APPLIED
MATH
202

PARTIAL
DIFFERENTIAL
EQUATIONS

AM 202

AM 202

APPLIED MATHEMATICS 202
BOUNDARY VALUE PROBLEMS

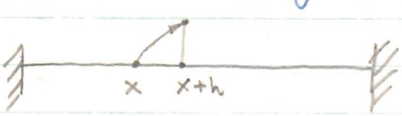
LECTURE I 2-6-61
Instructor: Carrier
Office: 306 Pierce

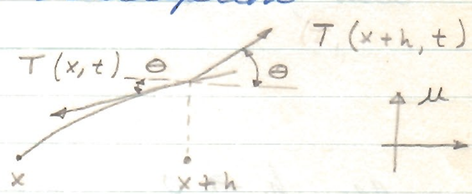
Texts:
- Titchmarsh: Eigenfunctions & Eigenvalue Expansions
- Churchill: Modern Op. Methods
- Sneddon: The Fourier Transform
- Lighthill: General Functions
- Morgenau-Murphy: Methods of Phys. & Chem.

Check Eulerian system: motion described by time and cartesian coordinates
Lagrangian system: motion described by time and original position of particles

Difference is that Eulerian uses immediate coordinates of particle and Lagrangian uses initial coordinates of system.

Vibrating string: Lagrangian Description





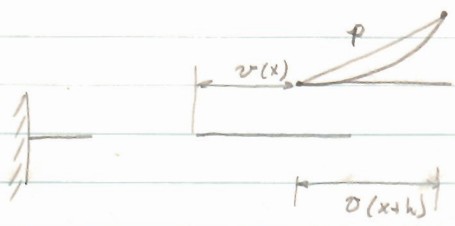
Now: $T(x+h,t)\cos\theta(x+h,t) - T(x,t)\cos\theta(x,t) = \rho A h \ddot{v}$

$$T(x+h,t)\sin\theta(x+h,t) - T(x,t)\sin\theta(x,t) = \rho A h \ddot{u}$$

We now need some physical facts:
Assume an elastic material:

$$T = \underset{\substack{\uparrow \\ \text{initial} \\ \text{tension}}}{T_0} + E A \underset{\substack{\uparrow \\ \text{elongation}}}{\epsilon}$$

The elongation and θ are geometrically related to u and v.



The length of the chord is:

$$p = \left\{ \left[u(x+h) - u(x)\right]^2 + \left[h + v(x+h) - v(x)\right]^2 \right\}^{1/2}$$

Now the elongation is $\lim_{h\to 0} \frac{p-h}{h}$, that is:

$$\frac{\left\{ \left[h + v(x+h) - v(x)\right]^2 + \left[u(x+h) - u(x)\right]^2 \right\}^{1/2} - h}{h}$$

Taking $\lim_{h\to 0}$; we get: $\epsilon = \sqrt{(1+v')^2 + (u')^2} - 1$

Also, $\sin\theta = \frac{\text{vert.}}{\text{hyp.}} = \dfrac{\frac{u(x+h) - u(x)}{h}}{\frac{\sqrt{\quad}\ h}{h}} = \dfrac{u'(x)}{\sqrt{\quad}}$

as $h \to 0$

Dividing by h on the first two equations and taking $\lim_{h\to 0}$:

$$(T\cos\theta)' = \rho A \ddot{v}$$

$$(T\sin\theta)' = \rho A \ddot{u}$$

and also:

$$T = T_0 + EA\left[\sqrt{(1+v')^2 + u'^2} - 1\right]$$

$$\sin\theta = \frac{u'}{\sqrt{\quad}} \quad ; \quad \cos\theta = \frac{1+v'}{\sqrt{\quad}}$$

which completes our set of equations of motion which we have derived rigourously. The boundary conditions are $u, v = 0$ at $x = 0, L$. We should also specify initial conditions on velocity $\dot{u}(0)$, $\dot{v}(0)$.

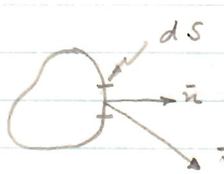
LECTURE II 2-8-61

Conservation of Momentum in a Fluid.

We assume that the fluid density is dependent on space coordinates and time. This is also true of the pressure and fluid velocity. That is:

$$\left.\begin{array}{l} \rho(x_i, t) \\ p(x_i, t) \\ u_i(x_j, t) \equiv \bar{V} \end{array}\right\} \text{Eulerian}$$

Approach to problem:



Conservation of momentum states that buildup of momentum inside the volume element is equal to the rate of momentum coming in.

The amount of mass flow coming thru dS is $\rho\bar{V}\cdot\bar{n}\,dS$. The amount of momentum is then: $(\rho\bar{V}\cdot\bar{n}\,dS)\bar{V}$. Then the total momentum convecting from the body is:

$$-\int_S (\rho\bar{V}\cdot\bar{n}\,dS)\bar{V}$$

now the contribution of the pressure on dS is $p\bar{n}\,dS$ and the total is:

$$-\int_S p\bar{n}\,dS$$

If an external field exists acting at a distance:

$$+\int_V \rho\bar{F}\,dV$$

All these terms add up to the net increase of momentum inside which is:

$$\frac{\partial}{\partial t}\int_V \rho\bar{v}\,dV$$

which is the rate of change of momentum inside. Finally:

$$-\int_S (\rho\bar{v}\cdot\bar{n}\,dS)\bar{v} - \int_S p\bar{n}\,dS + \int_V \rho\bar{F}\,dV = \frac{\partial}{\partial t}\int_V \rho\bar{v}\,dV$$

We would like to involke the divergence theorem but will have to take components of each integral along some direction. The first integral on the LHS then is:

$$v_n \operatorname{div}\rho\bar{v} + (\rho\bar{v}\cdot\operatorname{grad})v_n$$

which back in vector form becomes:

$$\bar{v}\operatorname{div}\rho\bar{v} + (\rho\bar{v}\cdot\operatorname{grad})\bar{v}$$

Then the integral equation becomes:

$$\boxed{\bar{v}\operatorname{div}\rho\bar{v} + \rho(\bar{v}\cdot\operatorname{grad})\bar{v} + \operatorname{grad}p - \rho\bar{F} + (\rho\bar{v})_t = 0}$$

Rearranging, we have the more usual form.

$$\boxed{\rho \bar{v}_t + \rho(\bar{v}\cdot \text{grad})\,\bar{v} + \text{grad}\, p = \rho \bar{F}}$$

If flow is isentropic, there is an equation between ρ and the pressure. This fact, in conjunction with conservation of mass:

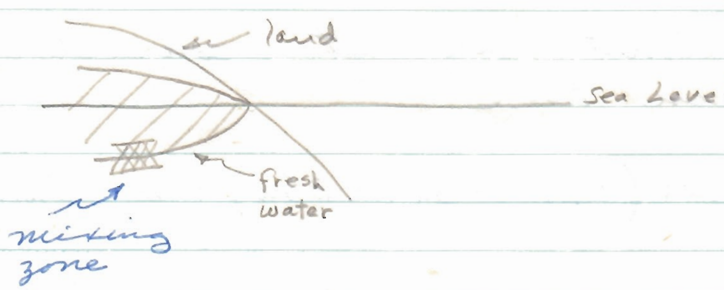
$$\text{div}(\rho \bar{v}) + \rho_t = 0$$

provides the information to solve the problem. In tensor notation:

$$\boxed{\rho u_{i,t} + \rho u_j u_{i,j} + P_{j,i} = \rho F_i}$$

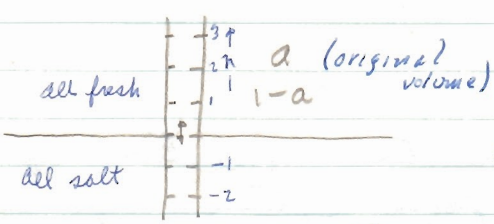
sum implied on j; $i = 1,2,3$

Seepage Problem:



problem is to find amount of mixing fresh and salt water. Problem exists with deep wells on volcanic islands

Model:



We denote salinity as:

Eulerian: $S_{n,m}$ — cell # ; time interval #

diffusion of sea water occurs in discreet intervals of time into discreet cells. Assume after a certain amount of time complete mixing takes place in each cell.

If the velocity of salt water is up or positive:

$$\underbrace{S_{n,m+1}}_{\text{time interval change}} = \underbrace{a\, S_{n,m}}_{\text{original salinity}} + \underbrace{(1-a)\, S_{n-1,m}}_{\text{added salinity from cell below}}$$

If negative: $S_{n,m+1} = a\, S_{n,m} + (1-a)\, S_{n+1,m}$

Combining:

$$S_{n,m+1} = a\, S_{n,m} + \frac{|w|+w}{2|w|}(1-a)\, S_{n-1,m}$$

$$+ \frac{|w|-w}{2|w|}(1-a)\, S_{n+1,m} \quad , \quad w \text{ is velocity.}$$

which we see reduces to either of the above equations, depending on the direction of velocity.

Rearranging:

$$S_{n,m+1} - S_{n,m} = \tfrac{1}{2}(1-a)\left(S_{n+1,m} + S_{n-1,m} - 2S_{n,m}\right)$$

$$- \frac{w}{2|w|}(1-a)\left(S_{n+1,m} - S_{n-1,m}\right)$$

We have, $|w|\, A\,(t_{n+1} - t_n) = (1-a)\, V$ (continuity equation).

or $\Delta t = \dfrac{(1-a)h}{|w|}$ as the time interval

We take the limit in Δt considering S continuous in time:

$$\frac{\partial S_n}{\partial t} = \frac{1}{2h}\left\{|w|\left(S_{n+1} + S_{n-1} - 2S_n\right) - w\left(S_{n+1} - S_{n-1}\right)\right\}$$

LECTURE III 2-10-61

Consider the momentum equation of a compressible fluid free from external forces:

$$\rho\, u_{i,t} + \rho\, u_j\, u_{i,j} + P_{,i} = 0 \qquad \sum_j \quad ; \; i = 1, 2, 3$$

We assume: $\rho(x_i, t) = \rho_0 + \epsilon\, \rho'(x_i, t)$

where ϵ characteristic of size of perturbation.

Also: $p(x_i, t) = p_0 + \epsilon\, p'(x_i, t)$

and: $u_i(x_i,t) = \epsilon\, u_i'(x_i,t)$

ϵ displays the magnitude of the perturbation. Actually quantities should be dimensionless, for example, the velocities should be compared with the speed of sound in the fluid, in order to obtain what order of the magnitude they are. We now form:

$$\frac{(\rho_0+\epsilon\rho')(\epsilon u_{i,t}) + (\rho_0+\epsilon\rho')\,\epsilon u_j'\,\epsilon u_{i,j}' + \epsilon p_{,i}' = 0}{\epsilon}$$

Taking $\lim_{\epsilon\to 0}$ and get:

$$\rho_0 u_{i,t}' + p_{,i}' = 0 \quad , \quad \rho_0 \frac{\partial \vec{v}'}{\partial t} + \operatorname{grad} p' = 0$$

Point is now to do same thing to conservation of mass equation to complete problem. Called linearization of problem. This procedure is useful in homework problems.

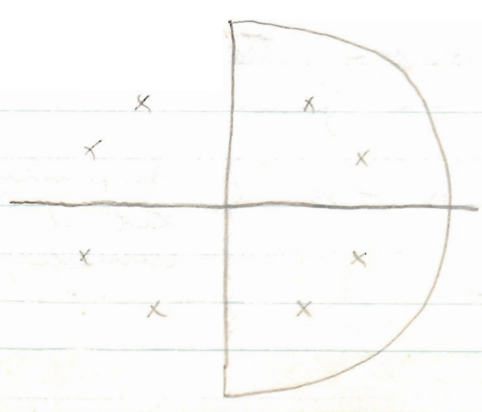
Laplace Transforms

Define: $\bar{u}(s) = \int_0^\infty u(t)\, e^{-st}\, dt$

We need an inversion formula: Consider

$$f(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{st}\,\bar{u}(s)\, ds$$

Path of Integration:



path must pass to the right of all singularities and to the right of the origin.

note that: $f(\tau) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{st}\int_0^\infty u(t)\,e^{-st}\,dt\,ds$

$$= \frac{1}{2\pi i}\frac{\partial}{\partial t}\int_{-i\infty}^{i\infty}\frac{e^{st}}{s}\int_0^\infty u(t)\,e^{-st}\,dt\,ds$$

This makes it possible to invert order of integration. Thus:

$$f(\tau) = \frac{1}{2\pi i}\frac{\partial}{\partial \tau}\int_0^\infty u(t)\left\{\int_{-i\infty}^{i\infty}\frac{e^{s(\tau-t)}}{s}\,ds\right\}dt$$

integrated over the proper paths.

$$= \frac{\partial}{\partial\tau}\int_0^\infty u(t)\,S(\tau-t)\,dt$$

Heaviside unit function

$$= \frac{\partial}{\partial\tau}\int_0^\tau u(t)\,dt = u(\tau)$$

Thus we have shown the inversion formula:

$$u(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{st}\,\bar{u}(s)\,ds$$

$$\boxed{\begin{aligned}\bar{u}(s) &= \int_0^\infty u(t)\,e^{-st}\,dt\\ u(t) &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{st}\,\bar{u}(s)\,ds\end{aligned}}$$

which is an <u>identity</u> which follows from the <u>definition</u>.

Suppose a linear differential equation:

$$u'' + a u' + b u = f(t), \qquad \begin{aligned}u(0) &= \alpha\\ u'(0) &= \beta\end{aligned}$$

we always assume that Laplace transforms exist. Then we form; and find:

$$\left.\begin{aligned}\int_0^\infty u''\,e^{-st}\,dt &= s^2\bar{u} - s\alpha - \beta\\ \int_0^\infty u'\,e^{-st}\,dt &= s\bar{u} - \alpha\end{aligned}\right\}$$

Implicitly assume that all values vanish at ∞.

Thus:

$$(s^2 + as + b)\,\bar{u} = \bar{f} + s\alpha + \beta + a\alpha$$

We can now evaluate inversion integral, at least in principle.

