2 l^2}$

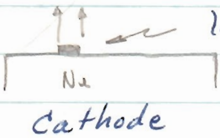
so that the PSD is proportional to I^2 and not I as in vaccuum diodes. Assomption made here is that $\tau_2 \gg \tau$ throughout derivation.



"Excess" noise is usually due to defects in material. These occur at frequencies of 10^{-3} cps which is just right for diffusion of imperfections. Analogous to "flicker" noise in vacuum tubes due to emission from semiconducting oxide cathodes.

Another type of excess noise is contact noise due to semiconducting oxide solder joints. Hard to theorize these effects as there is little knowledge of details.

Schottky's Theory of Flicker Noise:



lowering of work function $\delta\psi$ over small area A by adsorbed impurity.

From theory of thermionic emission; $J_0 \propto e^{-\frac{\psi}{kT}}$

Then increase in J due to decrease in ψ:

$$e^{-\frac{\psi - \Delta\psi}{kT}} - e^{-\frac{\psi}{kT}}$$

and $\Delta i = A J_0 \left(e^{+\frac{\Delta\psi}{kT}} - 1 \right)$ due to one imperfection

Noise comes from varying number of imperfections. Then

$$\Delta i = A J_0 \left(e^{+\frac{\Delta\psi}{kT}} - 1 \right) \Delta N$$

For correlation:

$$\overline{\Delta i(t)\, \Delta i(t+\tau)} = A^2 J_0^2 \left(\frac{\Delta\psi}{kT_c} \right) \overline{\Delta N^2}\, e^{-\tau/\tau_a}$$

where we have used $\Delta\psi$ small and the correlation function for shot current in semiconductor. Because of different adsoubtion times τ_a which are due to the many different ways of diffusion, we have a distribution in τ_a, so PSD becomes:

$$G_i^+(f) \longrightarrow \int_{\text{lower}}^{\text{upper}} \frac{4\,\tau_a \; g(\tau_a)}{1+\omega^2\tau_a^2}\, d\tau_a$$

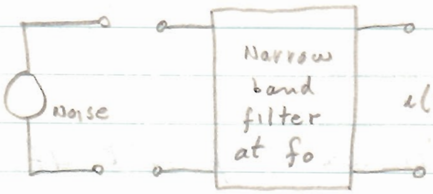
with $g(\tau_a) \propto \frac{1}{\tau_a}$ to give $\frac{1}{f}$ dependency of excess noise.

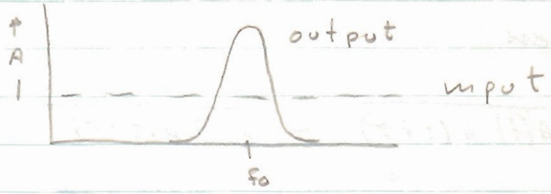
In metallic resistors, little fluctuation in number of carriers except maybe near the fermi surface, so there is no shot noise nor excess noise.

Reference: van der Ziel, Ch. 8, Noise

LECTURE XXX 12-13-60

Noise in Non-Linear Devices:





Can always make F-series expansion of noise current:

$$i(t) = \sum_{n=1}^{\infty} \left\{ a_n \cos 2\pi \frac{n}{T} t + b_n \sin 2\pi \frac{n}{T} t \right\}$$

We can also write

$$i(t) = A(t) \cos 2\pi f_0 t + B(t) \sin 2\pi f_0 t$$

where:

$$A(t) = \sum_{n=1}^{\infty} \left\{ a_n \cos 2\pi \left(\frac{n}{T} - f_0\right) t + b_n \sin 2\pi \left(\frac{n}{T} - f_0\right) t \right\}$$

$$B(t) = \sum_{n=1}^{\infty} \left\{ -a_n \sin 2\pi \left(\frac{n}{T} - f_0\right) t + b_n \cos 2\pi \left(\frac{n}{T} - f_0\right) t \right\}$$

A and B represent random gaussian variables because a_n and b_n are independent gaussian variables. We can write complete distribution if we just know all the linear moments, that is:

$$\left.\begin{array}{l} \overline{A(t)\, A(t+\tau)} \\ \overline{B(t)\, B(t+\tau)} \\ \overline{A(t)\, B(t+\tau)} \end{array}\right\} \text{ these give } W\left\{ A(t), B(t), A(t+\tau), B(t+\tau) \right\}$$

In forming distribution we make use of the properties of the a_n, b_n:

$$\lim \overline{a_n^2}\, T = \lim \overline{b_n^2}\, T = G\left(f = \frac{n}{T}\right)$$

$$\text{with} \quad \overline{a_n b_n} = \overline{a_n a_m} = \overline{a_n b_m} = 0$$

Then:

$$\overline{A(t)\, A(t+\tau)} = \overline{B(t)\, B(t+\tau)} = \int_0^{\infty} G^+(f) \cos 2\pi (f - f_0)\tau \, df$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} G(f) \left\{ e^{2\pi i (f - f_0)\tau} + e^{2\pi i (f + f_0)\tau} \right\} df$$

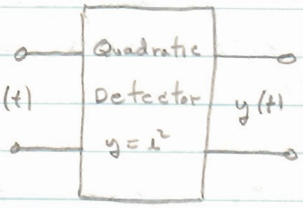
Also:

$$\overline{A(t)\, B(t+\tau)} = \int_0^{\infty} G^+(f) \sin 2\pi (f - f_0)\tau \, df$$

The total power is again:

$$\frac{1}{2}\;\overline{A^2(t) + B^2(t)} = \int_{-\infty}^{\infty} G(f)\,df$$

We now assume $i(t)$ is the input to a quadratic detector whose output is $y = i^2$



What is output frequency distribution?

Consider writing $i(t)$ as envelope - phase:

$$i(t) = V(t)\cos\{2\pi f_0 t + \varphi(t)\}$$

where $V^2(t) = A^2(t) + B^2(t)$

$$\varphi = \arctan\frac{B(t)}{B(t)}$$

We now write the appropriate distribution function:

$$W\{A(t), B(t)\} = \frac{1}{2\pi\sigma^2}\,e^{-\frac{A^2+B^2}{2\sigma^2}}$$

$$\text{or}\quad W(V, \varphi) = \frac{1}{2\pi\sigma^2}\,V\,e^{-\frac{V^2}{2\sigma^2}}$$

Now φ is uniformly distributed in $0 \leq \varphi \leq 2\pi$ as the $W(V, \varphi)$ is independent of φ. The quantity $V e^{-V^2/2\sigma^2}$ is called the Rayleigh distribution. Now the output of the detector is:

$$y(t) = \sqrt{a^2}\,i^2(t) = \sqrt{a^2}\,V^2(t)\cos^2\{2\pi f_0 t + \varphi(t)\}$$

$$= \tfrac{1}{2}\sqrt{a^2}\,V^2(t) + \tfrac{1}{2}\sqrt{a^2}\,V^2(t)\cos\{4\pi f_0 t + 2\varphi(t)\}$$

We now connect a low-pass filter the output:

Thus $z(t) = \frac{1}{2} a V^2(t)$

with consequent distribution:

$$W(z)\,dz = \frac{1}{a\sigma^2} e^{-\frac{z}{a\sigma^2}} dz$$

From gaussian distribution on $i(t)$, we get Rayliegh distribution on the envelope $V(t)$ and an exponential distribution on $z(t)$.

For fixed φ at input, $B(t) = 0$, then $z(t)$ would have been:

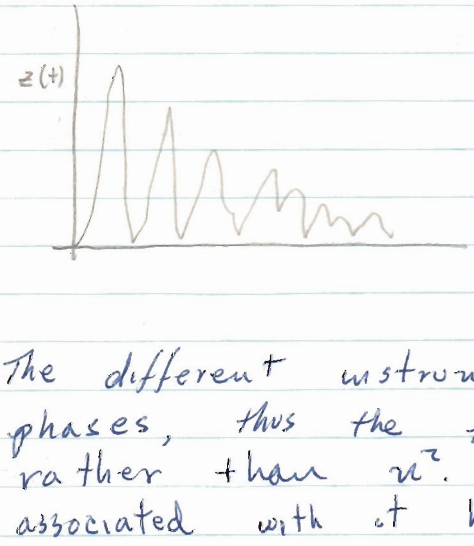
$z'(t) = \frac{1}{2} a A^2(t)$ where A is gaussian distributed

or $z'(t) = \frac{1}{\sqrt{2\pi\sigma^2}} \frac{1}{\sqrt{2z'a}} e^{-\frac{z'}{a\sigma^2}} dz'$ which is χ^2 distribution.

A remarkable propertie of the exponential distribution is that the r.m.s. deviation is equated to the mean That is:

$$\overline{z(t)} = a\sigma^2 \;;\quad \overline{z^n(t)} = n!\, a^n \sigma^{2n}$$

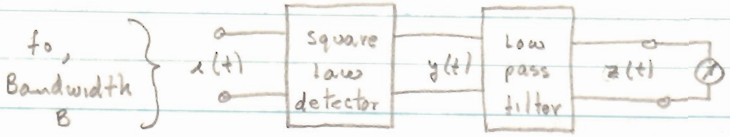
Variance: $\overline{\Delta z^2} = 2a^2\sigma^4 - a^2\sigma^4 = a^2\sigma^4 = \left(\overline{z(t)}\right)^2$



form of distribution for noise in $z(t)$. Most probable value is $z = 0$.

A physical example of this distribution would be n instruments playing the same frequency note. The different instruments will have rapidly varying phases, thus the power is proportional to the number rather than n^2. Any problem that has random phase associated with it has this distribution.

LECTURE XXXI   12-17-60



From last time:

$$w(z)\,dz = \frac{1}{a\sigma^2} e^{-\frac{z}{a\sigma^2}} dz$$

Correlation:

$$\overline{y(t)\,y(t+\tau)} = \alpha^2\, \overline{i^2(t)\, i^2(t+\tau)}$$

with $R_i(\tau) = \overline{i(t)\, i(t+\tau)}$

then (from homework problem),

$$\overline{y(t)\,y(t+\tau)} = \alpha^2 \left[R_i^2(0) + 2R_i^2(\tau) \right]$$

$$= \alpha^2 \sigma^4 + 2\alpha^2 R_i^2(\tau)$$

For PSD of y, take F transform:

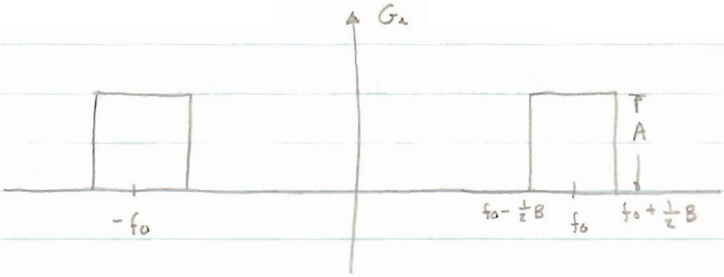
$$G_y(f) = \alpha^2 \sigma^4 \delta(f) + 2\alpha^2 \int_{-\infty}^{\infty} R_i^2(\tau) e^{-2\pi i f \tau} d\tau$$

We can write the last integral as:

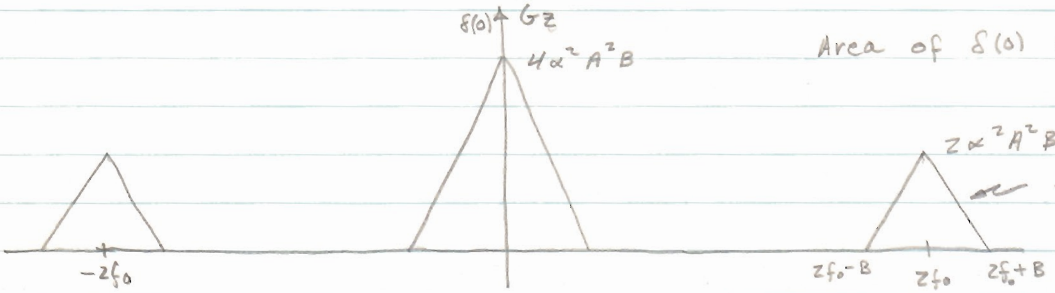
$$\iint_{-\infty}^{\infty} R_i(\tau)\, G(f')\, e^{2\pi i f' \tau}\, e^{-2\pi i f \tau}\, df'\, d\tau$$

or $$G_y(f) = \alpha^2 \sigma^4 \delta(f) + 2\alpha^2 \underbrace{\int_{-\infty}^{\infty} G(f')\, G(f-f')\, df'}_{\text{convolution of input PSD.}}$$

Assume a rectangular input PSD:



Then the output PSD is; from the convolution integral:



Area of $\delta(0) = 4\alpha^2A^2B^2$

these not seen after low pass filter

Let the time of meter indecation be t_m
then $t_m = \frac{1}{B'}$ with

$$\overline{(\Delta Z_r)^2} = \frac{4\alpha^2A^2B}{2t_m} \propto BB'$$

Now assume signal at f_0 into input. We can represent this by δ functions superposed on the input noise PSD. We write for the input:

$$s(t) + n(t) \quad , \quad \text{where} \quad s(t) = U\cos(\omega_0 t + \vartheta)$$

where ϑ is distributed uniformly in $0 - 2\pi$ so that input is stationary:

Output: $y(t) = \alpha\left[s^2(t) + 2s(t)n(t) + n^2(t)\right]$

$$\overline{y(t)} = \alpha \underbrace{\overline{s^2(t)}}_{\text{average output signal}} + \underbrace{\overline{n^2(t)}}_{\text{average noise output}}$$

For the correlation function:

$$R_y(\tau) = \alpha^2 \left[R_{s^2}(\tau) + 4 R_s(\tau) R_n(\tau) + 2 R_s(0) R_n(0) + R_{yn}(\tau) \right]$$

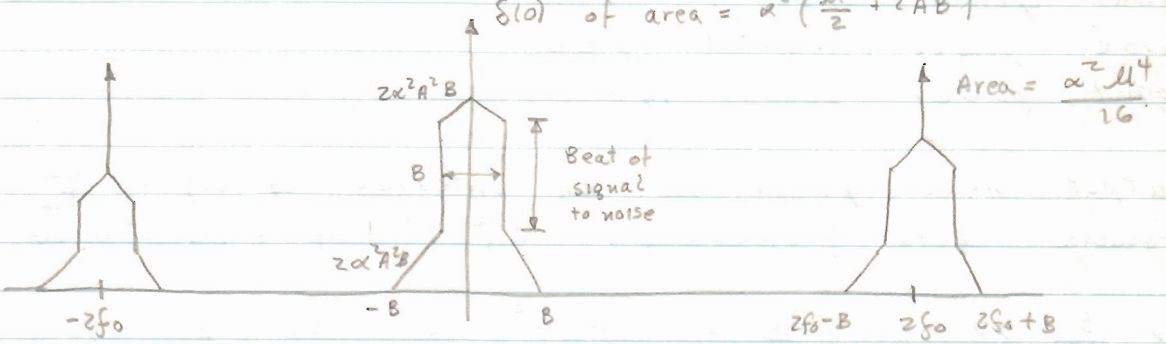
Now $R_{s^2}(\tau) = \overline{\mathcal{U}^2 \cos^2(\omega_0 t + \vartheta)\, \mathcal{U}^2 \cos^2[\omega_0 t + \omega_0 \tau + \vartheta]}$

$$= \frac{1}{4} \mathcal{U}^4 + \frac{1}{8} \mathcal{U}^4 \cos 2\omega_0 \tau$$

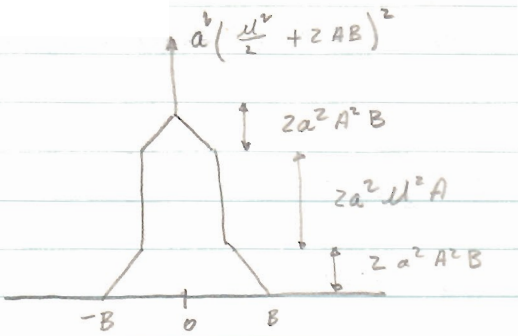
$R_s(\tau) = \overline{\mathcal{U} \cos(\omega_0 t + \vartheta)\, \mathcal{U} \cos(\omega_0 t + \omega_0 \tau + \vartheta)}$

$$= \frac{1}{2} \mathcal{U}^2 \cos \omega_0 \tau$$

What is new output PSD ?



LECTURE XXXII 12-20-60



Output PSD: No sidebands as signal has been passed thru low-pass filter.

Input PSD with signal superposed on noise.

Signal power input: $\frac{1}{2}\mu^2$
Noise power input: $2AB$
Signal power out: $1/4\, a^2\mu^4$

Output noise power in an interval of $\Delta f = \frac{1}{T_m}$ around zero frequency: $2a^2\mu^2A\,\Delta f + 4a^2A^2B\,\Delta f$

For $\frac{S_i}{N_i} \gg 1$: $\frac{S_o}{N_o(\Delta f)} = \frac{1/4\,\mu^2}{2A\Delta f} - \frac{S_i}{2N_i}\frac{B}{\Delta f}$

For $\frac{S_i}{N_i} \ll 1$: $\frac{S_o}{N_o(\Delta f)} = \frac{1/4\,\mu^4}{4A^2B\Delta f} = \left(\frac{S_i}{N_i}\right)^2\frac{B}{\Delta f}$

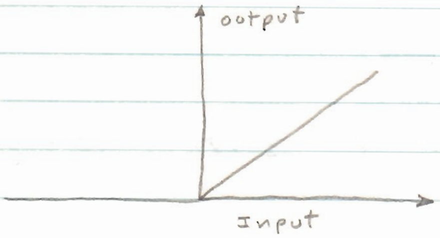
More important case: $\frac{S_i}{N_i} \gg 1$: If we use integrated noise input and output, $\frac{S_o}{N_o} = \frac{1}{2}\frac{S_i}{N_i}$. Notice that this is independent of B.
It is only bandwidth in audio section that matters.

For $\frac{S_i}{N_i} \ll 1$: $\frac{S_o}{N_o} \propto \frac{1}{B\Delta f}$, not independent of B.

Small signals in quadratic detectors are further suppressed which is general for all detectors.

This analysis can be repeated for modulated signals, taking care to include Fourier components. Result is similar to results obtained here.

Linear Detector:



The transfer function is:

$$z = b\, i(t) \quad ; \quad i > 0$$
$$z = 0 \quad ; \quad i < 0$$

Let $p(i_1, i_2)$ be the joint distribution of input at two times $i(t)$, $i(t+\tau)$. Then:

$$\overline{z(t)\, z(t+\tau)} = b^2 \int_0^\infty \int_0^\infty i_1(t)\, i_2(t+\tau)\, p(y_1, y_2)\, dy_1\, dy_2$$

The problem is essentially done. Must take inverse Fourier transform to find PSD. However, if input is gaussian with variance σ^2, we can go further and the moments of the output are:

$$\overline{z^{2n}} = \frac{1}{2} b^{2n} \sigma^{2n}\, 1 \cdot 3 \cdot 5 \cdots (2n-1)$$

$$\overline{z^{2n+1}} = \frac{b^{2n+1}}{\sqrt{2\pi\sigma^2}} \int_0^\infty y^{2n+1} e^{-y^2/2\sigma^2}\, dy$$

$$= \frac{2^n\, n!}{\sqrt{2\pi}}\, b^{2n+1}\, \sigma^{2n+1}$$ by partial integration.

We get for the joint moment:

$$\overline{z(t)\, z(t+\tau)} = \frac{b^2}{2\pi\sigma^2 \left(1-\rho^2(\tau)\right)^{1/2}} \int_0^\infty\int_0^\infty y_1 y_2 \exp\left[-\frac{y_1^2 + y_2^2 - 2\rho(\tau)\, y_1 y_2}{2\sigma^2 (1-\rho^2)}\right] dy_1\, dy_2$$

For solution, see Rice Section III and Davenport and Root Chapter 12.

The output psd for the linear detector is not much different than that of the quadratic detector when only noise is present at the input.

If we had cubic detector, we would have 3rd harmonic components in output as seen from expanding $\cos^3\theta$ and $\sin^3\theta$.

Case of General Non-Linear Device : $z = g(y)$; $i \rightarrow g$

Introduce Laplace transfer function :

$$f(u) = \int_0^{\infty} g(y)\, e^{-yu}\, dy$$

$$\text{Then} \quad z = \frac{1}{2\pi i} \int_{u'-i\infty}^{u'+i\infty} e^{uy} f(u)\, du$$

where the integration is in the complex "u" plane.

Now we can formally write the correlation function:

$$\overline{z(t)\, z(t+\tau)} = \iint_{-\infty}^{\infty} g(y_1)\, g(y_2)\, p(y_1, y_2)\, dy_1\, dy_2$$

$$= \frac{1}{(2\pi i)^2} \int_C f(u_1)\, du_1 \int_C f(u_2)\, du_2 \underbrace{\iint_{-\infty}^{\infty} e^{u_1 y_1 + u_2 y_2}\, p(y_1 y_2)\, dy_1\, dy_2}_{\text{joint characteristic function of input.}}$$

If the signal and noise are independent, we have the product of the characteristics $M_S(t_1, t_2)\, M_N(t_1, t_2)$ See Rice, section IV, Davenport and Root Ch. 13.

The important point is that all components of input PSD must be convoluted with each other to find PSD around f_0. For $\nu \neq$ integer in ν^{th} law device, we have infinite number of convolutions.

LECTURE XXXIII 1-5-61

Fluctuations in the EM Field.

This is essentially a problem of sorting particles out in boxes, each with non-equal a priori probability. Governed by M-B distribution:

$$(1) \quad \frac{e^{-E_i/kT}}{\sum_i e^{-E_i/kT}}$$

i representing a particle. We can also throw this into a phase space form:

$$(2) \quad C \, e^{-E(p_x p_y p_z x y z)/kT} \quad dx\, dy\, dz\, dp_x\, dp_y\, dp_z$$

We will find that the normalization constant contains Planck's constant.

Now, the average energy of the system is:

$$(3) \quad \overline{E} = \frac{\sum_i E_i e^{-E_i/kT}}{\sum_i e^{-E_i/kT}}$$

We will calculate fluctuations from the mean. Photons in Black box is in contact with a Temperature reservoir. This means that photon gas interacts with the outside world, and fluctuations in the mean will occur. Calculate $\overline{E^2}$ and show that this is not the same as $\overline{E}^2$.

$$(4) \quad \overline{E^2} = \frac{\sum_i E_i^2 e^{-E_i/kT}}{\sum_i e^{-E_i/kT}}$$

Define, $\alpha = -\frac{1}{kT}$, $z = \sum_i e^{-E_i/kT} = \sum_i e^{\alpha E_i}$

Then :

$$(5) \quad \overline{E} = \frac{1}{Z}\frac{\partial Z}{\partial \alpha} \quad , \quad \overline{E^2} = \frac{1}{Z}\frac{\partial^2 Z}{\partial \alpha^2}$$

Z is called the partition function.
Therefore

$$(6) \quad \overline{\Delta E^2} = \frac{1}{Z}\frac{\partial^2 Z}{\partial \alpha^2} - \frac{1}{Z^2}\left(\frac{\partial Z}{\partial \alpha}\right)^2 = \frac{\partial \overline{E}}{\partial \alpha}$$

We now note that the definition of specific heat is: $C_v = \frac{\partial \overline{E}}{\partial T}$,

Since all macroscopic thermodynamic quantities assume that $\overline{\Delta E^2}$ is small, that is; $\frac{\overline{\Delta E^2}}{\overline{E^2}}$ should be of the order 10^{-6} which is true when we have 10^6 particles. Now during calculation, we require that quantum states (i) doe not change. This means for EM waves that volume of box does not change as eigenstates depend on nodes which are governed by volume of box. Continuing:

$$(7) \quad \overline{\Delta E^2} = \frac{\partial \overline{E}}{\partial \alpha} = \frac{\partial T}{\partial \alpha}\frac{\partial \overline{E}}{\partial T} = kT^2 C_v \quad \text{(due to Einstein)}$$

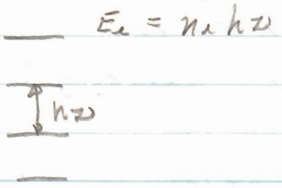
If we have large number of particles, it is clear that $\frac{\overline{\Delta E^2}}{\overline{E^2}}$ goes as N^{-1} since C_v depends on N^{+1}.

1st Example: Ideal Gas with N atoms. $\overline{E} = \frac{3}{2}NkT$ and $C_v = \frac{3}{2}Nk$. Then:

$$\frac{\overline{\Delta E^2}}{\overline{E}^2} = \frac{3/2\ N k^2 T^2}{9/4\ N^2 k^2 T^2} = \frac{2}{3}\frac{1}{N}$$

so that in an ideal gas with large N, the fluctuations are negligible. However, for small N and T, fluctuations will be large.

2nd Example: Quantized Harmonic Oscillator



We will neglect zero point energy.

What is Z?

$$Z = \sum_{n_i=0}^{\infty} e^{-\frac{n_i h\nu}{kT}} = \frac{1}{1-e^{-h\nu/kT}}$$

$$\overline{E} = \frac{\sum_{n_i=0}^{\infty} n_i h\nu\, e^{-n_i\frac{h\nu}{kT}}}{\sum_{n_i=0}^{\infty} e^{-\frac{n_i h\nu}{kT}}} = \frac{h\nu\, e^{-h\nu/kT}}{(1-e^{-h\nu/kT})^2}\cdot(1-e^{-h\nu/kT})$$

$$= \frac{h\nu}{e^{h\nu/kT}-1}$$

$$C_V = \frac{\frac{(h\nu)^2}{kT^2}\, e^{h\nu/kT}}{(e^{h\nu/kT}-1)^2}$$

and we get for $\overline{\Delta E^2}$

$$\overline{\Delta E^2} = (h\nu)^2 \frac{e^{h\nu/kT}}{(e^{h\nu/kT}-1)^2}$$

Classically we know that $\overline{E} = kT$ for harmonic oscillator. We have the same here for $\overline{E}$ when T is large, that is, for $kT \gg h\nu$. Also, we find that $\overline{E}$ goes as $e^{-h\nu/kT}$ when $kT \ll h\nu$.

Classically, we should have infinite number of modes in a black box each carrying energy kT which would mean infinite energy coming out of hole in black box. This is problem, is what faced Planck, and is resolved by $e^{h\nu/kT}$ cutting off energy at $h\nu$.

We now make connection with photons. $\bar{n}_i$ can be considered as the number of photons in mode $h\nu$ in black box. $h\nu$ is conversion from harmonic oscillator energy to number of photons. Then:

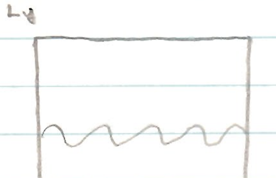
$$(8) \quad \bar{n}_i = \frac{1}{e^{h\nu/kT}-1} \; ; \quad \overline{\Delta n_i^2} = \frac{e^{h\nu/kT}}{(e^{h\nu/kT}-1)^2}$$

$$= \bar{n}_i(\bar{n}_i+1) \text{, since } \bar{n}_i+1 = \frac{e^{h\nu/kT}}{e^{h\nu/kT}-1}$$

Thus if $\bar{n}_i$ is large, the fluctuations increase. This is the same as EM wave going thru quadratic detector.

LECTURE XXXIV 1-7-61

How many modes will exist in given volume? Analogous to transmission lines. These modes should be independent of boundary conditions provided $\lambda^3 <<< V$. Also independent of whether boundary absorbs or reflects. However, initially need BC to facilitate counting.



Assume reflecting walls. E vanishes at boundary, thus we have sine waves.

$$E = A \sin k_x x \sin k_y y \sin k_z z \; e^{i\omega t}$$

This is harmonic solution of Maxwell's equation. Now wave only vanishes at both boundaries if:

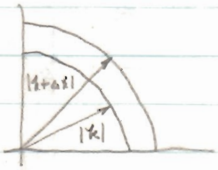
$$k_x L_x = n_x \pi$$
$$k_y L_y = n_y \pi$$
$$k_z L_z = n_z \pi$$

Consider now the number of waves possible in an interval of the wave vector. That is, the number of standing waves with wave vector k in a small element $\Delta k_x, \Delta k_y, \Delta k_z$

$$\Delta n_x \Delta n_y \Delta n_z = \frac{L_x L_y L_z}{\pi^3} \Delta k_x \Delta k_y \Delta k_z$$

$$|k| = \frac{2\pi}{\lambda} = \frac{2\pi\nu}{c}$$

For number of standing waves with frequency between ν and $\nu + \Delta\nu$ $= 4\pi k^2 \cdot \frac{L_x L_y L_z}{\pi^3} \cdot \frac{1}{8} \cdot dk$



Only first octant because change of sign of wave number gives same standing wave.

Finally, the number of standing waves is given by

$$\boxed{V \frac{4\pi\nu^2}{c^3} \Delta\nu}$$

making transformation from k to ν.

This is independent of boundary conditions in the limit of large volume. Also same result given by Born-von Karman boundary conditions.

Black Body Radiation Field:

We can integrate mode distribution from $\nu = 0$ to $\nu = \infty$ as there will be only small number of long wavelengths in EM considerations. We also assert that each mode carries average energy of the harmonic oscillator; $\frac{h\nu}{e^{h\nu/kT} - 1}$

Then the average energy density:

$$\bar{u} = \int_0^\infty \frac{h\nu}{e^{h\nu/kT}-1} \cdot \underset{\uparrow}{2} \cdot \frac{4\pi\nu^2}{c^3}\, d\nu$$

two waves (for each k) of different polarization

$$= \frac{8\pi^5\, k_{Boltz.}^4\, T^4}{15\, h^3 c^3}$$

Evalute by substitution $x = \frac{h\nu}{kT}$ and get:

$$\int_0^\infty \frac{x^3\, dx}{e^x - 1} = \int_0^\infty x^3 \sum_{n=1}^\infty e^{-nx}\, dx = \sum_{n=1}^\infty \int_0^\infty x^3 e^{-nx}\, dx$$

$$= \sum_{n=1}^\infty \frac{1}{n^4}\; 3!$$

Item of real interest is fluctuations $\overline{\Delta u^2}$. We consider oscillators as independent of each other. We consider creation and absorbtion of photons is due to one oscillator and each frequency is independent of others. We can then use expression of last time.

$$\overline{\Delta u^2} = \int_0^\infty \frac{(h\nu)^2\, e^{h\nu/kT}}{(e^{h\nu/kT}-1)^2}\, \frac{8\pi\nu^2}{c^3}\, d\nu$$

However: $\overline{\Delta u^2} = C_v k T^2$

where: C_ν = $\frac{\partial \bar{u}}{\partial T}$ = $\frac{32\pi^5 k^4 T^3}{15 h^3 c^3}$

specific heat of black body radiation

$\overline{\Delta u^2} = C_v k T^2$ is very generally good for any system of energy levels.

We now ask for the average number of photons in radiation field:

$$\overline{N} = \sum_{k} \overline{n}_k$$

k is index of modes and polarization
$\overline{n}_k$ is average number of photons per mode.

$$\overline{n}_k = \frac{1}{e^{\frac{h\nu(k)}{kT}} - 1}$$

Average number of photons in one cc. in a frequency interval $\Delta\nu$:

$$\boxed{\overline{N} = \frac{1}{e^{h\nu/kT} - 1} \frac{8\pi\nu^2}{c^3} \Delta\nu}$$

Mean square deviation from average number of photons:

$$\overline{\Delta N^2} = \overline{\Delta n_k^2} \frac{8\pi\nu^2}{c^3} \Delta\nu = \frac{\frac{h\nu}{kT^2} e^{h\nu/kT}}{\left(e^{h\nu/kT} - 1\right)^2} \frac{8\pi\nu^2}{c^3} \Delta\nu$$

$$= \overline{n}_k (\overline{n}_k + 1) \frac{8\pi\nu^2}{c^3} \Delta\nu$$

Relate $\overline{\Delta N^2}$ to $\overline{N}$: At high frequencies, low temperatures, $\frac{h\nu}{kT} \gg 1$

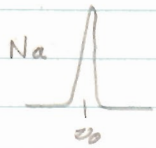
Then: $\overline{\Delta N^2} = \overline{N}$ as before at beginning of course.

At low frequencies, $h\nu \ll kT$, then:

$$\overline{\Delta N^2} = \frac{\overline{N}^2}{\frac{8\pi\nu^2}{c^3} \Delta\nu}$$

LECTURE XXXV 1-10-61

Spectral Lines as Narrow-Band Gaussian signal.



Spectral lines can be thought of as many atoms radiating with separate phase, like instruments in orchestra.

Atoms do not emit together, but with random phase and create a narrow band process. [Devices can be made to emit light coherently (Laser)]. Thus we can consider light to be made up of random amplitude and phase.

$$\sum_i a_i(t) \cos\{w_0 t + \varphi(t)\}$$

with Power $\propto \left[\sum_i a_i(t) \cos\{w_0 t + \varphi(t)\}\right]^2$

Various light detectors detect power and are thus square law detectors. We write for the correlation:

$$\overline{P(t)\, P(t+\tau)} = \overline{P}^2 \left\{1 + 2\rho^2(\tau)\right\}$$

In analogy with previous results:

$$P \rightarrow y(t) \quad \text{from} \quad y(t) = \alpha\, i^2(t)$$

$$\rho(\tau) = \frac{\overline{\left[\sum_i a_i(t) \cos\{w_0 t + \varphi(t)\}\right]\left[\sum_i a_i(t+\tau) \cos\{w_0(t+\tau) + \varphi(t+\tau)\}\right]}}{\overline{\left[\sum_i a_i(t) \cos\{w_0 t + \varphi(t)\}\right]^2}}$$

Recall for quantum fluctuations:

$$\overline{\Delta n_\nu^2} = \overline{n}_\nu(\overline{n}_\nu + 1) = \underset{\substack{\uparrow \\ \text{wave}}}{\overline{n}_\nu^{\,2}} + \underset{\substack{\uparrow \\ \text{quantum}}}{\overline{n}_\nu}$$

Would only have wave portion usually. Other stochastic process is probability of liberation of photoelectron in detector.