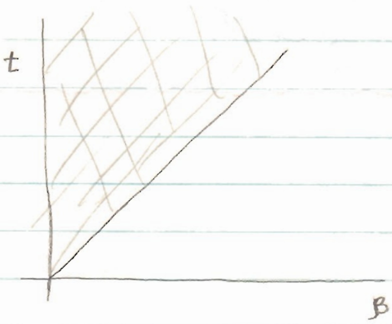
Short cut for some answers: If the transform is of form:

$$\bar{w}(s) = \bar{u}(s)\,\bar{v}(s)$$

we can use convolution integral:

$$w(t) = \int_0^t u(\beta)\, v(t-\beta)\, d\beta$$

Proof: consider $\displaystyle\int_0^\infty e^{-st} \int_0^t u(\beta)\, v(t-\beta)\, d\beta\, dt$



Form:

$$\int_0^\infty u(\beta)\, e^{-s\beta}\, d\beta \underbrace{\int_\beta^\infty v(t-\beta)\, e^{-s(t-\beta)}\, dt}_{\int_0^\infty v(t')\, e^{-st'}\, dt'}$$

$$\therefore\ \bar{w} = \bar{u}\,\bar{v}$$

Return to differential equation:

$$\bar{u} = \frac{\bar{f}}{s^2+as+b} + \underbrace{\frac{\alpha s}{s^2+as+b} + \frac{\beta - a\alpha}{s^2+as+b}}_{\text{consider known}}$$

$$\downarrow$$

$u(t) = \displaystyle\int_0^t f(t')\, v(t-t')\, dt'$ where $v(t) =$ inverse of $\dfrac{1}{s^2+as+b}$

Bessell's Equation:

$$(x u')' + x u = 0$$

Take Laplace transform: In first term, we have:

$$s \,(\text{tranform of } (x u'))$$

now: $\int e^{-sx} \, x u' \, dx = -\frac{\partial}{\partial s} \int e^{-sx} \, u' dx \; V(t-t') dt'$

We find:

$$-s \frac{\partial}{\partial s} (s \bar{u} - 1) - \frac{\partial}{\partial s} \bar{u} = 0$$

or $\quad -s\bar{u} - (s^2+1)\,\bar{u}' = 0$

which can be solved for $\bar{u}'$ by elementary methods. Thus Laplace transform can used in a limited sense on d.e. with non-constant coefficients. See Churchill.

LECTURE IV 2-20-61

Separation of Variables:

Consider the linear differential equation:

$$L(u) = 0$$

Must use coordinate system that has boundaries:

$$x_1 = \text{constant}$$
$$x_2 = \text{constant}$$
$$x_3 = \text{constant}$$

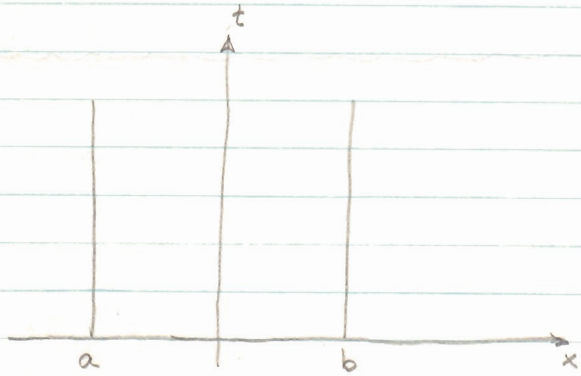
Assume a solution of the form $A(x_1)\, B(x_2)\, C(x_3)$

and we hope that u is of the form:

$$u = \sum_n a_n A_n(x_1) B_n(x_2) C_n(x_3)$$

Example: Diffusion Type Equation:

$$u_{xx} - x u_t = 0$$



Initial and Boundary Conditions:

$$u(a,t) = u(b,t) = 0$$

$$u(x,0) = f(x)$$

f is constantly continuous in $a < x < b$ and $f(a) = f(b) = 0$

Now define the Laplace Transform over the specific independent variable t:

$$\bar{u}(x,s) = \int_0^\infty e^{-st} u(x,t)\,dt$$

Now, operating on the equation:

$$\int_0^\infty e^{-st} u_{xx}\,dt = \frac{\partial^2}{\partial x^2} \int_0^\infty e^{-st} u(x,t)\,dt = \frac{\partial^2 \bar{u}}{\partial x^2} = \bar{u}_{xx}$$

$$\int_0^\infty x u_t e^{-st}\,dt = x \int_0^\infty e^{-st} u_t\,dt = x\left[s\bar{u} - u(x,0)\right]$$

$$= x\left[s\bar{u} - f(x)\right]$$

or, the new equation is:

$$\bar{u}_{xx} - x s \bar{u} = -x f(x)$$

Consider: $w_{xx} + a x w = 0$

Show that solutions are of the form:

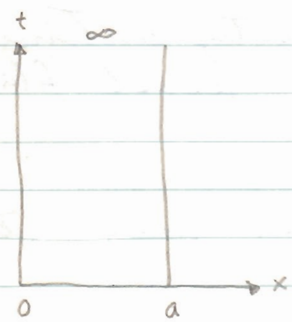
$$x^{1/2} J_{1/3}\left(\tfrac{2}{3} a^{1/2} x^{3/2}\right)$$

$$x^{1/2} J_{-1/3}\left(\tfrac{2}{3} a^{1/2} x^{3/2}\right) \quad \rightarrow \quad N_{1/3}$$

LECTURE V 2-24-61

Review of Bessel Functions:

Consider:



$$u_{xx} - u_t = 0$$
$$u(0,t) = u(a,t) = 0$$
$$u(x,0) = f(x)$$

Now: $\bar{u}(x,s) = \int_0^\infty e^{-st} u(x,t)\, dt$

which gives:

$$\bar{u}_{xx} - s\bar{u} = f(x)$$

First, let $f(x)$ be the Dirac delta function; and solve:

$$V_{xx} - sV = \delta(x - x')$$

We demand that V be continuous: $V(x'-) = V(x'+)$.
Then, integrating the equation:

$$V_x \Big]_{x'-\epsilon}^{x'+\epsilon} - sV\,2\epsilon = 1$$

or, as $\epsilon \to 0$, $V_x(x'+) - V_x(x'-) = 1$

Clearly, the solutions of $V_{xx} - sV = 0$ are cosh or sinh functions.

Involving the boundary conditions:

$$V = \begin{cases} A \sinh(x\sqrt{s}) & x < x' \\ B \sinh[(a-x)\sqrt{s}] & x > x' \end{cases}$$

From continuity:

$$A \sinh(x'\sqrt{s}) = B \sinh[(a-x')\sqrt{s}]$$

and, from the nature of the derivative:

$$-\sqrt{s}\, B \cosh[(a-x')\sqrt{s}] - \sqrt{s}\, A \cosh(x'\sqrt{s}) = 1$$

which gives:

$$V = V(x, x', s) = \begin{cases} \dfrac{\sinh[(a-x')\sqrt{s}] \sinh(x\sqrt{s})}{\sqrt{s} \sinh(a\sqrt{s})} & x < x' \\[2ex] \dfrac{\sinh[(a-x)\sqrt{s}] \sinh(x'\sqrt{s})}{\sqrt{s} \sinh(a\sqrt{s})} & x > x' \end{cases}$$

We will see that the original solution is:

$$\bar{u} = \int_0^a V(x, x', s)\, f(x')\, dx'$$

It is left to the student to show this by plugging in:

$$\bar{u}_{xx} - s\bar{u} = f(x)$$

and getting an identity: $V(x, x', s)$ is called the Greens' function.

Returning to the original equation:

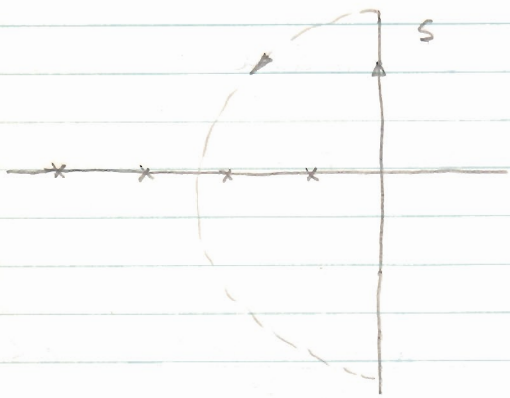
$$u(x,t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \bar{u}(x, s)\, e^{st}\, ds$$

Plugging in for $\bar{u}$:

$$u(x,t) = \frac{1}{2\pi i}\int_0^x f(x')\,dx' \left[\int_{-i\infty}^{i\infty} \frac{\sinh[(a-x)\sqrt{s}]\,\sinh(x'\sqrt{s})}{\sqrt{s}\,\sinh(a\sqrt{s})} e^{st}\,ds \right]$$

$$+ \frac{1}{2\pi i}\int_x^a f(x')\,dx' \left[\quad x' > x \quad \right]$$

The poles are at: $a\sqrt{s} = i n\pi$, $s_n = -\dfrac{n^2\pi^2}{a^2}$



Now, the residue at each pole is:

$$\left[\frac{\sinh\{(a-x)\sqrt{s_n}\}\,\sinh(x'\sqrt{s_n})\,e^{s_n t}}{a\cosh a\sqrt{s_n}} \right].$$

It can be shown by means of the Wronskian that the solution $[x' > x]$ is linearly dependent on $[x' < x]$ and thus is essentially the same. Now, we must sum the residues and finally get:

$$u(x,t) = \frac{2}{a}\sum_{n=1}^{\infty}\int_0^a f(x')\,dx' \,\sin\frac{n\pi x}{a}\,\sin\frac{n\pi x'}{a}\,e^{-\frac{n^2\pi^2}{a^2}t}$$

Now: $f(x) = u(x,0)$ so:

$$f(x) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{a}\left\{\frac{2}{a}\int_0^a \sin\frac{n\pi x'}{a} f(x')\,dx'\right\}$$

$$= \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{a} \qquad \text{(there could possibly be a sign error somewhere)}$$

if and only if a bounded solution exists in the domain and approaches the boundary conditions continuously. At points of discontinuity the series converges to the mean. For discontinuities of $f(x)$ at boundary, situation uncertain.

Problem: Generalize preceding problem, by changing boundary conditions such that:

$$u(0,t) = 0$$

$$\text{or } u(0,t) = u_x(a,t) + \sigma\, u(a,t) = 0$$

LECTURE VI   2-27-61

Recapitulation:

1) Method of solving inhomogeneous differential equation by Greens function.
2) Use of Laplace transforms.
3) Obtainment of identity

Example of Vibrating String: Separation of Variables

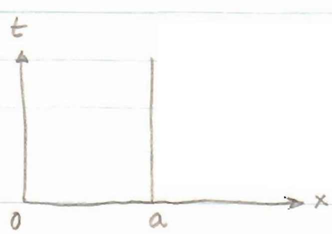
$$u_{xx} - u_{tt} = 0$$

with boundary conditions: $u(0,t) = u(a,t) = 0$
$u(x,0) = f(x)$
$u_t(x,0) = g(x)$

Assume: $u_n = X_n(x)\, T_n(t)$

$$X'' T - T'' X = 0$$

or $\dfrac{X''}{X} = \dfrac{T''}{T} = c$ as must be if solution is to hold for all x and t. For convenience, call $c = -\lambda^2$.



We usually do X equation first:

$X'' + \lambda^2 X = 0$, then: $X = A \sin \lambda x$

because of BC at $x=0$. For BC at $x=a$, $\lambda a = n\pi$ and then:

$$X_n = A_n \sin \frac{n\pi x}{a}$$

now: $T'' + \lambda T = 0$

$$T = \begin{cases} \sin \lambda t \\ \cos \lambda t \end{cases}$$

We are now at stage where we anticipate:

$$u = \sum_1 \alpha_n \sin \frac{n\pi x}{a} \sin \frac{n\pi t}{a}$$

$$+ \sum_1 \beta_n \sin \frac{n\pi x}{a} \cos \frac{n\pi t}{a}$$

as the most general equation for the given geometry. We now adjust this solution to fit initial conditions: We get:

$$f(x) = \sum_1 \beta_n \sin \frac{n\pi x}{a}$$

$$g(x) = \sum_1 \frac{n\pi}{a} \alpha_n \sin \frac{n\pi x}{a}$$

We are not yet assured that the functions can be expanded in the above series. For certain orthogonal functions, this is not possible and should always check using the Green's function method of previous lecture.

To calculate the coefficients:

$$\int_0^a f(x) \sin \frac{n\pi x}{a} dx = \int_0^a \sum_1 \beta_n \sin \frac{n\pi x}{a} \sin \frac{m\pi x}{a} dx$$

$$= \beta_n \frac{a}{2}$$

Sturm – Liouville Theory

Consider the equation:

$$[p(x)\, u']' + q(x)\, u + \lambda h(x)\, u = 0$$

with the BC: $u(0) + \alpha u'(0) = 0$, $u'(a) = 0$

$h(x)$ cannot change sign, as can be found usually from the physics of the application.

Any second order differential equation can be put into this form.
Now suppose we have found: λ_n, $u_n(x)$ where there is only one $u_n(x)$ for each λ_n.
We shall see if this is useful: We can then write:

$$[p(x) u_n']' + q u_n + \lambda_n h u_n = 0$$

$$[p(x) u_m']' + q u_m + \lambda_m h u_m = 0$$

Multiply by u_m and u_n respectively:
Subtract:

$$\left\{ p(x) \left[u_n' u_m - u_m' u_n \right] \right\}' = -(\lambda_n - \lambda_m) h(x) u_n u_m$$

Integrate over the interval a:

$$p(u_n' u_m - u_m' u_n) \Big|_0^a = (\lambda_m - \lambda_n) \int_0^a u_n(x) u_m(x) h(x) \, dx$$

Now; forming:

$$u_n'(u_m + \alpha u_m') - u_m'(u_n + \alpha u_n') \Big]_0^a = 0$$

which gives:

$$(\lambda_m - \lambda_n) \int_0^a u_n(x) u_m(x) h(x) \, dx = 0 \quad \text{(Orthogonality)}$$

or:

$$\int_0^a h(x) u_n(x) u_m(x) \, dx = 0 \qquad n \neq m$$

If $n = m$, the integral does not vanish only when $h(x)$ is one signed. We can always arrange u such that:

$$\int_0^a h u_n^2 = 1 \quad \text{(normalized)}$$

We have proved that the solutions of this equation for an orthonormal set. We have not proved that they form a complete set.

Consider: $\quad [p(x)\, u']' + q(x)\, u - h(x)\, u_t = 0$

with $u(x,0) = f(x)$. Now take Laplace transform:

$$[p(x)\, \bar{u}']' + q(x)\, \bar{u} - h(x)\, s\, \bar{u} = -h(x)\, f(x)$$

We must find the homogeneous solution. We call the homogeneous solution for each BC:

$\bar{Q}(x,s)$ for $u(0) + \alpha u'(0) = 0$

$\bar{P}(x,s)$ for $u'(a) = 0$

Program: Form integral of Green's function and then examine special p's and q's thus finding particular forms of functions.

LECTURE VII $\quad$ 3-1-61

Recall: $\quad [p(x)\, u']' + q(x)\, u - h(x)\, u_t = 0 \quad ; 0 < x < a, \; t > 0$

with the BC: $\quad a\, u'(0,t) + b\, u(0,t) = 0$
$\quad c\, u'(a,t) + d\, u(a,t) = 0$
$\quad u(x,0) = f(x)$

Transforming:

$$[p(x)\, \bar{u}']' + q(x)\, \bar{u} - s\, h(x)\, \bar{u} = -h(x)\, f(x)$$

which gives us a well defined ordinary differential equation (non-homogeneous). We select as the solutions to the homogeneous equation:

$w_1(x,s)$: 1st BC , $x < x'$
$w_2(x,s)$: 2nd BC , $x > x'$

These are generally linear independent except possibly at certain points.

We assume that we have solved for the case where the RHS is $\delta(x-x')$. Using Green's functions, we take as the expected final solution:

$$\bar{u}(x,s) = \int_0^x \frac{w_1(x',s)\, w_2(x,s)\, f(x')\, h(x')\, dx'}{Q(x')}$$

$$+ \int_x^a \frac{w_2(x',s)\, w_1(x,s)\, f(x')\, h(x')\, dx'}{Q(x')}$$

and : $$u(x,t) = \int_{-i\infty}^{i\infty} \bar{u}(x,s)\, e^{st}\, ds$$

When we plug in differential equation:

$$h f p \left[\frac{w_1' w_2 - w_2' w_1}{Q(s)} \right]$$

Thus form of w_1 and w_2 must be chosen. The Wronskian is of the form const./p, so that if $-Q(s)$ is this constant, we obtain the required identity.

Example :
$$u_{xx} + \lambda x u = 0$$
$$u(0) = u(\infty) = 0$$

Then examine :
$$\left.\begin{array}{l} u_{xx} - x u_t = 0 \\ u(0,t) = u(\infty,t) = 0 \\ u(x,0) = f(x) \end{array}\right\} \quad \bar{u}_{xx} - s x \bar{u} = -x f(x)$$

w_1 and w_2 are the well-known Bessel functions.

$$w_1 = x^{1/2}\, J_{1/3}\left(\tfrac{2}{3} s^{1/2} x^{3/2}\right)$$

$$w_2 = x^{1/2}\, J_{-1/3}\left(\tfrac{2}{3} s^{1/2} x^{3/2}\right)$$

Thus; we must have:

$$\frac{P}{Q}\left(w_1' w_2 - w_2' w_1\right) = 1$$

near $x = 0$: $w_1 = C_1 x^{1/2}\left(\frac{2}{3} s^{1/2} x^{3/2}\right)^{1/3}$

$w_1' = C_1 \left(\frac{2}{3} s^{1/2}\right)^{1/3}$

$$w_2 w_1' = C_1 \left(\frac{2}{3} s^{1/2}\right)^{1/3} C_2 x^{1/2}\left(\frac{2}{3} s^{1/2} x^{3/2}\right)^{-1/3}$$

We find that $Q = C_1 C_2$, and then:

$$\bar{u}(x,s) = \int_0^x \frac{x^{1/2} J_{1/3}\left(\frac{2}{3} s^{1/2} x^{3/2}\right)(x')^{1/2} J_{-1/3}\left(\frac{2}{3} s^{1/2} x'^{3/2}\right) x' f(x')\,dx'}{C_1 C_2}$$

$$+ \int_x^\infty \frac{x^{1/2} J_{-1/3}\left(\frac{2}{3} s^{1/2} x^{3/2}\right)(x')^{1/2} J_{1/3}\left(\frac{2}{3} s^{1/2} x'^{3/2}\right) x' f(x')\,dx'}{C_1 C_2}$$

However, there are no singularities in the integrands and hence an inverse transform won't work. Thus we must choose something different for $J_{-1/3}$. In general, Bessel's equation has solutions:

$J_n(s)$

$J_{-n}(s)$

$H_n^{(1)}(s) \sim C\, e^{is}/\sqrt{s}$

$H_n^{(2)}(s) \sim C\, e^{-is}/\sqrt{s}$

LECTURE VIII 3-3-61

Bessel Functions: Definitions:

$$J_\nu(z) = \frac{(z/2)^\nu}{\nu!}\left[1 + \sum_1^\infty a_n \left(\frac{z}{2}\right)^{2n}\right]$$

$$H_\nu^{(1)}(z) = \frac{J_{-\nu}(z) - e^{-\nu\pi i} J_\nu(z)}{i \sin \nu\pi}$$

$$J_\nu(z e^{i\pi}) \equiv e^{\nu\pi i} J_\nu(z) \quad , \quad \nu!\,(-\nu)!\,\sin\nu\pi \equiv \nu\pi$$

Then: $w_1 = x^{1/2} J_{1/3}\left[(-s)^{1/2} \frac{2}{3} x^{3/2}\right]$, $w_2 = x^{1/2} H^{(1)}_{1/3}[\;]$

Recall: $u_{xx} - x u_t = 0$

$\bar{u}_{xx} - x s \bar{u} = -x f(x)$

Can show: $w_1 w_2' - w_2 w_1' = \frac{3}{\pi i}$

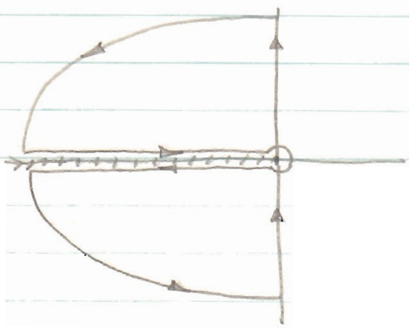
$$\bar{u}(x,s) = \int_x^\infty \frac{w_1(x)\, w_2(x')}{3/\pi i} x' f(x')\,dx' + \int_0^x \frac{w_2(x)\, w_1(x')}{3/\pi i} x' f(x')\,dx'$$

$$u = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \bar{u}\, e^{st}\, ds$$

We then form:

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{(x')^{1/2}}{3/\pi i} H^{(1)}_{1/3}\left[(se^{i\pi})^{1/2} \frac{2}{3} x'^{3/2}\right] e^{st}\, ds\; J_{1/3}\left[(se^{i\pi})^{1/2} \frac{2}{3} x^{3/2}\right]$$

Refer to Watson for properties of Bessel functions and asymptotic expansion of Hankel functions. Also Courant and Hilbert.



The integrand is of the form: $J_{1/3}(J_{1/3} + J_{-1/3})$ which makes it $J_{1/3} J_{-\frac{1}{3}}$ an entire function. The integral becomes:

$$\frac{-\left(\frac{3}{2}\right)^{-1} e^{-\nu\pi i}}{i \sin \pi\nu} \int J_{1/3} J_{1/3}\, e^{st}\, ds$$

Make substitution $s = \alpha^2 e^{i\pi}$ and get; for the upper side of the plane:

$$\int_0^\infty J_{1/3}\left[\alpha e^{i\pi} \frac{2}{3} (x')^{3/2}\right] J_{1/3}\left[\alpha e^{i\pi} \frac{2}{3} x^{3/2}\right] e^{-\alpha^2 t} (-2\alpha)\, d\alpha$$

Upon a similar operation for the lower plane, we obtain using identities:

$u(x,t) =$

Make the substitutions: $\frac{2}{3}(x')^{3/2} = \beta$, $\frac{f(x')}{\sqrt{x'}} = g(\beta)$

and get:

$$g(r) = \int_0^\infty \alpha\, J_{1/3}(\alpha r) \underbrace{\left\{ \int_0^\infty \beta\, J_{1/3}(\alpha\beta)\, g(\beta)\, d\beta \right\}}_{\text{called Hankel transform } \bar{g}(\alpha)} d\alpha$$

We can form a set of transforms:

$$\bar{g}(\alpha) = \int_0^\infty \beta\, J_{1/3}(\alpha\beta)\, g(\beta)\, d\beta$$

$$g(\beta) = \int_0^\infty \alpha\, J_{1/3}(\alpha\beta)\, \bar{g}(\alpha)\, d\alpha$$

The reason why there is an integral is because there was no poles but had branch points instead which led to integrals and Hankel Transform.

Now Consider: <u>Fourier Transforms</u>

$$u_{xx} - u_t = 0$$
$$u(-\infty,t) = u(+\infty,t) = 0 \quad , \quad u(x,0) = f(x)$$
$$\bar{u}_{xx} - s\bar{u} = -f(x)$$

The Green's function gives:

$$\bar{u} = \frac{1}{2\sqrt{s}} \left[\int_{-\infty}^{x} e^{\sqrt{s}(x'-x)} f(x')\, dx' + \int_x^\infty e^{-\sqrt{s}(x'-x)} f(x')\, dx' \right.$$

Taking inverse transform, setting $t = 0$:

$$\bar{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-i\xi x}\, dx \quad , \quad f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{f}(\xi)\, e^{i\xi x}\, d\xi$$

which are the Fourier Transform pair.

If we make the substitution $i\xi = s$, we get Laplace transform pair.

LECTURE IX   3-6-61

Resume': $\quad L(u) + \lambda h(x) u = 0 \quad , \quad L(u) = (p u')' + q u$

We found that solution to this equation given homogeneous boundary conditions would be a series expansion or integral transform. Also, the solutions form a complete set of eigenfunctions belonging to a complete set of eigenvalues.

Now suppose we have degeneracy present:

$r(x)$, $s(x)$ belonging to λ:

$$\left.\begin{array}{l} L(r) + \lambda h r = 0 \\ L(s) + \lambda h s = 0 \end{array}\right\} \quad r, s \text{ not necessarily orthogonal, but linearly independent.}$$

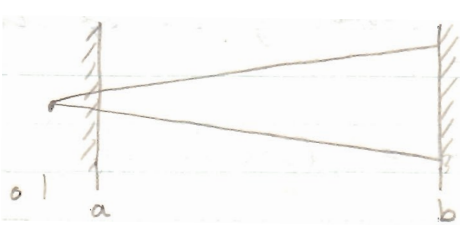
However, they can be made orthogonal:

$$\int_0^L h r \{ar + bs\} dx = 0 \quad , \quad r, (ar + bs) \text{ orthogonal}$$

or $\quad a \int_0^L h r^2 dx + b \int_0^L h r s = 0$

thus determining the ratio a/b which makes r and s orthogonal. Will always talk about them being orthogonal since we can make them so.

Applications:



$T(x) = T \quad$ (constant)

$\rho A(x) = K x$

The equation of motion is:

$$(T u_x)_x - \rho A u_{tt} = 0$$

Then: $u_{xx} - Q^2 x\, u_{tt} = 0$, $Q^2 = \frac{\kappa}{\tau}$

subject to: $u(a,t) = u(b,t) = 0$
$u(x,0) = (b-x)(x-a)$
$u_t(x,0) = 0$

Use separation of variables: solution is:

$$u = \sum a_n f_n(x)\, g_n(t)$$

$$f''g - Q^2 x f g'' = 0 \quad \text{or} \quad \frac{f''}{xf} - \frac{Q^2 g''}{g} = 0$$

$$\frac{f''}{xf} = \frac{Q^2 g''}{g} = -\lambda^2$$

which separates to: $f'' + \lambda^2 x f = 0$
$Q^2 g'' + \lambda^2 g = 0$

For g: $g = \begin{matrix}\sin \\ \cos\end{matrix} \frac{\lambda}{Q} t$

For f: $f = \alpha x^{1/2} J_{1/3}\left(\frac{2}{3}\lambda x^{3/2}\right) + \beta x^{1/2} J_{-1/3}\left(\frac{2}{3}\lambda x^{3/2}\right)$

For BC: $\alpha a^{1/2} J_{1/3}\left(\frac{2}{3}\lambda a^{3/2}\right) + \beta a^{1/2} J_{-1/3}\left(\frac{2}{3}\lambda a^{3/2}\right) = 0$
with the same at b.

Result: $\left[J_{1/3}\left(\frac{2}{3}\lambda a^{3/2}\right) J_{-1/3}\left(\frac{2}{3}\lambda b^{3/2}\right) - J_{1/3}\left(\frac{2}{3}\lambda b^{3/2}\right) J_{-1/3}\left(2/3\,\lambda a^{3/2}\right)\right]$

$= 0$ which is a transcendental equation which does have roots $\lambda_1, \lambda_2, \ldots$ We have then found the eigenvalues and eigenfunctions.

Now: $u(x,0) = \sum a_n f_n(x)\, g_n(0) = (b-x)(x-a)$
$u_t(x,0) = \sum a_n f_n(x)\, g_{n,t}(0) = 0$ or $g_n'(0) = 0$

from which we deduce that g is of the form: $g_n = \cos \frac{\lambda_n}{Q} t$

Thus: $u(x,0) = \sum a_n f_n(x) = (b-x)(x-a)$

We now form:

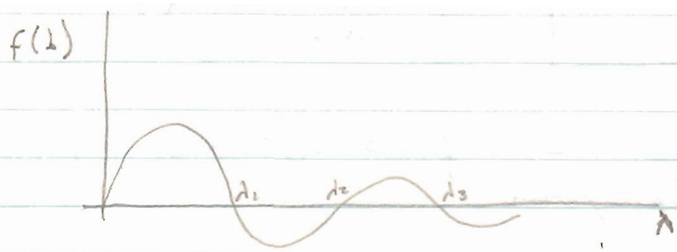
$$\sum_n a_n \int_a^b x\, f_n(x)\, f_m(x)\, dx = \int_a^b x\, f_m(x)\, (b-x)(x-a)\, dx$$

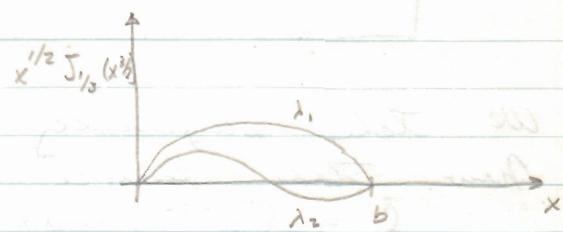
or
$$a_m \int_a^b x\, f_m^2(x)\, dx = \int_a^b x\, f_m(x)\, (b-x)(x-a)\, dx$$

which determines a_m and solves the problem in principle, $u = \sum_n a_n f_n(x) \cos \frac{\lambda_n}{a} t$.
It is hard to get physical picture from solution. However, a_n decreases with n increase, since series must converge. Make silly assumption that $a = 0$, or that string converges to point at a. Get for trancendental equation:

$b^{1/2} J_{1/3}\left(\frac{2}{3} \lambda b^{3/2}\right) = 0$ determines $\lambda_1, \lambda_2, \ldots$

Plot $x^{1/2} J_{1/3}\left(\frac{2}{3} \lambda_n x^{3/2}\right) = 0$

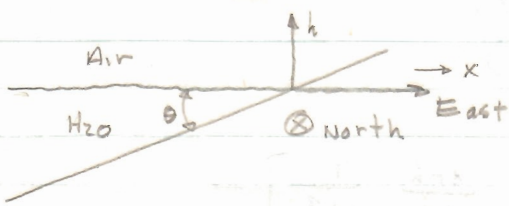




$a_n f_n(x) \cos \frac{\lambda_n}{a} t$ is a normal mode of oscillation of the system.