Probability for photoelectrons to be liberated in time $dt = \alpha\, P\, dt$

Probability to have exactly N photoelectrons in time t is:

$$p_N(t) = \frac{1}{N!}\left[\alpha \int_0^t P(t')\,dt'\right]^N e^{-\alpha \int_0^t P(t')\,dt'}$$

(Poisson Distribution)

Average number of photoelectrons in time t is:

$$N_t = \sum_{N=0}^{\infty} N p_N(t) = \alpha P t$$

We now want statistical average over various ensembles, or:

$$\overline{N_t} = \alpha \overline{P} t$$

We can now do N_t^2:

$$\overline{N_t^2} = \overline{\sum_{N=0}^{\infty} N^2 p_N(t)} = \overline{N_t(N_t+1)}$$

$$= \overline{\alpha \int_0^t P(t')\,dt'} + \overline{\alpha^2 \int_0^t\int_0^{t'} P(t')P(t'')\,dt'\,dt''}$$

$$= \overline{N_t} + \alpha^2 \int_0^t\int_0^{t'} \overline{P}^2\,dt'\,dt'' + 2\alpha^2 \int_0^t\int_0^t \overline{P}^2 \rho^2(t'-t'')\,dt'\,dt''$$

$$= \overline{N_t} + \underbrace{\alpha^2\,\overline{P}^2 t^2}_{\overline{N_t}^2} + 2\alpha^2\,\overline{P}^2 \int_0^t\int_0^t \rho^2(t'-t'')\,dt'\,dt''$$

or:

$$\boxed{\begin{array}{c} \overline{\Delta N_t^2} = \overline{N_t} + 2\alpha^2 \bar{P}^2 t \displaystyle\int_{-\infty}^{+\infty} \rho^2(\tau)\, d\tau \\ \text{for } t \gg \tau_c \end{array}}$$

Suppose all atoms emit exactly in phase.
Then:

$$\rho(\tau) = \cos \omega_0 \tau$$

which gives infinitely sharp spectral line.
Plugging in the equation for $\overline{N_t^2}$:

$$2\alpha^2 \bar{P}^2 \int_0^t \int_0^t \rho^2(t'-t'')\, dt'\, dt'' \rightarrow \alpha^2 \bar{P}^2 t^2$$

Thus, for coherent light source:

$$\overline{\Delta N_t^2} = \overline{N_t} + \overline{N_t}^2$$

or the fluctuation of a single electromagnetic source. The above treatment brings in the stochastic process of the detector, unlike the treatment before.

However, (except in Lasers) width of spectral line is $\Delta\nu \geq 10^8$ cps, so case of independent emitters is more physical.

Suppose we call spectral distribution $g(\nu)$. Can write:

$$\iint_{-\infty}^{\infty} \rho(\tau)\, g(\nu)\, e^{2\pi i \nu \tau}\, d\nu\, d\tau = \int_{-\infty}^{\infty} g^*(\nu)\, g(\nu)\, d\nu$$

and

$$\overline{\Delta N_t^2} = \overline{N_t} + 2\alpha^2 \bar{P}^2 t \underbrace{\int_{-\infty}^{\infty} g^2(\nu)\, d\nu}_{\text{can be shown to be proportional to } \frac{1}{2\Delta\omega} \leftarrow \text{spectral width}}$$

Then we can write:

$$\overline{\Delta N_t^2} = \overline{N_t} + \frac{\overline{N_t^2}^{\,2}}{\underbrace{(\Delta\omega)t}_{\text{number of oscillators that must be considered.}}}$$

When this is done, results are same as before. The more oscillators taking part, the wider the line. Must not make $(\Delta\omega)t < 1$, limitation of uncertainty principle. If $(\Delta\omega)t = 1$, we have coherent light source.

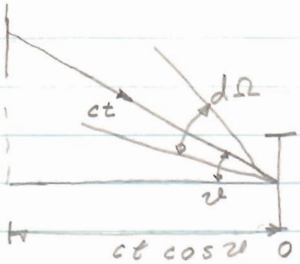
We have been considering point sources and polarized. If spatially distributed and unpolarized, must modify $(\Delta\omega)t$ term. We only write result here for spatial coherence factor.

$$\underset{\substack{\uparrow\\ \text{polarization}}}{\frac{1}{2}} \; \frac{\lambda^2 \leftarrow \text{light wavelength}}{\underset{\substack{\uparrow\\ \text{area of detector}}}{A}\; d\Omega \leftarrow \text{solid angle}}$$

Reference: L. Mandel, Proc Phys Soc London 72, 1037 (1958)

LECTURE XXXVI 1-12-61

Fluctuations in the Radiation Flux:



The EM radiation passing thru 0 in t seconds in direction ϑ is contained in the volume $V = ct\, O \cos\vartheta$.

The number of oscillators in volume V in direction ϑ over the solid angle $d\Omega$ in frequency $d\nu$ is:

$$\frac{\underset{\uparrow}{2}\, d\Omega\, \nu^2\, d\nu}{e^3}$$

polarization

The appropriate intensity:

$$\mathcal{I}(\nu)\, d\nu = 2\, \frac{h\nu}{e^{h\nu/kT} - 1}\, \frac{\nu^2}{c^3}\, d\nu \int_{\vartheta=0}^{\pi/2} \int_{\varphi=0}^{2\pi} c \cos\vartheta \sin\vartheta\, d\vartheta\, d\varphi$$

This is the energy omitted by a one cm^2 black body in 1 sec in $d\nu$ at temperature T. The average energy of the oscillators is used. The intensity after integrations is (integration over $d\Omega$):

$$\mathcal{I}(\nu)\, d\nu = 2\pi\, \frac{h\nu}{e^{h\nu/kT} - 1}\, \frac{\nu^2}{c^2}\, d\nu = \underset{\uparrow}{\frac{c}{4}}\, \underset{\uparrow}{u(\nu)}$$

angular integrations energy density

Because we have fluctuations in ν, we will have fluctuations in $\mathcal{I}(\nu)$. The integrated intensity is:

$$\mathcal{I}\cancel{(\nu)} = \int \mathcal{I}(\nu)\, d\nu = \sigma T^4$$

where $\sigma = \dfrac{2\pi^5 k^4}{15 h^3 c^3}$, Stephan-Boltzmann Constant.

Consider the mean square fluctuations in energy of a black body at temperature T with area O and in time t. We find that these fluctuations are:

$$\overline{\Delta W^2} = \frac{c}{4} O t \underset{\substack{\text{msf} \\ \text{in energy} \\ \text{density}}}{\overline{\Delta u^2}}$$

which comes from:

$$\overline{\Delta W^2} = \int_{\nu=0}^{\infty}\int_{\vartheta=0}^{\pi/2}\int_{\varphi=0}^{2\pi} O c t \cos\vartheta \sin\vartheta \, d\vartheta \, d\varphi \, \frac{2\nu^2}{c^3} \frac{(h\nu)^2 e^{h\nu/kT}}{(e^{h\nu/kT}-1)^2}$$

$$= 4 O t \sigma k T^5$$

from $u(\nu) = \frac{4}{c} \cdot \sigma T^4$, $\overline{\Delta u^2} = \frac{16}{c} \sigma k T^5$

The total radiated flux from a black body of area O is $\sigma O T^4 = W/t$ and the msf in this flux:

$$\overline{\Delta \phi_t^2} = \frac{\overline{\Delta W^2}}{t^2} = \frac{4 O \sigma k T^5}{t}$$

This is a statistical average over a time average.

As Black bodies are seldom isolated, there is ingoing and outgoing flux so the total radiated flux is:

$$\phi = \sigma O (T^4 - T_0^4)$$

$$\overline{\Delta \phi_t^2} = \frac{\overline{\Delta W^2}}{t^2} = \frac{4 O \sigma k (T^5 + T_0^5)}{t}$$

Of course, this is the integrated result over all frequencies and directions.

Consider a Black Body Bolometer:

We can write a Langevin equation for the temperature of the body knowing that eventually it will be brought into equilibrium via either radiation or conduction from external heat reservoir.

$$\underset{\substack{\uparrow \\ \text{heat} \\ \text{capacity}}}{C_v} \frac{d\Theta}{dt} = -\alpha\,\Theta + F(t) = -\Phi$$

where $\overline{F(t)} = 0$, $\overline{F(t')F(t'')} = \delta(t'-t'')\,\overline{F^2}$

α is given by $\Phi = \sigma O (T^4 - T_0^4) = \underset{\overset{\shortparallel}{\alpha}}{4\sigma O T^3}\,\Theta$

We must require that in the steady state:

$$C_v^2\,\overline{\Theta^2}_{t=\infty} = C_v\,k T^2$$

The solution is:

$$\Theta = \Theta_0\, e^{-\frac{\alpha}{C_v}t} + \frac{1}{C_v}\, e^{-\frac{\alpha}{C_v}t} \int_0^t F(t')\, e^{+\frac{\alpha}{C_v}t'}\, dt'$$

$$\overline{\Theta^2} = \Theta_0^2\, e^{-\frac{2\alpha}{C_v}t} + \frac{\overline{F(t')F(t'')}}{2\alpha C_v}\left(1 - e^{-\frac{2\alpha}{C_v}t}\right)$$

$$\overline{\Theta(t)\,\Theta(0)} = \Theta_0^2\, e^{-\frac{\alpha}{C_v}t}$$

Thus $\frac{C_v}{\alpha}$ is the system time constant.

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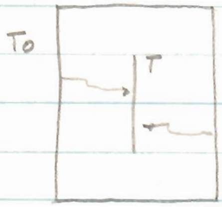


$$C_v \frac{d\theta}{dt} = -\alpha\,\theta + F_\theta(t)$$

Fluctuations due to heat conduction or radiation must satisfy:

$$\overline{\Delta E^2} = C_v k T^2$$

Assume times much greater than characteristic time C_v/α. The solution is:

$$\overline{\theta^2} = \theta_0^2\, e^{-\frac{2\alpha}{C_v}t} + \frac{1}{2\alpha C_v}\left(1 - e^{-\frac{2\alpha}{C_v}t}\right)\overline{F^2}$$

$$\overline{F(t')F(t'')} = \overline{F_2}\;\delta(t'-t''), \quad \overline{F^2} = 2\alpha k T^2$$

Now $\Delta E = C_v \Delta T = C_v\,\theta$

Then: $\overline{\theta_\infty^2} = \frac{\overline{\Delta E^2}}{C_v^2} = \frac{kT^2}{C_v}$

A problem arises because Einstein equation assumes constant temperature while we use θ which is time dependent. The point is that the body can be considered its own reservoir and heat transferred to it is immediately distributed over its degrees of freedom.

Calculate PSD. Assume wide band.

$$G_\theta^+(f) = \frac{G_F^+}{|j\omega C_v + \alpha|^2} = \frac{4\alpha k T^2}{\omega^2 C_v^2 + \alpha^2}$$

$$\overline{\theta^2} = \int_0^\infty G_\theta^+(f)\, df = \int_0^\infty \frac{4\alpha k T^2}{4\pi^2 f^2 c_v^2 + \alpha^2}\, df = \frac{k T^2}{C_v}$$

$$\overline{\Delta\theta_t^2} = \frac{G_\theta^+(0)}{2t} = \frac{2\alpha k T^2}{\alpha^2 t}$$

Radiation :

$$E = \sigma O\,(T^4 - T_0^4) = \sigma O \frac{d}{dT} T^4 \theta = 4\sigma O T_0^3 \theta$$

$\uparrow$ power

$\uparrow$ area of body

$\uparrow$ Taylor series expansion

$$\overline{\Delta\varphi_t^2} = \frac{(4t\overset{*}{\sigma} O T_0^3)^2}{t^2}\,\overline{\Delta\theta_t^2} = (4\sigma O T_0^3)^2\,\overline{\Delta\theta_t^2}$$

$$= (4\sigma O T_0^3)\,\frac{2kT_0^2}{\alpha t}$$

from: $E_t = 4t\sigma O T_0^3\,\theta$

$$\overline{\Delta E_t^2} = (4t\sigma O T_0^3)^2\,\overline{\Delta\theta^2}$$

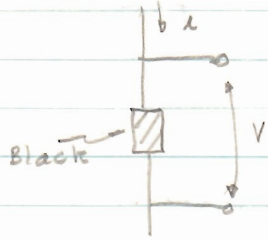
Finally, using the value for α:

$$\overline{\Delta\varphi_t^2} = \frac{8\sigma O k T_0^5}{t}$$

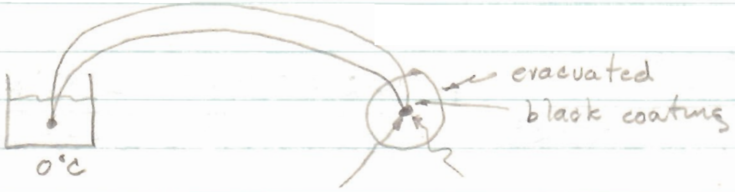
this was derived from the heat conduction equation and is exactly the same as that from considering EM oscillators directly, viz.,

$$\overline{\Delta\varphi_t^2} = \frac{4\sigma O k T_0^5 + 4\sigma O k T_0^5}{t} = \frac{8\sigma O k T_0^5}{t}$$

Radiation Measurements:



Temperature rises as radiation is incident on element.



Wires must be made thin, so conduction will not be much. However, the thinner the wire, the higher the resistance and hence Johnson noise. We also have Peltier heat. The total equation is:

$$C_v \frac{d\theta}{dt} + \alpha\theta + i\underbrace{S(T+\theta)}_{\text{Peltier heat}} = F_\theta(t)$$

The circuit equation is:

$$iR = S\theta + \underset{\overset{\uparrow}{\text{Johnson noise}}}{F_v(t)}$$

These equations are coupled and can be combined:

$$C_v \frac{d\theta}{dt} + \alpha\theta + \frac{S^2 T\theta}{R} = \underbrace{F_\theta(t) - \frac{ST}{R} F_v(t)}_{\text{uncorrelated}}$$

We can write immediatly:

$$G_\theta^+(f) = \frac{4\alpha k T^2 + \frac{S^2T^2}{R^2} 4RkT}{\left| j\omega C_v + \alpha + \frac{S^2 T}{R}\right|^2}$$

However, we really measure the current fluctions instead of the temperature fluctuations. Replace $\frac{d\theta}{dt}$ by $j\omega$ and substitute for θ in the current equation and get:

$$i\left[R + \frac{S^2 T}{\alpha + j\omega C_v}\right] = \frac{S\,F_\theta(f)}{\alpha + j\omega C_v} + F_v(f)$$

with:

$$G_i^+(f) = \frac{4\alpha k T^2 S^2}{(R\alpha + S^2 T)^2 + \omega^2 C_v^2 R^2} + \frac{4kRT\,(\alpha^2 + \omega^2 C_v^2 R^2)}{(R\alpha + S^2 T)^2 + \omega^2 C_v^2 R^2}$$

In the steady state, <u>using</u> P as the average incident flux $= \overline{F_0(t)}$. Get:

$$\frac{\alpha \bar{i} R}{S} + \bar{i}\,ST = P$$

or $$\bar{i} = \frac{P_{\text{average flux}}}{\left(\frac{\alpha R}{S} + ST\right)}$$

and $$\overline{\Delta i_t^2} = \frac{G_i(0)}{2t} \simeq \frac{\overline{\Delta \varphi_t^2}}{\left(\frac{\alpha R}{S} + ST\right)^2}$$

The problem is to minimize the fluctuations by choosing proper materials (c_v, α). The optimum case would be when the only fluctuations were that of the incident flux

Reference: Smith, Chasmar, Jones.

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$$G_i(f) = \frac{4\alpha k T^2 S^2}{(R\alpha + S^2 T)^2 + \omega^2 C_v^2 R^2} + \text{Johnson Noise}$$

$$\overline{\Delta i_t^2} = \frac{G_i(0)}{2t} = \frac{4\alpha k T^2}{\left(ST + \frac{\alpha R}{S}\right)^2 2t}$$

The DC response is:

$$\bar{i} = \frac{\bar{P}}{ST + \frac{\alpha R}{S}}$$

where $\alpha = 4\sigma O T^3$

$\overline{\Delta i_t^2}$ is output fluctuations

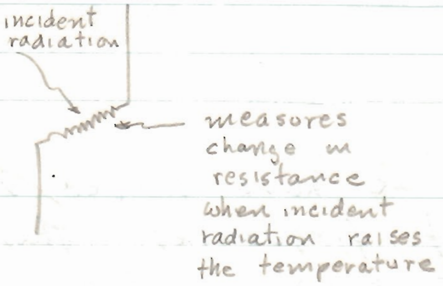
noise fluctuations at input:

$$\overline{\Delta \varphi_t^2} = \frac{8\sigma O T^5 k}{t}$$

$$\left(\frac{\text{noise}}{\text{signal}}\right)_{out} = \frac{\overline{\Delta i_t^2}}{(\bar{i})^2} = \frac{\overline{\Delta \varphi_t^2}}{(\bar{\varphi})^2} = \left(\frac{\text{noise}}{\text{signal}}\right)_{\substack{\text{incident}\\\text{radiation}}}$$

Thus the noise figure is unity. This analysis is for bolometer. (Natural Precision)

↓



Golay Cell

Incident Radiation expands gas

If we plug in numbers :

$$\sigma = 5.67 \cdot 10^{-12} \text{ watt/cm}^2$$

$$k = 1.38 \cdot 10^{-23} \text{ Joules}$$

we find that $\left[\overline{\Delta \varphi_t^2}\right]^{1/2} = 4.5 \cdot 10^{-11}$ watts

if $t = 1 \text{ sec}$, $O = 1 \text{ cm}^2$, $T = 300°K$

the relative fluctuations in flux is:

$$\left[\frac{\overline{\Delta \varphi_t^2}}{(\varphi)^2}\right]^{1/2} = \left[\frac{8 \sigma O k T^5}{t\,(\sigma O T^4)^2}\right]^{1/2} = \left[\frac{8k}{\sigma O t T^3}\right]^{1/2}$$

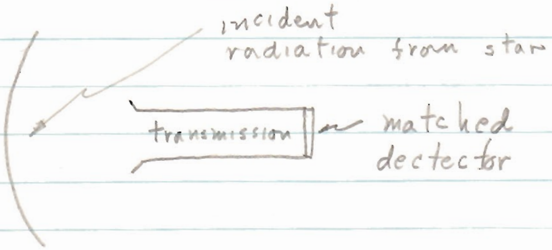
$$\approx 8 \cdot 10^{-10}$$

We claim that the uncertainty in temperature of the black body is 1/4 of this is :

$$\frac{\Delta T}{T} = 2 \cdot 10^{-10}$$

this analysis has been for black body radiation of all frequencies.

We now compare with noise thru transmission line.



Different from BB in that we have limited solid angle and limited set of frequencies. This device accepts only a narrow band of frequencies. Fluctuations are analogous to those previously considered for transmission line.

We are in the region where $h\nu \ll kT$, $T \geqslant 10\,^\circ K$

$$\overline{n}_\nu = \frac{1}{e^{h\nu/kT}-1} = \frac{kT}{h\nu}, \quad q = kT \sum_{\text{oscillators}} = kT\Delta\nu$$

$$\overline{\Delta q_t^2} = \frac{1}{t^2}\sum_{\text{oscillators}} h^2\nu^2\,\overline{\Delta n_\nu^2} = \frac{1}{t^2}\sum_{\text{oscillators}} h^2\nu^2\left(\overline{n}_\nu\right)^2$$

$$= \frac{k^2T^2}{t^2}\sum_{\text{oscillators}} = \frac{k^2T^2}{t^2}\,t\,\Delta\nu$$

and:

$$\left[\frac{\overline{\Delta q_t^2}}{(\overline{q})^2}\right]^{1/2} = \left[\frac{k^2T^2\Delta\nu}{k^2T^2\Delta\nu^2\,t}\right]^{1/2} = \left[\frac{1}{\Delta\nu\,t}\right]^{1/2} = 10^{-4}$$

If we accept $\Delta\nu = 100$ Mc/sec, over a time of integration of one second. This is used to measure temperature of stars within .01 °K. The rms in q is:

$$\Delta q_{rms} = kT\left(\frac{\Delta\nu}{t}\right)^{1/2}$$ which is the same as detector considered previously (ABB')

If $t = 1$ sec, $O = 1\text{ cm}^2$, $T = 300\,^\circ K$

The number of EM oscillators entering guide is $ct\,\frac{4\pi\nu^2\Delta\nu}{c^3}\,A\,d\Omega$

Now $\frac{4\pi A}{\lambda^2}\,d\Omega = 1$ Thus we see that the flow of EM oscillators is in agreement with previous results.

This concludes the formal lecturing.

BASIC COURSE OUTLINE

I. Elementary Probability Theory and Definitions

A.(1) Random or chance phenomena: situations not uniquely determined because of lack of knowledge of all variables and initial conditions.

(2) Sample space: set of all possible outcomes.

(3) Probability: $P(k) = \lim_{N \to \infty} \frac{\# \text{ of successes}}{\# \text{ of trials } N}$

(4) Mutually Exclusive events: A is independent of B. $P(A \text{ or } B) = P(A) + P(B)$

(5) Certainty $P(S) = 1$

(6) Impossibility: $P = 0$. Impossible event has probability zero, but probability zero does not mean impossible. Example: probability of gas molecules having given velocity.

(7) Joint Probability: probability that events A and B occur together, $P(A,B)$.

(8) Conditional Probability: probability that B occurs when one knows A occurs, $P(B|A) = P(A,B)/P(A)$

B. Statistical Independence: B is independent of A if $P(B|A) = P(B)$ in which case we have the product rule $P(A,B) = P(A)P(B)$. More generally, $P(A_1, A_2, \ldots A_N) = P(A_1)P(A_2) \ldots P(A_N)$

C. Bernoulli Distribution:

1) Two mutually exclusive events: probability $P(A) \to p$, $P(B) \to q$ with $q = 1-p$

(2) Perform N trials, get n successes of (A) and N-n failures (B). The probability of one sequence is $p^n q^{N-n}$.

(3) The possible sequences = N!. Sequences of successes = n! of failures = (N-n)!. Therefore, the probability of n successes in N trials is:

$$P_N(n) = \frac{N!}{n!\,(N-n)!} p^n q^{N-n}$$

C. (4) Calculate moments with $[py+q]^N = \sum_{n=0}^{N} y^n p^n q^{N-n} \cdot \frac{N!}{n!(N-n)!}$ and take derivatives, knowing that $p+q=1$ and letting $y \to 1$ after derivative.

(5) Properties: $\bar{n} = Np$, $\overline{\Delta n^2} = Npq \doteq \bar{n}$ for $p \ll 1$.

(6) Fluctuations from the mean $= \overline{\Delta n^2}$. Relative fluctuations $= (\overline{\Delta n^2})^{1/2}/\bar{n}$.

D. Poisson Distribution

(1) Take limit of B distribution such that
$$\lim_{\substack{N\to\infty \\ p\to 0}} = \text{constant} = \bar{n}$$

(2) $p = \frac{\bar{n}}{N}$, $\therefore$ $P_N(n) = \left(\frac{\bar{n}}{N}\right)^n \left(1 - \frac{\bar{n}}{N}\right)^{N-n} \frac{N!}{n!\,(N-n)!}$

$$= \frac{(\bar{n})^n}{n!} \; \frac{N(N-1)\cdots(N-n+1)}{N^n} \cdot \left(1 - \frac{\bar{n}}{N}\right)^{N-n}$$

$$= \frac{(\bar{n})^n}{n!} \cdot 1\left(1-\frac{1}{N}\right)\cdots\left(1-\frac{n-1}{N}\right)\left(1-\frac{\bar{n}}{N}\right)^{N-n}$$

(3) $$\lim_{N\to\infty} P_N(n) = \frac{(\bar{n})^n}{n!} e^{-\bar{n}}$$

(4) Properties: $\bar{n} = \bar{n}$, $\overline{\Delta n^2} = \bar{n}$. Density $\bar{n}$ is not constant but has small fluctuations. Limiting case of Bernoulli distribution. In application we must determine $\bar{n}$ initially or p. All particles independent, have no interaction.

E. Random Walk:

(1) Given equal probability of making steps of length l either to left or right, what is position ml after N steps?

(2) Could be anywhere between $+Nl$ and $-Nl$. Must take $\frac{N+m}{2}$ steps to right and $\frac{N-m}{2}$ steps to left. This will give the Bernoulli distribution:

$$P_m(N) = \frac{N!}{\left(\frac{N+m}{2}\right)!\left(\frac{N-m}{2}\right)!}\left(\frac{1}{2}\right)^N$$

(3) Make approximation, letting $N \to \infty$, $\lim pN \to \infty$, using Stirling's approximation $N! \doteq \sqrt{2\pi N}\left(\frac{N}{e}\right)^N$

E. (4) Plug in (2), expand, neglect terms in N^{-2} on, and use $\frac{m}{N} << 1$

(5) We get $P(m,N) = \left(\frac{2}{\pi N}\right)^{1/2} e^{-\frac{m^2}{2N}}$ which is a gaussian with which we can replace B distribution when neglecting tails. Good for integral m. Can get diffusion equation by making $x \to ml$, $\Delta x \to 2\Delta m l$.

F. Random Variables

(1) The random variable x is a variable in the sample space, taken on discreet or continuous values.

(2) $f(x) \equiv \frac{\partial P}{\partial x}$ (probability density)

(3) Joint Probability Density: $f(x,y) = f(x) f(y)$ for statistical independence

(4) Conditional Probability: $\frac{f(x,y)\,dx\,dy}{f(x)\,dx}$ = probability of y when we know probability of x.

(5) Functions of random variables are also random variables.

(6) Transformation of Probability Densities:

Given $f(x,y)$, $\left.\begin{array}{l} x = x(u,v) \\ y = y(u,v) \end{array}\right\}$ $\begin{array}{l} u = u(x,y) \\ v = v(x,y) \end{array}$ we want $f(u,v)$

$$f(x,y)\,dx\,dy = f\left(x = x(u,v),\, y = y(u,v)\right) J\,du\,dv$$

where:

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial(x,y)}{\partial(u,v)}$$

$$\therefore \quad f(u,v) = f\left[x = x(u,v),\, y = y(u,v)\right] J$$

For one variable: $f(u) = f\left[x = x(u)\right] \frac{dx}{du}$

Must usually renormalize.

G. Averages, Joint Moments, Characteristic functions.

(1) $\overline{g(x)} = \int_{-\infty}^{\infty} g(x)\, f(x)\, dx$

(2) Joint Moment: $\overline{x^m y^n} = \iint_{-\infty}^{\infty} x^m y^n f(x)\, dx\, dy$

(3) Central Moments are with respect to $\bar{x}$. First is zero, second is variance $\sigma^2 = \overline{x^2} - \bar{x}^2$, or $\overline{x^2}$

G. (4) Covariance: $\overline{(x-\bar{x})(y-\bar{y})} = \iint_{-\infty}^{\infty} (x-\bar{x})(y-\bar{y}) f(x,y)\,dx\,dy$

where $\bar{x} = \iint_{-\infty}^{\infty} x f(x,y)\,dx\,dy$

(5) Characteristic functions: definition

$$M_x(iu) = \overline{e^{iux}} = \int_{-\infty}^{\infty} e^{iux} f(x)\,dx$$

or the Fourier transform of the probability density function.

(6) Moment generation: $(-i)^n \frac{d^n}{du^n} M_x(iu)\Big]_{u=0} = \overline{x^n}$

and then:

$$M_x(iu) = \sum_0^\infty \overline{x^n}\,\frac{(iu)^n}{n!}$$

(7) Joint Charateristic function

$$M_{xy}(iu,iv) = \overline{e^{iux+ivy}} = \iint_{-\infty}^{\infty} e^{iux+ivy} f(x,y)\,dx\,dy$$

and

$$\overline{x^m y^n} = (-i)^{n+m} \frac{\partial^{m+n}}{\partial u^m \partial v^n} M_{xy}(iu,iv)\Big]_{\substack{u=0\\v=0}}$$

(8) Statistical Independence: covariance $= 0$, $\overline{x^m y^n} = \overline{x^m}\,\overline{y^n}$, $M_{xy}(iu,iv) = M_x(iu)\,M_y(iv)$, and we say x and y are uncorrelated.

H. Gaussian Distribution.

(1) Physical basis in Maxwell-Boltzmann statistics:

(2) $f(p_x,p_y,p_z)\,dp_x\,dp_y\,dp_z = \frac{1}{(2\pi m kT)^{3/2}} e^{-\frac{(p_x^2+p_y^2+p_z^2)}{2mkT}}\,dp_x\,dp_y\,dp_z$

(3) In spherical coordinates:

$$f(p) = \frac{1}{(2m\pi kT)^{3/2}} e^{-\frac{p^2}{2mkT}} p^2 \sin\vartheta\,dp\,d\vartheta\,d\varphi$$

(4) $F(E) = \frac{2\pi}{(\pi kT)^{3/2}} E^{1/2} e^{-E/kT}$

(5) In general: $f(x,y,z,p_x,p_y,p_z) = C\, e^{-\frac{V(x,y,z)}{kT}}\, e^{-\frac{(p_x^2+p_y^2+p_z^2)}{2mkT}}$

(6) Principle of Equipartition of energy. When the energy of a system of particles depends on an additive quadratic term, such as momentum, the Maxwell-Boltzmann distribution gives

for the average energy of the system the value $1/2\, kT$. Each quadratic term of the Hamiltonian contributes this amount to the average energy. As the quadratic term is usually momentum, it is associated with the directions of motion of the particle. In 3-D, $E = 3/2\, kT$, hence the equipartition theorem states that each degree of freedom will contribute $1/2\, kT$ to the mean energy.

(7) The general one dimensional gaussian is:

$$W(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-x_m)^2}{2\sigma^2}}$$

with $\overline{(x-x_m)^n} = 0$ if n odd

$\overline{(x-x_m)^n} = 1 \cdot 3 \cdot 5 \dots (n-1)\, \sigma^n$ if n even

Can show by partial integration.

(8) $$M_x(iu) = e^{imu - \frac{u^2\sigma^2}{2}} \cdot e^{-\frac{u^2\sigma^2}{2}} = \sum_0^\infty \left(-\frac{\sigma^2}{2}\right) \frac{u^{2n}}{n!}$$

if $x_m = 0$.

(9) Bivariate Distribution with zero mean:

(1) $$W(x,y) = \frac{1}{2\pi\sigma_1\sigma_2} \exp\left\{-\frac{x^2}{2\sigma_1^2} - \frac{y^2}{2\sigma_2^2}\right\}$$, no correlation

(2) $$M_{xy}(iu, iv) = \exp\left\{-\frac{u^2\sigma_1^2}{2} - \frac{v^2\sigma_2^2}{2}\right\}$$

(3) $$W(y_1, y_2) = \frac{1}{2\pi\sigma\tau(1-\rho^2)^{1/2}} \exp - \left\{\frac{1}{2(1-\rho^2)}\left(\frac{y_1^2}{\sigma^2} + \frac{y_2^2}{\tau^2} - \frac{2\rho}{\sigma\tau} y_1 y_2\right)\right\}$$

with $\sigma^2 = \overline{y_1^2}$, $\tau^2 = \overline{y_2^2}$, $\overline{y_1 y_2} = \rho\sigma\tau$

(10) Multivariate Gaussian:

For n independent variables: $W(x_1, x_2, \dots x_n) = \prod_{k=1}^{N} \frac{1}{\sqrt{2\pi\sigma_i^2}} e^{-\frac{x_i^2}{2\sigma_i^2}}$

Generally:

$$W(x_1 \dots x_N) = \frac{\exp\left[-\frac{1}{2|\Lambda|} \sum_{n=1}^{N} \sum_{m=1}^{N} |\Lambda|_{nm} x_n x_m\right]}{(2\pi)^{N/2} |\Lambda|^{1/2}}$$

$$M_x(jV) = \exp\left(-\tfrac{1}{2} V' \Lambda V\right)$$

where $V = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{pmatrix}$, $\Lambda = \begin{pmatrix} \lambda_{11} & \cdots & \lambda_{1N} \\ \vdots & & \\ \lambda_{N1} & \cdots & \lambda_{NN} \end{pmatrix}$, $\lambda_{nm} = E(x_n, x_m)$

I. Markoff Process:

(1) A Markovian process is a process such that the conditional probability that y lies in $y_N, y_N + dy_N$ at time t_N given that $y = y_1, y_2, \ldots y_{N-1}$ at times $t_1, t_2, t_3, \ldots t_{N-1}$ depends only on the value of y at the previous time t_{N-1}. In a physical process, this means that the process and its distribution do not have any past history beyond the immediately previous state.

(2) Smoluchowsky Equation:

$$P(x_3 t_3 | x_1 t_1) = \int P(x_3 t_3 | x_2 t_2)\, P(x_1 t_1 | x_2 t_2)\, dx_2$$

(3) Examples: Random walk, position after n steps depends only on position of step before. If one step away not known, two steps helps.

J. Stationary Random Processes; Ergotic Processes

(1) Choice of time coordinates does not matter.

(2) All probability functions are dependent only on time interval and not on absolute position in time.

(3) Examples: noise voltage across resistors and diodes.

(4) Correlations: $\overline{x(t_1)\,x(t_2)} = R_x(\tau)$ if stationary where τ is time interval. For stationary processes $\rho_x(\tau) = \dfrac{R_x(\tau) - (\bar{x})^2}{\sigma^2}$

K. Definition of Various Terms (Kittel)

(1) Stochastic or Random Variable: This is defined if set of possible values is given and if probability of attaining each value is given. The number of points on cast die is random variable with 6 values, each having $p = 1/6$.

(2) Central Limit Theorem: The sum of a large number of independent stochastic is itself a stochastic variable. The central limit theorem says that the distribution of this sum tends to a gaussian in the limit of large numbers.

K. (3) Random Process or Stochastic Process: This is a process in which the variable x does not depend on the time in a completely definite way. All we can do is examine the system at different times and derive a certain probability distribution. These processes are quite simple when stationary.

(4) Gaussian Random Process: This is one in which all the basic distribution functions $f(x_n)$ are gaussian distributed. The distribution of a finite number of gaussian random variables is also gaussian.

II. Correlation Functions and Power Spectral Density

A. Correlation Functions (Statistical)

(1) Cross Correlation: $R_{xy} = \overline{x_1 y_2^*}$, x_1 at t_1, y_2 at t_2

(2) auto correlation: $R_x(t_1 t_2) = \overline{x_1 x_2^*}$

B. Time Averages and Correlation Functions

(1) Time average: $\langle x \rangle = \lim_{T\to\infty} \frac{1}{T} \int_0^T x(t)\,dt$

(2) Cross Correlation: $R_{x,y}(t_1, t_2) = \lim_{T\to\infty} \frac{1}{T} \int_0^T x_1(t_1)\, y_2(t_2)\,dt$

(3) Auto Correlation: $R_x(t_1 t_2) = \lim_{T\to\infty} \frac{1}{T} \int_0^T x_1(t_1)\, x_2(t_2)\,dt$

C. Power Spectral Density:

(1) Correlation functions

Statistical: $R(\tau) = \overline{x(t)\, x(t+\tau)}$ (stationary)

Time: $R(\tau) = \lim_{T\to\infty} \int_0^T x(t)\, x(t+\tau)\,dt$

If we have statistical over a interval, we can write:

$$R_x(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \overline{x(t)\, x(t+\tau)}\,dt$$

useful if $\overline{x(t)\, x(t+\tau)}$ not independent of t.

(2) Ergotic processes: $\langle x \rangle = \overline{x}$, $R(\tau) = \mathcal{R}(\tau)$ with probability one.

(3) Wide-sense stationary: A process whose probability distribution is not invariant under a shift of the time origin, but whose mean and correlation functions are independent of time.

(4) Weiner - Khinctine Theorem:

$$G(f) = \int_{\infty}^{\infty} R_y(\tau) e^{2\pi i f \tau} d\tau \Bigg\} \; G^+(f) = 4 \int_0^{\infty} R_y(\tau) \cos 2\pi f \tau \, d\tau$$

$$R_y(\tau) = \int_{-\infty}^{\infty} G(f) e^{-2\pi i f \tau} df \Bigg\} \; R_y(\tau) = \int_0^{\infty} G^+(f) \cos 2\pi f \tau \, df$$

$$G(f) = \frac{1}{2} G^+(f) , \quad G(f) \equiv \lim_{T \to \infty} \frac{S(f) S^*(f)}{T} , \quad S(f) = \int_{-\infty}^{\infty} y(t) e^{2\pi i f t} dt$$

and $y(t) = \int_{-\infty}^{\infty} S(f) e^{-2\pi i f t} dt$

(5) Very usually, a random process will involve a trigonometric function of time and a uniformly distributed phase angle:

$$x(t) = A \cos\{\omega(t) + \varphi]$$

$$\overline{x(t)\, x(t+\tau)} = \frac{\overline{A^2}}{2} \cos \omega_0 \tau$$

The following relations are useful in these problems.

$$\overline{\cos [\omega_0 t + \varphi] \cos \{\omega_0 (t+\tau) + \varphi)} = \frac{1}{2} \cos \omega_0 \tau$$

$$\overline{+ \frac{1}{2} \cos (2\omega_0 t + 2\varphi + \omega_0 \tau)}$$

$\overset{\shortparallel}{0}$ because the cosine has mean value zero over an period.

$$\int_0^{\infty} \cos 2\pi f \tau \, d\tau = \frac{1}{2} \delta(f)$$

$$\int_0^{\infty} \cos 2\pi f_0 \tau \cos 2\pi f \tau \, d\tau = \frac{1}{4} \delta(f - f_0)$$

$\Bigg\}$ $\delta(f+f_0) = 0$ since $f > 0$, $f_0 > 0$.

$$\int_{-\infty}^{\infty} e^{i 2\pi f t} dt = \int_{-\infty}^{\infty} e^{-i 2\pi f t} dt = \delta(f)$$

(6) Non-stationary Random Processes: If the statistical auto-correlation function is not wide sense stationary, we can use time auto-correlation function and take PSD. In this way the PSD is defined for each sample. Actually Davenport and Root take this as definition of PSD.

D. Example: Random Telegraph Signal:

(1) Suppose the average # of 0 crossings per second is a. The number of zero crossings k in time interval T is given by Poisson distribution

$$P(k) = \frac{(aT)^k}{k!} e^{-aT}$$

(2) Now $R_y(\tau) - y(t)\,y(t+\tau) = +1$ if k even
$= -1$ if k odd

(3) Then:

$$R_y(\tau) = \sum_{\substack{k \\ \text{even}}}^{\infty} \frac{(aT)^k}{k!} e^{-aT} - \sum_{\substack{k \\ \text{odd}}}^{\infty} \frac{(aT)^k}{k!} e^{-aT}$$

$$= e^{-aT} \sum_{\substack{\text{all} \\ k}}^{\infty} \frac{(-aT)^k}{k!} = e^{-2aT}$$

(4) Take Fourier transform for PSD.

E. Fourier Series Representation of PSD:

(1) $$s(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{2\pi n t}{T} + b_n \sin\frac{2\pi n t}{T}\right)$$

$$a_n = \frac{2}{T}\int_0^T s(t)\cos\frac{2\pi n t}{T}\,dt$$
$$b_n = \frac{2}{T}\int_0^T s(t)\sin\frac{2\pi n t}{T}\,dt$$

(2) $\overline{a_n} = \overline{b_n} = 0$, $\overline{a_n^2} = \overline{b_n^2}$, $\overline{a_n a_m} = \overline{b_n b_m}$, $\overline{a_n b_m} = 0$

(3) $$\overline{a_n a_m} = \frac{2}{T}\int_{-\infty}^{\infty} R_y(\tau)\cos\frac{2\pi n \tau}{T}\;\delta_{nm} = \frac{1}{T}\,G^+\!\left(f = \frac{n}{T}\right)\delta_{nm}$$

(4) $$\therefore\ \overline{a_n^2} = \overline{b_n^2} = \frac{1}{T}\,G^+\!\left(f = \frac{n}{T}\right)$$

(5) Form $$G^+(f)\,df = \frac{T}{2}\left(\overline{a_n^2} + \overline{b_n^2}\right)df$$

F. Wide-Band PSD:

(1) This is PSD that is independent of frequency over a wide range of frequencies.

G. Narrow band PSD and Other Relations:

(1) A narrow band process is one whose bandwidth Δf of the significant part of its spectrum is small compared to the center frequency f_c.

(2) One can calculate the second moment of a random variable using the PSD of the variable:

$$\overline{x^2} = \int_0^\infty G^+(f)\,df = \int_{-\infty}^{\infty} G(f)\,df$$

H. Linear Fixed Parameter Applications.

(1) RLC, T are not functions of time

(2) $G(f)_{out} = |A(f)|^2\, G(f)_{in}$

$A(f)$ is system transfer function and may be an amplification, impedance, or admittance of the electrical or mechanical varieties.

(3) Extension to n-pair terminal networks

$$G_{out}(f) = \sum_{\lambda=1}^{N}\sum_{\lambda'=1}^{N} A_\lambda(f)\, A^*_{\lambda'}(f)\; G_{in\,\lambda\lambda'}(f)$$

The $G_{in\,\lambda\lambda'}(f)$ are cross spectral densities of other inputs which are correlated.

For uncorrelated inputs:

$$G_{out}(f) = \sum_{\lambda=1}^{N} |A_\lambda(f)|^2\; G_\lambda(f)_{in}$$

(4) Many times the signal source is a noisy resistor or a noisy diode. The spectral densitys of these devices are often involved in problems using the above equations.

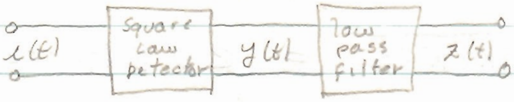
(5) Noise in circuits, noise figure.

$$F \equiv \frac{\left(\frac{\text{signal power}}{\text{noise power}}\right)_{\text{available in}}}{\left(\frac{\text{signal power}}{\text{noise power}}\right)_{\text{available out}}}$$

H. (5) This analysis consists in reflecting all noise in the circuit back to the input and then dividing by PSD of noiseless amplifier.

I. Non-linear Systems:

(1) Quadratic Detector:



$$y(t) = \alpha\, i^2(t)$$

$$\overline{y(t)\, y(t+\tau)} = \alpha^2\, \overline{i^2(t)\, i^2(t+\tau)} \quad , \quad R_i(\tau) = \overline{i(t)\, i(t+\tau)}$$

$$= \alpha^2 \left[R_i^2(0) + 2R_i^2(\tau) \right] = \sigma^4 \alpha^2 + 2\alpha^2 R_i^2(\tau)$$

The PSD is: $G_y(f) = \alpha^2 \sigma^4 \delta(f) + 2\alpha^2 \int_{-\infty}^{\infty} R_i^2(\tau)\, e^{i2\pi f\tau}\, d\tau$

or $G_y(f) = \alpha^2 \sigma^4 \delta(f) + 2\alpha^2 \underbrace{\int_{-\infty}^{\infty} G(f')\, G(f-f')\, df'}_{\text{convolution of input PSD}}$

and where $\sigma^2 = \int_{-\infty}^{\infty} G(f)\, df$

The low pass filter eliminates the sidebands.

III. Langevin Equation and Fokker-Planck methods

A. Properties of Langevin equation

(1) We begin with the equation of motion in a field of force which is a purely random process.

$$J \frac{d\psi}{dt} + B\psi = F(t)$$

where J is an inertial constant, B a damping constant and $F(t)$ the purely random process forcing function. We define $J/B = T_0$ as the time constant of the process.

A. (2) We rewrite the equation as:

$$\frac{d\psi}{dt} + \frac{1}{\tau_0}\psi = K(t)$$

where we assume $K(t) = \frac{F(t)}{J}$ to be a purely random process whose PSD is 4D. We also assume it to be a random gaussian process with zero mean. Therefore:

$$\overline{K(t)} = 0, \qquad \overline{K(t_1)\,K(t_2)} = 2D\,\delta(t_1 - t_2)$$

(3) The general solutions of equation (2) are:

$$\psi = \psi_0 e^{-t/\tau_0} + e^{-t/\tau_0}\int_0^t K(\xi)\, e^{\xi/\tau_0}\, d\xi$$

$$\overline{\Psi} = \psi_0 e^{-t/\tau_0}$$

$$\overline{\Psi^2} = \psi_0^2 e^{-2t/\tau_0} + e^{-2t/\tau_0}\int_0^t\!\!\int_0^t \underbrace{\overline{K(\xi)\,K(\eta)}}_{2D\,\delta(\xi-\eta)}\, e^{\frac{\xi+u}{\tau_0}}\, d\xi\, d\eta$$

$$= \psi_0^2 e^{-2t/\tau_0} + 2D\, e^{-2t/\tau_0}\int_0^{2t} e^{v/\tau_0}\cdot\frac{1}{2}\, dv$$

$$= \psi_0^2 e^{-2t/\tau_0} + 2D e^{-2t/\tau_0}\left\{\frac{1}{2}\tau_0 e^{2t/\tau_0} - \frac{1}{2}\tau_0\right\}$$

$$= D\tau_0 + e^{-2t/\tau_0}\left\{\psi_0^2 - D\tau_0\right\}$$

(4) The limiting forms as $t \to \infty$ are:

$$\overline{\Psi}_{t\to\infty} = 0$$

$$\overline{\Psi^2}_{t\to\infty} = D\tau_0$$

(5) As this is a gaussian process, we have:

$$m = \overline{\Psi} = \psi_0 e^{-t/\tau_0}$$

$$\sigma^2 = \overline{\Psi^2} - \overline{\Psi}^2 = D\tau_0\left(1 - e^{-2t/\tau_0}\right)$$

A. (6) We can then write for the distribution function:

$$W(\varphi, \varphi_0, t) = \frac{1}{\left[2\pi D \tau_0 (1-e^{-2t/\tau_0})\right]^{1/2}} \exp\left\{ \frac{-(\varphi - \varphi_0 e^{-t/\tau_0})^2}{2D\tau_0(1-e^{-2t/\tau_0})} \right\}$$

(7) We can also find the spectral density and correlation function of φ.

$$G_\varphi^+(f) = \frac{4\tau_0^2 D}{(2\pi f)^2 \tau_0^2 + 1}$$

$$R_\varphi(\tau) = 4\tau_0^2 D \int_0^\infty \frac{\cos 2\pi f \tau \, df}{(2\pi f)^2 \tau_0^2 + 1}$$

$$= \tau_0 D e^{-\tau/\tau_0}$$

(8) If we have a second inertial element A in the physical system, the Langevin equation is:

$$J \frac{d^2\varphi}{dt^2} + B \frac{d\varphi}{dt} + A\varphi = \frac{d}{dt}[F(t)]$$

which we rewrite as:

$$\frac{d^2\varphi}{dt^2} + \frac{1}{\tau_0}\frac{d\varphi}{dt} + \omega_0^2 \varphi = \frac{d}{dt}\{K(t)\}$$

where $\omega_0 = \sqrt{\frac{A}{J}}$

$$G_\varphi^+(f) = \frac{4D\tau_0^2\omega^2}{\omega^2 + \tau_0^2(\omega_0^2 - \omega^2)^2}$$

which can be integrated to give either $R_\varphi(\tau)$ or $\overline{\varphi^2_{t\to\infty}}$.

This equation is useful essentially in electric circuits.

B. Fokker-Planck Equation

(1) The FP equation is derived from the Smoluchowsky equation as a Markoff process assuming that the conditional probability of jumping out of one interval to another is slowly varying. This assumption is made on the master equation, viz:

$$\frac{\partial W(x,t)}{\partial t} = \int \left\{ W(x',t)\, P(x|x') - W(x,t)\, P(x'|x) \right\} dx'$$

(2) The Fokker-Planck equation is then:

$$\frac{\partial W(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left\{ A(x)\, W(x,t) \right\} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left\{ B(x)\, W(x,t) \right\}$$

where $A(x) = \lim_{\Delta t \to 0} \frac{\overline{y}}{\Delta t}$, $B(x) = \lim_{\Delta t \to 0} \frac{\overline{y^2}}{\Delta t}$

(3) For the Langevin equation of before, where

$$\overline{\Psi} = \psi_0 e^{-t/\tau_0}, \quad \overline{\psi^2} = D\tau_0 + e^{-2t/\tau_0}\left\{ \psi_0^2 - D\tau_0 \right\}$$

Then $\lim_{\Delta t \to 0} \frac{\overline{\psi}}{\Delta t} = -\frac{1}{\tau_0}\psi$

$\lim_{\Delta t \to 0} \frac{\overline{\Delta\psi^2}}{\Delta t} = 2D$

} assume Δt gets small but greater than τ_0

Writing the Langevin equation as in ψ: We write the F-P equation as:

$$\frac{\partial W}{\partial t} = -\frac{\partial}{\partial \psi}\left\{ A(\psi)\, W \right\} + \frac{1}{2}\frac{\partial^2}{\partial \psi^2}\left\{ B(\psi)\, W \right\}$$

we have: $$\frac{\partial W}{\partial t} = \frac{1}{\tau_0}\frac{\partial}{\partial \psi}\left\{ \psi W \right\} + D\frac{\partial^2 W}{\partial \psi^2}$$

(4) The steady state solution is:

$$D\frac{\partial W}{\partial \psi} + \frac{\psi}{\tau_0} W = 0, \quad W(\psi) = \text{constant}\; e^{-\frac{(\psi-\psi_0)^2}{D\tau_0}}$$

B. (5) The general steady state solution, using method of least descent and expanding $A(x)$ in Taylor series, keeping first term, is:

$$W_{ss}(x) = \frac{\text{constant}}{B(x_m)} e^{-\frac{(x-x_m)^2}{2\sigma^2}} , \quad \sigma^2 = \frac{-B(x_m)}{2\left(\frac{\partial A}{\partial x}\right)_{x_m}}$$

C. Relations of Constants in General Langevin equation to Physical Problems.

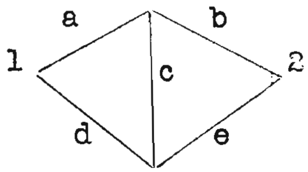
Problem	ψ	τ_0	D	PSD of driving fluctuation
R-L Circuit	i	L/R	$\frac{4RkT}{L}$	$4RkT$
Brownian motion	v	m/β	$\frac{4\beta kT}{m}$	$4\beta kT$
Heat Transport	θ Temperature Difference	$\frac{C_v}{\alpha}$	$\frac{4\alpha kT^2}{C_v}$	$4\alpha kT^2$

Applied Physics 215
Problem Set 1
Due October 18, 1960

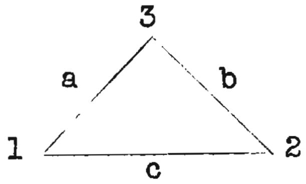
1. Consider a box containing 5 balls, three white and two black.
    - a. What is the probability of drawing a white ball first, then a black ball?
    - b. What is the probability of drawing a black ball on a second draw without knowing the color of the first ball taken out.

2. Consider a family with 4 children, and assume that each child has probability 0.51 of being a boy. Find the conditional probability that all the children will be boys, given that a) the eldest child is a boy b) at least one of the children is a boy.

3. Consider the diagram. Each of the five links can be either open or closed. If the probability of each link being closed is 1/2, what is the probability that 1 and 2 are connected?



4. Consider the diagram. Each link can be closed with a probability p, all links being independent. What is the probability that the three terminals are connected. What would the probability be if a and b are closed with probability p and c closed with probability p^1?



5. Six persons are to meet in the restaurant of a hotel. The hotel has however three equally attractive restaurants. What is the probability that three persons are waiting in one restaurant, two in another and one in the third?

6. The probability to make a step forward of length L is p; the probability to make a step backward of length l is q = 1-p. What is the average displacement and variance after N steps?

Applied Physics 215
Problem Set 2
Due November 1, 1960

7. The Maxwell-Boltzmann distribution over space coordinates is given by $\exp\left[-V(x,y,z)/kT\right]$, where V is the potential energy.

   What is the average potential energy of a linear harmonic oscillator?

   What is the average total (potential and kinetic) energy of a three-dimensional harmonic oscillator?

8. Two pendula are connected by a spring to represent two coupled harmonic oscillators,

$$V(x_1,x_2) = \tfrac{1}{2}\alpha\, x_1^{\,2} + \tfrac{1}{2}\beta\, x_2^{\,2} + \tfrac{1}{2}\gamma\,(x_1 - x_2)^2$$

   What is the average potential energy if the system is suspended in a gas at temperature T?

9. Let x and y be statistically independent random variables with the probability density functions

$$p(x) = \frac{1}{\pi\sqrt{1 - x^2}} \qquad \text{and } p(y) = \begin{cases} y \exp\left(\dfrac{-y^2}{2}\right) & \text{for } y \geq 0 \\ 0 & \text{for } y < 0 \end{cases}$$

   Show that their product has a gaussian probability density function.

10. The random variable x has an exponential probability density function

$$p(x) = a \exp(-2a|x|)$$

   a. Determine the mean and variance of x.

   b. Determine the nth moment of x.

11. Let x_1, x_2, x_3, and x_4 be real random variables with a gaussian joint probability density function, and let their means all be zero. If E denotes statistical average, show that

$$E(x_1x_2x_3x_4) = E(x_1x_2)E(x_3x_4) + E(x_1x_3)E(x_3x_4) + E(x_1x_4)E(x_2x_3)$$

12. Let x(t) be a sample function of a stationary real gaussian random process with a zero mean. Let a new random process be defined with the sample functions

$$y(t) = x^2(t)$$

Show that

$$R_y(\tau) = R_x{}^2(0) + 2R_x{}^2(\tau)$$

13. Consider the random process defined by the sample functions

$$y(t) = a \cos (t + \phi)$$

where a and ϕ are statistically independent random variables and where

$$p(\phi) = \begin{cases} \frac{1}{2\pi} & \text{for } 0 \leq \phi \leq 2\pi \\ 0 & \text{otherwise} \end{cases}$$

a. Derive an expression for the autocorrelation function of this process.

b. Show that $E(y_t) = \langle y(t) \rangle$

Applied Physics 215
Problem Set III
Due November 15, 1960

14. In each interval τ_a a voltage can assume one of the values +1, o or -1 with equal probability of $\frac{1}{3}$. What are the correlation function and spectral density for this random voltage signal, which can change only at times n τ_a.

15. Let a random process have sample functions

$$y(t) = x(t) \cos (\omega_o t + \theta)$$

where ω_0 is a constant, θ is a random variable uniformly distributed over the interval $0 \leq \theta \leq 2\pi$, and x(t) is a wide-sense stationary random process which is independent of θ. Show that the y(t) process is wide-sense stationary and determine its autocorrelation function and spectral density in terms of those for x(t).

16. Let y(t) be as given in 15. Let

$$w(t) = x(t) \cos \left[(\omega_o + \delta)t + \theta\right]$$

This represents y(t) heterodyned up in frequency by an amount δ. Show that w(t) is wide-sense stationary and find its autocorrelation function and spectral density in terms of those for x(t). Show that the cross-correlations between y(t) and w(t) are not stationary, and show that y(t) + w(t) is not wide-sense stationary. Show that if the heterodyning is done with random phase, i.e.,

$$w(t) = x(t) \cos \left[(\omega_o + \delta)t + \theta + \theta'\right]$$

where θ' is uniformly distributed over $0 \leq \theta' \leq 2\pi$ and is independent of θ and x(t), then y(t) + w(t) is wide-sense stationary.

17. Prove explicitly that the shot noise spectral density in a temperature limited plane parallel diode with transit time τ_a is given by

$$G(f) = \frac{\delta e\bar{i}_a}{(2\pi f\tau_a)^4}\left[(2\pi f\tau_a)^2 + 2 - 2\cos(2\pi f\tau_a) - 4\pi f\tau_a \sin(2\pi f\tau_a)\right]$$

18. A photocurrent i_{ph} is amplified by consecutive secondary emission stages. In each stage the average multiplication is $\bar{p}$ with a mean square deviation $\overline{\Delta p^2}$. What is the shot noise fluctuation in the amplified photocurrent after n stages. By how much does the amplification process increase the relative fluctuations in the photocurrent in the limit $n >> 1$.

Applied Physics 215

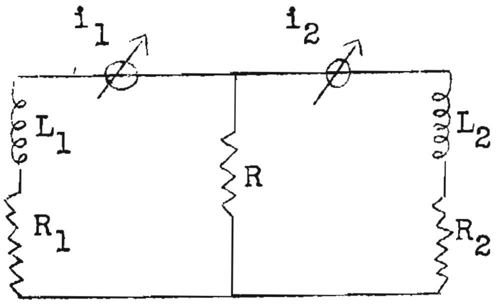
Problem Set 4, due December 13, 1960

19. Two equal resistors at temperatures T_1 and T_2 respectively, are connected by a matched lossless transmission line, which has a rectangular passband of 10000 cps. width. Calculate the net power transfer in watts.

If the heat capacity of each resistor is 0.1 cal/degree, how rapidly is thermal equilibrium reached?

20. Consider an RC circuit. What is the time-auto-correlation function for the voltage across the capacitance? Is there a cross-correlation between the current in the circuit and the voltage across C?

21 Consider the two-mesh circuit of the diagram, where all resistors are at the same temperature T.



Express the mean square fluctuations of i_1 and i_2 in terms of integrals of the power spectral density.

Are the fluctuations of i_1 and i_2 statistically independent?

Discuss the two limiting cases $R \longrightarrow 0$, and $R_1 = R_2 \longrightarrow 0$.

22. A temperature limited diode has a saturation current i_a. A parallel RLC combination connects the cathode with the plate via a battery. Calculate the voltage fluctuations across the RLC combination, if the temperature of the resistance is T.

23. A diode operates in the exponential region with a retarding potential at the plate. In this region the current in the diode is given by

$$i = i_a \exp(-eV_a/kT_{cat.})$$

The diode current passes through a resistance R at temperature T_R. Calculate the voltage fluctuations across the diode ΔV_a. Express the result in terms of the dynamic internal resistance $R_i = \partial V_a/\partial i$.

24. A signal source is connected via a lossy transmission line (attenuation L) at temperature T to an amplifier with an effective noise temperature T_{eff}. What is the effective noise temperature at the source end of the transmission line?

Applied Physics 215

Problem Set 5, due January 10th, 1961

25. Consider spherical particles of aluminum suspended in a column of water 10 cm high. Give the order of magnitude of the radius of aluminum particles, at which they would remain suspended indefinitely. If the radius of the particles is 10^{-4} cm, at what initial rate will sedimentation proceed from a homogeneous suspension? The density of aluminum is 2.7 and the viscosity of water at 25° C 0.9.

26. Consider a critically damped RLC-series combination, $R = 2(L/C)^{1/2}$. Calculate the auto-correlation function of the thermal noise current. Also calculate the cross-correlation function between the current and the voltage across C.

27. An intrinsic semiconductor has a pair creation rate $\mathcal{G}$ and an average number of electron-hole pairs n_o per unit volume. Calculate the spectral density of the current shot noise in a piece of cross section A and length ℓ .

28. Consider the "hard-collision" model of a monatomic gas. This means that regardless of the initial velocity of the atom the velocity after a collision is distributed according to the Maxwell-Boltzmann distribution. Let $F_i = \int K_i dt_i$ be the impulse during the ith collision. Calculate the correlation function $\overline{F_i F_j}$ for two consecutive impulses, and also for the case that the collisions are not consecutive.

29. Let x(t) be a sample function of a stationary narrow-band real gaussian random process. Consider a new random process defined with the sample functions

$$y(t) = x(t) \cos \omega_o t$$

where $f_o = \omega_o/2\pi$ is small compared to the center frequency f_c of the original process, but large compared to the spectral width of the original process. If we write

$$x(t) = V(t)\cos\left[\omega_c t + \phi(t)\right]$$

then we may define

$$y_L(t) = \frac{V(t)}{2}\cos\left[(\omega_c - \omega_o)t + \phi(t)\right]$$

to be the sample functions of the "lower sideband" of the new process, and

$$y_U(t) = \frac{V(t)}{2}\cos\left[(\omega_c + \omega_o)t + \phi(t)\right]$$

to be the sample functions of the "upper sideband" of the new process.

a. Show that the upper and lower sideband random processes are each stationary random processes even though their sum is nonstationary.

b. Show that the upper and lower sideband random processes are not statistically independent.

30. Let V_t be the envelope of a stationary narrow-band real gaussian random process. Show that

$$E(V_t) = (\tfrac{\pi}{2})^{1/2}\,\delta_x$$

and

$$\delta^2(V_t) = (2 - \tfrac{\pi}{2})\,\sigma_x^{\,2}$$

where $\sigma_x^{\,2}$ is the variance of the gaussian random process.

$$\int K_i \, dt_i = m_i (v_i' - v_i'')$$

non-consecutive $\rightarrow \overline{F_i F_j} = 0$

$$F_i F_j = m^2 (v_i' - v_i'')(v_j' - v_j'')$$

$$\overline{F_i F_j} = m^2 \int\int\int\int p(v_i') p(v_j') p(v_i'') p(v_j'') (v_i' - v_i'')(\quad) \, dv_i' \, dv_j'$$

For consecutive,

$$v_i'' = v_j'$$

31. Consider a synchronous or phase-sensitive detector consisting of a local oscillator, mixer and low-pass filter.
The input signal is $S \cos\omega_o t$.
The input noise has a spectral density A in the interval $f_o - \frac{B}{2} < f < f_o + \frac{B}{2}$ and vanishes outside this band.
The local oscillator output is $L \cos\omega_o t$.
The output of the mixer is given by the product $L \cos\omega_o t \times$ input.
Calculate the signal to noise ratio at the output of the low-pass filter. Make a comparison with the square-law detector.

Assignment #1
Problem Set #1

Paul Grant
DEAP-AP-G1
AP 215
10-5-60

1. a. (1)

We assume that the probability of drawing any ball is 1/5. The probability that W occurs first is:

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(2) $P(W) = 3/5$

The probability that B occurs when we know W has occured is:

(3) $P(B|W) = \frac{2}{4} = \frac{1}{2}$

Therefore the probability that W and B both occur when we know that W occurs and then B is:

(4) $P(W,B) = P(W)\ P(B|W) = \frac{3}{5} \cdot \frac{1}{2} = .3$ ✓

b. The probability that W or B (mutually exclusive events) occur is:

(1) $P(W \text{ or } B) = P(W) + P(B) = 1$ or certainty

Now, the probability that first W, then B occurs is .3. The other alternative is first B, then B again, which is:

(2) $P(B,B) = P(B)\ P(B|B) = \frac{2}{5} \cdot \frac{1}{4} = .1$

The probability of either W, B or B, B (mutually exclusive events) is:

(3) $P(W,B \text{ or } B,B) = P(W,B) + P(B,B) = .4$ ✓

which is the probability of getting B on the second draw, which is also the probability of getting B on the first draw.

2\. a. (1) Each event is statistically independent. If we know with certainty that the first child is a boy, it influences in no way the outcome of the next three trials. Therefore, by the product rule for statistically independent events, we have:

10

(2) $P(4 \text{ Boys}) = (.51)^3 = .132$

b. (1) Here we know that *at least one* of the children is a boy, hence there is no chance that they are all girls. However, we may assume that four girls are possible in order to calculate the probability that at least one boy occurs, that is:

(2) $P(\text{at least } 1 \text{ boy}) = 1 - P(4 \text{ girls}) = 1 - (.49)^4$

(3) We know what the probability is for 4 boys *and* at least one boy is: $(.51)^4$. Now the problem becomes one in conditional probability: What is the probability that 4 boys occur when it is known that at least one boy occurs?

(4) $$P(4 \text{ boys} \mid \text{at least one boy}) = \frac{(.51)^4}{1 - (.49)^4} = .0725$$

Assignment #1
Continued

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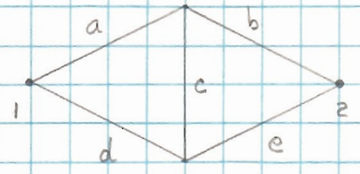
3.



The probability of being open or closed is one-half. We will consider the successful configurations, denoting open branches with a bar under the appropriate letter

Successful Configuration	Probability	Successful Configuration	Probability
abcde	p^5	abcd$\underline{e}$	p^5
ab$\underline{c}$de	"	a$\underline{b}$cde	"
ab$\underline{c}\underline{d}$e	"	$\underline{a}$bcde	"
$\underline{a}\underline{b}\underline{c}$de	"	abc$\underline{d}$e	"
a$\underline{b}$c$\underline{d}$e	"	$\underline{a}$b$\underline{c}$de	"
$\underline{a}$bcd$\underline{e}$	"	ab$\underline{c}$d$\underline{e}$	"
abc$\underline{d}\underline{e}$	"	ab$\underline{c}\underline{d}$e	"
$\underline{a}\underline{b}$cde	"	a$\underline{b}\underline{c}$de	"

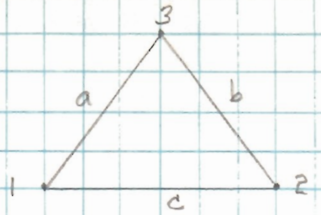
All configurations have equal probability because the probability of each branch being open or closed is equal. The probability of success is the sum of the probabilities of each mutually exclusive event.

(1) $P = 16 p^5 = \frac{16}{32} = 1/2$

It is interesting to note that the sample space contains $2^5 = 32$ possibilities. Since the probability of any configuration is the same, the probability of a success is the total number of successes divided by the size of the sample space.

(2) $P = \frac{16}{32} = 1/2$

4. a.



The probability of a side being closed is p and is independent of the state of the other sides.

10/

Successful Configuration	Probability
abc	p^3
$\underline{a}bc$	$p^2(1-p) = p^2 - p^3$
$a\underline{b}c$	$p^2(1-p) = p^2 - p^3$
$ab\underline{c}$	$p^2(1-p) = p^2 - p^3$

The probability of success is:

(1) $P = 3p^2 - 2p^3 = p^2(3-2p)$

b. The probability of a, b being closed is p, that of c is p'

Successful configurations	Probability
abc	$p^2 p'$
$\underline{a}bc$	$p\,p'(1-p)$
$a\underline{b}c$	$p\,p'(1-p)$
$ab\underline{c}$	$p^2(1-p')$

The probability of success is:

(2) $P = 2pp' - 2p^2p' + p^2$

$= p(p + 2p' - 2pp')$

Assignment #1
Problem Set #1
Continued

Paul Grant
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AP 215
10-5-60

5. We make the following assumptions:
   1) Each person must go into a restaurant
   2) The probability of a person going into any one restaurant is the same, i.e., 1/3
   3) Each person is indistinguishable from the others.