LECTURE X 3-8-61

Waves on the Continental Shelf:



This is a case for the linearized shallow waves equation:

$$(\eta_x h)_x + (h\eta_y)_y - \frac{1}{g}\eta_{tt} = 0$$

$$h = \alpha x$$

Assume for answer: $\eta = f(x)\, e^{i(ky-\omega t)}$

$\cos(ky-\omega t)$ is a wave traveling in the y direction with speed $\frac{\omega}{k}$ which is speed an observer must travel to see the same wave constantly.

Upon substitution:

$$(\alpha x f')' - \alpha k^2 x f + \frac{\omega^2}{g} f = 0$$

or $$\boxed{(xf')' - k^2 x f + \lambda f = 0}\ , \quad \frac{\omega^2}{\alpha g} = \lambda$$

We take physically as BC, $f=0$ at $x=\infty$.
Now take Laplace Transform:

$$\bar{f} = \int_0^\infty e^{-sx} f(x)\, dx$$

$$\int_0^\infty (xf')' e^{-sx}\, dx = \underbrace{(xf')e^{-sx}\Big]_0^\infty}_{0} + s\int_0^\infty x f' e^{-sx}\, dx$$

$$= -s\frac{\partial}{\partial s}\int_0^\infty f' e^{-sx}\, dx = -s\frac{d}{ds}\left\{s\bar{f} - f(0)\right\}$$

Result: $$-s\frac{d}{ds}(s\bar{f}) + k^2\frac{\partial}{\partial s}\bar{f} + \lambda\bar{f} = 0$$

or $$(k^2 - s^2)\bar{f}' + (\lambda - s)\bar{f} = 0$$

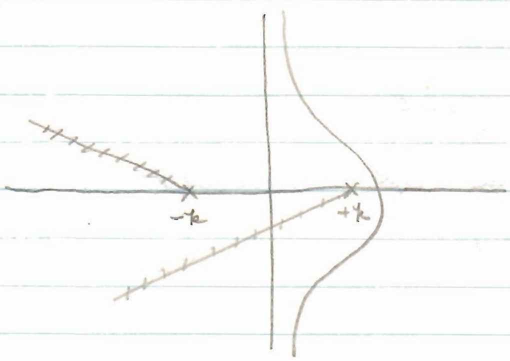
$$\bar{f} = \exp\left\{-\int^s \frac{\lambda - s}{k^2 - s^2}\, ds\right\}$$

$$= \exp\left\{+\int^s \left[\frac{\lambda - k}{2k}\frac{1}{s-k} - \frac{\lambda + k}{2k}\frac{1}{s+k}\right] ds\right.$$

$$\mathcal{F} = \exp\left\{ \ln (s-k)^{\frac{\alpha-k}{2k}} - \ln (s+k)^{\frac{\alpha+k}{2k}} \right\}$$

$$= \frac{(s-k)^{\frac{\alpha-k}{2k}}}{(s+k)^{\frac{\alpha+k}{2k}}}$$

$$f(x) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{(s-k)^{\frac{\alpha-k}{2k}}}{(s+k)^{\frac{\alpha+k}{2k}}} e^{sx}\, ds$$



Because poles in RHP cause rising exponentials, we must restrict α to $\alpha_n = (2n+1)k$, Then:

$$f(x) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{(s-k)^n e^{sx}}{(s+k)^{n+1}}\, ds$$

$$= \frac{1}{2\pi i} \int \frac{G(s)}{(s+k)^{n+1}}\, ds = \frac{G^{(n)}(-k)}{n!}$$

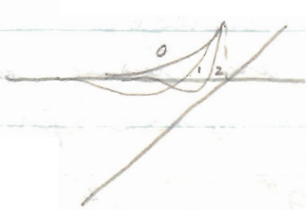
Thus the result for $f(x)$ is a polynomial found by repetitive differentiation of $(s-k)^n e^{sx}$ and plugging in $s = -k$.

Recap:

$$\eta = f(x)\, e^{i(ky-\omega t)}, \qquad \alpha_n = \frac{\omega_n^2}{g\alpha} = (2n+1)k$$

$$= R_n(x)\, e^{-kx}\, e^{i(ky-\omega t)}$$

Usual wavelengths are in miles. Waves travel in y direction with no change in x cross-section.



LECTURE XI 3-13-61

Non-Homogeneous Problems:

Stretched membrane with tension T per unit length in all directions. Differential equation is:

$$\frac{1}{r}(r u_r)_r + \frac{1}{r^2} u_{\theta\theta} - u_{tt} = f(r,\theta,t)$$

BC: $u(a,\theta,t) = 0$

Let us take $f(r,\theta,t) = g(r,\theta)\, e^{i\omega t}$

Anticipate that time dependence of result is of form $e^{i\omega t}$, that is,

$u = W(r,\theta)\, e^{i\omega t}$

and get:

$$L(W) + \lambda W = g(r,\theta) \quad ; \quad \lambda = \omega$$

We implicitly assume that the final solution is independent of the initial conditions or we are taking the steady-state solution. We first look at the homogeneous equation:

$$L(y) + \lambda y = 0 \quad , \quad y(a,\theta) = 0$$

$$\frac{1}{r}(r y_r)_r + \frac{1}{r^2} y_{\theta\theta} + \lambda y = 0$$

$$y = R(r)\, \Theta(\theta)$$

$$\frac{1}{r}(rR')'\Theta + \frac{1}{r^2} R\Theta'' + \lambda R\Theta = 0$$

$\Theta = \cos \alpha\theta$ where α is an integer.
actually, $\Theta = e^{i\alpha\theta}$, but we are taking even solutions for simplicity. This comes from:

$$\frac{r^2 \frac{1}{r}(rR')'}{R} + \lambda r^2 = -\frac{\Theta''}{\Theta} = \alpha^2$$

For the R equation:

$$R'' + \frac{1}{r}R' + \lambda R - \frac{\alpha^2}{r^2} R = 0$$

The finite solution is:

$$R = J_\alpha(\sqrt{\lambda}\, r) \quad \text{subject to} \quad J_\alpha(a\sqrt{\lambda}) = 0$$

We label the roots $\lambda_{m\alpha}$. Thus, we have for y:

$$y_{m\alpha} = J_\alpha(r\sqrt{\lambda_{m\alpha}}) \cos \alpha\theta$$

This is complete in r and θ. This implies that any function can be written:

$$g(r, \theta) = \sum_{m=0}^{\infty} \sum_{\alpha=0}^{\infty} b_{m\alpha} J_\alpha(r\sqrt{\lambda_{m\alpha}}) \cos\alpha\theta \;;\; g(\theta) = g(-\theta)$$

Now return to original equation and assert that the solution takes the form:

$$W = \sum_m \sum_\alpha q_{m\alpha} J_\alpha(\sqrt{\lambda_{m\alpha}}\, r) \cos\alpha\theta$$

We now plug in and match the unknown coefficients $q_{m\alpha}$ with the known equation coefficients $b_{m\alpha}$.

The meaning of eigenvalue is that of self-sustaining oscillations. This means that if forcing function is near λ, coefficients corresponding to λ will be very large.

Another Way: $W = \sum_\alpha l_\alpha(r) \cos\alpha\theta$

$$g = \sum_\alpha g_\alpha(r) \cos\alpha\theta$$

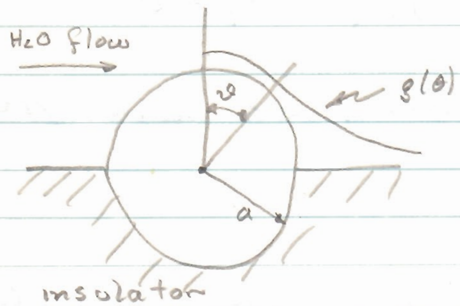
where $g_\alpha(r)$ can be found in the expansion of Bessel functions. $l_\alpha(r)$ can then be found in terms of $g_\alpha(r)$.

Example: Take $g(r,\theta) = \cos 2\theta$

By Green's functions : $\quad l(r) = \int_0^1 K(r,r')\,dr'$

$$\frac{1}{r}(r\,l')' - \frac{4}{r^2}\,l + \lambda l = 1$$

LECTURE XII   3-15-61



Radioactive sphere, uniform generation of heat. Rate of decrease of T due to H_2O flow $\propto T$.

Boundary Conditions : $\quad 0 \le |\vartheta| < \pi/2 , \quad T(a,\vartheta) + \beta T_r(a,\vartheta) = 0$

$\qquad \pi \geq |\vartheta| > \pi/2 , \quad T_r(a,\vartheta) = 0$

The heat equations are : $\quad -k\,T_r = \alpha T$

$\qquad -\operatorname{div} k \operatorname{grad} T = q_0$

Define normalized temperature : $\quad T = \dfrac{Tk}{a^2 q_0}$

therefore $\nabla^2 T = -1$ or :

$$(r^2 T_r)_r + \frac{1}{\sin\vartheta}(\sin\vartheta\, T_\vartheta)_\vartheta = -r^2$$

Take for homogeneous equation: $\quad T^* = R(r)\,\Theta(\vartheta) :$

$$-\frac{(r^2 R')'}{R} = \frac{(\sin\vartheta\,\Theta')'}{\sin\vartheta\,\Theta} = -\lambda$$

Introduce $t = \cos\vartheta$, then get Legendre's equation :

$$\left[(1-t^2)\,\Theta_t\right]_t + \lambda\,\Theta = 0$$

Has two regular singular points, $t = 1, -1$.
We can only expect one valid solution. We can use Frobenius method to get two solutions, one odd and one even series. Will find convergence if $t < 1$.

However, for certain values of λ, the recurrence formula terminates and we have a polynomial which is good for all values of t. These λ are:

$\lambda_n = n(n+1)$ and the polynomials are $P_n(\cos\vartheta)$

$n = 1$	$P_n = 1$	these polynomials are orthogonal on $-1 < t < 1$ and they also form a complete set.
$= 2$	t	
$= 3$	$1 - 3t^2$	

If we take $T(r, \vartheta) = \sum f_n(r) P_n(\cos\vartheta)$ and carry thru procedure of last lecture and find differential equation for $f(r)$. The trouble is that boundary condition depend on ϑ and there is no ϑ dependence in $f(r)$. Thus separation of variables is a foolish approach for this problem. This will never work when mixed boundary conditions exist as in this problem.

To proceed, assume that for $0 \leq |\vartheta| < \pi/2$, $T_r(a, \vartheta) = g(\vartheta)$. Now solve:

$$(r^2 f_n')' - n(n+1) f_n = \delta_{n0} r^2$$

Homogeneous solutions are r^n, r^{-n-1} ← excluded.
Thus:

$$f_0 = \frac{r^2}{6} + a_0$$

$$f_1 = a_1 r$$

$$f_2 = a_2 r^2$$

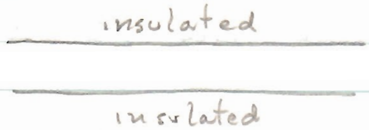
Then: $T_r(1, \vartheta) = \sum P_n(\cos\vartheta) f_n'(1) = G(\vartheta)$

$$= \sum G_n P_n(\cos\vartheta) \quad \Big\} \quad G(\vartheta) = \begin{cases} g(\vartheta), & 0 \leq \vartheta < \pi/2 \\ 0, & \pi \geq \vartheta > \pi/2 \end{cases}$$

which will give equations for determining the coefficients of f_n: $a_0, a_1, a_2, \ldots$
Can check assumption for $g(\vartheta)$ by plugging $T(r, \vartheta)$ back into original boundary condition.

LECTURE XIII   3-17-61

Integral Transforms.



$$k T_{xx} - \rho c_p T_t = 0$$
$$T(x,0) = T_0 e^{-ax^2}$$

$$\nu t = \tau = \frac{\rho c}{k} t$$

$$T_{xx} - T_\tau = 0$$

If we take: $\bar{T}(\xi,t) = \int_{-\infty}^{\infty} T(x,t) e^{-i\xi x} dx$

$$T(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{T}(\xi,t) e^{i\xi x} d\xi$$

The Fourier transforms are convenient to use as range of x is from $-\infty$ to ∞.
Multiply by $e^{-i\xi x}$ and integrate: get terms of form:

$$e^{-i\xi x} T \Big]_{-\infty}^{\infty} \text{ or } e^{-i\xi x} T_x \Big]_{-\infty}^{\infty}$$

which we tacitly assume vanishes at $\pm\infty$.

get:

$$-\xi^2 \bar{T} - \bar{T}_\tau = 0$$

$$\bar{T}(\xi,\tau) = A e^{-\xi^2 \tau}$$

$$\bar{T}(\xi,0) = \int T e^{-ax^2} e^{-i\xi x} dx = f(\xi)$$

$$\bar{T} = f(\xi) e^{-\xi^2 \tau}$$

How do we solve $I = \int_{-\infty}^{\infty} e^{-ax^2 - i\xi x} dx$?

Form:

$$-2ia\, I_\xi = \int (-2ia)(ix) e^{-ax^2 - i\xi x} dx$$

subtract $-i\xi I$ and get:

$$-2ia\, I_\xi - i\xi I = \int_{-\infty}^{\infty} (-2ax - i\xi) e^{-ax^2 - i\xi x} dx$$

or $\quad -2ia\,I_\xi - i\xi I = e^{-ax^2+i\xi x}\Big]_{-\infty}^{\infty} = 0$

$$I_\xi + \frac{\xi}{2a} I = 0 \quad ; \quad I = \underset{I(0)}{\overset{}{?}}\; e^{-\xi^2/4a}$$

$$I(0) = \int_{-\infty}^{\infty} e^{-ax^2}\,dx = \sqrt{\frac{\pi}{a}}$$

$$\therefore \quad I = \sqrt{\frac{\pi}{a}}\, e^{-\xi^2/4a} = \frac{f(\xi)}{T_0}$$

and:

$$\bar{T}(\xi,\tau) = \sqrt{\frac{\pi}{a}}\, e^{-\xi^2/4a}\, T_0\, e^{-\xi^2\tau}$$

$$T(x,t) = \frac{1}{2\pi} T_0 \sqrt{\frac{\pi}{a}} \int_{-\infty}^{\infty} e^{-\xi^2(\tau+\frac{1}{4a})+i\xi x}\,d\xi$$

Note that this is the same type of integral as before. Get:

$$T(x,t) = \frac{1}{2\pi} T_0 \sqrt{\frac{\pi}{a}} \sqrt{\frac{\pi}{T+1/4a}}\; e^{-x^2/(4\tau+1/a)}$$

$$= \frac{T_0}{\sqrt{1+4a\tau}}\, e^{-ax^2/(1+4a\tau)}$$

Gaussian-type function.