Successful Configuration

3	2	1
1	3	2
2	1	3
1	2	3
3	1	2
2	3	1

10

Sample Space

Groups	No. of Configurations
6, 0, 0	3
5, 1, 0	6
4, 2, 0	6
3, 3, 0	3
1, 2, 3	6
2, 2, 2	1
4, 1, 1	3

Thus there are 28 possibilities in the sample space of which 6 are satisfactory. Thus the probability of finding them distributed in 3, 2, 1 groups is:

$$P = \frac{6}{28} = 3/14$$

This presupposes indistinguishable objects and a uniform sample space where the probabilities of the 28 points are the same.

If the objects (persons) are distinguishable, we may use the following equation (From Feller):

(1) $$P(k) = \frac{r!}{k_1!\, k_2! \cdots k_n!} n^{-r} \quad ; \quad k_1 + k_2 + \cdots + k_n = r$$

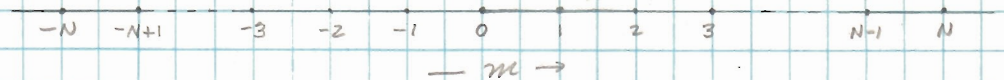
with r the number of objects (persons), n the number of cells (restaurants), $k_1, k_2, \ldots, k_n$ is the number in each cell. Therefore:

(2) $$P(3,2,1) = \frac{6!}{3!\,2!\,1!} \left(\frac{1}{3}\right)^6 = \frac{20}{243}$$

However, there are 6 arrangements, thus

(3) $$P = 6 \left(\frac{20}{243}\right) = \frac{40}{81}$$

6. Let N steps be taken from some arbitrary origin either to the right or left with final arrival at position m



If N is even, m must be even, and if N is odd, m is odd. To arrive at m, there must be taken $\frac{N+m}{2}$ steps to the right and $\frac{N-m}{2}$ steps to the left. The probability of a step to the right is p and the probability of a step to the left is $q = 1-p$. Clearly, the probability of any given sequence is the probability of steps to the right times the probability of the number of steps to the left. This must be multiplied by the number of such distinct sequences to form the distribution function.

10 ✓

$$(1) \quad W(m,N) = \frac{N!}{\left(\frac{N+m}{2}\right)!\left(\frac{N-m}{2}\right)!} \, p^{\frac{N+m}{2}} q^{\frac{N-m}{2}}$$

which is immediately identifiable as a Bernoulli distribution of the type:

$$(2) \quad p^{\mu} q^{N-\mu} \frac{N!}{\mu!\,(N-\mu)!}$$

with the well known relations : $\overline{\mu} = Np$

$$\overline{(\Delta\mu)^2} = Npq = Np - Np^2$$

We make the identification $\mu \rightarrow \frac{N+m}{2}$ in our random walk problem. We thus find from direct substitution:

$$(3) \quad \overline{m} = N(2p-1)$$

$$\overline{m^2} = 4\overline{\mu}^2 - 4\overline{\mu}N + N^2$$

$$\overline{(\Delta m)^2} = \overline{m^2} - \overline{m}^2 = 4Npq$$

In the case of the usual random walk, with equal probability in both directions, we see that $\overline{m} = 0$ and $\overline{(\Delta m)^2} = N$ which one expects. Also $\overline{m}$ lies to the right for $p > 1/2$ and to the left for $p < 1/2$, also as one expects. Of course, one can find these values in terms of length by the identification $x \rightarrow ml$, thus

$$(4) \quad \overline{x} = \overline{m}\,l$$

$$\overline{(\Delta x)^2} = \overline{(\Delta m)^2}\,l^2$$

Assignment #1
Continued

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AP 215
10-5-60

Problem 6
Continued

For the case of unequal steps in each direction, we will define the new random variable:

(5) $x = nL - (N-n)l = n(L+l) - Nl$

where L is the length of a step taken in the forward direction and n is their number. $N-n$ is the number taken in the reverse direction of length l.

(6) $\bar{x} = \bar{n}(L+l) - Nl$

(7) $\overline{(\Delta x)^2} = \overline{x^2} - \bar{x}^2 = (L+l)^2(\overline{n^2} - \bar{n}^2)$

Now the probability for n steps to be taken in the forward direction is:

(8) $$\frac{N!}{n!\,(N-n)!}\, p^n q^{N-n},$$

the Bernoulli distribution with:

$$\bar{n} = Np$$
$$\overline{(\Delta n)^2} = Npq = Np - Np^2$$

Thus we have

(9) $\bar{x} = Np(L+l) - Nl$

(10) $\overline{(\Delta x)^2} = Np(L+l)^2(1-p)$

We see that for $L=l$, we pass to the previous case. For $L=l$, $p=1/2$ we have our usual random walk mean and variance.

60/60

Assignment #2

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DEAP-AP-1G
AP 215
10-28-60

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7. (1) Given the Maxwell-Boltzmann distribution: $\exp\left[-V(x,y,z)/kT\right]$

(2) The potential energy of the classical, anisotropic, three dimensional harmonic oscillator is:

$V = \frac{1}{2}(\alpha x^2 + \beta y^2 + \gamma z^2)$ where α, β, γ are three independent force constants.

(3) $\therefore$ The Maxwell-Boltzmann distribution for this system is:

$\exp\left[-\frac{1}{2}(\alpha x^2 + \beta y^2 + \gamma z^2)/kT\right]$ which demand to be normalized, viz.,

$$\eta \iiint_{-\infty}^{\infty} \exp\left[-\tfrac{1}{2}(\alpha x^2 + \beta y^2 + \gamma z^2)/kT\right] dx\, dy\, dz$$

$$= 8\eta \iiint_{0}^{\infty} \exp\left[-\tfrac{1}{2}(\alpha x^2 + \beta y^2 + \gamma z^2)/kT\right] dx\, dy\, dz = 1$$

$$= 8\eta \iint_{0}^{\infty} \frac{\sqrt{\pi}}{2\sqrt{\frac{\alpha}{2kT}}} \exp\left[-\tfrac{1}{2}(\beta y^2 + \gamma z^2)/kT\right] dy\, dz$$

$$= \eta \frac{[2\pi kT]^{3/2}}{\sqrt{\alpha\beta\gamma}} = 1 \, ; \quad \therefore \eta = \frac{\sqrt{\alpha\beta\gamma}}{[2\pi kT]^{3/2}}$$

(4) $$p(V) = \frac{\sqrt{\alpha\beta\gamma}}{[2\pi kT]^{3/2}} \exp\left[-\tfrac{1}{2}(\alpha x^2 + \beta y^2 + \gamma z^2)/kT\right]$$

(5) For the linear harmonic oscillator, clearly:

$$p\{V(x)\} = \sqrt{\frac{\alpha}{2\pi kT}} \exp\left[-\tfrac{1}{2}\alpha x^2/kT\right]$$

(6) Now $V(x) = \frac{1}{2}\alpha x^2$; $\overline{V(x)} = \frac{1}{2}\alpha \overline{x^2}$

We recognize that (5) is a one dimensional Gaussian and $\bar{x}^2$ is merely the variance:

$$\overline{x^2} = \frac{kT}{\alpha} \; ; \quad \overline{V(x)} = \tfrac{1}{2} kT$$

(7) $\overline{V(x,y,z)} = 1/2\,\alpha\,\overline{x^2} + 1/2\,\beta\,\overline{y^2} + 1/2\,\gamma\,\overline{z^2}$

$$p(V) = \sqrt{\frac{\alpha}{2\pi kT}}\,\exp\left[-1/2\,\alpha x^2/kT\right]\cdot\sqrt{\frac{\beta}{2\pi kT}}\,\exp\left[-1/2\,\beta y^2/kT\right]\cdot\sqrt{\frac{\gamma}{2\pi kT}}\,\exp\left[-1/2\,\gamma z^2/kT\right]$$

which is a trivariate Gaussian distribution, with

$$\overline{x^2} = \frac{kT}{\alpha}, \quad \overline{y^2} = \frac{kT}{\beta}, \quad \overline{z^2} = \frac{kT}{\gamma}$$

$\therefore \quad \overline{V(x,y,z)} = 3/2\,kT$

(8) It is well known that the total energy of the statistical harmonic oscillator is twice the average kinetic or potential energy. Statistically we see this by averaging the Hamiltonian for the system obeying Maxwell-Boltzmann statistics:

$$H = \frac{p_x^2 + p_y^2 + p_z^2}{2m} + 1/2\left(\alpha x^2 + \beta y^2 + \gamma z^2\right)$$

The average of the momentum term is just another Gaussian variance, viz,

$$\overline{p^2} = 3mkT \quad ; \quad \overline{T} = \frac{3}{2}kT$$

$\therefore \quad \overline{H} = 3kT$

Assignment #2
Continued

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8. (1) Given: a system in which the potential energy is described by:

$$V(x_1,x_2) = 1/2\,\alpha\,x_1^2 + 1/2\,\beta\,x_2^2 + \tfrac{1}{2}\gamma\,(x_1-x_2)^2$$

(2) We hold that the system in the gas is described by Maxwell-Boltzmann statistics for which the potential energy is distributed in the form $\exp\left[-V(x,y,z)/kT\right]$. We then have for the distribution function:

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$$\rho(V) = \eta \exp\left[-1/2\left(\alpha x_1^2 + \beta x_2^2 + \gamma x_1^2 + \gamma x_2^2 - 2\gamma x_1 x_2\right)/kT\right]$$

$$= \eta \exp\left[-1/2\left(\{\alpha+\gamma\}x_1^2 + \{\beta+\gamma\}x_2^2 - 2\gamma x_1 x_2\right)/kT\right]$$

which we immediately recognize as the form of the bivariate Gaussian distribution with zero mean and non-zero correlation; that is:

$$\rho(V) = \frac{1}{2\pi\sigma\tau(1-\rho^2)^{1/2}}\exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{x_1^2}{\sigma^2} + \frac{x_2^2}{\tau^2} - \frac{2\rho}{\sigma\tau}x_1x_2\right)\right]$$

(3) Making proper identifications:

$$(1-\rho^2)\sigma^2 = \frac{kT}{\alpha+\gamma} \; ; \quad (1-\rho^2)\tau^2 = \frac{kT}{\beta+\gamma} \; ; \quad \frac{\rho}{\sigma\tau(1-\rho^2)} = \frac{\gamma}{kT}$$

Then: $\dfrac{\sigma^2}{\tau^2} = \dfrac{\beta+\gamma}{\alpha+\gamma}$, $\rho^2 = 1 - \dfrac{kT}{\sigma^2(\alpha+\gamma)}$, $\rho^2 + \dfrac{kT}{\gamma\sigma\tau}\rho - 1 = 0$

and, after more manipulation:

$$\sigma^2 = \frac{kT\;(\beta+\gamma)}{\beta\alpha+\beta\gamma+\alpha\gamma} \; ; \quad \tau^2 = \frac{kT\;(\alpha+\gamma)}{\beta\alpha+\beta\gamma+\alpha\gamma}$$

$$\rho^2 = \frac{\gamma^2}{(\alpha+\gamma)(\beta+\gamma)}$$

(4) The average potential energy is:

$$\overline{V(x_1, x_2)} = 1/2 \left(\{\alpha+\gamma\} \overline{x_1^2} + \{\beta+\gamma\} \overline{x_2^2} - 2\gamma \overline{x_1 x_2} \right)$$

where

$$\overline{x_1^2} = \sigma^2$$

$$\overline{x_2^2} = \tau^2$$

$$\overline{x_1 x_2} = \rho\sigma\tau = \left[\frac{\gamma^2}{(\alpha+\gamma)(\beta+\gamma)} \right]^{1/2} \left[\frac{kT\,(\beta+\gamma)}{\beta\alpha+\beta\gamma+\alpha\gamma} \right]^{1/2} \left[\frac{kT\,(\alpha+\gamma)}{\beta\alpha+\beta\gamma+\alpha\gamma} \right]^{1/2}$$

$$= kT \frac{\gamma}{\beta\alpha+\beta\gamma+\alpha\gamma}$$

(5) $$\overline{V(x_1, x_2)} = 1/2 \left[\frac{kT\,(\alpha+\gamma)(\beta+\gamma)}{\beta\alpha+\beta\gamma+\alpha\gamma} + \frac{kT\,(\beta+\gamma)(\alpha+\gamma)}{\beta\alpha+\beta\gamma+\alpha\gamma} - \frac{2\gamma^2\,kT}{\beta\alpha+\beta\gamma+\alpha\gamma} \right]$$

$= kT$, as one would expect with two degrees of freedom.

Assignment #2
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9. (1) Given: $p(x) = \dfrac{1}{\pi\left[1-x^2\right]^{1/2}}$; $p(y) = \begin{cases} y \exp\left(-y^2/2\right), & y \geq 0 \\ 0, & y < 0 \end{cases}$

with $p(x,y) = p(x)\,p(y)$, that is, x and y are statistically independent.

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(2) Let $u = g_1(x,y) = xy$ and find $p(u)$

(3) In general, if: $\left.\begin{aligned} u &= g_1(x,y) \\ v &= g_2(x,y) \end{aligned}\right\}$ $\begin{aligned} x &= f_1(u,v) \\ y &= f_2(u,v) \end{aligned}$

Then: $p(u,v) = p(x=f_1,\ y=f_2)\left|\dfrac{\partial(x,y)}{\partial(u,v)}\right|$

(4) For our problem, we choose:

$$\left.\begin{aligned} u &= g(x,y) = xy \\ v &= g(x,y) = y \end{aligned}\right\} \begin{aligned} x &= f_1(u,v) = u/v \\ y &= f_2(u,v) = v \end{aligned}$$

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1/v & 0 \\ -u/v^2 & 1 \end{vmatrix} = \frac{1}{v} = \frac{1}{y}$$

(5) $\therefore\ p(u,y) = p\left(x = \frac{u}{y},\ y\right)\cdot\frac{1}{y}$

If x and y are statistically independent;

$$p(u,y) = p\left(x = \frac{u}{y}\right)p(y)\cdot\frac{1}{y}$$

(6) finally $p(u) = \displaystyle\int_{-\infty}^{\infty} p\left(x=\frac{u}{y}\right)p(y)\,\frac{dy}{y}$; if $u = xy$

(7) $p\left(x = \dfrac{u}{y}\right) = \dfrac{1}{\pi\left[1-\dfrac{u^2}{y^2}\right]^{1/2}} = \dfrac{y}{\pi\left[y^2-u^2\right]^{1/2}}$

(8) Then: $p(u) = \displaystyle\int_{u}^{\infty} \frac{y\,e^{-y^2/2}}{\pi\left[y^2-u^2\right]^{1/2}}\,dy$, since y must be greater than u for the integrand to exist.

(9) We propose the following substitutions:

Let $z^2 = y^2 - \mu^2$, $y^2 = z^2 + \mu^2$; $2z\,dz = 2y\,dy$

(10) $$p(\mu) = \frac{1}{\pi}\int_0^\infty e^{-\frac{1}{2}(z^2+\mu^2)}\,dz = \frac{e^{-1/2\,\mu^2}}{\pi}\int_0^\infty e^{-1/2\,z^2}\,dz$$

$$= \frac{e^{-\mu^2/2}}{\pi}\cdot\frac{\sqrt{2}}{2}\sqrt{\pi} = \frac{1}{\sqrt{2\pi}}\,e^{-\mu^2/2}$$

which is a Gaussian density function.

$$p(xy) = \frac{1}{\sqrt{2\pi}}\,e^{-\frac{x^2y^2}{2}}$$

Note: when performing the integration (8), the limits must be fixed over the range of y from μ to ∞ as $p(x)$ contains a singularity at $x = \pm 1$, instead of the range of definition of y, viz., 0 to ∞.

Assignment #2
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10. a. (1) Given: $p(x) = a \exp(-2a|x|)$

we note that $p(x) = p(-x)$ or $p(x)$ is an even function, and that a is the normalization constant.

(2) $\overline{x} = a \int_{-\infty}^{\infty} x \exp(-2a|x|)\, dx = 0$, by inspection since the integrand is an odd function.

(3) $\overline{x^2} = a \int_{-\infty}^{\infty} x^2 \exp(-2a|x|)\, dx = 2a \int_0^{\infty} x^2 e^{-2ax}\, dx$

$$= 2a \cdot \frac{2}{(2a)^3} = \frac{1}{2a^2}$$

b. (1) Consider $\overline{x^m} = a \int_{-\infty}^{\infty} x^m \exp(-2a|x|)\, dx$:

It obvious that the integral will vanish whenever m is odd because the integrand will be odd. Therefore, it is useful only to consider even moments, viz.:

(2) $\overline{x^{2n}} = a \int_{-\infty}^{\infty} x^{2n} \exp(-2a|x|)\, dx$

$$= 2a \int_0^{\infty} x^{2n} \exp(-2ax)\, dx = 2a \cdot \frac{(2n)!}{(2a)^{2n+1}}$$

$$= \frac{(2n)!}{(2a)^{2n}}$$

11. Consider the following multivariate gaussian characteristic function:

$$(1) \qquad M_x(iV) = \exp\left[-\tfrac{1}{2} V' \Lambda V\right]$$

where the mean values of the x_n's vanish and

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$$V = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{pmatrix} ; \quad \Lambda = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1N} \\ \lambda_{21} & & & \\ \vdots & & & \\ \lambda_{N1} & \cdots & \cdots & \lambda_{NN} \end{pmatrix}$$

$$\lambda_{nm} = E(x_n x_m)$$

Now:

$$(2) \qquad E(x_1 x_2 \cdots x_N) = (-i)^N \left. \frac{\partial^N M_x(iV)}{\partial V} \right|_{V=0}$$ ✓

Consider: $V' \Lambda V$ for order 4. ✓

$$(v_1\ v_2\ v_3\ v_4) \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} & \lambda_{14} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} & \lambda_{24} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} & \lambda_{34} \\ \lambda_{41} & \lambda_{42} & \lambda_{43} & \lambda_{44} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}$$

$$= (v_1\ v_2\ v_3\ v_4) \begin{pmatrix} \lambda_{11} v_1 + \lambda_{12} v_2 + \lambda_{13} v_3 + \lambda_{14} v_4 \\ \lambda_{21} v_1 + \lambda_{22} v_2 + \lambda_{23} v_3 + \lambda_{24} v_4 \\ \lambda_{31} v_1 + \lambda_{32} v_2 + \lambda_{33} v_3 + \lambda_{34} v_4 \\ \lambda_{41} v_1 + \lambda_{42} v_2 + \lambda_{43} v_3 + \lambda_{44} v_4 \end{pmatrix}$$

$$= \lambda_{11} v_1^2 + \lambda_{12} v_2 v_1 + \lambda_{13} v_3 v_1 + \lambda_{14} v_4 v_1 + \lambda_{21} v_1 v_2 + \lambda_{22} v_2^2 + \lambda_{23} v_3 v_2 + \lambda_{24} v_4 v_2$$

$$+ \lambda_{31} v_1 v_3 + \lambda_{32} v_2 v_3 + \lambda_{33} v_3^2 + \lambda_{34} v_4 v_3 + \lambda_{41} v_1 v_4 + \lambda_{42} v_2 v_4 + \lambda_{43} v_3 v_4 + \lambda_{44} v_4^2$$

$$= \lambda_{11} v_1^2 + \lambda_{22} v_2^2 + \lambda_{33} v_3^2 + \lambda_{44} v_4^2 + (\lambda_{12} + \lambda_{21}) v_1 v_2 + (\lambda_{13} + \lambda_{31}) v_1 v_3$$

$$+ (\lambda_{14} + \lambda_{41}) v_1 v_4 + (\lambda_{23} + \lambda_{32}) v_2 v_3 + (\lambda_{24} + \lambda_{42}) v_2 v_4 + (\lambda_{34} + \lambda_{43}) v_3 v_4$$

$$= \lambda_{11} v_1^2 + \lambda_{22} v_2^2 + \lambda_{33} v_3^2 + \lambda_{44} v_4^2 + 2 \lambda_{12} v_1 v_2 + 2 \lambda_{13} v_1 v_3$$

$$+ 2 \lambda_{14} v_1 v_4 + 2 \lambda_{23} v_2 v_3 + 2 \lambda_{24} v_2 v_4 + 2 \lambda_{34} v_3 v_4$$

since $E(x_n x_m) = E(x_m x_n)$

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Problem 11
Continued

(3) Consider $\frac{\partial^4 M_x}{\partial v_1 \partial v_2 \partial v_3 \partial v_4}$

$$\frac{\partial M_x}{\partial v_4} = -(\lambda_{44} v_4 + \lambda_{14} v_1 + \lambda_{24} v_2 + \lambda_{34} v_3) \exp\left[-\tfrac{1}{2} V' \Lambda V\right]$$

$$\frac{\partial^2 M_x}{\partial v_3 \partial v_4} = -\lambda_{34} \exp\left[-\tfrac{1}{2} V' \Lambda V\right] - (\lambda_{44} v_4 + \lambda_{14} v_1 + \lambda_{24} v_2 + \lambda_{34} v_3)$$
$$\cdot (-1)(\lambda_{33} v_3 + \lambda_{13} v_1 + \lambda_{23} v_2 + \lambda_{34} v_4) \exp\left[-\tfrac{1}{2} V' \Lambda V\right]$$

$$\frac{\partial^3 M_x}{\partial v_2 \partial v_3 \partial v_4} = \lambda_{34} (\lambda_{22} v_2 + \lambda_{12} v_1 + \lambda_{23} v_3 + \lambda_{24} v_2) \exp\left[-\tfrac{1}{2} V' \Lambda V\right]$$

$$+ \lambda_{24} (\lambda_{33} v_3 + \lambda_{13} v_1 + \lambda_{23} v_2 + \lambda_{34} v_4) \exp\left[-\tfrac{1}{2} V' \Lambda V\right]$$
$$+ \lambda_{23} (\lambda_{44} v_4 + \lambda_{14} v_1 + \lambda_{24} v_2 + \lambda_{34} v_3) \exp\left[-\tfrac{1}{2} V' \Lambda V\right]$$
$$- (\lambda_{22} v_2 + \lambda_{12} v_1 + \lambda_{23} v_3 + \lambda_{24} v_4)(\lambda_{44} v_4 + \lambda_{14} v_1 + \lambda_{24} v_2 + \lambda_{34} v_3)$$
$$\cdot (\lambda_{33} v_3 + \lambda_{13} v_1 + \lambda_{23} v_2 + \lambda_{34} v_4) \exp\left[-\tfrac{1}{2} V' \Lambda V\right].$$

$$\frac{\partial^4 M_x}{\partial v_1 \partial v_2 \partial v_3 \partial v_4} = \lambda_{34} \lambda_{12} \exp\left[-\tfrac{1}{2} V' \Lambda V\right] + \lambda_{24} \lambda_{13} \exp\left[-\tfrac{1}{2} V' \Lambda V\right]$$
$$+ \lambda_{23} \lambda_{14} \exp\left[-\tfrac{1}{2} V' \Lambda V\right] +$$ other terms which vanish when $V \to 0$.

(4) $$(-\imath)^4 \frac{\partial^4 (\imath V)}{\partial V}\Big]_{V \to 0} = E(X_1 X_2 X_3 X_4) = \lambda_{34} \lambda_{12} + \lambda_{24} \lambda_{13} + \lambda_{23} \lambda_{14}$$

$$= E(X_1 X_2) E(X_3 X_4) = E(X_1 X_3) E(X_2 X_4) + E(X_1 X_4) E(X_2 X_3)$$ QED!

12. (1) Given the random process $y(t) = x^2(t)$ where $x(t)$ is a sample function of a stationary real gaussian process with zero mean.

(2) $$R_y(\tau) = \overline{y(t)\, y(t+\tau)} = \overline{x^2(t)\, x^2(t+\tau)}$$

Since the process is stationary, let us make the change to:

$$x(t) \rightarrow x_1$$
$$x(t+\tau) \rightarrow x_2\,,$$

Keeping in mind only the time interval matters. Now, x_1, x_2 will be distributed according to a bivariate gaussian distribution, viz.

(3) $$p(x_1, x_2) = \frac{1}{2\pi \sigma_1 \sigma_2 (1-\rho^2)^{1/2}} \exp\left[-\frac{1}{2(1-\rho^2)} \left\{ \frac{x_1^2}{\sigma_1^2} + \frac{x_2^2}{\sigma_2^2} - \frac{2\rho x_1 x_2}{\sigma_1 \sigma_2} \right\} \right]$$

(4) $$\therefore \quad R_y(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1^2 x_2^2 \; p(x_1, x_2)\, dx_1\, dx_2$$

We may calculate this with the help of the characteristic function for the bivariate gaussian process:

$$M_x(\imath v_1, \imath v_2) = \exp\left[-\frac{1}{2}\, \sigma_1^2 v_1^2 + \sigma_2^2 v_2^2 + 2\sigma_1\sigma_2 \rho\, v_1 v_2 \right].$$

with $$E(x_1^2 x_2^2) = (-\imath)^4 \left. \frac{\partial^4 M_x(\imath v_1, \imath v_2)}{\partial v_1^2\, \partial v_2^2} \right]_{v_1, v_2 = 0}$$

(5) $$\frac{\partial M_x}{\partial v_2} = -(\sigma_2^2 v_2 + \sigma_1\sigma_2\rho\, v_1) \exp[\quad].$$

$$\frac{\partial^2 M_x}{\partial^2 v_2} = -\sigma_2^2 \exp[\quad] + (\sigma_2^2 v_2 + \sigma_1\sigma_2\rho\, v_1)^2 \exp[\quad].$$

$$\frac{\partial^3 M_x}{\partial v_1\, \partial^2 v_2} = \sigma_2^2 (\sigma_1^2 v_1 + \sigma_1\sigma_2\rho\, v_2) \exp[\quad]$$

$$+ 2\sigma_1\sigma_2\rho\, (\sigma_2^2 v_2 + \sigma_1\sigma_2\rho\, v_1) \exp[\quad]$$

$$- (\sigma_1^2 v_1 + \sigma_1\sigma_2\rho\, v_2)(\sigma_2^2 v_2 + \sigma_1\sigma_2\rho\, v_1)^2 \exp[\quad].$$

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Problem 12
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(6) $$\frac{\partial^4 M_x}{\partial^2 v_1\,\partial^2 v_2} = \sigma_2^2\sigma_1^2 \exp[\;] + 2\sigma_1^2\sigma_2^2\rho^2 \exp[\;]$$
+ terms which will vanish when $v_1, v_2 \to 0$.

(7) $$\therefore\; E(x_1^2x_2^2) = \left.\frac{\partial^4 M_x}{\partial^2 v_1\,\partial^2 v_2}\right]_{\substack{v_1\\ v_2=0}} = \sigma_2^2\sigma_1^2 + 2\sigma_1^2\sigma_2^2\rho^2$$

Now: $R_x(\tau) = E(x_1x_2) = \overline{x_1x_2} = \sigma_1\sigma_2\rho$

$R_x(0) = E(x_1)^2 = E(x_2)^2$, since $x_1 = x_2$ when $\tau = 0$,

Therefore, $R_x(0) = \sigma_1^2 = \sigma_2^2$

(8) Finally: $R_y(\tau) = R_x^2(0) + 2R_x^2(\tau)$

---

13. a. (1) $y(t) = a\cos(t+\varphi)$; $R_y(\tau) = \overline{y(t)\,y(t+\tau)}$

(2) $$R_y(\tau) = \overline{a^2\cos(t+\varphi)\cos(t+\varphi+\tau)}$$

$$= \tfrac{1}{2}\bar{a}^2\cos\tau + \tfrac{1}{2}\overline{a^2}\,\underbrace{\overline{\cos(2t+2\varphi+\tau)}}_{\substack{\parallel\\ 0}}$$ (distributed uniformly)

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$$= \tfrac{1}{2}\overline{a^2}\cos\tau$$

(3) $$R_y(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_0^T y(t)\,y(t+\tau)\,dt = \lim_{T\to\infty}\frac{1}{T}\left[\int_0^T\frac{a^2}{2}\cos\tau\,dt + \frac{a^2}{2}\int_0^T\cos(2t+2\varphi+\tau)\,dt\right]$$

$$= \lim_{T\to\infty}\frac{1}{T}\left[\frac{a^2}{2}T\cos\tau + \frac{a^2}{4}\left\{\sin(2T+2\varphi+\tau) - \sin(2\varphi+\tau)\right\}\right]$$

$$= \frac{a^2}{2}\cos\tau + \frac{a^2}{4}\lim_{T\to\infty}\left\{\frac{\sin(2T+2\varphi+\tau)}{T} - \frac{\sin(2\varphi+\tau)}{T}\right\}$$

(4) $R_y(\tau) = \frac{a^2}{2} \cos \tau$

(5) $R_y(\tau) = \bar{R}_y(\tau)$

b. (1) $\langle y(t) \rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} y(t)\, dt = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} a \cos(t+\varphi)\, dt$

$$= \lim_{T\to\infty} \frac{1}{2T} \left[a \sin(T+\varphi) - a \sin(-T+\varphi) \right] = 0 \checkmark$$

(2) $E(y_t) = \overline{y(t)} = \bar{a}\ \underbrace{\overline{\cos(t+\varphi)}}_{\overset{\shortparallel}{0}}$ ✓ (uniformly distributed)

$\therefore E(y_t) = \langle y(t) \rangle$

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Assignment #3
Problems 14 – 18

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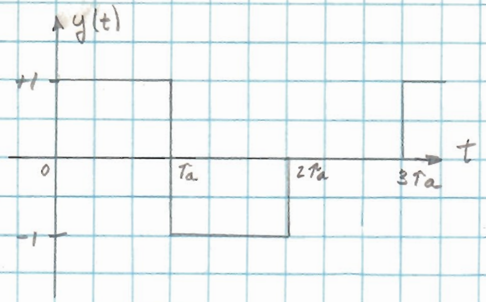
14.



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(1) $P(1) = P(-1) = P(0) = \frac{1}{3}$

We wish to form $\overline{y(t)\,y(t+\tau)}$. We may work in the first interval as the others are statistically alike.

Each interval is statistically independent of the others.

(2) For $t < T_a$, $t+\tau > T_a$, $\overline{y(t)\,y(t+\tau)} = \overline{y(t)}\;\overline{y(t+\tau)} = 0$ from the above statement.

(3) For $0 < t < T_a$; $0 < t+\tau < T_a$:

$$\overline{y(t)\,y(t+\tau)} = (+1)(+1)\,P(+1,+1) + (0)(0)\,P(0,0) + (-1,-1)\,P(-1,-1)$$

(4) Other combinations, like +1, 0 or +1, -1 are impossible since the signal can switch only at nT_a.

$$P(+1,+1) = P(+1)\,P(+1|+1)$$

Now $P(+1|+1)$ is certainly one since we know that given the current at one time in the interval, the current at all other times must be the same.

$\therefore\ P(+1,+1) = P(0,0) = P(-1,-1) = 1/3$

(5) $\overline{y(t)\,y(t+\tau)} = 2/3$

Now this correlation function is not independent of T as it holds over only one interval.

(6) $$R_y(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_0^T \overline{y(t)\,y(t+\tau)}\,dt$$

This integral is composed of $\frac{T}{T_a}$ integrals over 0 to $T_a - \tau$, therefore:

(7) $$R_y(\tau) = \frac{1}{T_a}\int_0^{T_a-\tau} \overline{y(t)\,y(t+\tau)}\,dt = \frac{2/3}{T_a}\int_0^{T_a-\tau} dt = \frac{2}{3}\left(1 - \frac{|\tau|}{T_a}\right)$$

(8) Since the process is wide sense stationary, we have for the PSD:

$$G_+(f) = 4\int_0^\infty R_y(\tau)\cos 2\pi f\tau \, d\tau$$

$$= \frac{8}{3}\left\{ \int_0^{\tau_a} \cos 2\pi f \tau \, d\tau - \frac{1}{\tau_a}\int_0^{\tau_a} \tau \cos 2\pi f \tau \, d\tau \right\}$$

$$= \frac{8}{3}\left\{ \frac{\sin 2\pi f \tau_a}{2\pi f} - \frac{1}{\tau_a}\left[\frac{\cos 2\pi f \tau_a + 2\pi f \tau_a \sin 2\pi f \tau_a - 1}{4\pi^2 f^2} \right] \right\}$$

$$= \frac{8}{3}\left\{ \frac{1 - \cos 2\pi f \tau_a}{4\pi^2 f^2 \tau_a} \right\} = \frac{2}{3}\tau_a \left\{ \frac{\sin \pi f \tau_a}{\pi f \tau_a} \right\}^2$$

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15. (1) A wide-sense stationary process must satisfy the following relation:

$$R_x(t, t+\tau) = R_x(\tau) = E(x_t, x^*_{t+\tau})$$

(2) We take the process to be real: $y(t) = x(t)\cos(\omega_0 t + \theta)$

$$\overline{y(t)\,y(t+\tau)} = \overline{x(t)\,x(t+\tau)\cos(\omega_0 t+\theta)\cos(\omega_0 t + \omega_0\tau + \theta)}$$

Since $x(t)$ and θ are independent and $x(t)$ is wide sense stationary, we have:

(3) $$\overline{y(t)\,y(t+\tau)} = \tfrac{1}{2}R_x(\tau)\cos\omega_0\tau + \tfrac{1}{2}R_x(\tau)\cos\overline{(2\omega_0 t + 2\theta + \omega_0\tau)}$$

As θ is distributed uniformly

(4) $$\overline{y(t)\,y(t+\tau)} = \tfrac{1}{2}R_x(\tau)\cos\omega_0\tau = R_y(\tau)$$

Since the process is now shown to be wide-sense stationary,

$$G_y^+(f) = 4\int_0^\infty R_y(\tau)\cos 2\pi f\tau\,d\tau$$
$$= 2\int_0^\infty R_x(\tau)\cos\omega_0\tau\cos 2\pi f\tau\,d\tau$$

(5) We write $R_x(\tau) = \int_0^\infty G_x^+(f')\cos 2\pi f'\tau\,df'$

(6) $$G_y^+(f) = 2\int_0^\infty\int_0^\infty G_x^+(f')\cos\omega_0\tau\cos 2\pi f\tau\cos 2\pi f'\tau\,d\tau\,df'$$

We shall integrate over τ first:

(7) $$\cos\omega_0\tau\cos 2\pi f\tau\cos 2\pi f'\tau = \Big[\tfrac{1}{2}\cos\{2\pi(f+f_0)\tau\}$$
$$+\tfrac{1}{2}\cos\{2\pi(f-f_0)\tau\}\Big]\cos 2\pi f'\tau$$
$$= \tfrac{1}{4}\cos\{2\pi(f+f_0+f')\tau\} + \tfrac{1}{4}\cos\{2\pi(f+f_0-f')\tau\}$$
$$+\tfrac{1}{4}\cos\{2\pi(f-f_0+f')\tau\} + \tfrac{1}{4}\cos\{2\pi(f-f_0-f')\tau\}$$

(8) We have shown: $\int_0^\infty \cos 2\pi f \tau \, d\tau = \frac{1}{2} \delta(f)$

Using this, the first and third integrals vanish as arguments of their integrands exist everywhere from 0 – ∞. The second and forth will give δ functions at f_0+f and f_0-f.

$$\therefore \int_0^\infty [(7)] \, d\tau = \frac{1}{8} \left[\delta(f+f_0-f') + \delta(f-f_0-f') \right]$$

Then (6) becomes:

(9) $$G_y^+(f) = \frac{1}{4} \int_0^\infty G_x^+(f') \left[\delta(f+f_0-f') + \delta(f-f_0-f') \right] df'$$

$$= \frac{1}{4} \left[G_x^+(f+f_0) + G_x^+(f-f_0) \right]$$

This result looks like a very close analogy to the sideband frequencies of a amplitude modulated wave. This is what one would expect $y(t)$ appears to be a noise signal modulated by a cosine wave.

ASSIGNMENT #3
Continued

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16. (1) The procedure here is essentially the same as in problem 15. We construct the correlation factor thus:

$$\overline{w(t)\,w(t+\tau)} = \overline{x(t)\,x(t+\tau)\cos\left[(\omega_0+\delta)t+\theta\right]\cos\left[(\omega_0+\delta)t+(\omega_0+\delta)\tau+\theta\right]}$$

$$= \frac{1}{2}R_x(\tau)\cos(\omega_0+\delta)\tau + \frac{1}{2}R_x(\tau)\,\overline{\cos\left[2(\omega_0+\delta)t+2\theta+(\omega_0+\delta)\tau\right]}$$

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(2) As θ is distributed uniformly:

$$\overline{w(t)\,w(t+\tau)} = R_w(\tau) = \frac{1}{2}R_x(\tau)\cos(\omega_0+\delta)\tau$$

In finding the PSD, we follow the same method as in problem 15, setting $f_0 \to f_0+\Delta$, $y \to w$.

(3) $$G^+_w(f) = \frac{1}{4}\left[G^+_x(f+f_0+\Delta) + G^+_x(f-f_0-\Delta)\right]$$

We now set up the cross-correlation between $y(t)$ and $w(t)$:

(4) $$\overline{w(t)\,y(t+\tau)} = \overline{x(t)\,x(t+\tau)\cos\left[(\omega_0+\delta)t+\theta\right]\cos(\omega_0t+\omega_0\tau+\theta)}$$

$$\cos\left\{\left[(\omega_0+\delta)t+\theta\right]\pm(\omega_0t+\omega_0\tau+\theta)\right\}$$

$$= \cos\left[(\omega_0+\delta)t+\theta\right]\cos(\omega_0t+\omega_0\tau+\theta) \mp \sin[\;]\sin(\;).$$

(5) $$\overline{w(t)\,y(t+\tau)} = \frac{1}{2}R_x(\tau)\,\overline{\cos\{\delta t-\omega_0\tau\}} + \frac{1}{2}R_x(\tau)\,\overline{\cos\{2\omega_0t+2\theta+\delta t+\omega_0\tau\}}$$

$$= \frac{1}{2}R_x(\tau)\,\overline{\cos\{\delta t-\omega_0\tau\}}$$

It is easily seen that the cross-correlation function is not independent of t and dependent only on τ which is necessary to the wide-sense stationary condition. We might also form:

(6) $$\overline{y(t)\,w(t+\tau)} = \overline{x(t)\,x(t+\tau)\cos(\omega_0t+\theta)\cos\left[(\omega_0+\delta)t+(\omega_0+\delta)\tau+\theta\right]}$$

$$= \frac{1}{2}R_x(t)\,\overline{\cos(\delta t+\omega_0\tau+\delta\tau)}$$

which again is not stationary in the wide sense.

(7) Let $z(t) = w(t) + y(t)$. Then, $R_z(t, t+\tau) = \overline{[w(t) + y(t)][w(t+\tau) + y(t+\tau)]}$

$$= \overline{w(t)\,w(t+\tau)} + \overline{y(t)\,y(t+\tau)} + \overline{w(t)\,y(t+\tau)} + \overline{y(t)\,w(t+\tau)}$$

We have already shown that $R_w(t, t+\tau)$ and $R_y(t, t+\tau)$ are wide sense stationary. However, we have also ✓ just shown that $R_{wy}(t, t+\tau)$ and $R_{yw}(t, t+\tau)$ are not wide sense stationary, therefore $R_z(t, t+\tau)$ is not wide sense stationary.

(8) Given: $w(t) = x(t) \cos[(\omega_0 + \delta)t + \theta + \theta']$, with θ, θ' independent and uniformly distributed. It is immediately obvious from previous results and the fact that θ, θ' are uniformly distributed over the same interval that:

$$R_w(t, t+\tau) = R_w(\tau) = \tfrac{1}{2} R_x(\tau) \cos(\omega_0 + \delta)\tau$$

$$R_y(t, t+\tau) = R_y(\tau) = \tfrac{1}{2} R_x(\tau) \cos \omega_0 \tau$$

(9) Now $R_{wy}(t, t+\tau) = \overline{x(t)\,x(t+\tau) \cos[(\omega_0 + \delta)t + \theta + \theta'] \cos(\omega_0 t + \omega_0 \tau + \theta)}$

$$= \tfrac{1}{2} R_x(\tau)\, \overline{\cos(\delta t + \theta' - \omega_0 \tau)} + \tfrac{1}{2} R_x(\tau)\, \overline{\cos[(2\omega_0 + \delta)t + \omega_0 \tau + 2\theta + \theta']}$$

$$= 0$$

(10) $R_{yw}(t, t+\tau) = \overline{x(t)\,x(t+\tau) \cos(\omega_0 t + \theta) \cos[(\omega_0 + \delta)(t+\tau) + \theta + \theta']}$

$$= \tfrac{1}{2} R_x(\tau)\, \overline{\cos(\delta t + \delta \tau + \omega_0 \tau + \theta')} + \tfrac{1}{2} R_x(\tau)\, \overline{\cos[\omega_0 t + (\omega_0 + \delta)(t+\tau) + 2\theta + \theta']}$$

$$= 0$$

(11) From (7): $R_z(t, t+\tau) = R_w(\tau) + R_y(\tau) = \tfrac{1}{2} R_x(\tau)[\cos(\omega_0 + \delta)\tau + \cos \omega_0 \tau]$

$$= R_z(\tau)$$ ✓

and thus $z(t) = w(t) + y(t)$ is wide-sense stationary.

Assignment #3
Continued

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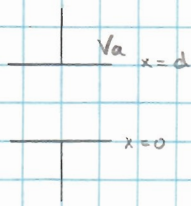
17. (1) The anode current pulse in a temperature limited parallel plate diode as a function of the transit time for one electron is well known and given by:

$$m\bar{a} = e \,\mathrm{grad}\, V \; ; \quad \frac{d^2x}{dt^2} = \frac{e}{m}\frac{V_a}{d}$$



$$x = \frac{eV_a}{2md} t^2 \; ; \quad \tau_a = \left(\frac{2m}{eV_a}\right)^{1/2} d$$

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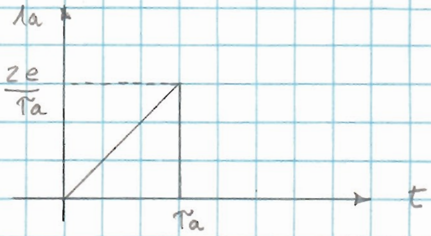
$$v = \frac{eV_a}{md} t \; ; \quad v_a = \frac{eV_a}{md} \tau_a$$

$J = nev$; $i = nevA$, one electron in unit area per unit volume

$$i = \frac{ev}{d} = \frac{e^2 V_a}{md^2} t \; , \quad \frac{eV_a}{2m} = \frac{d^2}{\tau_a^2} \; , \quad \frac{eV_a}{md^2} = \frac{2}{\tau_a^2}$$

$$\therefore \; i_a = \frac{2e}{\tau_a^2} t$$



(2) $$R_i(\tau) = \bar{n} \int_{-\infty}^{+\infty} i_a(t)\, i_a(t+\tau)\, dt$$ ✓

where $\bar{n}$ is the average number of electrons per second, $\bar{n} = \frac{\bar{I}}{e}$

(3) $$R_i(\tau) = \frac{\bar{I}}{e} \int_0^{\tau_a} \frac{2e}{\tau_a^2} t \cdot \frac{2e}{\tau_a^2} (t+\tau)\, dt$$

$$= \frac{4\bar{I}e}{\tau_a^4} \int_0^{\tau_a - \tau} (t^2 + \tau t)\, dt = \frac{4\bar{I}e}{\tau_a^4} \left(\frac{t^3}{3} + \tau\frac{t^2}{2}\right)\Bigg]_0^{\tau_a - \tau}$$

$$= \frac{4\bar{I}e}{\tau_a^4}\left[\frac{\tau_a^3}{3} - \tau\tau_a^2 + \tau_a\tau^2 - \frac{\tau^3}{3} + \frac{\tau\tau_a^2}{2} - \tau^2\tau_a + \frac{\tau^3}{2}\right]$$

$$= \frac{4e\bar{I}}{\tau_a^4}\left[\frac{\tau_a^3}{3} - \frac{\tau\tau_a^2}{2} + \frac{\tau^3}{6}\right] = \frac{4}{3}\frac{e\bar{I}}{\tau_a}\left[1 - \frac{3\tau}{2\tau_a} + \frac{\tau^3}{2\tau_a^3}\right]$$

for $0 < \tau < \tau_a$ and 0 otherwise because the currents due to different electrons are uncorrelated, and each electron's motion is independent of the others.

should also consider
τ negative

(4) $$G^+(f) = 4\int_0^\infty R_a(\tau)\cos 2\pi f\tau\, d\tau$$

$$= \frac{16\, e\bar{I}}{\tau_a^4}\int_0^{\tau_a}\left[\frac{\tau_a^3}{3} - \frac{\tau\tau_a^2}{2} + \frac{\tau^3}{6}\right]\cos 2\pi f\tau\, d\tau$$

(5) $$\frac{\tau_a^3}{3}\int_0^{\tau_a}\cos 2\pi f\tau\, d\tau = \frac{\tau_a^3}{3\cdot 2\pi f}\sin 2\pi f\tau_a$$

(6) $$\frac{\tau_a^2}{2}\int_0^{\tau_a}\tau\cos 2\pi f\tau\, d\tau = \frac{\tau_a^2}{2}\left[\frac{\cos 2\pi f\tau_a + 2\pi f\tau_a\sin 2\pi f\tau_a - 1}{4\pi^2 f^2}\right]$$

(7) $$\frac{1}{6}\int_0^{\tau_a}\tau^3\cos 2\pi f\tau\, d\tau = \frac{1}{6}\frac{\tau_a^3}{2\pi f}\sin 2\pi f\tau_a - \frac{3}{6\cdot 2\pi f}\int_0^{\tau_a} x^2\sin 2\pi f\tau\, d\tau$$

$$= \frac{1}{6}\frac{\tau_a^3}{2\pi f}\sin 2\pi f\tau_a - \frac{3}{6\cdot 2\pi f}\left[\frac{4\pi f\tau_a\sin 2\pi f\tau_a - (4\pi^2 f^2\tau_a^2 - 2)\cos 2\pi f\tau_a - 2}{(2\pi f)^3}\right]$$

$$= \Big[\tfrac{1}{6}(2\pi f\tau_a)^3\sin 2\pi f\tau_a - 2\pi f\tau_a\sin 2\pi f\tau_a + \tfrac{1}{2}(2\pi f\tau_a)^2\cos 2\pi f\tau_a$$
$$- \cos 2\pi f\tau_a + 1\Big](2\pi f)^{-4}$$

(8) (6) → $$\frac{\frac{1}{2}(2\pi f\tau_a)^2\cos 2\pi f\tau_a + \frac{1}{2}(2\pi f\tau_a)^3\sin 2\pi f\tau_a - \frac{1}{2}(2\pi f\tau_a)^2}{(2\pi f)^4}$$

(9) $$\Big[-\tfrac{1}{3}(2\pi f\tau_a)^3\sin 2\pi f\tau_a - 2\pi f\tau_a\sin 2\pi f\tau_a - \cos 2\pi f\tau_a + 1 + \tfrac{1}{2}(2\pi f\tau_a)^2$$
$$+ \tfrac{1}{3}(2\pi f\tau_a)^3\sin 2\pi f\tau_a\Big](2\pi f)^{-4}$$

(10) $$G^+(f) = \frac{16\, e\bar{I}}{(2\pi f\tau_a)^4}\left[\tfrac{1}{2}(2\pi f\tau_a)^2 + 1 - \cos 2\pi f\tau_a - 2\pi f\tau_a\sin 2\pi f\tau_a\right]$$

$$= \frac{8\, e\bar{I}}{(2\pi f\tau_a)^4}\left[(2\pi f\tau_a)^2 + 2 - 2\cos 2\pi f\tau_a - 4\pi f\tau_a\sin 2\pi f\tau_a\right]$$

Assignment #3
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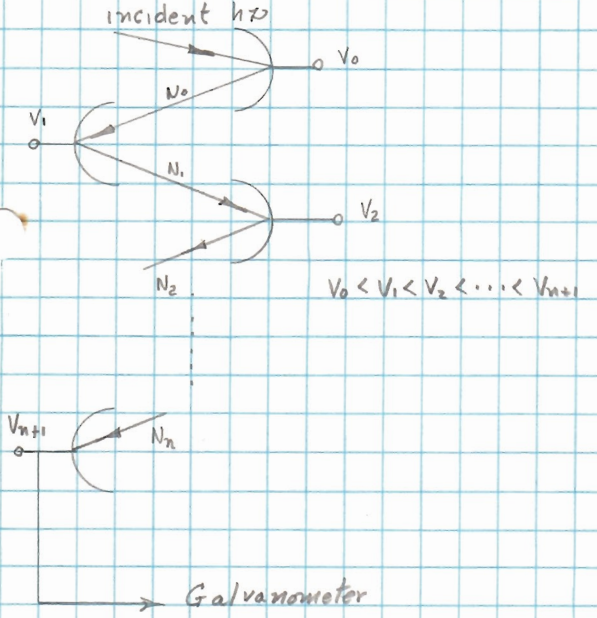
18. (1) The basic relation between the mean square deviation of the number of secondary electrons emitted from a surface and the mean square deviation in the incident electrons in terms of an average multiplicative constant $\bar{p}$ and its mean square deviation $\overline{\Delta p^2}$ is:

$$\overline{\Delta N^2_{sec}} = \overline{\Delta N^2_{pri}}\ \bar{p}^2 + \bar{N}_{pri}\ \overline{\Delta p^2}$$

with $\bar{N}_{sec} = \bar{N}_{pri}\ \bar{p}$

Let us construct the following model for a photomultiplier:



(2) We may write for equations beginning with the 0th anode:

$$\overline{\Delta N_1^2} = \overline{\Delta N_0^2}\ \bar{p}^2 + \bar{N}_0\ \overline{\Delta p^2}$$

$$\overline{\Delta N_2^2} = \overline{\Delta N_1^2}\ \bar{p}^2 + \bar{N}_1\ \overline{\Delta p^2}$$

$$= \overline{\Delta N_0^2}\ \bar{p}^4 + \bar{N}_0\ \overline{\Delta p^2}\ \bar{p}^2 + \bar{N}_1\ \overline{\Delta p^2}$$

$$\overline{\Delta N_3^2} = \overline{\Delta N_2^2}\ \bar{p}^2 + \bar{N}_2\ \overline{\Delta p^2}$$

$$= \overline{\Delta N_0^2}\ \bar{p}^6 + \bar{N}_0\ \overline{\Delta p^2}\ \bar{p}^4 + \bar{N}_1\ \overline{\Delta p^2}\ p^2 + \bar{N}_2\ \overline{\Delta p^2}$$

$$\overline{\Delta N_4^2} = \overline{\Delta N_3^2}\ \bar{p}^2 + \bar{N}_3\ \overline{\Delta p^2}$$

$$= \overline{\Delta N_0^2}\ \bar{p}^8 + \bar{N}_0\ \overline{\Delta p^2}\ \bar{p}^6 + \bar{N}_1\ \overline{\Delta p^2}\ \bar{p}^4 + \bar{N}_2\ \overline{\Delta p^2}\ \bar{p}^2 + \bar{N}_3\ \overline{\Delta p^2}$$

(3)
$$\overline{\Delta N_n^2} = \overline{\Delta N_0^2}\ \bar{p}^{2n} + \left(\bar{N}_0\ \bar{p}^{2n-2} + \bar{N}_1\ \bar{p}^{2n-4} + \bar{N}_3\ \bar{p}^{2n-6} + \cdots \right.$$
$$\left. \cdots + \bar{N}_{n-3}\ \bar{p}^4 + \bar{N}_{n-2}\ \bar{p}^2 + \bar{N}_{n-1}\right) \overline{\Delta p^2}$$
$$= \overline{\Delta N_0^2}\ \bar{p}^{2n} + \overline{\Delta p^2} \sum_{k=0}^{n-1} \bar{N}_k\ \bar{p}^{2n-2-2k}$$

(4) Now:
$\bar{N}_1 = \bar{p}\ \bar{N}_0$
$\bar{N}_2 = \bar{p}\ \bar{N}_1 = \bar{p}^2\ \bar{N}_0$
$\bar{N}_3 = \bar{p}\ \bar{N}_2 = \bar{p}^3\ \bar{N}_0$
$\vdots$
$\bar{N}_n = \bar{p}^n\ \bar{N}_0$

(5) $$\overline{\Delta N_n^2} = \overline{\Delta N_0^2}\,\bar{p}^{2n} + \bar{N}_0\,\bar{p}^{2n-2}\,\overline{\Delta p^2}\sum_{k=0}^{n-1}\bar{p}^{-k}$$

$$= \bar{p}^{2n}\left[\overline{\Delta N_0^2} + \bar{N}_0\,\overline{\Delta \bar{p}^2}\,\bar{p}^{-2}\left\{\frac{1-\left(\frac{1}{\bar{p}}\right)^n}{1-\left(\frac{1}{\bar{p}}\right)}\right\}\right]$$

$$= \bar{p}^{2n}\left[\overline{\Delta N_0^2} + \bar{N}_0\,\overline{\Delta p^2}\left\{\frac{1-(\bar{p})^{-n}}{\bar{p}^2-\bar{p}}\right\}\right]$$

(6) We now divide by $\bar{N}_n^2 = \bar{p}^{2n}\,\bar{N}_0^2$:

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$$\frac{\overline{\Delta N_n^2}}{\bar{N}_n^2} = \frac{\overline{\Delta N_0^2}}{\bar{N}_0^2} + \frac{\overline{\Delta p^2}}{\bar{N}_0}\left\{\frac{1-(\bar{p})^{-n}}{\bar{p}^2-\bar{p}}\right\}$$

In the limit of large n :

(7) $$\frac{\overline{\Delta N_n^2}}{\bar{N}_n^2} = \frac{\overline{\Delta N_0^2}}{\bar{N}_0^2} + \frac{\overline{\Delta p^2}}{\bar{N}_0\,\bar{p}}\left\{\frac{1}{\bar{p}-1}\right\}$$

Rearranging:

(8) $$\frac{\left[\overline{\Delta N_n^2}\right]^{1/2}}{\bar{N}_n} \Big/ \frac{\left[\overline{\Delta N_0^2}\right]^{1/2}}{\bar{N}_0} = \left[\frac{\overline{\Delta N_1^2} - \bar{p}\,\overline{\Delta N_0^2}}{\overline{\Delta N_0^2}\,\bar{p}\,(\bar{p}-1)}\right]^{1/2}$$

$$= \left[\frac{\overline{\Delta N_1^2}}{\overline{\Delta N_0^2}\,\bar{p}} - 1\right]^{1/2}\left[\frac{1}{\bar{p}-1}\right]^{1/2}$$

This ratio is the same for the current. It seem as if a maximum develops in the ratio of the relative fluctuations for $2 > \bar{p} > 1$.

Assignment # 4
Problems 19 - 24

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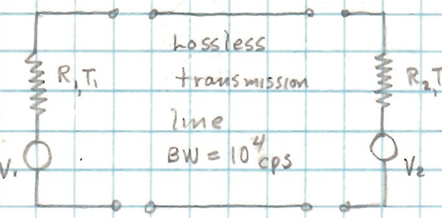
19.



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(1) Power delivered by V_1 into R_1 $= \dfrac{V_1^2 \; R_2^2}{(R_1 + R_2)^2} \cdot \dfrac{1}{R_2}$

$$= G_{V_1}(f) \, \Delta f \, \frac{R_2}{(R_1+R_2)^2}$$

$$= \frac{4 k T_1 R_1 R_2 \Delta f}{(R_1+R_2)^2}$$

(2) Similarly, the power delivered by V_2 into R_2 is:

$$\frac{4 k T_2 R_2 R_1 \Delta f}{(R_1+R_2)^2}$$

(3) If $T_2 < T_1$, The net power delivered to R_2 is $P = \dfrac{4 k R_2 R_1 \Delta f}{(R_1 + R_2)^2} (T_1 - T_2) = \dfrac{4 k R_1 R_2 \Delta f \, \Delta T}{(R_1+R_2)^2}$

(4) Now $\underset{\uparrow}{2} Q = mc \, \Delta T$; $P = -J \dfrac{dQ}{dt} = -\dfrac{Jmc}{2} \dfrac{d\,(\Delta T)}{dt}$

because of two resistors (-) because Q decreases as time increases.

(5) $$\left[\frac{d}{dt} + \frac{8 k R_1 R_2 \Delta f}{J mc \, (R_1+R_2)^2} \right] \Delta T = 0$$

(6) $$\Delta T = K \exp\left[- \frac{8 k R_1 R_2 \Delta f}{J mc \, (R_1+R_2)^2} \right] t = K e^{-t/\tau}$$

(7) In this problem, $R_1 = R_2$, $\Delta f = 10^4$ cps, $mc = 10^{-1}$ cal/degree, $J = 4.186$ joules/cal., $k = 1.38 \cdot 10^{-23}$ joule/deg.

$\therefore P = k \Delta f \, \Delta T = (1.38 \cdot 10^{-23})(10^4) \Delta T = 1.38 \cdot 10^{-19} \Delta T$ watts

(8) $$\tau = \frac{J mc \, (R_1+R_2)^2}{8 k R_1 R_2 \Delta f} = \frac{J mc}{2 k \Delta f}$$

$$= \frac{(4.186)(10^{-1})}{2\,(1.38 \cdot 10^{-19})} \; \frac{(\text{joule/cal})(\text{cal/deg})}{(\text{joule/deg})(\text{sec}^{-1})} = 1.51 \cdot 10^{18} \text{ sec}$$

$$= (1.51 \cdot 10^{18})(3.169 \cdot 10^{-8}) = 4.8 \cdot 10^{10} \text{ years}$$

Remarks: It would take $4.8 \cdot 10^{10}$ years for the temperature difference to decrease to 36% of its original value. For $K = 100°K$, $4.8 \cdot 10^{10}$ years later, ΔT would be $36°K$. This shows the tremendously long time it would take for thermal equilibrium to be set up by radiation alone. Also, even for $\Delta T = 10^{6} °K$, only $1.38 \cdot 10^{-13}$ would be transferred. This is fortunate, as it shows that the universe is proceeding towards equilibrium at a rate slow enough such that we will not be bothered by a large increase in entropy during our lifetimes.

Assignment # 4
Continued

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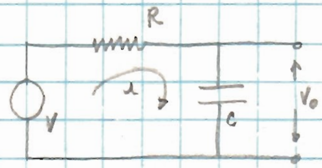
20. Consider the following simple circuit consisting of a resistor at temperature T and a capacitor.



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$G^+_V = 4kRT$ $\qquad$ $G^+_I = \frac{4kT}{R}$

(1) $I = V_0\left(sC + \frac{1}{R}\right) = V_0\left(\frac{sRC+1}{R}\right) = V_0\left(\frac{s+\frac{1}{RC}}{1/C}\right)$

(2) $V_0 = \left(\frac{1/C}{s+1/RC}\right) I \rightarrow \left(\frac{1/C}{j\omega+\omega_0}\right) I \; ; \quad \omega_0 = \frac{1}{RC}$

(3) $G^+_{V_0} = \frac{1/C^2}{\omega^2+\omega_0^2} G^+_I = \frac{\omega_0^2}{\omega^2+\omega_0^2} \cdot 4kTR$

(4) $\overline{V_0(t)\,V_0(t+\tau)} = \frac{1}{2\pi}\int_0^\infty G^+_{V_0}\cos\omega\tau\, d\omega = \frac{2kTR\,\omega_0^2}{\pi}\int_0^\infty \frac{\cos\omega\tau\, d\omega}{\omega^2+\omega_0^2}$

(5) Consider $\int_{-\infty}^{\infty}\frac{e^{mix}\,dx}{x^2+a^2} = 2\pi i \sum R^+\left[\frac{e^{miz}}{z^2+a^2}\right]$

$\lim_{z\to ia}\left\{\frac{e^{miz}}{z+ia}\right\} = \frac{e^{miai}}{2ia} = \frac{e^{-ma}}{2ia}$

(6) $\int_{-\infty}^{\infty}\frac{e^{mix}}{x^2+a^2}\,dx = \frac{\pi e^{-ma}}{a} = \int_{-\infty}^{\infty}\frac{\cos mx\, dx}{z^2+a^2}$

(7) $\therefore \int_0^\infty \frac{\cos\omega\tau\, d\omega}{\omega^2+\omega_0^2} = \frac{\pi e^{-\tau\omega_0}}{2\,\omega_0}$

(8) $\overline{V_0(t)\,V_0(t+\tau)} = kTR\,\omega_0\, e^{-\tau\omega_0}$

(9) $i = \frac{V}{R+1/sC} = \frac{sC\,V}{sRC+1} = \frac{s/R\;V}{s+\frac{1}{RC}} \rightarrow \frac{\frac{j\omega}{R}V}{j\omega+\omega_0}$

$= j\omega C\, V_0 = \frac{j\omega}{j\omega+\omega_0} I$

(10) $$\overline{i V_0^*} = \frac{j\omega/c}{\omega^2 + \omega_0^2} \quad \overline{I^2} = \frac{j\omega/c}{\omega^2 + \omega_0^2} \; \frac{4kT}{R} \; \frac{d\omega}{2\pi}$$

$$= \frac{j\,2kT\omega_0}{\pi} \cdot \frac{\omega}{\omega^2 + \omega_0^2} \; d\omega$$

The current and voltage are correlated as they are both functions of the same noise source.

This can also be shown more formally by the following:

(11) $$R_{iV_0}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-\infty}^{\infty} i(t)\, V_0^*(t+\tau)\, dt$$

$$= \lim_{T \to \infty} \frac{1}{T} \int_{-\infty}^{\infty} C \iint_{-\infty}^{\infty} -j\omega\, S(\omega)\, S^*(\omega')\, e^{j\omega'\tau} e^{j(\omega'-\omega)t} \frac{d\omega\, d\omega'\, dt}{2\pi}$$

$$= \lim_{T \to \infty} \frac{C}{2\pi T} (-j) \iint_{-\infty}^{\infty} \omega\, S(\omega)\, S^*(\omega')\, e^{j\omega'\tau}\, \delta(\omega'-\omega)\, d\omega\, d\omega'$$

$$= \frac{-jC}{4\pi} \int_{-\infty}^{\infty} \omega\, G_{V_0}^{+}\, e^{j\omega\tau}\, d\omega = \frac{-jC\, 4kTR\, \omega_0^2}{4\pi} \int_{-\infty}^{\infty} \frac{\omega\, e^{j\omega\tau}}{\omega^2 + \omega_0^2}\, d\omega$$

$$= (2\pi j)\left(\frac{-j\, kT\, \omega_0}{\pi}\right) \lim_{\omega \to j\omega_0} \left\{ \frac{\omega\, e^{j\omega\tau}}{\omega + j\omega_0} \right\}$$

$$= kT\omega_0\, e^{-\omega_0 \tau}$$

so that the voltage and current are correlated.

Assignment #4
Continued

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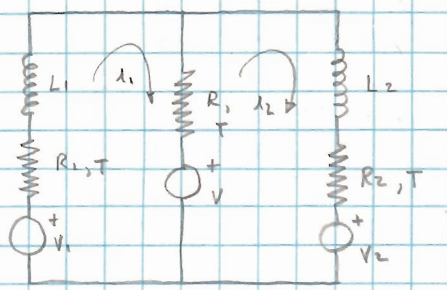
21.



we have replaced all resistors by noiseless resistors and noise generators. The power spectral densities are:

$G^+_{V_1} = 4R_1kT$; $G^+_V = 4RkT$; $G^+_{V_2} = 4R_2kT$
defined as wide PSD, independent of f

We use Laplace transform analysis throughout:

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(1) $$\begin{vmatrix} R_1 + R + sL_1 & -R \\ -R & R + R_2 + sL_2 \end{vmatrix} \begin{vmatrix} i_1 \\ i_2 \end{vmatrix} = \begin{vmatrix} V_1 - V \\ V - V_2 \end{vmatrix}$$

(2) $$\Delta = R_1R + R_2R_1 + sR_1L_2 + RR_2 + sL_2R + sL_1R + sL_1R_2 + s^2L_1L_2$$

$$= L_1L_2s^2 + (R_1L_2 + L_2R + L_1R + L_1R_2)s + (R_1R + R_2R_1 + RR_2)$$

(3) $$i_1 = \frac{(V_1 - V)(R + R_2 + sL_2) + R(V - V_2)}{\Delta}$$

$$= \frac{(R + R_2 + sL_2)}{\Delta} V_1 - \frac{R}{\Delta} V_2 - \frac{(R_2 + sL_2)}{\Delta} V$$

(4) $$i_2 = \frac{(V_1 - V_2)(R_1 + R + sL_1) + R(V_1 - V)}{\Delta}$$

$$= \frac{R}{\Delta} V_1 - \frac{(R_1 + R + sL_1)}{\Delta} V_2 + \frac{(R_1 + sL_1)}{\Delta} V$$

(5) In the steady state, $s \to j\omega$:

$$\Delta \to + j\omega(R_1L_2 + L_2R + L_1R + L_1R_2) + (R_1R + R_2R_1 + RR_2 - \omega^2 L_1L_2)$$

$$i_1 = \frac{(R + R_2) + j\omega L_2}{\Delta} V_1 - \frac{R}{\Delta} V_2 - \frac{(R_2 + j\omega L_2)}{\Delta} V$$

$$i_2 = \frac{R}{\Delta} V_1 - \frac{(R_1 + R) + j\omega L_1}{\Delta} V_2 + \frac{(R_1 + j\omega L_1)}{\Delta} V$$

(6) As the noise sources are all statistically independent, we use the the theorem for the PSD:

$$G_i^+ = \sum_k |Y_k|^2 \, G_{V_k}^+$$

(7) $$G_{i_1}^+ = 4kT\left[\left\{\frac{(R+R_2)^2+\omega^2 L_2^2}{|\Delta|^2}\right\}R_1 + \left\{\frac{R^2}{|\Delta|^2}\right\}R_2 + \left\{\frac{R_2^2+\omega^2 L_2^2}{|\Delta|^2}\right\}R\right]$$

$$G_{i_2}^+ = 4kT\left[\left\{\frac{R^2}{|\Delta|^2}\right\}R_1 + \left\{\frac{(R_1+R)^2+\omega^2 L_1^2}{|\Delta|^2}\right\}R_2 + \left\{\frac{R_1^2+\omega^2 L_1^2}{|\Delta|^2}\right\}R\right]$$

where $|\Delta|^2 = (R_1R + R_2R_1 + RR_2 - \omega^2 L_1 L_2)^2 + \omega^2(R_1L_2 + L_2R + L_1R + L_1R_2)^2$

(8) Now $\overline{i_1^2} = \frac{1}{2\pi}\int_0^\infty G_{i_1}^+ \, d\omega$, $G_{i_1}^+$ as above
$\overline{i_2^2} = \frac{1}{2\pi}\int_0^\infty G_{i_2}^+ \, d\omega$, $G_{i_2}^+$ as above
} solve by contour integration

(9) If we form $\overline{i_1 i_2^*}$ from equations (5), we see that we involve auto and cross-correlations among the voltages. Of course the cross-correlations vanish as the noise sources are statistically independent; however, the terms of $\overline{V^2}$, $\overline{V_1^2}$, and $\overline{V_2^2}$ remain so the currents are not statistically independent as one can guess intuitively.