We now solve the same problem using two transforms:

$$T_{xx} - T_\tau = 0 \quad ; \text{ take Laplace transform:}$$

$$T_{xx}^* - s\,T^* = -T_0 e^{-ax^2} \quad ; \text{ take Fourier transform:}$$

$$-\xi^2\, T^{**} - s\, T^{**} = -T_0 \sqrt{\frac{\pi}{a}}\, e^{-\xi^2/4a}$$

Then: $\quad T^{**}(\xi,s) = T_0 \sqrt{\frac{\pi}{a}}\, \dfrac{e^{-\xi^2/4a}}{\xi^2+s}$

Taking inverse:

$$\frac{T_0}{2\pi}\,\frac{\sqrt{\frac{\pi}{a}}}{2\pi i}\iint \frac{e^{i\xi x + st - \xi^2/4a}}{\xi^2 + s}\,d\xi\,ds$$

Do with respect to s first and get same equation for the taking of the inverse Fourier equation. Note that Fourier and Laplace transforms are only good for constant coefficient differential equations.

Next time: Poisson's Equation: $\nabla^2\varphi = f(x,y,z)$

Use triple Fourier transform:

$$\bar{\varphi}(\xi,\eta,\zeta) = \iiint e^{-i(\xi x+\eta y+\zeta z)}\,\varphi(x,y,z)\,dx\,dy\,dz$$

$$\varphi(x,y,z) = \frac{1}{(2\pi)^3}\iiint e^{i(\xi x+\eta y+\zeta z)}\,\bar{\varphi}(\xi,\eta,\zeta)\,d\xi\,d\eta\,d\zeta$$

Take: $\bar{\varphi} = \dfrac{-\bar{f}}{\xi^2+\eta^2+\zeta^2}$, then:

$$\varphi = \frac{1}{(2\pi)^3}\iiint e^{i(\xi x+\eta y+\zeta z)}\,\frac{\bar{f}}{\xi^2+\eta^2+\zeta^2}\,d\xi\,d\eta\,d\zeta$$

$$= \frac{1}{(2\pi)^3}\iiiint\!\!\iint \frac{e^{i\{\xi(x-x')+\eta(y-y')+\zeta(z-z')\}}}{\xi^2+\eta^2+\zeta^2}\,d\xi\,d\eta\,d\zeta\,f(x',y',z')\,dx'\,dy'\,dz'$$

LECTURE XIV   3-20-61

Recall:

$$\varphi(x,y,z) = \frac{-1}{8\pi^3}\iiint dx'\,dy'\,dz'\, f(x',y',z')\iiint e^{\,i\left[\xi(x-x')+\eta(y-y')+\zeta(z-z')\right]}\,\frac{d\xi\,d\eta\,d\zeta}{\xi^2+\eta^2+\zeta^2}$$

Define a vector: $\bar{r} = \bar{i}(x-x') + \bar{j}(y-y') + \bar{k}(z-z')$

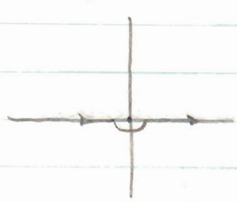
and also a spherical coordinate system: R, ϑ, γ with the vector $\bar{R}(\xi,\eta,\zeta)$ expressed in this system. We can write,

$$\frac{-1}{8\pi^3}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{e^{\,i\bar{r}\cdot\bar{R}}}{R^2}\, R^2 \sin\vartheta \, dR\, d\vartheta\, d\gamma$$

$$= \frac{1}{4\pi^2}\int_0^{\infty}\left[\frac{2e^{\,irR\cos\vartheta}}{2irR}\right]_0^{\pi} dR = \int_0^{\infty}\frac{-1}{2\pi^2 r}\,\frac{\sin(rR)}{R}\,dR$$

$$= -\frac{1}{4\pi^2 r}\int_{-\infty}^{\infty}\frac{\sin rR}{R}\,dR$$

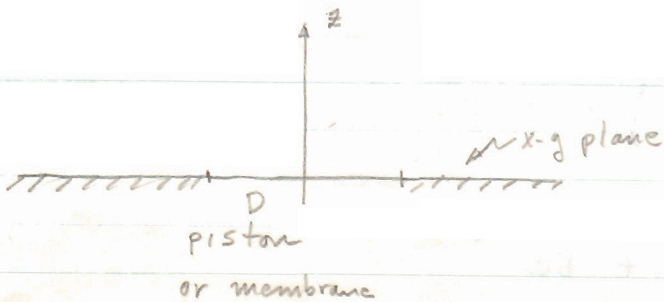
Consider $\int \frac{e^{\,irR}}{R}\,dR$ and evaluate by contour integration. We get then:



$$\varphi(x,y,z) = \frac{-1}{4\pi}\iiint \frac{f(x',y',z')\,dx'\,dy'\,dz'}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}$$

This is called Newton's Potential function which is a general solution of Poisson's equation.

Sonic Wave Problem:



Solution for acoustic field above piston:

$W = f(x,y)\,e^{\,i\omega t}$

Derivation of differential equation:

Mass: $(\rho u_i)_{,i} + \rho_t = 0$

Momentum: $\rho u_{i,t} + \rho u_j u_{i,j} + P_{,i} = 0$

Thermodynamics: $P/\rho^\gamma = P_0/\rho_0^\gamma$

Use perturbation methods to linearize equations (see homework assignment #1) and get:

$$\nabla^2 \varphi - \frac{1}{c^2}\varphi_{tt} = 0 \quad ; \quad c^2 = \gamma \frac{P_0}{\rho_0}$$

We assume that sonic field vanishes at ∞. Also, we have only waves which go outward; Sommerfeld radiation postulate. For further BC:

$$\text{on } z = 0 \quad \left\{ \begin{array}{ll} \varphi_z = 0 & \text{off } D \\ = f(x,y)\, e^{i\omega t} & \text{on } D \end{array} \right\}$$

We can look toward the solution $\varphi = \psi(x,y,z)\, e^{i\omega t}$ and get on substitution: Helmholtz equation:

$$\nabla^2 \psi + k^2 \psi = 0$$

Take Fourier transforms in x and y directions only because we get reduction to ordinary differential equation.

$$\bar{\psi}(\xi,\eta) = \iint_{-\infty}^{\infty} e^{-i(\xi x + \eta y)}\, \psi(x,y)\, dx\, dy$$

and get:

$$\left\{ \frac{\partial^2}{\partial z^2} + k^2 - \xi^2 - \eta^2 \right\} \bar{\psi} = 0$$

$$\bar{\psi} = A e^{-\sqrt{\xi^2+\eta^2-k^2}\, z} + B e^{+\sqrt{}\, z}$$

Because $\bar{\Phi}$ must vanish as $z \to \infty$, $B = 0$, and

$$\bar{\Phi} = A(\xi,\eta)\, e^{-\sqrt{\xi^2+\eta^2-k^2}\, z}$$

or $\bar{\Phi}_z(\xi,\eta,0) = -A\sqrt{\xi^2+\eta^2-k^2}$

comparing with $\bar{\Phi}_z(\xi,\eta,0) = \bar{f}(\xi,\eta)$
we finally obtain:

$$\bar{\Phi} = \frac{-\bar{f}(\xi,\eta)}{\sqrt{\xi^2+\eta^2-k^2}}\, e^{-\sqrt{\xi^2+\eta^2-k^2}\, z}$$

LECTURE XV 3-22-61

Acoustic Radiation Problem:

Recall: $\varphi = \phi(x,y,z)\, e^{i\omega t}$; $\phi(x,y,0) = f(x,y)$

$$\bar{\Phi}_{zz} - (\xi^2+\eta^2-k^2)\bar{\Phi} = 0$$

$$\bar{\Phi} = \frac{-\bar{f}\, e^{-\sqrt{\xi^2+\eta^2-k^2}\, z}}{\sqrt{\xi^2+\eta^2-k^2}}$$

which is the formal solution for any f.

Consider for f a rigid piston moving up and down. Piston is circular. Then:

$$f(x,y) = \begin{cases} 1 & x^2+y^2 < R \\ 0 & x^2+y^2 > R \end{cases}$$

Now; $\bar{f} = \iint_R e^{-i(\xi x + \eta y)}\, dx\, dy$

Transfer to polar form: Define $\alpha^2 = \xi^2 + \eta^2$
$r^2 = x^2 + y^2$

Then: $\bar{F} = \iint e^{-i\alpha r \cos\theta}\, r\, dr\, d\theta$

Recall from 201: $\int_0^{2\pi} e^{\pm i \zeta \cos\theta}\, d\theta = 2\pi J_0(\zeta)$

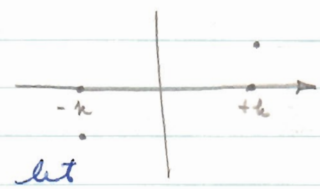
Then:

$$\bar{F} = 2\pi \int_0^R J_0(\alpha r)\, r\, dr = \frac{2\pi}{\alpha} R\, J_1(\alpha R)$$

And, then:

$$\psi = \frac{1}{4\pi^2} \iint_{\substack{-\infty \\ (0,0)}}^{\substack{(\infty, 2\pi) \\ \infty}} \frac{\frac{2\pi}{\alpha} J_1(\alpha R)\, R\, e^{-z\sqrt{\alpha^2 - k^2}}}{\sqrt{\alpha^2 - k^2}}\, e^{i\alpha r \cos\theta'}\, \alpha\, d\alpha\, d\theta$$

$$= -\int_0^\infty \frac{J_1(\alpha R)\, R\, e^{-z\sqrt{\alpha^2 - k^2}}}{\sqrt{\alpha^2 - k^2}}\, J_0(\alpha R)\, d\alpha$$

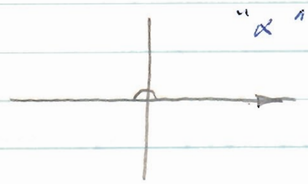


Go into complex α plane:
Branch points at $\alpha = \pm k$, however if take k complex and then let become real in limit so indentation is not important to consider now:

Theorem from Freddem:

$$\int_0^\infty \varphi(\alpha)\, J_0(\alpha r)\, d\alpha = \frac{1}{2} \int_{-\infty}^\infty \varphi(\alpha)\, H_0^{(1)}(\alpha r)\, d\alpha$$

"α" plane ($H_0^{(1)}(\alpha r)$)



This is about as far as we can carry the problem without approximation methods.

Consider again: $\nabla^2\psi + k^2\psi = 0$

in polar form: $(r\psi_r)_r + r\psi_{zz} + r k^2 \psi = 0$

Form of Bessels equation. Try Bessel transforms.

Multiply by $J_0(\alpha r)$ and integrate from 0 to ∞.

$$\int_0^\infty J_0(\alpha r)\,(r\,\psi_r)_r\,dr = J_0\,(r\,\psi_r)\Big]_0^\infty - \alpha\int (r\,\psi_r)\,J_0'(\alpha r)\,dr$$

$$-\alpha\,\psi\,r\,J_0' + \alpha\int \psi\,(r\,J_0'(\alpha r))_r\,dr$$

$$-\alpha^2\int r\,\psi\,J_0\,dr = -\alpha^2\,\bar{\psi}$$

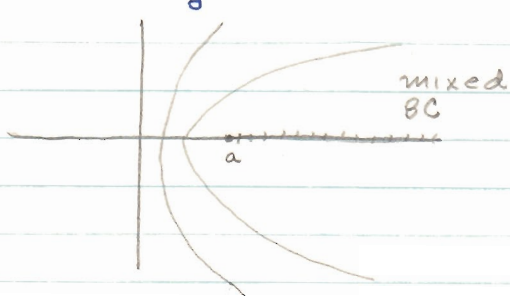
The whole equation becomes in terms of the Bessel transformed $\bar{\psi}$:

$$\bar{\psi}_{zz} - (\alpha^2 - \lambda^2)\,\bar{\psi} = 0$$

Thus we can use Bessel transform on equations of the type:

$$[u\,J_0']' + u\,\alpha r\,J_0 = 0$$

Another example: $u_{xx} + x\,u_{\alpha\alpha} = 0$
The transform function in this case is $x^{1/2}\,J_{1/3}\left(\frac{2}{3}\alpha x^{3/2}\right)e^{i\beta\alpha}$
Boundary Condition:



A better Transform would be:

$$\int_{-\infty}^{\infty} f(\alpha)\,x^{1/2}\,J_{1/3}\left(\frac{2}{3}\alpha\,x^{3/2}\right)e^{-|\alpha|\,|z|}\,d\alpha$$

The use of integral representations is sometimes better than Fourier transform analysis when mixed or complex boundary conditions exist. Usually it is hard to find the "expansion" coefficient $f(\alpha)$.

LECTURE XVI   3-24-61

Acoustic Potential:

$$\varphi = \psi e^{i\omega t}, \quad f(x,y)$$

$$\psi = \frac{1}{4\pi^2} \iint \frac{\bar{f}(\xi,\eta)\, e^{i(\xi x + \eta y)}\, e^{-z\sqrt{\xi^2+\eta^2-k^2}}}{\sqrt{\xi^2+\eta^2-k^2}}\, d\xi\, d\eta$$

$$= \frac{1}{4\pi^2} \iint f(x',y')\, dx'\, dy' \iint \frac{e^{i\xi(x-x') + i\eta(y-y') - z\sqrt{\xi^2+\eta^2-k^2}}}{\sqrt{\xi^2+\eta^2-k^2}}\, d\xi\, d\eta$$

Use the transform: $\displaystyle\int_{-\infty}^{\infty} e^{-\Gamma|z|}\, e^{-i\gamma z}\, dz = \frac{2\Gamma}{\Gamma^2+\gamma^2}$

Then:

$$\psi = \frac{1}{8\pi^3} \iint f(x',y')\, dx'\, dy' \iiint \frac{2e^{iR\cdot r}}{R^2-k^2} R^2 \sin\theta\, dR\, d\theta\, d\phi$$

$$\frac{1}{4\pi^2 r} \int_0^{\infty} \left[\frac{2e^{iRr\cos\theta}}{(R^2-k^2)\, iR} \right]_0^{\pi} R^2\, dR$$

$$\longrightarrow \frac{-1}{2\pi^2 r} \int_0^{\infty} \frac{2 \sin rR}{R^2-k^2}\, R\, dR$$

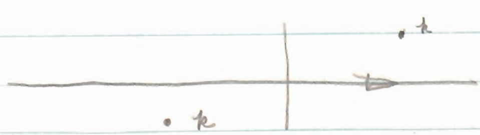
Consider k to be complex for the moment

note: $\displaystyle\frac{R}{R^2-k^2} = \frac{1}{R-k} + \frac{1}{R+k}$

$$\sin rR = \frac{e^{irR} - e^{-irR}}{2i}$$

Consider first: $\displaystyle\frac{1}{2\pi i} \int \frac{e^{irR}}{R+k}\, dR = e^{ikr}$

from UHP contour



For $\frac{1}{2\pi i}\int \frac{e^{-irR}}{R-k}\,dR$, use LHP contour to get another e^{ikr}. Get $\frac{e^{ikr}}{2\pi r}$

However, this does not give outgoing wave. If we change sign of imaginary part of k, we would get:

$$\frac{e^{-ikr}}{2\pi r} \rightarrow \frac{e^{-ikr+i\omega t}}{2\pi r}$$

which is all right.