(10) As $R \to 0$, $V \to 0$ (no more noise generated): $\overline{i_1 i_2^*} = \frac{(R_2 + j\omega L_2)(R_1 - j\omega L_1)}{|\Delta(R\to 0)|^2}\,\overline{V_1 V_2} = 0$

so the currents become statistically independent, as one expects since a short circuit develops across the network.

$$\overline{i_1^2} = \frac{2kTR_1}{\pi}\int_0^\infty \frac{R_2^2+\omega^2 L_2^2}{|\Delta(R\to 0)|^2}\,d\omega, \qquad \overline{i_2^2} = \frac{2kTR_2}{\pi}\int_0^\infty \frac{R_1^2+\omega^2 L_1^2}{|\Delta(R\to 0)|^2}\,d\omega$$

with $|\Delta(R\to 0)|^2 = (R_1R_2 - \omega^2 L_1 L_2)^2 + \omega^2(R_1L_2 + L_1R_2)^2$

(11) As $R_1, R_2 \to 0$, $V_1, V_2 \to 0$; $|\Delta(R_1,R_2\to 0)|^2 = \omega^4 L_1^2 L_2^2 + \omega^2 R^2 (L_1+L_2)^2$

$$\overline{i_1^2} = \frac{2kTR}{\pi}\int_0^\infty \frac{\omega^2 L_2^2\,d\omega}{|\Delta(R_1,R_2\to 0)|^2}\;; \qquad \overline{i_2^2} = \frac{2kTR}{\pi}\int_0^\infty \frac{\omega^2 L_1^2\,d\omega}{|\Delta(R_1,R_2\to 0)|^2}$$

$$\overline{i_1 i_2^*} = \frac{\omega^2 L_1 L_2}{|\Delta(R_1,R_2\to 0)|^2}\,\overline{V^2} \neq 0$$

so thus there is a correlation between the currents which one would expect they are both supplied by the same noise source.

ASSIGNMENT #4
Continued

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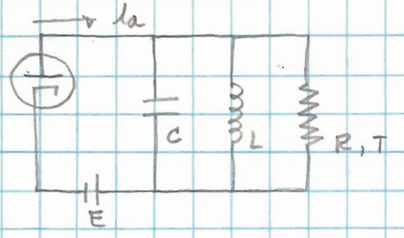
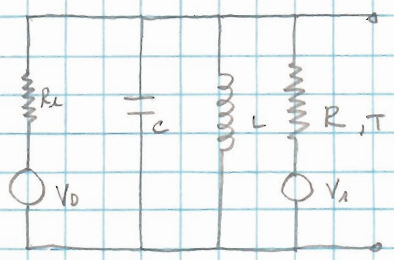
22.





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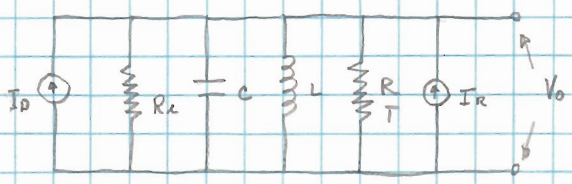


Small signal equivalent circuits. R_i = internal resistance of diode; noiseless. $G^+_{I_R} = \frac{4kT}{R}$

$G^+_{I_D} = 2e\bar{I}_a$

(1) $$\left[\left(\frac{1}{R_i}+\frac{1}{R}\right)+sC+\frac{1}{sL}\right]V_o = I_D + I_R$$

(2) Define $$G = \frac{1}{R_i}+\frac{1}{R} = \frac{1}{R'} = \frac{R+R_i}{RR_i}$$

(3) $$\left[\frac{1}{R'}+sC+\frac{1}{sL}\right] = \frac{s^2+\frac{s}{R'C}+\frac{1}{LC}}{s/C}$$

$$V_o = (I_D+I_R)\left[\frac{s/C}{s^2+\frac{s}{R'C}+\frac{1}{LC}}\right]$$

$$\rightarrow (I_D+I_R)\left[\frac{j\omega/C}{\left(\frac{1}{LC}-\omega^2\right)+j\,\omega/R'C}\right]$$

(4) Define: $\omega_0^2 = \frac{1}{LC}$; $2\xi\omega_0 = \frac{1}{R'C}$; $\xi = \frac{1}{2R'}\sqrt{\frac{L}{C}}$

$$\therefore\ G^+_{V_o} = \left[\frac{2e\bar{I}_aR+4kT}{RC^2}\right]\left[\frac{\omega^2}{(\omega_0^2-\omega^2)^2+4\xi^2\omega_0^2\omega^2}\right]$$

$$= 2\pi A\left[\frac{\omega^2}{\omega^4+2\omega_0^2[2\xi^2-1]\omega+\omega_0}\right] ;\quad 2\pi A = \left[\frac{2e\bar{I}_aR+4kT}{RC^2}\right]$$

$$= 2\pi A\left[\frac{\omega^2}{(\omega^2-\alpha^2)(\omega^2-\beta^2)}\right] = 2\pi A\left[\frac{\omega^2}{(\omega+\alpha)(\omega-\alpha)(\omega+\beta)(\omega-\beta)}\right]$$

where $\alpha^2 = [1-2\xi^2]\omega_0^2 + j2\xi[1-\xi^2]^{1/2}\omega_0^2 = \rho e^{j\theta}$

$\beta^2 = (\alpha^2)^* = \rho e^{-j\theta}$

taking $\xi < 1$ for the more physically interesting underdamped case.

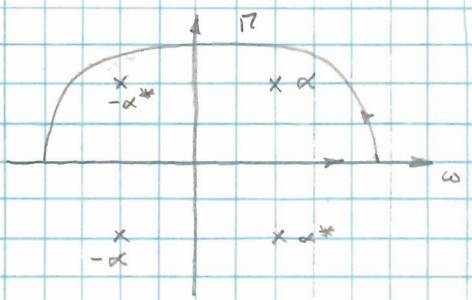
(5) $\Omega = \omega_0^2, \quad \theta = \tan^{-1} \dfrac{2\xi\left[1-\xi^2\right]^{1/2}}{1-2\xi^2}$

$\therefore \quad \alpha = \omega_0 e^{i\theta/2}$

$\beta = \omega_0 e^{-i\theta/2} = \alpha^*$

(6) Now: $$\overline{V_0^2} = A\int_0^\infty \frac{\omega^2\, d\omega}{(\omega-\alpha)(\omega+\alpha)(\omega-\beta)(\omega+\beta)} = \frac{A}{2}\int_{-\infty}^{\infty} \frac{\omega^2\, d\omega}{(\omega-\alpha)(\omega+\alpha)(\omega-\alpha^*)(\omega+\alpha^*)}$$

Taking residues: $$\int_\Gamma \frac{z^2\, dz}{(z-\alpha)(z+\alpha)(z-\alpha^*)(z+\alpha^*)}$$



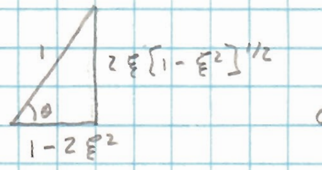
$$\lim_{z\to\alpha}\left\{\frac{z^2}{(z+\alpha)(z-\alpha^*)(z+\alpha^*)}\right\} \frac{\alpha}{2(\alpha^2-\alpha^{*2})}$$

$$= \frac{e^{i\theta/2}}{2\omega_0(e^{i\theta}-e^{-i\theta})} = \frac{e^{i\theta/2}}{4i\omega_0 \sin\theta}$$

$$\lim_{z\to-\alpha^*}\left\{\frac{z^2}{(z-\alpha)(z+\alpha)(z-\alpha^*)}\right\} = \frac{\alpha^*}{2(\alpha^2-\alpha^{*2})} = \frac{e^{-i\theta/2}}{4i\omega_0 \sin\theta}$$

$$\Sigma \text{ Residues} = \frac{1}{2i\omega_0}\frac{\cos\theta/2}{\sin\theta} = \frac{1}{4i\omega_0 \sin\theta/2}$$

(7) $\therefore \quad \overline{V_0^2} = \dfrac{\pi A}{4\omega_0 \sin\theta/2}$;



$\cos\theta = 1-2\xi^2$

$$\sin\frac{\theta}{2} = \left[\frac{1-\cos\theta}{2}\right]^{1/2}$$
$$= \xi$$

$$\therefore \quad \overline{V_0^2} = \frac{A\pi}{4\omega_0\xi} = \frac{1}{8\omega_0\xi}\left[\frac{2e\bar{I}_a R + 4kT}{RC^2}\right] = \frac{R'}{C}\left[e\bar{I}_a + \frac{2kT}{R}\right]$$

(8) The mean voltage out, $\overline{V_0}$, is zero as the inductor shorts the resistor and the average value of the noise voltage is zero. Therefore:

✓

(9) $$\overline{\Delta V_0^2} = \overline{V_0^2} = \frac{RR_1}{(R_1+R)C}\left[e\bar{I}_a + \frac{2kT}{R}\right]$$

✓ It is seen that this result is generally valid independent of ξ, so that the assumption of underdamping was only a crutch to do the calculation.

Assignment #4
Continued

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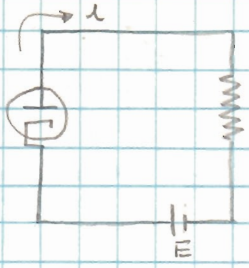
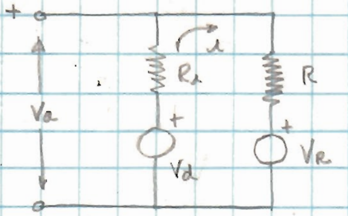
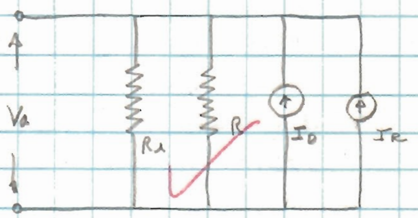
23.



Small signal circuit:





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(1) $V_a = V_d - iR_i \quad ; \quad V_d - V_R = i(R_i + R)$

(2) $$V_a = V_d - \left(\frac{V_d - V_R}{R_i + R}\right) R_i = V_d\left[1 - \frac{R_i}{R_i + R}\right] + \frac{V_R R_i}{R_i + R}$$

$$= \frac{V_d R}{R_i + R} + \frac{V_R R_i}{R_i + R}$$

(3) $$i = i_a e^{-eV_a/kT_c} \quad ; \quad \ln\frac{i}{i_a} = -\frac{eV_a}{kT_c}$$

$$V_a = -\frac{kT_c}{e}\ln\frac{i}{i_a}$$

(4) $$R_i = \left(\frac{\partial V_a}{\partial i}\right)_{i=\bar{i}} = -\frac{\cancel{i_a}\, kT_c}{\bar{i}e}$$

(5) Writing the node equation:

$$I_0 + I_R = \left(\frac{1}{R_i} + \frac{1}{R}\right) V_a \quad ; \quad V_a = \frac{R_i R}{R + R_i}\left[I_0 + I_R\right]$$

(6) $$S^{+}_{V_a} = \left(\frac{R_i R}{R + R_i}\right)^2 \left[S^{+}_{I_0} + S^{+}_{I_R}\right] = \left(\frac{R_i R}{R + R_i}\right)^2 \left[2e\bar{i}\,\Gamma^2 + \frac{4kT_R}{R}\right]$$

$$\underset{\frac{2kT_c}{R_i}}{\parallel}$$

$$= \left[\frac{-\frac{i_a kT_c R}{\bar{i}e}}{R - \frac{i_a kT_c}{\bar{i}e}}\right]^2 \left[2e\bar{i}\,\Gamma^2 + \frac{4kT_R}{R}\right]$$

$$= \left[\frac{i_a kT_c R}{\bar{i}eR - i_a kT_c}\right]^2 \left[2e\bar{i}\,\Gamma^2 + \frac{4kT_R}{R}\right]$$

(7) $$\overline{V_a^2} = \left[\frac{i_a kT_c R}{\bar{i}eR - i_a kT_c}\right]^2 \left[2e\bar{i}\,\Gamma^2 + \frac{4kT_R}{R}\right] \Delta f$$

(8) $\overline{V}^{2} = 0 \quad , \text{ as } \overline{I_d} = 0 \, , \overline{I_R} = 0 \quad , \overline{I_d I_R} = 0$

(9) $\therefore \quad \overline{\Delta V_a^2} = \overline{V_a^2} - \overline{V_a}^{2} = \overline{V_a^2} = \left[\frac{\lambda_a k T_c R}{\bar{i} e R - \lambda_a k T_c} \right]^2 \left[2 e \bar{i} \Gamma^2 + \frac{4 k T_R}{R} \right] \Delta f$

As there are no reactive components in the circuit, the integral of $S^{+}_{V_a}$ cannot be taken unless one specifies a desired passband of frequencies Δf as above which can be thought of as the integral over the passband Δf.

ASSIGNMENT #4
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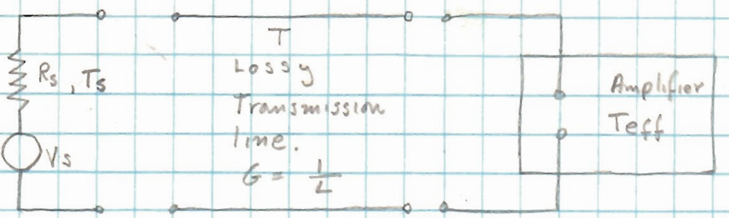
24.



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(1) The noise power at the input to the amplifier is:

$$G k T_s \Delta f + (1 - G) k T \Delta f + k T_{eff} \Delta f$$

(2) We can now reflect this back to the source via the transfer function of the transmission line and find the noise power at the source:

$$\frac{G k T_s \Delta f + (1 - G) k T \Delta f + k T_{eff} \Delta f}{G^2}$$

(3) We now define an effective noise temperature at the source, T', that is:

$$\Delta f \, k T' = \frac{G k T_s \Delta f + (1 - G) k T \Delta f + k T_{eff} \Delta f}{G^2}$$

or
$$T' = \frac{G T_s + (1 - G) T + T_{eff}}{G^2}$$

$$= \left\{ T_s/L + (1 - 1/L) T + T_{eff} \right\} L$$

60

Assignment #5
Problems 25-31

Paul Grant
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25. (1) The Fokker-Planck equation for Brownian motion in a gravitational field is:

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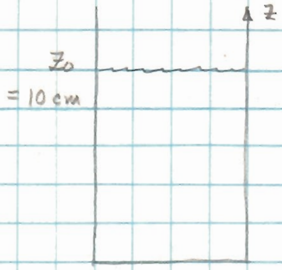
$$\frac{\partial W}{\partial t} = c\,\frac{\partial W}{\partial z} + D\,\frac{\partial^2 W}{\partial z^2}$$

where $C = \dfrac{\frac{4\pi a^3}{3}(\rho-\rho_0)g}{\beta}$



$D = \dfrac{kT}{\beta}$, $\beta = 6\pi\eta a$

(2) We seek the steady state solution of:

$$D\,\frac{d^2W}{dz^2} + c\,\frac{dW}{dz} = 0$$

(3) $W = n\,e^{-\frac{c}{D}z}$ which is the equilibrium distribution.

Normalizing, $\int_0^\infty W\,dz = n\int_0^\infty e^{-\frac{c}{D}z}\,dz = n\frac{D}{c} = 1$

or $W = \frac{c}{D}\,e^{-c/D\,z}$

(4) We now make the physical argument that we initially assume the height of the liquid to be infinite and let the particles come to equilibrium. We then have a 1/e probability for a particle to be at z_0. If we remove the liquid above z_0, this probability will remain the same, that is, a particle will have a probability of .36 of being at z_0. We take this for the purpose of finding the order of magnitude of the maximum radius of the particles:

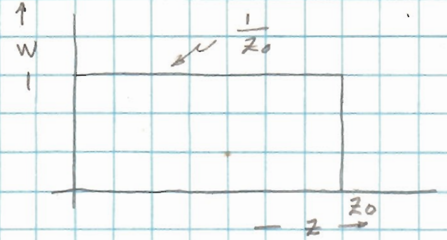
$$z_0 = \frac{D}{c} = \frac{3kT}{4\pi a^3(\rho-\rho_0)g}$$

$$a^3 \sim \frac{3kT}{4\pi\, z_0\,(\rho-\rho_0)\,g} = \frac{(.239)(1.38\cdot 10^{-16})(3\cdot 10^{2})\ \text{erg}}{(10)(1.7)(9.81\cdot 10^{2})\ \frac{\text{cm}^2}{\text{sec}^2}\cdot\frac{\text{gm}}{\text{cm}^3}}$$

$$\doteq .6\cdot 10^{-18}\ \text{cm}^3 \sim 10^{-18}\ \text{cm}^3$$

$\therefore a \sim 10^{-6}$ cm, this approximation assumes an interchange across z_0 but is good enough for an order of magnitude.

(5) If the initial condition is a homogeneous suspension, w has the following form at $t=0$



Normalizing:

$$\int_0^{z_0} n\,dz = n z_0 = 1$$

$$\therefore \; W(t=0) = \frac{1}{z_0}, \quad 0 < z < z_0$$

$$, \quad 0 \quad \text{otherwise.}$$

Using the Fokker-Planck equation, $\frac{\partial W}{\partial t} = C\frac{dW}{dz} + D\frac{d^2W}{dz^2}$

we have $\frac{\partial W}{\partial t} = 0$ at $t=0$ as $\frac{\partial W}{\partial z} = \frac{\partial^2 W}{\partial z^2} = 0$ everywhere except at $z = z_0$, where the boundary cannot be crossed.

(6) Another way of thinking about this is to argue that we consider only times greater than the damping under which we obtain the terminal velocity. this average velocity will go to zero with time as equilibrium sedimentation is reached. Starting with the Langevin equation:

$$m\frac{dv}{dt} + \beta v = F(t) + F_G$$

Since we consider $t \gg \frac{m}{\beta}$: $\bar{v} = \frac{F_G}{\beta}$, neglecting $\frac{m}{\beta}\frac{dv}{dt}$

or $$\bar{v} = \frac{4/3\,\pi a^3(\rho-\rho_0)g}{6\pi\eta a} = \frac{2}{9}\frac{a^2(\rho-\rho_0)g}{\eta}$$

$$= \frac{(.222)(10^{-8})(1.7)(9.81\cdot 10^2)}{.9} = 4.12\cdot 10^{-6} \text{ cm/sec}$$

This can be considered as the initial sedimentation rate. We can convert this to more proper units by considering a flow density made up of the original number of particles per cc times the average velocity which will have the units of particles/cm² sec:

$\therefore$ Sedimentation Rate $= 4.12\cdot 10^{-6}\, n_0$ particles/cm²·sec

Assignment #5
Problems 25-31
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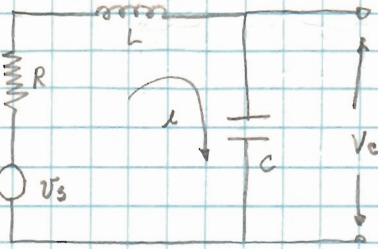
26.



(1) $G_{v_s} = 4 R k T$

10

(2) $v_s = i \left(sL + \frac{1}{sC} + R \right) i$

(3) $$i = \frac{s/L \; v_s}{s^2 + s\,R/L + \frac{1}{LC}}$$

$$= \frac{s/L \; v_s}{s^2 + 2 \zeta \omega_n s + \omega_n^2} \; ; \quad \omega_n = \frac{1}{\sqrt{LC}} \; , \quad \zeta = \frac{R}{2} \sqrt{\frac{C}{L}}$$

(4) Under critical damping $R = 2 \sqrt{\frac{L}{C}}$, $\therefore$ $\zeta = 1$

(5) $$i = \frac{v_s \cdot s/L}{(s + \omega_n)^2} \longrightarrow \frac{j\omega}{L} \frac{v_s}{(j\omega + \omega_n)^2}$$

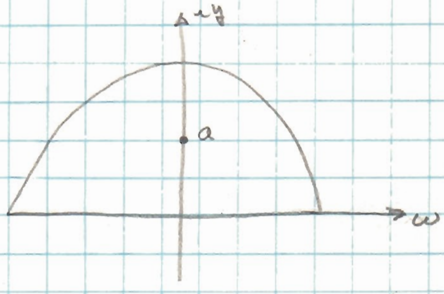
(6) $$G_i(\omega) \quad \frac{4kRT}{L^2} \frac{\omega^2}{(\omega^2 + \omega_n^2)^2}$$

(7) $$R_i(\tau) = \int_0^\infty G_i(f) \cos 2\pi f \tau \, df = \frac{1}{2\pi} \int_0^\infty G_i(\omega) \cos \omega \tau \, d\omega$$

$$= \frac{kRT}{\pi L^2} \int_{-\infty}^{\infty} \frac{\omega^2 \cos \omega \tau \, d\omega}{(\omega^2 + \omega_n^2)^2}$$

(8) Into the complex plane; consider

$$\int_C \frac{z^2 e^{imz} \, dz}{(z^2 + a^2)^2}$$



where C is:
There is a double pole at $z = ia$ inside the contour and the integral vanishes properly on the semicircle.

(9) The residue is: $$\lim_{z \to ia} \frac{d}{dz} \left\{ \frac{z^2 e^{imz}}{(z + ia)^2} \right\}$$

$$= \lim_{z \to ia} \Big] \frac{(z+ia)^2 \left\{ 2z e^{imz} + imz^2 e^{imz} \right\} - 2(z+ia) \cdot z^2 e^{imz}}{(z+ia)^4}$$

$$= \frac{-4a^2 \left\{ i2a e^{-ma} - i ma^2 e^{-ma} \right\} + 4ia \cdot \ a^2 e^{-ma}}{16 a^4}$$

$$= \frac{\left\{ -i2 - ima + i \right\}}{4a} e^{-ma} = \frac{1}{4a^2} (1 + ma) e^{-ma}$$

10. $\therefore$ $$\int_{-\infty}^{\infty} \frac{\omega^2 \cos \omega \tau \, d\omega}{(\omega^2 + \omega_n^2)^2} = \frac{\pi}{2\omega_n} (1 + \omega_n \tau) e^{-\omega_n \tau}$$

and $$R_i(\tau) = \frac{kRT}{2\omega_n L^2} (1 + \omega_n \tau) e^{-\omega_n \tau} \ ; \quad \omega_n \equiv \frac{1}{\sqrt{LC}}$$

sign wrong

$$= \frac{kT}{L} (1 + \omega_n \tau) e^{-\omega_n \tau}$$

(11) For the cross-correlation:

$$G_{iv_c}(f) = \lim_{T \to \infty} \frac{S_i(f) \, S^*_{v_c}(f)}{T}$$

$$S_i(f) = \int_{-\infty}^{\infty} \frac{1}{Z} v_s(t) e^{2\pi i f t} dt$$

$$S^*_{v_c}(f) = \int_{-\infty}^{\infty} A^* v^*_s(t') e^{-2\pi i f t'} dt'$$

$$S_i(f) \, S^*_{v_c}(f) = \frac{A^*}{Z} \iint_{-\infty}^{\infty} v_s(t) v^*_s(t') e^{2\pi i (t-t') f} dt \, dt'$$

$$= \frac{A^*}{Z} \int_{-\infty}^{\infty} e^{-2\pi i f \tau} \int_{-\infty}^{\infty} v(s) v^*_s(t+\tau) \, dt \, d\tau$$

letting $t' = t + \tau$

(12) $\therefore$ $$G_{iv_c}(f) = \lim_{T \to \infty} \frac{2 S_i(f) \, S^*_{v_c}(f)}{T}$$

$$= \frac{A^*}{Z} \int_{-\infty}^{\infty} e^{2\pi i f \tau} \underbrace{\left[\lim_{T \to \infty} \frac{1}{T} \int_0^T v_s(t) \, v^*_s(t+\tau) \, dt \right] d\tau}_{G_{v_c}(f)}$$

Assignment #5
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Problem 26
Continued

(13) $$G_{iv_c}(f) = \frac{A^*}{Z} G_{v_c}(f)$$

(14) $$\frac{1}{Z} = \frac{j\omega}{L} \frac{1}{(j\omega + \omega_n)^2}$$

$$v_c = \frac{1}{sC} \longrightarrow \frac{1}{j\omega C} = \frac{\omega_n^2 v_s}{(j\omega + \omega_n)^2}$$

$$A = \frac{v_c}{v_s} = \frac{\omega_n^2}{(j\omega + \omega_n)^2}$$

(15) $$G_{iv_c}(\omega) = \frac{j\omega_n^2}{L} \frac{\omega}{(\omega^2 + \omega_n^2)^2} \cdot 2kRT$$

(16) $$R_{iv_c}(\tau) = \frac{j\omega_n^2 kRT}{\pi L} \int_{-\infty}^{\infty} \frac{\omega e^{i\omega\tau} d\omega}{(\omega^2 + \omega_n^2)^2}$$

(17) Into the complex plane. The same conditions hold as before and the same contour is used:

$$\int_{-\infty}^{\infty} \frac{\omega e^{i\omega\tau} d\omega}{(\omega^2 + \omega_n^2)^2} = 2\pi j \sum R^+(I(z)) \quad ; \quad I(z) = \frac{z e^{imz}}{(z^2 + a^2)^2}$$

$$R^+ = \lim_{z \to ia} \left| \frac{d}{dz} \left\{ \frac{z e^{imz}}{(z + ia)^2} \right\} = \frac{(z+ia)^2 \{ e^{imz} + imze^{imz} \} - 2(z+ia) z e^{imz}}{(z+ia)^4} \right.$$

$$= \frac{-4a^2 \{ e^{-am} - mae^{-am} \} + 4a^2 e^{-am}}{16a^4} = \frac{me^{-am}}{4a}$$

Then: $$\int_{-\infty}^{\infty} \frac{\omega e^{j\omega\tau} d\omega}{(\omega^2 + \omega_n^2)^2} = \frac{j\pi\tau}{2\omega_n} e^{-\omega_n \tau}$$

(18) $$R_{iv_c}(\tau) = \frac{-\omega_n k R T \tau}{2L} e^{-\omega_n \tau} \quad ; \quad \omega_n \equiv \frac{1}{\sqrt{LC}}$$

$$= -\frac{kT}{LC} \tau \checkmark e^{-\omega_n \tau}$$

Please note that $R_{v_c i}(\tau) = -R_{iv_c}(\tau)$ because the second factor is pure imaginary and changes sign upon conjugating.

Assignment # 5
Continued

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27. (1) The following equations were derived in lecture for the fluctuation in carrier density in a semiconductor:

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$\alpha(n)$ = probability of absorption

$\beta(n)$ = probability of generation

Fluctuations in the steady state:

$$\overline{\Delta n^2(0)} = \frac{-\left(\alpha(n_0) + \beta(n_0)\right)}{2\left\{-\alpha'(n_0) + \beta'(n_0)\right\}}$$

Correlation Function:

$$\overline{\Delta n(0)\,\Delta n(\tau)} = \overline{\Delta n^2(0)}\; e^{-\tau/\tau_L}$$

where τ_L is the carrier lifetime:

$$\tau_L = \left[\alpha'(n_0) - \beta'(n_0)\right]^{-1}$$

(2) Consider the fluctuations in the current flowing in the following sample:

The current due to one electron during its lifetime is:

$$\frac{e}{A\tau_L}\cdot\frac{xA}{l} = \frac{e}{\tau_e}\,, \quad x = \mu E \tau_L$$

and $\frac{1}{\tau_e} = \frac{\mu_e E}{l}$ which is the transit time for electrons. Then, from the equations of the shot effect:

$$\bar{i} = n_0\,\frac{e}{\tau_e}\;; \qquad \overline{\Delta i^2} = \frac{e^2}{\tau_e^2}\,\Delta n^2$$

and

$$\overline{\Delta i(0)\,\Delta i(\tau)} = \frac{e^2}{\tau_e^2}\,\overline{\Delta n(0)\,\Delta n(\tau)}$$

is the correlation function.

(3) $\therefore \quad R_i(\tau) = \dfrac{e^2}{T_e^2}\,\overline{\Delta n^2(0)}\; e^{-\tau/\tau_{Le}}$

(4) $$G_i^+(\omega) = 4\int_0^\infty \frac{e^2}{T_e^2}\,\overline{\Delta n^2(0)}\,\cos\omega\tau\; e^{-\tau/\tau_{Le}}\,d\tau$$

$$= \frac{4\,e^2\,\overline{\Delta n^2(0)}}{T_e^2\,\tau_{Le}} \cdot \frac{1}{\frac{1}{\tau_{Le}^2} + \omega^2}$$

$$= \frac{4\,\tau_{Le}\,e^2\,\overline{\Delta n^2(0)}}{T_e^2\,(1+\tau_{Le}^2\omega^2)}$$

(5) It should now be pointed out that an exactly analogous relation holds for the holes; and the total PSD is:

$$G_i^+(\omega) = 4e^2\left\{\frac{\overline{\Delta n_e^2(0)}\;\tau_{Le}}{T_e^2\,(1+\tau_{Le}^2\omega^2)} + \frac{\overline{\Delta n_h^2(0)}\;\tau_{Lh}}{T_h^2\,(1+\tau_{Lh}^2\omega^2)}\right\}$$

(6) For an intrinsic semiconductor,

$\alpha(n) = a\,n^2 \;;\quad \beta(n) = b$ for both holes and electrons.

In the steady state: $a\,n_0^2 = b$, $a = \dfrac{b}{n_0^2}$; $\alpha(n_0) = b$

$$\overline{\Delta n^2(0)} = \frac{-2b}{2(-2b/n_0)} = \frac{1}{2}n_0$$

$$\tau_L = \left[\frac{2b}{n_0}\right]^{-1} = \frac{n_0}{2}\,b = b\,\overline{\Delta n^2(0)}$$

The lifetimes are the same for each carrier type (intrinsic). There will be a difference in the transit times due to the mobilities being different.

Define $$\frac{1}{T^2} = \frac{1}{T_e^2} + \frac{1}{T_h^2}$$

Then: $$G_i^+(\omega) = \frac{8e^2}{T^2}\;\frac{b\,n_0^2}{4}\;\frac{1}{\left(1+\frac{n_0^2b^2\omega^2}{4}\right)}$$

$$= \frac{8\,e^2\,b\,n_0^2}{T^2\left[4 + n_0^2b^2\omega^2\right]}$$

$$= \frac{8\,e^2}{T^2\,b\left[\frac{4}{n_0^2b^2} + \omega^2\right]}$$

Assignment #5
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28. (1) From a fundamental definition of mechanics, the impulse is equal to the change in momentum:

$$F_i = \int K_i \, dt_i = m(v_i'' - v_i')$$

(2) If the two impulses are consecutive, we have that the initial velocity before the second collision is equal to the final velocity after the first.

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$$F_j = \int K_j \, dt_j = m(v_j'' - v_j') = m(v_j'' - v_i'')$$

(3) The velocities before and after collision are taken to be given by independent Maxwell-Boltzmann distributions.

$$p(v_i', v_i'', v_j', v_j'') = p(v_i')\, p(v_i'')\, p(v_j')\, p(v_j'')$$

(4)
$$\overline{F_i F_j} = m^2 \iiiint p(v_i', v_i'', v_j', v_j'')\,(v_i'' v_j'' - v_i' v_j'' - v_j' v_i'' + v_i' v_j')\; dv_i'\, dv_i''\, dv_j'\, dv_j''$$

$$= m^2 \Big\{ \iint p(v_i'')\, p(v_j'')\, v_i'' v_j''\, dv_i''\, dv_j''$$

$$- \iint p(v_i')\, p(v_j'')\, v_i' v_j''\, dv_i'\, dv_j'' - \iint p(v_j')\, p(v_i'')\, v_j' v_i''\, dv_j'\, dv_i''$$

$$+ \iint p(v_i')\, p(v_j')\, v_i' v_j'\, dv_i'\, dv_j' \Big\}$$

(5) It is now clear that $\overline{F_i F_j} = 0$ if none of the velocities are equal, as each double integral has the same value as the others.

(6) If $v_j' = v_i''$:

$$p(v_i', v_i'', v_j'') = p(v_i')\, p(v_i'')\, p(v_j'')$$

where $p(v) \rightarrow f(p) = \frac{4\pi}{(2\pi m kT)^{3/2}}\, p^2 e^{-\frac{p^2}{2mkT}}$

(7) $\therefore \quad \overline{F_i F_j} = \iiint f(p_i')\, f(p_i'')\, f(p_j'')\, \left(p_i'' p_j'' + p_i'' p_i' - p_i' p_j'' - p_i''^{2} \right) \cdot dp_i'\, dp_i''\, dp_j''$

$$= \left[\int f(p)\, p\, dp \right]^2 - \int p^2 f(p)\, dp = -\sigma^2$$

The subscripts can be dropped as we are only interested in a "numerical result" from the definite integrals.