This is how we determine the contour which satisfies the problem boundary conditions. The positions of the complex k's above imply that imaginary part of k denotes dissipation.

The above result is the Green's function. It denotes the waves emanating from a point source. Note that phase velocity is ω/k. The Green's function denotes waves from point dipole. Whole problem can be considered as double piston or single piston with interfering barrier.

Consider Helmholtz's Equation:

$$\nabla^2\varphi + k^2\varphi = H(x,y)$$

and we want to find Green's function. Use transforms.

$$(\xi^2+\eta^2-k^2)\bar{\varphi} = -\bar{H}, \quad \bar{\varphi} = \frac{-\bar{H}}{\xi^2+\eta^2-k^2}$$

$$\varphi = \frac{-1}{4\pi^2}\iint H(x',y')\,dx'\,dy' \underbrace{\iint \frac{e^{i\xi(x-x')+i\eta(y-y')}}{\xi^2+\eta^2-k^2}\,d\xi\,d\eta}_{\text{Green's Function}}$$

First we solve $\dfrac{e^{i\xi a}}{\xi^2 + b^2}\, d\xi$

assuming $\eta^2 - \kappa^2$ is positive real, that is, $b^2 = \eta^2 - \kappa^2$ and $a = x - x'$

First, $a > 0$: get $2\pi i \dfrac{e^{-ab}}{2ib}$

and, $a < 0$: get $2\pi i \dfrac{e^{-|a|b}}{2ib}$

so we are left with after first integration:

$$\int_{-\infty}^{\infty} \frac{1}{4\pi\sqrt{\eta^2-\kappa^2}}\, e^{-(x-x')\sqrt{\eta^2-\kappa^2} + i\eta(y-y')}\, d\eta$$

which gives Hankel or Bessel function depending on where the branch lines and points are chosen. One of the homework problems will be:

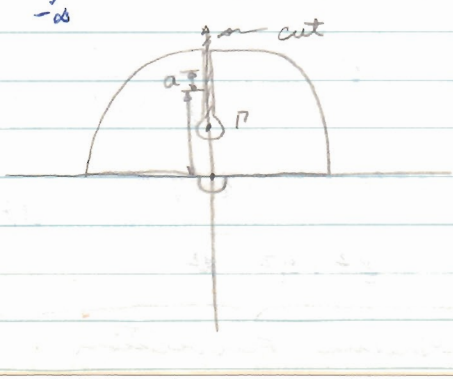
$$\nabla^2 \varphi - k^2 \varphi = H(x,y) \qquad \text{(two dimensions)}$$

and, if have time, $\nabla^2\nabla^2 \psi - \frac{\partial}{\partial x}\nabla^2\psi = f(x,y)$ in two dimensions.

LECTURE XVII 3-27-61

We consider some problems whose integrals are not simple:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi x} f(\xi)\, d\xi \quad ; \quad F(\xi) = \frac{e^{i\xi x}}{\xi\sqrt{\xi - i}}$$



The integral becomes:

$$\underbrace{e^{i\pi/4}}_{\text{from } 0} + \int_\Gamma$$

Consider contribution from Γ: For a little line segment a we have:

$$\frac{e^{-ax}\, i\,da}{ia\sqrt{i(a-1)}} \qquad \text{as } a \to \infty, \text{ this} \to 0$$

What we do is keep ξ in the denominator constant as it varies more slowly than the $\sqrt{\ }$ and the exponential. Fix ξ at i, then I becomes:

$$\frac{e^{i\xi x}}{\sqrt{\xi - i}} \quad \text{and is integrable,}$$

Formally, make expansion of rest of integrand about i:

$$\frac{1}{2\pi}\int \frac{e^{i\xi x}\,d\xi}{\xi\sqrt{\xi - i}} = \frac{1}{2\pi}\int \frac{e^{i\xi x}}{\sqrt{\xi - i}}\frac{1}{i}\left[1 - \frac{\xi - i}{i} + \left(\frac{\xi - i}{i}\right)^2\right.$$

$$\left. + \cdots + \left(\frac{\xi - i}{i}\right)^n + R_n\right] d\xi$$

$$= \sum_{n=0}^{N-1} a_n x^{-n-\frac{1}{2}} e^{-x} + \int \frac{R_N(\xi)\, e^{i\xi x}}{\sqrt{\xi - i}}\, d\xi$$

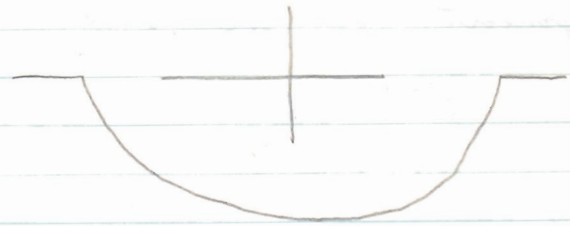
Now this is a semi-convergent series: Write:

$$F(x)\, e^{x} = \sum_{n=0}^{N-1} a_n x^{-n-\frac{1}{2}} + Q_N(x)$$

If $x^N Q_N(x) \to 0$ as $x \to \infty$
then series is semiconvergent.
There is a limit to the accuracy of these series, the first few terms give good results for high x, but taking more terms leads to more error. These are also called asymptotic series.

For small x, consider:

$$x^{1/2} \int \frac{e^{iu}\,du}{u\sqrt{u-x}}$$



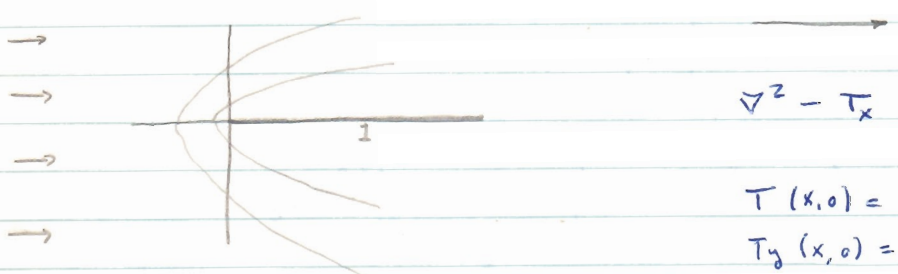
Expand:

$$\int \frac{e^{iu}\,du}{u^{3/2}} \left(1 + \frac{x}{2u} + \frac{3}{8}\frac{x^2}{u^2} + \dots \right)$$

which converges for small x, which is the reason for the large contour. These rules are the Tauberian theorems.

LECTURE XVIII 3-31-61

Wiener-Hopf Technique:



$$\nabla^2 - T_x = 0$$

$T(x,0) = 1 \quad x > 0$

$T_y(x,0) = 0 \quad x < 1$

Take Fourier transforms; with respect to x:

$$\left[\frac{\partial^2}{\partial y^2} - (\xi^2 + i\xi)\right]\bar{T} = 0$$

Then; the solution that vanishes at ∞

$$\bar{T} = A(\xi)\, e^{-y\sqrt{\xi^2 + i\xi}}$$

We pretend we know $T(x,0)$ completely.

$$T(x,0) = u(x) + v(x)$$

$$u(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}, \qquad v(x) = \begin{cases} 0 & x > 0 \\ ? & x < 0 \end{cases}$$

Then:

$$\bar{T}(\xi,0) = \bar{u}(\xi) + \bar{v}(\xi) = A(\xi)$$

And:
$$\bar{T}_y(\xi,0) = -A\sqrt{\xi^2 + i\xi}$$

which gives upon combination:

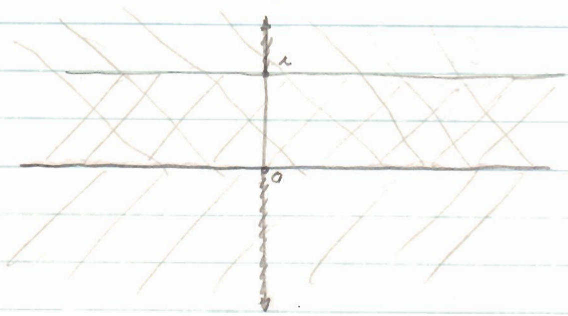
$$\frac{-\bar{T}(\xi,0)}{\sqrt{\xi^2+i\xi}} = \bar{u}(\xi) + \bar{v}(\xi)$$

Now comes the heart of W-H technique:
Note that:
$$\bar{u}(\xi) = \int_0^\infty e^{-i\xi x}\, u(x)\, dx$$

unless $u(x)$ blows up exponentially, the above integral converges. $\bar{u}(\xi)$ converge in the lower half-plane of the ξ plane: $\bar{u}_\ominus(\xi)$

Now:
$$\bar{v}(\xi) = \int_{-\infty}^0 e^{-i\xi x}\, v(x)\, dx$$

Using the same reasoning, we find that $\bar{v}(\xi)$ is analytic in the UHP: $\bar{V}_\oplus(\xi)$



$$\sqrt{\xi^2+i\xi} = \underset{\substack{\downarrow \\ \text{analytic} \\ \text{in LHP}}}{\sqrt{i\xi}}\ \underset{\substack{\downarrow \\ \text{analytic} \\ \text{above } i}}{\sqrt{1-i\xi}}$$

Then:
$$\frac{-\bar{T}_{y\ominus}}{\sqrt{i\xi}} = \bar{V}_\oplus(\xi)\sqrt{1-i\xi}_\oplus + \frac{\sqrt{1-i\xi}}{i\xi}$$

↑ F transform of step function

Make the change:
$$\frac{\sqrt{1-i\xi}}{i\xi} = \left(\frac{\sqrt{1-i\xi}-1}{i\xi}\right)_\oplus + \frac{1}{i\xi}_\ominus$$

What we have done is find a composite function by analytic continuation which is entire.

$$E(\xi) = -\frac{1}{i\xi} - \frac{\bar{T}_{y\ominus}}{\sqrt{i\xi}} = \bar{V}_\oplus \sqrt{1-i\xi}_\oplus + \left(\frac{\sqrt{1-i\xi}-1}{i\xi}\right)_\oplus$$

How does it behave at ∞? $E(\infty)=0$, and thus by Liouville's Theorem $E(\xi)=0$

$$\therefore \quad \bar{T}_y(\xi,0) = -\frac{1}{\sqrt{i\xi}}$$

Then: $$T_y(x,0) = -\frac{1}{2\pi}\int e^{i\xi x}\frac{1}{\sqrt{i\xi}}\,d\xi = \frac{-1}{\sqrt{\pi x}}$$

$$A = \frac{-1}{i\xi\sqrt{1-i\xi}}$$

and: $T(x,y) = \operatorname{erf}(c\eta)$; $(\xi+i\eta) = (x+iy)^{1/2}$

Recap: usually obtain things of the form:

$$\bar{u}_\ominus(\xi)\,\bar{K}(\xi) = \bar{f}_\ominus(\xi) + \bar{v}_\oplus(\xi)$$

Factor $\bar{K} = \bar{K}_- K_+$

Get: $$\bar{u}_\ominus(\xi)\,\bar{K}_-(\xi) = \frac{\bar{f}_\ominus(\xi)}{K_+} + \frac{\bar{v}_\oplus(\xi)}{K_+}$$

LECTURE XIX 4-10-61

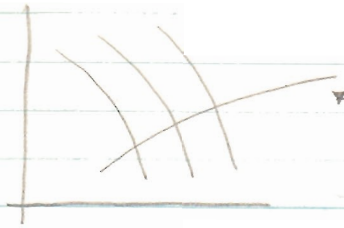
Classification of Partial Differential Equations:

Hyperbolic
Parabolic
Elliptic

$$a_{11}u_{1,x} + a_{12}u_{1,y} + a_{21}u_{2,x} + a_{22}u_{2,y} + a_1u_1 + a_2u_2 + a = 0$$
$$b_{11} \quad '' \quad '' \quad '' \quad '' \quad '' \quad '' \quad '' = 0$$

where $a_{ij} = a_{ij}(u_1, u_2, x, y)$
so that the equation is non-linear (quasi-linear).

Consider the family of interior solutions:



$n(x,y) =$ constant

$$u_{1,x} = u_{1,\alpha}\, \alpha_x + u_{1,n}\, n_x$$

where $n(x,y)$ is normal to $\alpha(x,y)$. Then:

$$a_{11} u_{1,\alpha} \alpha_x + a_{11} u_{1,n} n_x + a_{12} u_{1,\alpha} \alpha_y + a_{12} u_{1,n} n_y + \cdots$$

Write the same equation at another point and subtract, taking limit with discontinuities in the derivatives: [ ] means discontinuity in $\frac{du}{dn}$ across the curve α, but α is continuous along α. We get:

$$a_{11} n_x [u_{1,n}] + a_{12} n_y [u_{1,n}] + a_{21} n_x [u_{2,n}] + a_{22} n_y [u_{2,n}] = 0$$

$$b_{11} n_x [u_{1,n}] + b_{12} n_y [u_{1,n}] + b_{21} n_x [u_{2,n}] + b_{22} n_y [u_{2,n}] = 0$$

Now the determinant of the u's must vanish:

$$\begin{vmatrix} a_{11} n_x + a_{12} n_y & a_{21} n_x + a_{22} n_y \\ b_{11} n_x + b_{12} n_y & b_{21} n_x + b_{22} n_y \end{vmatrix} = 0$$

or

$$\begin{vmatrix} a_{11} \frac{n_x}{n_y} + a_{12} & a_{21} \frac{n_x}{n_y} + a_{22} \\ b_{11} \frac{n_x}{n_y} + b_{12} & b_{21} \frac{n_x}{n_y} + b_{22} \end{vmatrix} = 0$$

Now: $dn = n_x\, dx + n_y\, dy$ and $\frac{dy}{dx} = -\frac{n_x}{n_y}$

Now, according then to whether the roots of the quadratic determinant in $\frac{dy}{dx}$ are real and distinct, repeated real, or complex, the differential equation is hyperbolic, parabolic, or elliptic.