(8) $\int f(p)\, p\, dp \longrightarrow \int_{-\infty}^{\infty} p^3 e^{-\frac{p^2}{2mkT}} dp = 0$ as it is an odd moment of a gaussian.

(9) $\int_{-\infty}^{\infty} p^2 f(p)\, dp = \frac{4\pi}{(2\pi m kT)^{3/2}} \int_{-\infty}^{\infty} p^4 e^{-\frac{p^2}{2mkT}} dp$

$$= \frac{4\pi}{(2\pi m kT)^{3/2}} \cdot \cancel{2} \cdot \frac{3\,(2mkT)^2}{\cancel{8}} \sqrt{\pi}\,\sqrt{2mkT}$$

$$= 6\, m kT$$

(missed a "2" somewhere)

(10) $\therefore \quad \overline{F_i F_j} = \boxed{-6mkT}$ rather $-3mkT$

These results are reasonable as one would expect no correlation after several collisions, and some correlation between consecutive collisions. The minus sign indicates work done by particle between collisions. If one divides by $2m$, one gets:

$$\frac{\overline{p\, p'}}{2m} = -3kT \quad \text{or the equipartition energy.}$$

what about $i = j$

$$\overline{F_i^2} = 6mkT$$

Assignment #5
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29. a. (1) $x(t) = V(t) \cos\{\omega_c t + \varphi(t)\}$

and is a sample function of a stationary, real gaussian random process.

(2) $y(t) = x(t) \cos \omega_0 t$

$f_0 = \frac{\omega_0}{2\pi} << f_c$, but $f_0 >> $ bandwidth at f_0.

(3) $y_L(t) = \frac{V(t)}{2} \cos\{(\omega_c - \omega_0)t + \varphi(t)\}$

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$y_u(t) = \frac{V(t)}{2} \cos\{(\omega_c + \omega_0)t + \varphi(t)\}$

(4) $y_L(t)\, y_L(t+\tau) = \frac{1}{4} V(t) V(t+\tau) \cos\{(\omega_c - \omega_0)t + \varphi(t)\}$

$\cdot \cos\{(\omega_c - \omega_0)(t+\tau) + \varphi(t+\tau)\}$

$= \frac{1}{8} V(t) V(t+\tau) \cos\{2(\omega_c - \omega_0)t + (\omega_c - \omega_0)\tau + \varphi(t) + \varphi(t+\tau)\}$

$+ \frac{1}{8} V(t) V(t+\tau) \cos\{(\omega_c - \omega_0)\tau + \varphi(t+\tau) - \varphi(t)\}$

(5) Let $t \rightarrow 1$
$t+\tau \rightarrow 2$

$y_{L1} y_{L2} = \frac{1}{8} V_1 V_2 \cos\{2(\omega_c - \omega_0)t + (\omega_c - \omega_0)\tau + \varphi_1 + \varphi_2\}$

$+ \frac{1}{8} V_1 V_2 \cos\{(\omega_c - \omega_0)\tau + \varphi_2 - \varphi_1\}$

(6) Now, from (1), $R_x(\tau) = \overline{x(t)\, x(t+\tau)}$ as given.

$y_L(t)$ and $y_u(t)$ are of the same functional form as $x(t)$, only $\omega_c \rightarrow \omega_c - \omega_0,\ \omega_c + \omega_0$ is different and this just a constant. Thus we can write:

$\overline{y_{L1} y_{L2}} = R_{y_L}(\tau)$
$\overline{y_{u1} y_{u2}} = R_{y_u}(\tau)$

} Therefore, both sidebands are stationary processes.

(7) $y = y_{L1} + y_{u1} = x(t) \cos \omega_0 t$

(8) $\overline{y(t)\, y(t+\tau)} = \overline{x(t)\, x(t+\tau)} \cos \omega_0 t \cos \omega_0 (t+\tau)$

which is not independent of time and therefore not stationary.

b. (1) To show $y_L(t)$ and $y_u(t)$ are not statistically independent, we form:

$\overline{y_L(t)\, y_u(t)}$, $\overline{y_u(t)\, y_L(t)}$.

recalling $p(V_t, \varphi_t) = p(V_t)\, p(\varphi_t)$

where $p(U_t) = \frac{V_t}{\sigma_x^2} \exp\left\{ -\frac{V_t^2}{2\sigma_x^2} \right\}$, $V_t > 0$

$p(\varphi_t) = \frac{1}{2\pi}$, $0 \leq \varphi \leq 2\pi$

(2) $\overline{y_L(t)\, y_u(t)} = \frac{1}{8} \overline{V^2(t) \left[\cos(2\omega_c t + 2\varphi(t)) + \cos(2\omega_0 t) \right]}$

$= \frac{1}{8} \overline{V^2(t)} \cos(2\omega_0 t)$ as φ is distributed uniformly.

(3) $\left. \begin{array}{l} \overline{y_L(t)} = 0 \\ \overline{y_u(t)} = 0 \end{array} \right\}$ ✓ φ distributed uniformly

(4) $\overline{y_L(t)\, y_u(t)} \neq \overline{y_u(t)}\; \overline{y_L(t)}$

Thus $y_L(t)$ and $y_u(t)$ are not statistically independent.

Assignment # 5
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30. (1) It was shown in lecture that the distribution function for the envelope and phase of a narrow band gaussian process is given by:

$$W(V_t, \varphi_t) = W(V_t)\, W(\varphi_t) = \frac{1}{2\pi \sigma_x^2} V_t e^{-\frac{V_t^2}{2\sigma_x^2}}$$

where $W(\varphi_t)$ is uniformly distributed in $0 \leq \varphi_t < 2\pi$. σ_x^2 is gaussian variance.

$$W(V_t) = \frac{V_t}{\sigma_x^2} \exp\left\{-\frac{V_t^2}{2\sigma_x^2}\right\} \quad ; \quad V_t > 0$$

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$$W(\varphi_t) = \frac{1}{2\pi} \quad ; \quad 0 \leq \varphi_t < 2\pi$$

(2)

$$E(V_t) = \int_0^\infty V_t W(V_t)\, dV_t = \frac{1}{\sigma_x^2} \int_0^\infty V_t^2 e^{-\frac{V_t^2}{2\sigma_x^2}} dV_t$$

$$= 2 \int_0^\infty \frac{V_t^2}{2\sigma_x^2} e^{-\frac{V_t^2}{2\sigma_x^2}} dV_t = 2\sqrt{2}\,\sigma_x \int_0^\infty x^2 e^{-x^2} dx$$

$$x = \frac{V_t}{\sqrt{2}\,\sigma_x}$$

$$= \frac{\sqrt{2}}{2} \sqrt{\pi}\, \sigma_x = \sqrt{\frac{\pi}{2}}\, \sigma_x$$

(3)

$$E^2(V_t) = \frac{1}{\sigma_x^2} \int_0^\infty V_t^3 e^{-V_t^2/2\sigma_x^2} dV_t = 2\sigma_x^2 \int_0^\infty x\, e^{-x} dx$$

Let $x = \frac{V_t^2}{2\sigma_x^2}$,

$$dx = \frac{V_t\, dV_t}{\sigma_x^2}$$

$$= 2\sigma_x^2$$

(4)

$$\sigma^2(V_t) = E^2(V_t) - [E(V_t)]^2 = \left(2 - \frac{\pi}{2}\right) \sigma_x^2$$

Assignment #5
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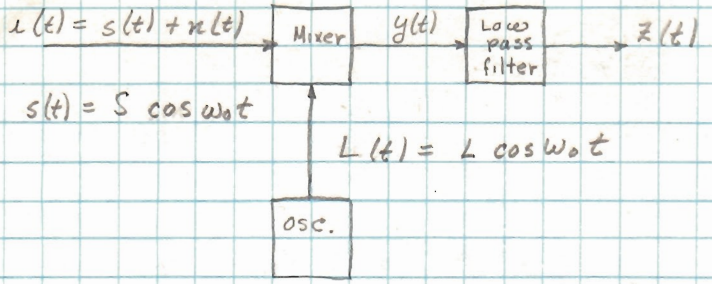
31.



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(1) $y(t) = L\,(s(t) + n(t)) \cos \omega_0 t$

(2) $\overline{y(t)} = L\,(\overline{s(t)} + \overline{n(t)}) \cos \omega_0 t$

(3) $y(t)\,y(t+\tau) = L^2 \cos \omega_0 t \cos \omega_0 (t+\tau) \left[s(t)s(t+\tau) + s(t)n(t+\tau) + n(t)s(t+\tau) + n(t)n(t+\tau) \right]$

$= L^2 s(t)s(t+\tau) \cos \omega_0 t \cos \omega_0 (t+\tau)$

$+ 2L^2 s(t)\,n(t+\tau) \cos \omega_0 t \cos \omega_0 (t+\tau)$

$+ L^2 n(t)\,n(t+\tau) \cos \omega_0 t \cos \omega_0 (t+\tau)$

(4) We now take the statistical average, knowing that s(t) is non-stationary, n(t) is stationary, and n(t), s(t) are uncorrelated:

$$\overline{y(t)\,y(t+\tau)} = L^2 \left\{ \overline{s(t)s(t+\tau)} \cos \omega_0 t \cos \omega_0 (t+\tau) + R_n(\tau) \cos \omega_0 t \cos \omega_0 (t+\tau) \right\}$$

(5) We now find the time auto-correlation function by the definition:

$$R_y(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \overline{y(t)\,y(t+\tau)}\, dt$$

using $\cos \omega_0 t \cos \omega_0 (t+\tau) = \frac{1}{2} \cos \omega_0 \tau + \frac{1}{2} \cos (2\omega_0 t + \omega_0 \tau)$

(6) $\lim_{T\to\infty} \frac{1}{T} \int_0^T \left\{ \frac{1}{2} \cos \omega_0 \tau + \frac{1}{2} \cos (2\omega_0 t + \omega_0 \tau) \right\} dt = \frac{1}{2} \cos \omega_0 \tau$

$+ \frac{1}{2} \lim_{T\to\infty} \frac{1}{T} \cdot \frac{1}{2\omega_0} \sin (2\omega T + \omega_0 \tau) = \frac{1}{2} \cos \omega_0 \tau$

(7) $$\lim_{T\to\infty} \frac{1}{T} \int_0^T \left\{ \cos w_0 t \, \cos w_0 (t+\tau) \right\}^2 dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_0^T \left\{ \frac{1}{4} \cos^2 w_0 \tau + \frac{1}{2} \cos w_0 \tau \, \cos (2 w_0 t + w_0 \tau) \right.$$

$$\left. + \frac{1}{4} \cos^2 (2 w_0 t + w_0 \tau) \right\} dt$$

$$= \frac{1}{4} \cos^2 w_0 \tau + \frac{1}{8}$$

(8) $$\therefore \quad R_y(\tau) = \frac{S^2 L^2}{4} \cos^2 w_0 \tau + \frac{1}{8} S^2 L^2 + \frac{1}{2} R_n(\tau) \cos w_0 \tau$$

(9) $$G_y^+(f) = 4 \int_0^\infty R_y(\tau) \cos w \tau \, d\tau$$

(10) $$\int_0^\infty \cos^2 w_0 \tau \, \cos w \tau \, d\tau = \frac{1}{2} \int_0^\infty \cos w \tau \, d\tau + \frac{1}{2} \int_0^\infty \cos 2 w_0 \tau \, \cos w \tau \, d\tau$$

$$= \frac{1}{4} \delta(f) + \frac{1}{8} \delta(f - 2 f_0)$$

(11) Now consider: $$\int_0^\infty R_n(\tau) \cos 2\pi f_0 \tau \, \cos 2\pi f \tau \, d\tau$$

From Definition: $$R_N^+(\tau) \equiv \int_0^\infty G_n^+(f) \cos 2\pi f \tau \, df$$

Then we have: $$\int_0^\infty \int_0^\infty G_n^+(f') \cos 2\pi f \tau \, \cos 2\pi f_0 \tau \, \cos 2\pi f' \tau \, df' \, d\tau$$

$$= \int_0^\infty G_n^+(f') \int_0^\infty \cos 2\pi f \tau \, \cos 2\pi f_0 \tau \, \cos 2\pi f' \tau \, d\tau \, df'$$

$$\frac{1}{2} \int_0^\infty \cos 2\pi (f - f_0) \tau \, \cos 2\pi f' \tau \, d\tau + \frac{1}{2} \int_0 \cos 2\pi (f + f_0) \tau \, \cos 2\pi f' \tau \, d\tau$$

$$= \frac{1}{8} \delta \left(f' - \{ f - f_0 \} \right) + \frac{1}{8} \delta \left(f' - \{ f + f_0 \} \right)$$

Finally: $$\int_0^\infty R_n(\tau) \cos 2\pi f_0 \tau \, \cos 2\pi f \tau \, d\tau$$

$$= \frac{1}{8} G_n^+(f - f_0) + \frac{1}{8} G_n^+(f + f_0)$$

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Problem 31
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(12) $$G_y^+(f) = \frac{S^2L^2}{4}\,\delta(f) + \frac{S^2L^2}{8}\,\delta(f-2f_0)$$

$$+ \frac{1}{4}S^2L^2\,\delta(f) + \frac{L^2}{4}G_n^+(f-f_0) + \frac{L^2}{4}G_n^+(f+f_0)$$

$$= \frac{S^2L^2}{2}\,\delta(f) + \frac{S^2L^2}{8}\,\delta(f-2f_0) + \frac{L^2}{4}G_n^+(f-f_0) + \frac{L^2}{4}G_n^+(f+f_0)$$

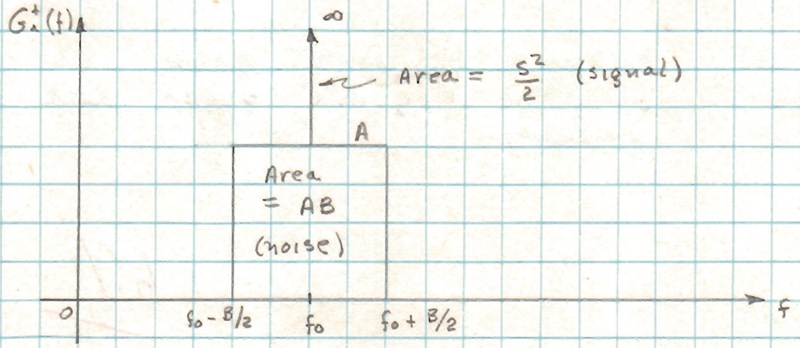
(13) For purposes of comparison, we compute the signal input PSD:

$$s(t)\,s(t+\tau) = S^2 \cos\omega_0 t \,\cos\omega_0(t+\tau)$$

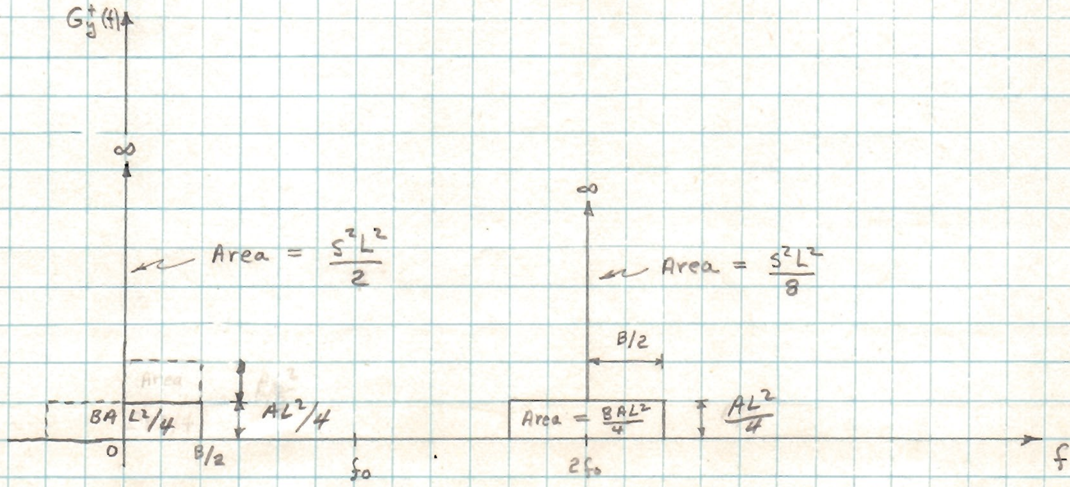
$$= \tfrac{1}{2}S^2\left\{\cos\omega_0\tau + \cos(2\omega_0 t + \omega_0\tau)\right\}$$

$$R_s(\tau) = \frac{1}{2}S^2\cos\omega_0\tau$$

$$G_s^+(\tau) = 4\int_0^\infty R_s(\tau)\cos\omega\tau\,d\tau = \frac{1}{2}S^2\,\delta(f-f_0)$$



$G_N^+(f) = A$
in $f_0 - B/2 < f < f_0 + B/2$



(14) We now pass through a low pass filter which eliminates the signals around $2f_0$.

We then have the following:

Signal power input $= S^2/2 = S_i$

Noise power input $= AB = N_i$

Noise power output $= L^2AB/4 = N_o$

Signal power output $= S^2L^2/4 = S_o$

must include L

Noise Pwr out $= \frac{AB}{4}L^2$

Sig. pwr out $= \frac{S^2L^2}{4}$

follows from $v_f(t) = \frac{LS}{2}$

$v^2 \rightarrow \frac{L^2S^2}{4}$

(15) $$\frac{S_o}{N_o} = \frac{S^2L^2/4}{L^2AB/4} = \frac{S^2}{AB} = 2\frac{S_i}{N_i}$$

$= S^2/2AB$

We see that the SNR of the output is related to the SNR of the input in a linear manner and that it is independent of the bandwidth for all input SNR unlike the quadratic detector for which this is only true for $S_i/N_i << 1$.
Also S_o/N_o depends on S_i/N_i via 2.

(Do not note! 12-29-62)
Note that mixing signal is not included as part of input signal but is part of output. If it is included, we get:

$\frac{S_o}{N_o} = 8\frac{S_i}{N_i}$, taking the input power as the product of the signal power and oscillator power.

$$\begin{array}{r} 9.5 \\ 70\overline{)67.0} \\ 63 \\ \hline 400 \end{array}$$

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Applied Physics 215

Hour Examination

November 17, 1960

1. (a) Two random variable x and y have zero means and equal variances σ. They have a correlation coefficient $\rho = 0.7$. Calculate the variance of their difference.

   (b) Two independent Gaussian random variables x and y have variances σ and τ respectively. Calculate the joint moment $\underline{x^m y^n}$.

2. A photo-multiplier tube counts incident light quanta at an average rate of one count per second. When counting is started at $t = 0$, what is the probability that the third count will occur after exactly 1 second? and after 3 seconds? and after 10 seconds?

3. A periodic voltage $V(t) = A\sin(\omega t + \phi)$ has a constant amplitude, but a random phase ϕ, which is uniformly distributed in the interval $0 \rightarrow 2\pi$. Calculate and sketch the probability density function $f(V)$ of the voltage at an arbitrary time.

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HARVARD UNIVERSITY

FACULTY OF ARTS and SCIENCES

Examination Book

Name Paul Grant

Date 11-17-60

Subject AP 215

Section

DO NOT REMOVE PAGES FROM BOOK
DO NOT REMOVE BOOK FROM ROOM

J. L. Hammett Co., Cambridge, Mass.

① a) (1) $\sigma_x^2 = \overline{x^2} - \bar{x}^2$, $\sigma_y^2 = \overline{y^2} - \bar{y}^2$

(2) $z = x - y$, $\bar{z} = \bar{x} - \bar{y}$, $\bar{z}^2 = \bar{x}^2 - \underset{\overset{\shortparallel}{0}}{2\bar{x}\bar{y}} + \bar{y}^2$

$z^2 = x^2 - 2xy + y^2$

$\overline{z^2} = \overline{x^2} - 2\overline{xy} + \overline{y^2}$

(3) $\sigma_z^2 = \overline{z^2} - \bar{z}^2 = (\overline{x^2} - \bar{x}^2) + (\overline{y^2} - \bar{y}^2) - 2\overline{xy}$

(4) $\rho\sigma^2 = \overline{xy}$

(5) $\sigma_z^2 = 2\sigma^2 - 2\rho\sigma^2 = 2(1-\rho)\sigma^2$

$= 2(.3)\sigma^2 = .6\sigma^2$

0 ✓

b) (1) If independent $\overline{x^m y^n} = \overline{x^m}\;\overline{y^n}$

(2) $\overline{x^m} = 0$ if m odd

$= 1\cdot 3\cdot 5\cdots(m-1)\,\sigma^{m}$ if m even

(3) $\overline{y^n} = 0$ if n odd

$= 1\cdot 3\cdot 5\cdots(n-1)\,\sigma^{n}$ if n even

(4) $\therefore$ $\overline{x^m y^n} = 0$, if either m, n odd

$= \left[1\cdot 3\cdot 5\cdots(m-1)\sigma^m\right]\left[1\cdot 3\cdot 5\cdots(n-1)\sigma^n\right]$

if m, n are even, and only if random variables are gaussian.

② (1) Use Poisson distribution $P_N(n) = \frac{(\bar{n})^n}{n!} e^{-\bar{n}}$

(2) $P_n(\tau) = \frac{(a\tau)^n}{n!} e^{-a\tau}$, $p = a\Delta\tau$

(3) Probability that no counts in time τ and first count in time $\tau + \Delta\tau$ is $a\Delta\tau\, e^{-a\tau}$

(4) Probability that two counts occur in time τ and third in $\tau + \Delta\tau$

$= P_2(\tau)\, P_1(\Delta\tau) = a\Delta\tau \left(\frac{(a\tau)^2}{2} e^{-a\tau}\right)$

OK $= \frac{a^3\tau^2}{2} e^{-a\tau} \Delta\tau$ (should possibly be a sum over n from $\sum_0^2$

(5) Replace $\Delta\tau$ by $d\tau$ and integrate over period of interest. $a = 1$

(6) $P(T) = \frac{1}{2}\int_0^T \tau^2 e^{-\tau}\, d\tau =$

$= -\frac{1}{2}\tau^2 e^{-\tau} + \int \tau e^{-\tau}\, d\tau$

$= -\frac{1}{2}\tau^2 e^{-\tau} + \left\{ e^{-\tau}(-\tau - 1) \right]$

$$= -\frac{1}{2}\tau^2 e^{-\tau} - \tau e^{-\tau} - e^{-T}$$

$$= -e^{-\tau}\left(\frac{1}{2}\tau^2 + \tau + 1\right)\Big]_0^T$$

$$-e^{-T}\left(\frac{1}{2}T^2 + T + 1\right) + 1$$

$$= 1 - e^{-T}\left(\frac{1}{2}T^2 + T + 1\right)$$

$T = 1$: $P(1) = 1 - \frac{5}{2}e^{-1}$

$T = 3$: $P(3) = 1 - e^{-3}\left(\frac{9}{2} + 3 + 1\right)$

$$= 1 - \frac{17}{2}e^{-3}$$

$T = 10$: $P(10) = 1 - e^{-10}(50 + 10 + 1)$

$$= 1 - 61e^{-10}$$

③ (1) $V = A \sin(\omega t + \varphi)$

$f(\varphi) = \frac{1}{2\pi} \quad 0 < \varphi < 2\pi$

$f(V) = ?$

(2) $f(V) = f\left[\varphi = \varphi(V)\right] \frac{d\varphi}{dV}$

I am not sure this can be done as $\varphi = \varphi(V) = \sin^{-1}\frac{V}{A} - \omega t$ is a multiple valued function. Perhaps if it is restricted to the range of definition of $\varphi \{0 < \varphi < 2\pi]$ it will be valid over this principal value.

(3) $\frac{d\varphi}{dV} = \frac{1/A}{\{1 - (\frac{V}{A})^2]^{1/2}} = \frac{1}{[A^2 - V^2]^{1/2}}$

(4) $f\left[\varphi = \varphi(V)\right] = \frac{1}{2\pi}$

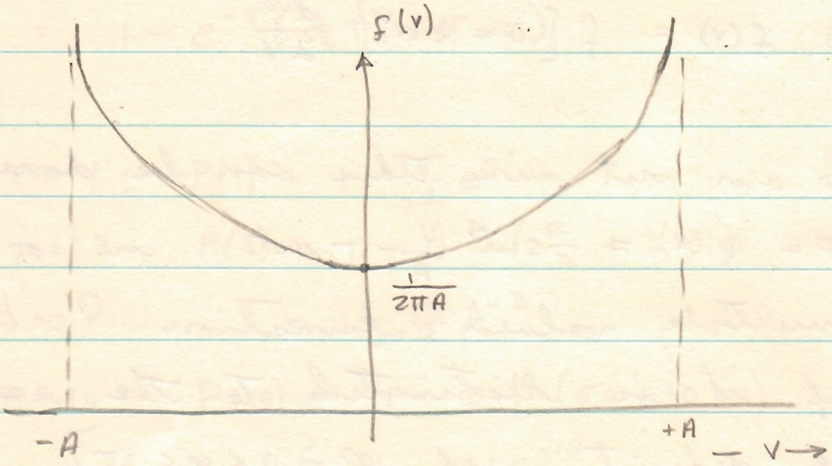
(5) $\therefore \quad f(V) = \frac{\overset{2}{1}}{2\pi A} \cdot \frac{1}{[1 - (\frac{V}{A})^2]^{1/2}}$

Check normalization

<u>over for sketch</u>

(6) $$f(V) = \frac{1}{2\pi A} \cdot \frac{1}{\left\{1 - \left(\frac{V}{A}\right)^2\right\}^{1/2}}$$

f(V) is valid only in $V < A$ or $\frac{V}{A} < 1$



(7) $$\frac{\cancel{2}}{2\pi A}\int_{\cancel{0}\,-A}^{A} \frac{dV}{\left\{1-\left(\frac{V}{A}\right)^2\right\}^{1/2}} = 1 \quad \cancel{\frac{2}{\pi}\sin^{-1}1 = \frac{2}{\pi}\,\frac{\pi}{2} = 1}$$

∴ normalized

Applied Physics 215

Final Examination

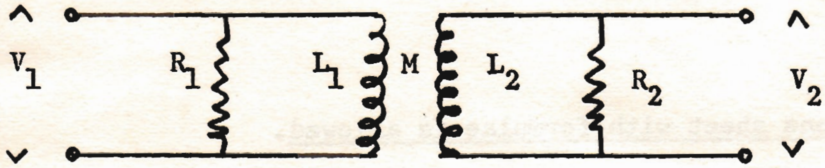
January, 1961

1. Define briefly and give one physical example of each of the following
    - a) Non-stationary random process
    - b) Narrow-band gaussian random process
    - c) A non-gaussian stationary process
    - d) A two-dimensional Markoff process
    - e) A non-Markoffian process

2. Describe all stochastic processes that may occur in a thermionic tetrode.

Answer <u>three</u> out of the following <u>four</u> problems

3. Consider two RL circuits, coupled by a mutual inductance M, as shown in the diagram. Both resistors are at the temperature T.



Derive an expression for the cross-correlation function of the noise voltages V_1 and V_2.

4. A radioactive source has exactly N_0 radioactive nuclei at t = 0. The half life of the source is 10 seconds. Calculate the mean square deviation in the number of radioactive nuclei in an ensemble of such sources, as a function of time. At what time does $\overline{\Delta N^2}$ reach a maximum?

(OVER)

5. Consider a random walk in two dimensions with steps of length ℓ. The steps are in the direction of the positive or negative x-axis, or along the diagonal making an angle ϕ of 45° or 225° with the positive x-axis. These four directions have equal a priori probability. Calculate the distribution in direction ϕ after a large number of steps when the random walk started at the origin. What is the most probable direction in which to observe the random walker?

6. A very long, lossless transmission line is terminated at both ends by a matched resistance R. The temperature of the resistors are T and T + ΔT respectively. If the transmission line transmits all frequencies, calculate the heat conduction between the two resistors. What is the spectral density of the temperature fluctuations of each resistor when its heat capacity is c_v?

<u>The use of one sheet with formulae is allowed.</u>

Sample Space: Set of poss. outcomes

Prob. $P(k) = \lim_{N\to\infty} \frac{\#\ \text{success}}{\#\ \text{trials}}$

M.E. $P(A) + P(B) = P(A \text{ or } B)$

J. P: $P(A \text{ and } B) = P(A,B)$

C. P: $P(B|A) = P(A,B)/P(A)$

Stat. Ind. $P(A_1,\ldots,A_N) = P(A_1)\cdots P(A_N)$

Bernoulli Dist.

Two M.E. events, $P(A)\to p$, $P(B)\to q$

$q = 1-p$

Take N trials, get n successes N-n failures. Prob. of one seq. $= p^n q^{N-n}$

Possible seq. $= N!$, Successes $= n!$, Fail. $= (N-n)!$

Prob. n successes, N trials:

$$P_N(n) = \frac{N!}{n!(N-n)!} p^n q^{N-n}$$

$$[py+q]^N = \sum_{n=0}^{N} y^n p^n q^{N-n} \cdot \frac{N!}{n!(N-n)!}$$

$p+q=1$

$\bar{n} = Np$, $\overline{\Delta n^2} = Npq \doteq \bar{n}$, $p \ll 1$

$(\overline{\Delta n^2})^{1/2}/\bar{n} = 1/\sqrt{\bar{n}}$

Poisson Dist. Limit of B dist.

$\lim_{\substack{N\to\infty\\ p\to 0}} = \bar{n}$, $p = \frac{\bar{n}}{N}$

$$P_N(n) = \left(\frac{\bar{n}}{N}\right)^n \left(1-\frac{\bar{n}}{N}\right)^{N-n} \frac{N!}{n!(N-n)!}$$

$$= \frac{(\bar{n})^n}{n!}\cdot 1\cdot\left(1-\frac{1}{N}\right)\cdots\left(1-\frac{n-1}{N}\right)\left(1-\frac{\bar{n}}{N}\right)^{N-n}$$

$$\lim_{N\to\infty} P_N(n) = \frac{(\bar{n})^n}{n!} e^{-\bar{n}}$$

Prop.: $\bar{n} = \bar{n}$, $\overline{\Delta n^2} = \bar{n}$

no interaction assumed (very wrong

Emission: $p = a\Delta\tau$, $N = \frac{\tau}{\Delta\tau}$

$\bar{n} = a\tau$, $P_n(\tau) = \frac{(a\tau)^n}{n!} e^{-a\tau}$

prob. of emitting in $\Delta\tau = a\Delta\tau$

Prob of n counts in τ and 1 in $\Delta\tau = a\Delta\tau \frac{(a\tau)^n}{n!} e^{-a\tau}$

Random Walk: given equal prob of steps of length l to left or right, what is prob. of being at ml after N steps.

B. Dist. $P_m(N) = \left(\frac{1}{2}\right)^N \frac{N!}{\left(\frac{N+m}{2}\right)!\left(\frac{N-m}{2}\right)!}$

Use Stirl. Approx.

let $N\to\infty$, and get:

$$P(m,N) = \left(\frac{2}{\pi N}\right)^{1/2} e^{-\frac{m^2}{2N}}$$

Can get diffusion equation by taking $x = ml$, $\Delta m = \frac{\Delta x}{2l}$, $N = N'\tau$

$N'\to\infty$, $\tau = 0$, $\lim N' l^2 = 2D$

Random Variable: varible in sample space, discreet or cont.

Prob. Density: $f(x) \equiv \partial P/\partial x$

JP: $f(x,y) = f(x)f(y)$ if SI

fns of r.v. also r.v.

Transformation of Prob. Den.

Given $f(x,y)$: $\left.\begin{array}{l} x = x(u,v)\\ y = y(u,v)\end{array}\right\}\ \begin{array}{l} u = u(x,y)\\ v = v(x,y)\end{array}$

$\therefore$ $f(x,y)\,dx\,dy = f(x = x(uv), y = y(uv))\,J\,du\,dv$

$$J = \begin{vmatrix} x_u & y_u\\ x_v & y_v\end{vmatrix} = \frac{\partial(x,y)}{\partial(u,v)}$$

$\therefore$ $f(uv) = f(x = x(uv), y = y(uv))\,J$

1 variable: $f(u) = f(x = x(u))\,dx/du$

Averages: $\overline{g(x)} = \int_{-\infty}^{\infty} g(x)f(x)\,dx$

Joint M. $\overline{x^m y^n} = \iint_{-\infty}^{\infty} x^m y^n f(x,y)\,dx\,dy$

Covariance: $\overline{(x-\bar{x})(y-\bar{y})} = \iint (x-\bar{x})(y-\bar{y}) f(x,y)\,dx\,dy$

Char. Fn: $M_x(iu) = \overline{e^{iux}} = \int_{-\infty}^{\infty} e^{iux} f(x)\,dx$

$\overline{x^n} = (-i)^n \frac{d^n}{du^n} M_x(iu)\Big]_{u=0}$, $M_x(iu) = \sum_0^\infty \overline{x^n}\frac{(iu)^n}{n!}$

Joint Char. $M_{xy} = \overline{e^{iux+ivy}}$

$\overline{x^m y^n} = (-i)^{n+m} \frac{\partial^{m+n}}{\partial u^m \partial v^n} M_{xy}(iu, iv)$

Stat. Ind. $\overline{x^m y^n} = \overline{x^m}\ \overline{y^n}$

Gaussian. - M-B Stat.

$$f = \frac{1}{(2\pi m k T)^{3/2}} e^{-\frac{(p_x^2+p_y^2+p_z^2)}{2mkT}}$$

$$= \frac{1}{(\ \ )^{3/2}} e^{-\frac{p^2}{2mkT}} p^2\,dp \cdot \sin\theta\,d\theta\,d\varphi$$

$$= \frac{2\pi}{(\pi k T)^{3/2}} E^{1/2} e^{-E/kT}$$

In potential, add $e^{-\frac{V(x,y,z)}{kT}}$

Equipartition: Due to quad. terms in exp. such as p^2 M-B gives ½ kT for $\bar{E}$.