The characteristics are the curves across which the first derivatives are discontinuous.

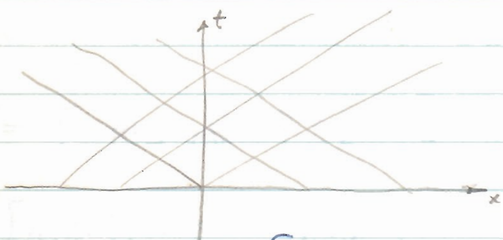
Example: $\left.\begin{aligned} u_x + p_t &= 0 \\ u_t + c^2 p_x &= 0 \end{aligned}\right\}$ $\quad u_{xx} - \frac{1}{c^2} u_{tt} = 0$

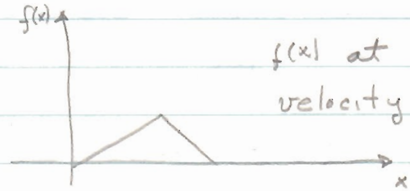
now we have: $a_{11} = 1 \qquad b_{11} = 0$
$a_{12} = 0 \qquad b_{12} = 1$
$a_{21} = 0 \qquad b_{21} = c^2$
$a_{22} = 1 \qquad b_{22} = 0$

so the determinant is:

$$\begin{vmatrix} n_x & n_t \\ n_t & c^2 n_x \end{vmatrix} = 0 \quad ; \quad \frac{n_x}{n_t} = -\frac{dt}{dx} = \pm \frac{1}{c^2}$$

$$\therefore \quad c^2 \left(\frac{n_x}{n_t}\right)^2 - 1 = 0$$





f(x) at t=0
velocity at t=0, = 0

$$u = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right]$$ (pulled from hat)

Thus propagation equation is hyperbolic with two characteristics. Assume that initial shape is propogated undistorted in the string with speed c.
For three dimensions, must use hyperspace. For wave propagation get cone: $r^2 - c^2 t^2 = 0$

Next time, find form of d.e. when characteristic are used as independent variables.

LECTURE XX   4-12-61

Recall: $\begin{vmatrix} a_{11} n_x + a_{12} n_y & a_{21} n_x + a_{22} n_y \\ - - - - - & - - - - - \end{vmatrix} = 0$

Given: x, y : n, m and n_x, m_x, n_y, m_y

Now: $n_x = \dfrac{y_m}{J}$, $n_y = \dfrac{-x_m}{J}$

Can get: $y_m = -[\ \ ]\, x_m$ (differential equation)

We will get another differential equation from the other root of the determinant.

Recall Wave Problem:

$$[u(\alpha x + h)]_x + h_t = 0$$

$$u_t + u u_x + g h_x = 0$$

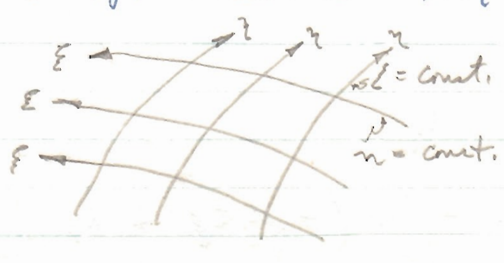
These equations are quasilinear:

$$(\alpha x + h) u_x + \alpha u + u h_x + h_t = 0$$
$$u_t + u u_x + g h_x = 0$$

Note: If: $A u_{xx} + B u_{xy} + C u_{yy} + \cdots = 0$

Then: $B^2 - 4AC \begin{cases} > 0 & \text{Hyperbolic} \\ = 0 & \text{Parabolic} \\ < 0 & \text{Elliptic} \end{cases}$

Choose two new independent variables via the curves $\xi(x,y) = \text{constant}$, $\eta(x,y) = \text{constant}$



Not necessarily orthogonal. ξ curves are curves across which discontinuities occur. Not η curves are just auxiliary

Later will find η curves also have discontinuities across them. Operating as before:

$$(\alpha x + h)[u_\xi]\xi_x + u[h_\xi]\xi_x + [h_\xi]\xi_t = 0$$

$$u[u_\xi]\xi_x + [u_\xi]\xi_t + g[h_\xi]\xi_x = 0$$

$$\begin{vmatrix} (\alpha x + h)\xi_x & u\xi_x + \xi_t \\ u\xi_x + \xi_t & g\xi_x \end{vmatrix} = 0$$

Define: $\dfrac{\xi_x}{\xi_t} = z$

$$(\alpha x + h) g z^2 = (u z + 1)^2$$

$$\sqrt{g(\alpha x + h)}\, z = \pm (u z + 1)$$ (Cannot have two solutions)

We arbitrarily choose the + sign as single solution. Now repeat for η: Choose − sign:

$$\sqrt{g(\alpha x + h)}\, Y = -(u Y + 1)$$

Choose minus sign to get different family of curves from z.

defining $\dfrac{\eta_x}{\eta_t} = Y$

now: $\xi_x = \dfrac{y_\eta}{y_\eta x_\xi - y_\xi x_\eta = J}$; $\xi_y = -\dfrac{x_\eta}{J}$

then: $\eta_x = -\dfrac{y_\xi}{J}$; $\eta_y = \dfrac{x_\xi}{J}$

where $y = t$.

This gives for z:

$$\sqrt{g(\alpha x + h)}\, t_\eta = u t_\eta - x_\eta$$

and for Y: $\sqrt{g(\alpha x + h)}\, t_\xi = +(-u t_\xi + x_\xi)$

Remember that u is a function of ξ, η.
Will get final result of u in terms of ξ, η and x, y in terms of ξ, η, so that we have parametric representation for u.
Now form:

$$(\alpha x + h)\left[u_\xi \xi_x + u_\eta \eta_x\right] + u\left[h_\xi \xi_x + h_\eta \eta_x\right]$$

$$+ h_\xi \xi_t + h_\eta \eta_t + \alpha u = 0$$

$$u\left[u_\xi \xi_x + u_\eta \eta_x\right] + u_\xi \xi_t + u_\eta \eta_t + g\left[h_\xi \xi_x + h_\eta \eta_x\right] = 0$$

LECTURE XXI 4-17-61

$$\left[(h+\alpha x)u\right]_x + h_t = 0 \qquad (u+c)\xi_x - \xi_t = 0$$

$$u_t + u u_x + g h_x = 0 \qquad (u-c)\eta_x + \eta_t = 0$$

$$(h+\alpha x)\left[u_\xi\right]\xi_x + u\left[h_\xi\right]\xi_x + \left[h_\xi\right]\xi_t$$

$$(u\xi_x + \xi_t)\left[u_\xi\right] + g\left[h_\xi\right]\xi_x = 0$$

Solve and get:

$$(u+c)\,t_\eta - x_\eta = 0 \qquad ①$$

$$(u-c)\,t_\xi - x_\xi = 0 \qquad ②$$

Now: $g(h+\alpha x) = c^2$

$$c^2\left(u_\xi \xi_x + u_\eta \eta_x\right) + u\left(c_\xi^2 \xi_x + c_\eta^2 \eta_x\right) + c_\xi^2 \xi_t + c_\eta^2 \eta_t = 0$$

$$u_\xi(\xi_t + u\xi_x) + u_\eta(\eta_t + u\eta_x) + c_\xi^2 \xi_x + c_\eta^2 \eta_x - g\alpha = 0$$

Try to eliminate ξ and η differentiated terms by multiplying by $\xi_t + u\xi_x$ and $c^2\xi_x$.

Result is:

$$\left[c^2(\xi_t + u\xi_x)\eta_x - c^2\xi_x(\eta_t + u\eta_x)\right]u_\eta + \left\{(\xi_t + u\xi_x)(u\xi_x + \xi_t) - c^2\xi_x^2\right\}c_\xi^2$$

$$+ \left\{(\xi_t + u\xi_x)(\eta_t + u\eta_x) - c^2\xi_x\eta_x\right\}c_\eta^2 - g\alpha c^2\xi_x = 0$$

Get: $-2c^3 u_\eta - 2c^2 c_\eta^2 - \dfrac{g\alpha c^2}{\eta_x} = 0$

From: $\eta_x = \dfrac{t_\xi}{t_\xi x_\eta - t_\eta x_\xi}$, use Jacobian

We have: $u_\eta + 2c_\eta + g\alpha t_\eta = 0$ (3)

In a similar manner, we can find the ξ equation the probable equation:

$u_\xi - 2c_\xi + g\alpha t_\xi = 0$ (4)

We now have enough equations to solve the problem when BC are given. Generally; integrate (3), (4):

$u + 2c + g\alpha t = \xi$ (5)

$u - 2c + g\alpha t = \eta$ (6)

If we take $\sigma = \dfrac{\xi+\eta}{2}$, $\tau = \dfrac{\xi-\eta}{2}$, $t = (\sigma\varphi')$

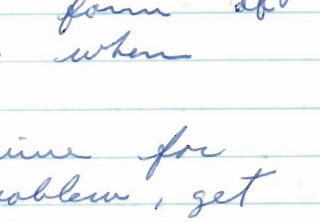
get:

$(\sigma\varphi_\sigma)_\sigma - \sigma\varphi_{\tau\tau} = 0$ (Bessell's Equation)

Must watch to see that results are physically meaningfull. Usually enters in the form of multiple valued results. Only occurs when Jacobian vanishes.

Suppose we now want to examine for small waves or linearize the problem, get same d.e. in the physical coordinates. A solution to the above d.e. is:

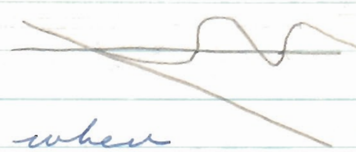
$J_0(k\sqrt{x})\,e^{i\omega t}$ (standing wave)



Waves actually look like Bessell functions except near shore.

However, now we cannot use linearized solutions to get waves breaking. Take non-linear form in σ and τ:

$$A\, H_0^{(1)}(k\sqrt{\sigma})\, e^{i\omega\tau}$$



Waves begin to break when Jacobian vanishes.

LECTURE XXII 4-19-61

Consider $\nabla^2\varphi - \frac{1}{c^2}\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)^2\varphi = 0$

$$\varphi_y(x,0) = \begin{cases} 0 & x>a \\ 0 & x<0 \\ e^{i\omega t} & 0<x<a \end{cases}$$

$$\varphi = \chi(x,y)\, e^{i\omega t}$$

$$\nabla^2\chi - \frac{1}{c^2}\left(i\omega + u\frac{\partial}{\partial x}\right)^2\chi = 0$$

Take Fourier transform in x:

$$\bar{\chi}_{yy} - \xi^2\bar{\chi} + \underbrace{\frac{1}{c^2}(\omega + u\xi)^2}_{(k+M\xi)^2}\bar{\chi} = 0$$

$$\bar{\chi} = A\, e^{-\sqrt{\xi^2-(k+M\xi)^2}\, y}$$

We must now take $\frac{u}{c} < 1$ to choose above answer. For $\frac{u}{c} > 1$, must include other solution; Now:

$$\bar{\chi}_y(\xi,0) = \frac{1}{i\xi} - \frac{e^{-ia\xi}}{i\xi} = \bar{f}(\xi) = -A\sqrt{\quad}$$

Then:

$$\bar{\chi} = \frac{-\bar{f}(\xi)}{\sqrt{\xi^2-(k+M\xi)^2}}\, e^{-\sqrt{\xi^2-(k+M\xi)^2}\, y}$$

now, let $\xi = \gamma + \frac{Mk}{1-M^2}$; $\bar{f}(\xi) = \bar{g}(\gamma)$

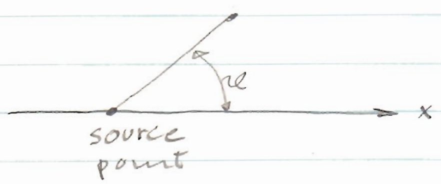
$$\bar{\chi} = - \frac{\bar{g}(\gamma)\, e^{-\sqrt{(1-M^2)\gamma^2 - \frac{k^2}{1-M^2}}\, y}}{\sqrt{(1-M^2)\gamma^2 - \frac{k^2}{1-M^2}}}$$

$$\chi = \frac{e^{-i\beta x}}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\gamma x}\, \bar{g}(\gamma)\, e^{-\sqrt{\quad}\, y}}{\sqrt{\quad}}\, d\gamma$$

Use convolution to get:

$$\chi = C\, e^{-i\beta x} \int_0^a g(x')\, H_0^{(1)}\left[\frac{k}{1-M^2}\left((x-x')^2 + (1-M^2)y^2\right)^{1/2}\right] dx'$$

Examine downstream point for $y = 0$



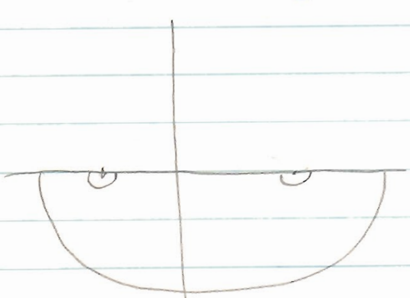
$$e^{-\frac{ik}{1-M^2}x + \frac{iMk}{1-M^2}x}$$

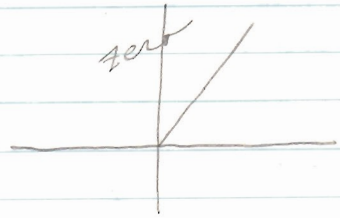
Upstream ; $e^{\frac{-ix\omega - i\omega t(c-u)}{c-u}}$

For $\frac{u}{c} > 1$, must invert forms like:

$$\frac{e^{\pm i\sqrt{M^2-1}\, y \sqrt{\gamma^2 - \frac{k^2}{(M^2-1)c}}}\; e^{i\gamma x}}{\sqrt{\quad}}$$

Will get zero when use minus sign, $\sqrt{M^2-1}\, y > x$, and following contour:





Otherwise get $J_0\left[\left((x-x')^2 - (M^2-1)y^2\right)^{1/2}\right]$

New path of integration is only over part of plate.