Gaussian 1-D: $W(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m)^2}{2\sigma^2}}$

$\overline{(x-m)^n} = 0$ n odd

$= 1.3.5\ldots(n-1)\sigma^n$ n even

$M_x(iu) = e^{imu - u^2\sigma^2/2}$

Bivariate: $W(y_1 y_2) = \frac{1}{2\pi\sigma\tau(1-\rho^2)^{1/2}}$

$\cdot \exp - \left\{\frac{1}{2(1-\rho^2)}\left(\frac{y_1^2}{\sigma^2} + \frac{y_2^2}{\tau^2} - \frac{2\rho}{\sigma\tau} y_1 y_2\right)\right\}$

$\sigma^2 = \overline{y_1^2}$, $\tau^2 = \overline{y_2^2}$; $\overline{y_1 y_2} = \rho\sigma\tau$

Mult. Var:

$$W(x_1\cdots x_N) = \frac{\exp\left[-\frac{1}{2|\Lambda|}\sum_{n=1}^{N}\sum_{m=1}^{N} |\Lambda|_{nm} x_n x_m\right]}{(2\pi)^{N/2}|\Lambda|^{1/2}}$$

$M_x(iV) = \exp\left(-\frac{1}{2}V'\Lambda V\right)$

$V = \begin{pmatrix} v_1\\ v_N\end{pmatrix}$, $\Lambda = (\lambda)$, $\lambda_{nm} = \overline{x_n x_m}$

Markoff Process: process such that cond. prob. y in y_N, $y_N + dy_N$ at time t_N given that $y = y_1, \ldots, y_{N-1}$ at times $t_1,\ldots t_{N-1}$ depends only on y at previous time t_{N-1} (Uhlenbeck)

Smol. Eq. $P(x_3 t_3|x_1 t_1)$

$= \int P(x_3 t_3|x_2 t_2)\,P(x_1 t_1|x_2 t_2)\,dx_2$

Stat. R.P., Erg. Pro.

density fns invariant to shift of time axis, depend only on interval.

$\overline{x_1 x_2} = R(\tau)$ if station.

$\rho_x(\tau) = \frac{R_x(\tau) - (\bar{x})^2}{\sigma^2}$

Ergotic: $R(\tau) \equiv \mathcal{R}(\tau)$, $\langle x\rangle = \bar{x}$

Definitions (Kittel)

stochastic or RV: if set of poss. values given and prob. of each one given, this is defined.

Cent. Limit. Theorem: Σ RV is also RV. CLT says that dist. of Σ tends to gaussian in limit of large numbers.

Random process: RV x does not depend on time in a completely definite way. All can have is prob. dist. fn. for RV.

Gaussian RP. each rv has gaussian dist. Also, finite # of gaussian RV has gaussian dist.

Corr. Fns:

Stat. $R_{xy} = \overline{x_1 y_2^*}$

$R_x(t_1 t_2) = \overline{x_1 x_2^*}$

Time: $R_x(t_1 t_2) = \lim_{T\to\infty}\frac{1}{T}\int_0^T x_1 x_2^*\,dt$

$\langle x\rangle = \lim_{T\to\infty}\frac{1}{T}\int_0^T x(t)\,dt$

PSD:

$R(\tau) = \overline{x(t)x(t+\tau)}$ station.

$\mathcal{R}(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_0^T x(t)x(t+\tau)\,dt$

Rice uses:

$R_x(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_0^T \overline{x(t)x(t+\tau)}\,dt$

Wide sense stationary, prob. dist. fn. not invariant under timeshift but means and correlations don't depend on time.

Useful Relation:

$\int_0^\infty \cos 2\pi f\tau\,d\tau = \frac{1}{2}\delta(f)$

$\int_0^\infty \cos 2\pi f_0\tau \cos 2\pi f\tau\,d\tau = \frac{1}{4}\delta(f-f_0)$

$\int_{-\infty}^{\infty} e^{i2\pi f\tau}\,d\tau = \delta(f)$

Wiener-Khintchine Theorem:

$G(f) = \int_{-\infty}^{\infty} R_y(\tau) e^{2\pi i f\tau}\,d\tau$ $\Big\}$ $G^+(f) = 4\int_0^\infty R_y(\tau)\cos 2\pi f\tau\,d\tau$

$R_y(\tau) = \int_{-\infty}^{\infty} G(f) e^{-2\pi i f\tau}\,d\tau$ $\Big\}$ $R_y(\tau) = \int_0^\infty G^+(f)\cos 2\pi f\tau\,df$

This pair forms W-K Theorem (Kittel)

$G(f) = \frac{1}{2}G^+(f)$, $G(f) \equiv \lim_{T\to\infty}\frac{S(f)S^*(f)}{T}$, $S(f) = \int_{-\infty}^{\infty} y(t)e^{2\pi i f t}\,dt$

Useful Relation:

$\cos\{\omega_0 t + \varphi\}\cos\{\omega_0(t+\tau) + \varphi\} = \frac{1}{2}\cos\omega_0\tau + \frac{1}{2}\cos(2\omega_0 t + 2\varphi + \omega_0\tau)$

Non-Stat. RP.

If $\overline{x(t)x(t+\tau)}$ not wide sense stat., can use $\mathcal{R}(\tau)$ and take PSD. In this way, PSD defined for each sample. D & R. use this for def.

Random Tel. # of 0 cross/sec $= a$. # k in time interval T is given by P dist.

$P(k) = \frac{(aT)^k}{k!} e^{-aT}$. $R_y(\tau) = \overline{y(t)y(t+\tau)} = +1$ k even, -1 k odd

$R_y(\tau) = \sum_{e} P(k) - \sum_{0} P(k) = e^{-a\tau}\sum_{all}\frac{(-a\tau)^k}{k!} = e^{-2a\tau}$

Take F transform for PSD.

F Series Rep. of PSD:

$i(t) = \frac{a_0}{2} + \Sigma\left(a_n\cos\frac{2\pi n t}{T} + \ldots\right)$

$a_n = \frac{2}{T}\int_0^T i(t)\cos\frac{2\pi n t}{T}\,dt$

$\bar{a}_n = \bar{b}_n = 0$ assume

$\overline{a_n^2} = \overline{b_n^2}$, $\overline{a_n a_m} = \frac{2}{T}\int_{-\infty}^{\infty} R_y(\tau)\cos\frac{2\pi n\tau}{T}\,\delta_{nm}$

$= \frac{1}{T} G^+\left(f = \frac{n}{T}\right)\delta_{nm}$

$G^+(f)\,df = \frac{T}{2}\left(\overline{a_n^2} + \overline{b_n^2}\right)$

More on PSD:

Wide band = ind. of f also called pure random process. $R(\tau) \sim \delta(\tau)$

Narrow Band = $\Delta f \ll f_c$

Useful: $\overline{x^2} = \int_0^\infty G^+(f)\,df = \int_{-\infty}^{\infty} G(f)\,df$

Linear Fixed Parameter Systems: RLC, $T \neq f$(time)

$G(f)_{out} = |A(f)|^2 G(f)_{in}$

For n pair:

$G_{out}(f) = \sum_{k}^{N}\sum_{k'}^{N} A_k(f)A_{k'}^*(f) \cdot G_{in\,kk'}(f)$

If inputs uncorrelated:

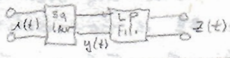
$G_{out}(f) = \sum_{k=1}^{N} |A_k(f)|^2 G_k(f)_{in}$

Noise Figure:

$$F \equiv \frac{\left(\frac{\text{sig. power}}{\text{noise ''}}\right)_{\text{avail. in}}}{\left(\frac{\text{sig. power}}{\text{noise ''}}\right)_{\text{avail. out}}}$$

Non-Linear Systems

Quad. Det. $y(t) = \alpha i^2(t)$



$R_y(\tau) = \alpha^2\,\overline{i^2(t)\,i^2(t+\tau)}$

$R_i(\tau) = \overline{i(t)\,i(t+\tau)}$

$= \alpha^2[R_i^2(0) + 2R_i^2(\tau)]$

$= \sigma^4\alpha^2 + 2\alpha^2 R_i^2(\tau)$

$G_y(f) = \alpha^2\sigma^4\delta(f) + 2\alpha^2\int_{-\infty}^{\infty} G(f')G(f-f')\,df'$

$\sigma^2 = \int_{-\infty}^{\infty} G(f)\,df$

For nth law device all components of input PSD must be convoluted with each other to find PSD around f_0. $\nu \neq$ integer, ∞ # of convolutions

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Langevin Equation and F-P Methods.

$J\frac{d\psi}{dt} + B\psi = F(t)$, $\frac{J}{B} \equiv \tau_0$

J - inertia, B - damping

$\frac{d\psi}{dt} + \frac{1}{\tau_0}\psi = K(t)$, $K(t) = \frac{F(t)}{J}$

K(t) is purely random process of zero mean, PSD = 4D, gaussian process. $\overline{K^2} = 2D\,\delta(t_1 - t_2)$

Gen'l Sol'n:

$\psi = \psi_0 e^{-t/\tau_0} + e^{-t/\tau_0}\int_0^t K(\xi)e^{\xi/\tau_0}\,d\xi$

$\bar{\psi} = \psi_0 e^{-t/\tau_0}$

$\overline{\psi^2} = D\tau_0 + e^{-2t/\tau_0}\{\psi_0^2 - D\tau_0\}$

As $t\to\infty$: $\bar{\psi}_{t\to\infty} = 0$, $\overline{\psi^2_{t\to\infty}} = D\tau_0$

Because this is gaussian process, we can find dist. fn.

Correl. & PSD:

$G_\psi^+(f) = \frac{4\tau_0^2 D}{(2\pi f)^2\tau_0^2 + 1}$, $R_\psi(\tau) = \tau_0 D e^{-\tau/\tau_0}$

Suppose $J\frac{d^2\psi}{dt^2} + B\frac{d\psi}{dt} + A\psi = \frac{d}{dt}F(t)$

$\frac{d^2\phi}{dt^2} + \frac{1}{\tau_0}\frac{d\psi}{dt} + \omega_0^2\phi = \frac{d}{dt}K(t)$

$\omega_0 = \sqrt{\frac{A}{J}}$, then $G_\psi^+(f) = \frac{4D\tau_0^2\omega^2}{\omega^2 + \tau_0^2(\omega_0^2 - \omega^2)^2}$

Useful in RLC circuits.

The F-P equation is derived from Sm. eq. as a Mark. Pro. assuming that cond. prob. of jumping from one interval to another is slowly varying. This assump. made on Master eq.

viz. $\frac{\partial W}{\partial t} = -\frac{\partial}{\partial x}\{A(x)W\} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\{B(x)W\}$

where $A = \lim_{\Delta t\to 0}\frac{\overline{\Delta y}}{\Delta t}$, $B = \lim_{\Delta t\to\infty}\frac{\overline{\Delta y^2}}{\Delta t}$

Using above Langevin eq.,

$\frac{\partial W}{\partial t} = -\frac{\partial}{\partial\psi}\{AW\} + \frac{1}{2}\frac{\partial^2}{\partial\psi^2}\{BW\}$

$A = -\frac{1}{\tau_0}\psi$, $B = 2D$

Assuming Δt gets small but stays larger than τ_0

SS sol'n of F-P eq., use method of least descent, expand A in Taylor series, keep first term.

$W_{SS} = \frac{\text{const.}}{B} e^{-\frac{(\psi - m)^2}{2\sigma^2}}$

$\sigma^2 = -B(m)/2\left(\frac{\partial A}{\partial\psi}\right)_m$

Prob.	φ	τ_0	D	(PSD)m
R-L circuit	i	L/R	$\frac{4RkT}{L}$	$4RkT$
Brown. Motion	v	m/β	$\frac{4\beta kT}{m}$	$4\beta kT$
Heat	θ	C_V/a	$\frac{4akT^2}{C_V}$	$4\alpha kT^2$

RLC	ψ	τ_0	ω_0	D
series	i	L/R	$1/\sqrt{LC}$	$4RkT/L$
parallel	v	$1/RC$	''	$4kT/RC$

Fluct. in EM Field:

Sort photons, use MB dist. $\frac{e^{-E_i/kT}}{\sum_i e^{-E_i/kT}}$

$\bar{E} = \frac{\sum_i E_i e^{-E_i/kT}}{\sum_i e^{-E_i/kT}}$

$\overline{E^2} = \frac{\sum_i E_i^2 e^{(\,)}}{\sum_i e^{(\,)}}$

$\alpha = -1/kT$, $Z = \sum_i e^{-\alpha E_i}$

$\bar{E} = \frac{1}{Z}\frac{\partial Z}{\partial \alpha}$, $\overline{E^2} = \frac{1}{Z}\frac{\partial^2 Z}{\partial \alpha^2}$

$\therefore \overline{\Delta E^2} = \frac{\partial \bar E}{\partial \alpha}$; $C_v \equiv \frac{\partial \bar E}{\partial T}$

QM states don't change means dimensions of box constant.

$\overline{\Delta E^2} = \frac{\partial \bar E}{\partial \alpha} = \frac{\partial T}{\partial \alpha}\frac{\partial \bar E}{\partial T} = kT^2 C_v$

Ideal Gas: $\bar E = \frac{3}{2}NkT$

$\overline{\Delta E^2}/\bar{E}^2 = 2/3\,\frac{1}{N}$, thus For large N, fluct. negligible

QM HO: Neglect 0 pt E.

$E_i = n_i h\nu$, $Z = \frac{1}{1-e^{-h\nu/kT}}$

$\bar E = \frac{\sum n_i h\nu\, e^{-n_i h\nu/kT}}{\sum e^{(\,)}}$

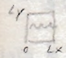
$= \frac{h\nu\, e^{-h\nu/kT}(1-e^{-h\nu/kT})}{(1-e^{-h\nu/kT})^2}$

$= \frac{h\nu}{e^{h\nu/kT}-1}$, can find Cv and $\overline{\Delta E^2}$. See that HO is cut off at hν, so ∞ # of modes can't exist thus resolving paradox. n_i is # of photons. $\bar n_i = \frac{1}{e^{h\nu/kT}-1}$

$\overline{\Delta n_i^2} = \frac{e^{h\nu/kT}}{(e^{h\nu/kT}-1)^2} = \bar n_i(\bar n_i+1)$

If $\bar n_i$ large, fluct. increase like EM waves in Quad det.

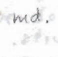
Density of States:



$L_x k_x = n_x \pi$, $E \sim \sin k_x x\, e^{i\omega t}$

$\Delta n_x \cdots = \frac{V}{\pi^3}\Delta k_x \cdots$

$|k| = \frac{2\pi}{\lambda} = \frac{2\pi\nu}{c}$



# of standing waves $= V\frac{4\pi\nu^2}{c^3}\Delta\nu$ ind. of BC

Could use BVK BC

Black Body Radiation

Each mode carries E of HO: $h\nu/e^{h\nu/kT}-1$

Avg E density $= \bar u = \int_0^\infty \frac{h\nu}{e^{h\nu/kT}-1} \cdot 2 \cdot \frac{4\pi\nu^2}{c^3}d\nu$ } involves B numbers
polarization

All oscillators radiate ind. of each, ∴ freq. are also ind.

$\overline{\Delta u^2} = \int_0^\infty \frac{(h\nu)^2 e^{h\nu/kT}}{(e^{h\nu/kT}-1)^2}\frac{8\pi\nu^2}{c^3}d\nu$

However, $\overline{\Delta u^2} = C_v kT$

$C_v = \frac{d\bar u}{dT} = \frac{32\pi^5 k^4 T^3}{15h^3c^3}$ specific heat of BB Rad.

from $\bar u = \frac{8\pi^5 k^4 T^4}{15h^3c^3}$

$\overline{\Delta u^2} = C_v kT^2$ is generally good for any system of E levels

Avg # of photons in Rad. field in 1cc in Δν:

$\bar N = \frac{1}{e^{h\nu/kT}-1}\frac{8\pi\nu^2}{c^3}\Delta\nu$

$\overline{\Delta N^2} = \overline{\Delta n_k^2}\frac{8\pi\nu^2}{c^3}\Delta\nu$

$= \bar n_n(\bar n_n+1)\frac{8\pi\nu^2}{c^3}\Delta\nu$

$\overline{\Delta N^2} = \bar N$, $h\nu/kT \gg 1$

$\overline{\Delta N^2} = \frac{\bar N^2}{8\pi\nu^2/c^3\,\Delta\nu}$, $\frac{h\nu}{kT} \ll 1$

Spectral Lines as NB Gauss. Proc.

Think of orchestra playing same note but all out of phase. same with light emitted from solid. $P\alpha\left[\sum_i a_i(t)\cos\{\omega_0 t + \varphi(t)\}\right]^2$ Light dectors detect power, thus $\overline{P(t)P(t+\tau)} = \overline{P^2}\{1+2\rho^2(\tau)\}$ like quad. dectector.

Would usually have just wave fluct., but get quant. also because of liberation of photon in detector. Prob for liberation = αPdt. Prob. for N in time t is P dist. Then calc. $\bar N_t$ and $\overline{N_t^2}$. Get: $\overline{\Delta N_t^2}$

$= \bar N_t + 2\alpha^2 \bar P^2 t\int_{-\infty}^{\infty}\rho^2(\tau)d\tau$

If all in phase get $\overline{\Delta N_t^2} = \bar N_t + \bar N_t^2$. $t \gg \tau_c$

Line width $\Delta\nu \cong 10^8$ cps

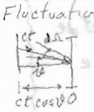
Define spectral dist. $\xi(\nu)$.

Get: $\overline{\Delta N_t^2} = \bar N_t + 2\alpha^2 \bar P^2 t\int_{-\infty}^{\infty}\xi^2(\nu)d\nu$

or: " $= \bar N_t + \bar N_t^2/(\Delta\omega)t$

The more oscillators take part, wider spectral line.

Fluctuations in Rad. Flux.



EM thru O in t seconds in direction ϑ is in volume $V = ctO\cos\vartheta$. # of osc. in V in direction ϑ in dΩ in dν is $\sum_{pol.}\frac{d\Omega\,\nu^2 d\nu}{c^3}$. The intensity is $k(\nu)d\nu = \frac{2h\nu}{e^{h\nu/kT}-1}\frac{\nu^2}{c^3}d\nu\iint d\Omega\cdot c$

$= 2\pi\frac{h\nu}{e^{h\nu/kT}-1}\frac{\nu^2}{c^2}d\nu = \frac{c}{4}\underset{\uparrow}{u(\nu)}$ E density

Because of fluct. in ν, will have fluct. in k(ν). Integrated k(ν) is: $K(\nu) = \int k(\nu)d\nu = \sigma T^4$ with $\sigma = \frac{2\pi^5 k^4}{15h^3c^3}$ S B constant

MSF in energy of BB with area O in time t. These are $\overline{\Delta W^2} = \frac{c}{4}Ot\overline{\Delta u^2}$ coming from

$\overline{\Delta W^2} = \iiint_{\nu\varphi\vartheta} Oct\cos\vartheta\sin\vartheta\,d\varphi\,d\vartheta\,\frac{\frac{2\nu^2}{c^3}(h\nu)^2 e^{h\nu/kT}}{(e^{h\nu/kT}-1)^2}$

$= 4Ot\sigma kT^5$

Since $u(\nu) = \frac{4}{c}\sigma T^4$, $\overline{\Delta u^2} = \frac{16}{c}\sigma kT^5$. The total flux from BB of area O is $\sigma O T^4 = W/t$ and the msf in flux is:

$\overline{\Delta\varphi_t^2} = \frac{\overline{\Delta W^2}}{t^2} = \frac{4\sigma OkT^5}{t}$

This is stat. avg over time avg. Since BB is non-isolated, Net φ is: radiated $\varphi = \sigma O(T^4 - T_0^4)$

$\overline{\Delta\varphi_t^2} = \frac{\overline{\Delta W^2}}{t^2} = \frac{4O\sigma k(T^5 - T_0^5)}{t}$

This has been integrated over all directions, and freq.

Heat Flow: α is given by $\Phi = \sigma O(T^4 - T_0^4) = \underbrace{4\sigma OT^3}_{\alpha}\Theta$

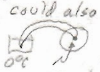
Fluctuations due to heat cond. or rad. must satisfy $\overline{\Delta E^2} = C_v kT^2$

$\overline{F^2} = 2\alpha kT^2$. $C_v\frac{d\Theta}{dt} = -\alpha\Theta + F(t) = -\Phi$. Can show from $E = 4\sigma OT_0^3\Theta$, $\overline{\Delta F_t^2} = (4\sigma OT_0^3)^2\overline{\Delta\Theta^2}$, and that $\overline{\Delta\varphi_t^2}$ is the same as before.

Radiation Measurements

Bolometer: radiation raises temp. hence resis.

could also be used as:



wires thin because of heat conductance. Still, not too thin because of Johnson Noise. $C_v\frac{d\theta}{dt} + \alpha\theta + iST = F_\theta(t)$.

$iR = S\theta + \underset{\text{Johnson noise}}{F_V(t)}$

Equations coupled, can write: $C_v\frac{d\theta}{dt} + \alpha\theta + \frac{S^2T\theta}{R} = F_\theta(t) - ST/R\,F_V(t)$, can find fluct. in temp and current.

$\left(\frac{N}{S}\right)_{ove} = \frac{\overline{\Delta i^2}}{(\bar i)^2} = \frac{\overline{\Delta\varphi_t^2}}{(\bar\varphi)^2} = \left(\frac{N}{S}\right)_{\substack{inc.\\rad.}}$

Noise figure is 1, natural precision.

At t=1sec, O=1cm², T=300°K $\left[\frac{\overline{\Delta\varphi_t^2}}{(\bar\Phi)^2}\right]^{1/2} = 8\cdot 10^{-10}$, we claim that Temp. fluct. is 1/4 or $\frac{\Delta T}{T} = 2\cdot 10^{-10}$

Comparison with Xmission Line

Different from BB in that we have limited sold angle and limited set of freq. Only accepts NB of frequencies. Analogous to those previously considered for Xmission line. We are in region where $h\nu \ll kT$, $T \gtrsim 10^0$K, $n_\nu \approx \frac{kT}{h\nu}$

$\varphi = kT\sum_{osc} = kT\Delta\nu$

$\overline{\Delta\varphi_t^2} = \frac{k^2T^2}{t^2}\sum_{osc} = \frac{k^2T^2}{t^2}t\Delta\nu$

$\Delta\varphi_{rms} = kT\left(\frac{\Delta\nu}{t}\right)^{1/2}$ which is same as previous considerations.

Shot Effect: No space charge

i for 1e → $\underset{t_k\ \ \tau_a}{\Delta} = F(t - t_k)$

Total current for K electrons $I_K(t) = \sum_{k=0}^{K}F(t-t_k)$.

$\overline{I_K(t)} = \frac{1}{T}\int_0^T dt_1\cdots dt_K\sum_0^K F(t-t_k)$

$= \frac{1}{T}\sum_0^K\int_0^T F(t-t_k)dt_k$

$= \frac{K}{T}\int_0^T F(t-t_k)dt_k$ which is current arriving at anode in time T for K elect.

Now K is P distributed;

$\overline{I(t)} = \sum_{K=0}^{\infty}P(K)\overline{I_K(t)}$

$= \sum_0^\infty \frac{P(K)}{T}K\int_0^T F(t-t_k)dt_k = Ne$ where N is avg # of elect/sec.

$\overline{I^2(t)} = \sum_{K=0}^{\infty}P(K)\int\cdots\int\frac{dt\cdots dt}{T}\sum_{k=0}^{K}\sum_{m=0}^{K}F(t-t_k)F(t-t_m)$

$= \sum_{K=0}^{\infty}\frac{P(K)K}{T}\int_0^T F^2(t-t_k)dt_k + \sum_{K=0}^{\infty}\frac{P(K)K(K-1)}{T}\iint F_k F_m dt_k dt_m$

$= N\int_0^T F^2 dt_k + N^2e^2$

$\therefore \overline{I^2(t)} - \overline{I(t)}^2 = N\int_0^\infty F^2 dt_k$

Campbells Theorem. Can do same for $R_I(\tau)$ and $G^+(f)$. We get:

$R_I(\tau) = N^2e^2 + N\int_{-\infty}^{\infty}F(t)F(t+\tau)d\tau$

For $f \ll \frac{1}{\tau_a}$, $G^+(f=0) = 2\overline{I(t)}e$

$[\langle\Delta I\rangle_{T_0}]^2 = \frac{1}{T_0}\int_0^{T_0}\int_{-\tau}^{T_0-\tau}\Delta I(t)\Delta I(t+\tau)dt\,d\tau$

For long time avg, $\int_{-\infty}^{\infty}\Delta I(t)\Delta I(t+\tau)dt\,d\tau = \frac{1}{2T_0}G^+(f=0) = \frac{eI}{T_0}$

$= (\langle\Delta I\rangle_{T_0})^2$

Space Charge Limited Case: $G^+(f=0) = 2\bar I e\Gamma^2$

$\Gamma < 1$ and is noise suppression factor

Pentodes: partition noise and shot noise. prob. to get to anode $= \frac{i_a}{i_a + i_s}$ with $n_a = pN$

$\overline{\Delta n_a^2} = pqN$. For $\Gamma = 0$

$\overline{\Delta I_a^2} = \frac{e^2\overline{\Delta n_a^2}}{T_0^2} = \frac{\bar i_a \bar i_s e}{T_0(\bar i_a + \bar i_s)}$

$G^+(f=0) = 2T_0\overline{\Delta I_a^2} = \frac{2\bar i_a\bar i_s}{T_0(\bar i_a + \bar i_s)}$

$\Gamma \neq 0$: partition and shot noise uncorr., can add variances

$\overline{\Delta n^2} = pqN + p^2N\Gamma^2$

$(\langle\Delta i_a\rangle)^2 = \frac{pqNe^2}{T_0^2} + \frac{p^2Ne^2\Gamma^2}{T_0^2}$

$G_{ia}^+(0) = 2e\bar I_a\left[\frac{\bar I_s + \bar I_a\Gamma^2}{I_s + I_a}\right]$

Phototubes: $\overline{\Delta N^2_{sec}} = \overline{\Delta N^2_{pri}}\,\bar p^2 + \bar N_{pri}\overline{\Delta p^2}$

Other fluctuations:
Secondary emission
PE Effect
Fluct. in grid current

Johnson Noise: Drude-Lorentz Model, momentum gauss. dist. $i_e = \frac{ev_x}{L}$, Use F series on current. $\frac{\Theta}{\tau}$ = period. $a_n = \frac{2e}{\Theta L}\sum_{k=1}^{\Theta N/\tau}\int_{t_k}^{t_k+\tau}V_{xk}\cos 2\pi f_n t\,dt$

$= \frac{2e\tau}{\Theta L}\sum^{\Theta N/\tau}V_{xk}\cos 2\pi f_n t_k$.

Get: $a_n^2 = \frac{4e^2\tau^2}{\Theta^2 L^2}\sum\overline{V_{xk}^2}\cos^2$

$\overline{V_{xk}^2} = \frac{kT}{m}$, $\overline{a_n^2} = \frac{2e^2\tau kT}{\Theta mL^2}$

$G_n^+(f_n) = \lim_{\Theta\to\infty}\{\Theta\,\overline{a_n^2}\}$

$= \frac{2e^2\tau kT}{mL^2}$ for 1e.

Mul. by N. Then # of elec/volum $= \frac{N}{LA} = n$

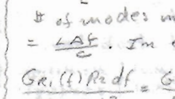
$\sigma = \frac{ne^2\tau}{2m}$, $R = \frac{\rho L}{A}$

Get: $G_i(f) = \frac{4kT}{R}$

$G_V(f) = 4kRT$.

Alt. proof using $\frac{1}{2}Li^2$ assuming wide PSD.

Nyquist Theorem, Xmission Lines



$\frac{N}{L} = \frac{1}{\lambda}$
$= f/c$

# of modes in Δf $= \frac{L\Delta f}{c}$. In equil.,

$\frac{G_{R_1}(f)R_2\,df}{(R_1+R_2)^2} = \frac{G_{R_2}R_1\,df}{(R_1+R_2)^2}$

Johnson noise cut off $f = \frac{kT}{h} \approx 10^{14}$ cps or capacity first

Brownian Motion:

$\overline{x - x_0} = \frac{mv_0}{\beta}(1 - e^{-\beta/mt})$

$\overline{(x-x_0)^2} = \frac{m^2v_0^2}{\beta^2}(1 - e^{-\frac{\beta}{m}t})^2 + \frac{2\beta kT}{\beta^2}t + \frac{2m\beta kT}{\beta^2}(-3 + 4e^{(\,)} - e^{-2(\,)})$

$\overline{(x-x_0)^2} = \frac{2mkT}{\beta^2}\left(\frac{\beta}{m}t - 1 + e^{-\beta/mt}\right)$

$= 2\frac{kT}{\beta}t$, $t \gg ($

$= \frac{kT}{m}t^2$, $t \ll \frac{m}{\beta}$

BM, consider $t \gg \frac{m}{\beta}$ For $t \to \infty$, we get Diff eq. $\frac{1}{(4\pi Dt)^{1/2}}\exp\left\{-\frac{(x-x_0)^2}{4Dt}\right\}$

$D = \frac{kT}{\beta}$. Diffusion obs. directly as $\overline{(x-x_0)^2}$ of eye power

Can also get Diff. eq. from F-P $A = 0$, $B = a^2N$ from random walk, F-P for field:

$\frac{\partial W}{\partial t} = -\frac{1}{\beta}\frac{\partial(KW)}{\partial x} + D\frac{\partial^2 W}{\partial x^2}$

$\beta = 6\pi\eta a$, $K = \frac{4\pi a^3}{3}(\rho - \rho_0)$

In SS: $W \sim e^{Kx/DB}$ which is MB.

Noise in Semicon.

Intrinsic $\alpha(n) = an^2$, $\beta(n)$ = constant.
α → absorption } prob.
β → generation

N-type: $\alpha(n) = a'n^2$
$\beta(n) = b(N_d - n)$

Radio-decay: $\alpha(n) = a'n$, $\beta(n) = 0$

In SS: $\alpha(n_0) = \beta(n_0)$

$\frac{1}{\tau} = (-\alpha' + \beta')_{n_0}$

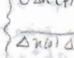
$n = n_0$

$\overline{\Delta n^2(0)} = \overline{(n - n_0)^2}$

$= -\frac{\alpha(n_0) +}{2(-\alpha'(n_0) + \beta'(n_0))}$

$G_{\Delta n}(f) = \overline{\Delta n^2(0)}\,\frac{2\tau_{decay}}{1 + \omega^2\tau^2_{dec.}}$

$\overline{\Delta n(0)\Delta n(t)} = \overline{\Delta n^2(0)}\,e^{-\tau/\tau}$

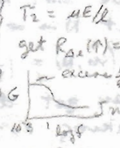


$\bar i = n_0 e^2/\tau_t$

$\overline{\Delta i^2} = \frac{e^2}{\tau_t^2}\overline{\Delta n^2}$

$T_2 = \mu E/l$

get $R_{\Delta i}(\tau) = \frac{e^2}{\tau_t^2}\overline{\Delta n^2}e^{-\tau/\tau_d}$



Excess noise due to imperfection analogous to flicker noise

Rayleigh Dist.

$i(t) = V(t)\cos(2\pi f_0 t + \varphi(t))$

$V^2(t) = A^2(t) + B^2(t)$ Freq. envelope

$\varphi = \tan^{-1}B/A$

$W(V,\varphi) = \frac{1}{2\pi\sigma^2}Ve^{-\frac{V^2}{2\sigma^2}}$

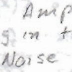
φ uniform dist → $\frac{1}{2\pi}$

Schot. Fick. Noise

$J_0\,\alpha\, e^{-e\phi/kT}$, $\Delta i = AJ_0(e^{+e\Delta V/kT} - 1)$

$R_{\Delta i}(\tau) = (AJ_0)^2\left(\frac{\Delta f}{kT}\right)\overline{\Delta N^2}\,e^{-\tau/\tau_N}$ ΔN

Galv. $I\ddot\vartheta + r\dot\vartheta + \alpha\vartheta = \gamma i + F(t)$



$-L\dot i + iR = V_c + V_{ind} + F_{el}(t)$
$-\gamma\dot\vartheta$

$\gamma = \frac{nHab}{c}$

Noise in circuits: Amp. add own noise. Everything in terms of input PSD. Eff. Noise Temp:

$G_{in}(f) = \frac{4k}{R_s}(T_S + T_N)$, $kT_N = (F-1)kT_S$.

$F = \frac{S/kT_SB}{GS/G(kT_SB + P_N)} = 1 + \frac{P_N}{kT_SB} =$

2 stages: $F_{1+2} = \frac{S/kT_SB}{G_1G_2S/G_1G_2(kT_SB + kT_{N_1}B + \frac{1}{G_1}kT_{N_2}B)}$

$= 1 + \frac{T_{N1}}{T_S} + \frac{T_{N2}}{G_1T_S}$ later stages contribute less.