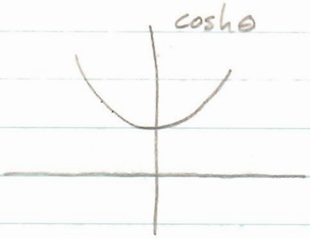
LECTURE XXIII 4-21-61

Method of Steepest Descent:

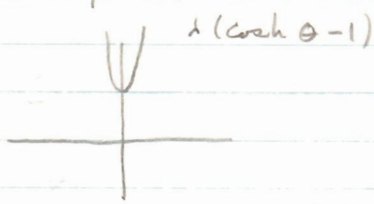
$$\int_a^b g(z)\, e^{\lambda f(z)}\, dz \qquad |\lambda| \gg 1$$

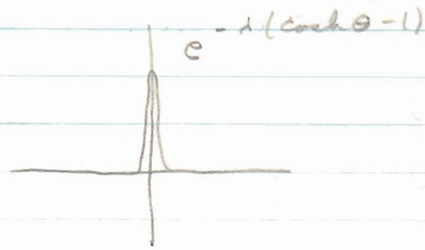
Put in form:

$$\int_{-\infty}^{\infty} e^{-\lambda \cosh\theta}\, d\theta \qquad \text{(Bessel function)}$$



$$e^{-\lambda}\int e^{-\lambda(\cosh\theta - 1)}\, d\theta$$





Expand; $e^{-\lambda\left[\frac{\theta^2}{2} + \frac{\theta^4}{24} + \cdots\right]}\, d\theta$

Take $\lambda^{1/2}\theta = u$: $\displaystyle\int \frac{e^{-\frac{u^2}{2} - \frac{u^4}{4\lambda}}}{\sqrt{\lambda}}\sqrt{2}\, du$

Can drop $\frac{u^4}{4\lambda}$ if λ small enough, don't know this at first, must check answer. Means range of integrand is accumulated in a single place, that is, it is of form:

$$\sqrt{\frac{2\pi}{\lambda}}\, e^{-\lambda}$$

Recall previous subsonic case with Mach number zero:

$$\int_{-\infty}^{\infty} \frac{e^{-y\sqrt{\xi^2-k^2}+i\xi x}}{i\xi\sqrt{\xi^2-k^2}}\,d\xi$$

Because $e^{i\xi x}$ is highly oscillatory, change path by Cauchy's integral theorem to get into form for method of steepest descent. Find some of this in Watson's Bessel Function or asymptotic expansions. Find minimum of above function. May not find true minimum but get saddle points. We consider path on saddle which gives maximum and which keeps imaginary part constant and is steep.

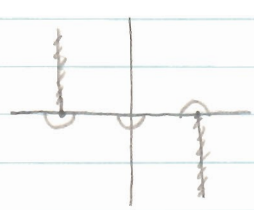
Call $\frac{x}{y} = \beta$. Now:

$$f(\xi) = -y\left[\sqrt{\xi^2-k^2} - i\beta\xi\right]$$ Take derivative:

$$\frac{\xi}{\sqrt{\xi^2-k^2}} = i\beta \quad , \quad \xi^2 = -\beta^2(\xi^2-k^2)$$

$$\xi = \frac{\pm k\beta}{\sqrt{1+\beta^2}}$$: Plug back in to get sign.

Recall:



Get; with + root:

$$\frac{k\beta/\sqrt{1+\beta^2}}{ik/\sqrt{1+\beta^2}} = \frac{\beta}{i} \neq i\beta$$

$\therefore$ + root is spurious, use −

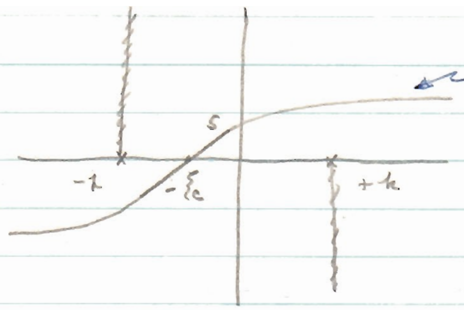
For second derivative:

$$\frac{1}{\sqrt{\xi^2-k^2}} - \frac{\xi^2}{(\xi^2-k^2)^{3/2}} = \frac{-k^2}{(\xi^2-k^2)^{3/2}} = \frac{-k^2}{i^{3/2}k^3}(1+\beta^2)^{3/2}$$

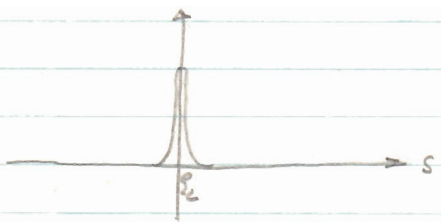
or $$f'' = \frac{(1+\beta^2)^{3/2}}{k} e^{i\pi/4}$$

Now we have:

$$e^{-y f(\xi_c)} \int e^{-y f(\xi_c) \frac{(\xi - \xi_c)^2}{2}} d\xi$$



← part of $f(\xi)$ which goes thru saddle point and whose imaginary part is constant

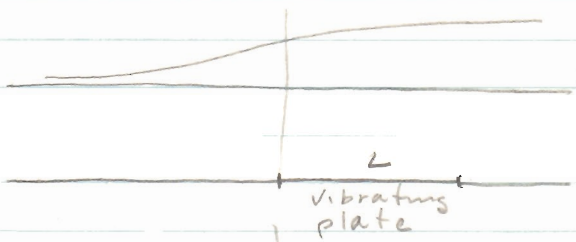


Replace ξ everywhere by ξ_c, integrate and get:

$$\frac{e^{-y f(\xi_c)}}{\imath \xi_c \sqrt{\xi_c^2 - k^2}} \frac{\sqrt{2\pi}}{\sqrt{y f''(\xi_c)}}$$

which is for large x. Can get two different answers. If we had $+\xi_c$, would include zero and would get residue at this point.

Physical Picture:



This is because of singularity at $x = 0$, since: $\xi = \dfrac{-k\beta}{\sqrt{1+\beta^2}}$

Can use this to find extent of beam: L^2/λ.

LECTURE XXIV 4-24-61

Perturbation theory: Consider:

$$u''(x) - u(x) - \epsilon e^{x} u^{2} = 0$$

$$u(0) = 1, \quad u'(0) = -1 - \epsilon, \quad 0 \le x \le 1$$

$$u(x,\epsilon) = u_0(x) + \epsilon u_1(x) + \cdots$$

$$u(0,\epsilon) = u_0(0) + \epsilon u_1(0) + \cdots = 1 + \epsilon 0 + \epsilon^2 0 + \cdots$$

$$u_0(0) = 1$$

$$\epsilon << 1: \quad u_n(0) = 0, \quad n > 0$$

$$u'(0) = u_0'(0) + \epsilon u_1'(0) + \cdots = -1 - \epsilon + \cdots$$

$$u_0'(0) = -1$$

$$u_1'(0) = -1$$

$$u_n'(0) = 0, \quad n > 1$$

$$u_0'' - u_0^{\cancel{3}} = 0$$

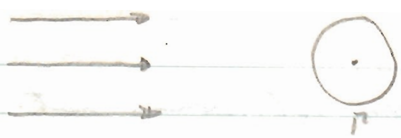
$$u_1'' - \cancel{3u_0^2}\, u_1 - e^{x} u_0^{2} = 0$$

$$u_2'' - u_2 - e^{x}\, 2 u_0 u_1 = 0$$

$$u_0 = e^{-x}, \quad u_1 = \frac{x e^{-x}}{2}$$

In partial differential equations:
Low speed compressible flow example:

$$\left\{\left[1 - \epsilon(\varphi_{,i})^2\right]^{\beta} \varphi_{,j}\right\}_{,j} = 0$$



$\varphi_{,y} \longrightarrow \delta_{iy}$ as $r \to \infty$

$\varphi_{,n} = 0$ on Γ

Write: $\varphi = \varphi_0(x,y) + \epsilon\,\varphi_1(x,y) + \dots$

$\nabla^2 \varphi_0 = 0$, $\nabla^2 \varphi_1 = F(\varphi_0, \varphi_{0,i})$

Use complex variables: $\varphi_0 + i\,\psi_0 = F_0(z)$

now $F(\psi_0, \varphi_{0,i})$ goes to:

$F(z,\bar z)$: Choose for example: $F(z,\bar z) = \sin z + z\bar z$

$\varphi_{1,z\bar z} = \sin z + z\bar z$

$\varphi_{1,z} = \bar z \sin z + z\,\frac{(\bar z)^2}{2} + p'(z)$

$\varphi_1 = -\bar z \cos z + \frac{z^2 \bar z^2}{4} + p(z) + q(\bar z)$

$p(z)$ and $q(\bar z)$ must satisfy boundary conditions. Use conformal mapping and solve Laplaces equation for each order of perturbation.

Van der Pol Oscillator:
Dimensionless equation:

$$u'' - \underbrace{\epsilon\, u'(1-u^2)}_{\text{damping term}} + u = 0$$

Are there periodic solutions? $u(t) = u(t+T)$
Anticipate:

$u = u_0(t) + \epsilon\, u_1(t) + \dots$

which gives: $u_0'' + u^0 = 0$, $u_0 = A\cos t$
choosing phase angle $= 0$.

$u'' + u = A_1 \sin t + A_2 \sin 3t$

This method is not adequate as it does not give the unknown period. Define new independent variable so that period is 2π:

$W(s) = u(t)$; $s = \frac{2\pi t}{T}$ (Method of Poincaré)

$T = 2\pi + \epsilon T_1 + \epsilon^2 T_2 + \ldots$

or writing $\omega = \frac{2\pi}{T}$:

$\omega = 1 + \epsilon \omega_1 + \omega_2 \epsilon^2 + \ldots$

The d.e. becomes:

$\omega^2 W'' - \epsilon \omega W'(1 - W^2) + W = 0$

Now: $\omega^2 = 1 + 2\epsilon \omega_1 + \epsilon^2(2\omega_2 + \omega_1^2) + \ldots$

$$\left[1 + 2\epsilon \omega_1 + \epsilon^2(2\omega_2 + \omega_1^2) + \ldots\right]\left[W_0'' + \epsilon W_1'' + \epsilon^2 W_2'' + \ldots\right]$$

$$- \epsilon \left(1 + \epsilon \omega_1 + \ldots\right)\left[W_0' + \epsilon W_1' + \ldots\right]\left[1 - W_0^2 - 2\epsilon W_0 W_1 + \ldots\right]$$

$$+ W_0 + \epsilon W_1 + \epsilon^2 W_2 + \ldots = 0$$

Get: $W_0'' + W_0 = 0$, $W_0 = A \cos s$

$W_1'' + W_1 = -\left[2\omega_1 W_0'' - W_0'(1 - W_0^2)\right]$

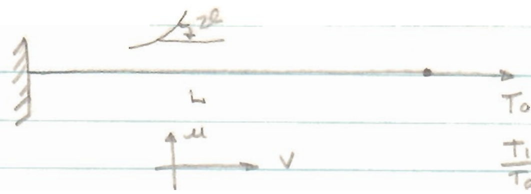
If we choose $A = 2$, $\omega_1 = 0$ we get periodic solution for W_1, of form:

$W_1 + A_1 \cos s + B_1 \sin s$

Then $W_2'' + W_2 = \left[\omega_2, A_1, B_1\right]$

LECTURE XXV   4-26-61

Forced Vibrations:



$$V_{xx} - \frac{1}{c^2} V_{tt} = 0$$

Assume that tension is function of time only and not of displacement along the string

From $\left[(T_0 + T_1 \cos\omega t) \sin v \; u_x \right]_x = \rho A \, u_{tt}$

we get for small displacements:

$$\frac{T_0 + T_1 \cos\omega t}{\rho A} u_{xx} = u_{tt}$$

Define the dimensionless independent variables:

$$\xi = \frac{x\pi}{L} \quad , \quad \eta = t\sqrt{\frac{T_0}{\rho A}}\,\frac{\pi}{L}$$

Then: $(1 + \epsilon \cos k\eta)\, u_{\xi\xi} = u_{\eta\eta}$

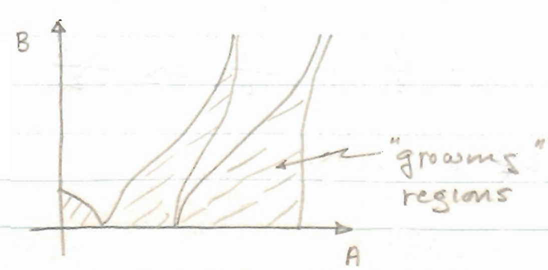
Do only for the fundamental mode and look for solution of form $u = \sin\xi \, f(\eta)$ and get:

$$f'' + (1 + \epsilon \cos k\eta) f = 0$$

which is a form of Mathieu's equation:

$$f''(v) + (A + B\cos v) f(v) = 0$$

For what values of A and B are there "growing" solutions??



Reference on this equation in Stokes: note existence of Floquet theorem. $A \sim k$, $B \sim \epsilon$

Now use perturbation methods for small tensions. Change variables again:

$\vartheta = k\eta \quad g(\vartheta) = f(\eta)$ and get:

$$k^2 g'' + (1 + \epsilon \cos \vartheta) g = 0$$

In the zeroth order: $g = P e^{i\vartheta/k}$

We now look for sub-harmonic solutions in $\vartheta/2$. Now:

$$g = g_0(\vartheta) + \epsilon g_1(\vartheta) + \dots$$

$$k^2 = k_0^2 + \epsilon a + \epsilon^2 b + \dots$$

$$(k_0^2 + \epsilon a + \epsilon^2 b + \dots)(g_0'' + \epsilon g_1'' + \dots) + (1 + \epsilon \cos \vartheta)(g_0 + \epsilon g_1 + \dots) = 0$$

$$k_0^2 = 4, \quad \therefore \quad g_0 = e^{i\vartheta/2} \quad \text{or} \quad g_0 = Re\, e^{i(\frac{\vartheta}{2} - \alpha)}$$

Now: $a g_0'' + 4 g_1'' + g_1 + g_0 \cos \vartheta = 0$

Then: $$4 g_1'' + g_1 = Re\left\{\frac{a}{4} e^{i(\frac{\vartheta}{2} - \alpha)} - \frac{e^{i(\frac{3}{2}\vartheta - \alpha)} + e^{i(\frac{\vartheta}{2} - \alpha)}}{2}\right\}$$

or $$e^{-i\alpha}\left[\frac{a}{4} e^{i\frac{\vartheta}{2}} - \frac{1}{2} e^{-i\frac{\vartheta}{2}}\right]$$

If we choose $\alpha = 0$ and $a = 2$ to get $e^{-i\alpha}\, 2i \sin\frac{\vartheta}{2}$

Using the values of the parameters:

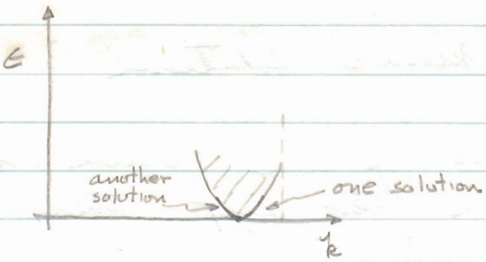
$$4 g_1'' + g_1 = -\tfrac{1}{2} Re\left[e^{i(\frac{3}{2}\vartheta - \alpha)}\right]$$

$$g_1 = K \cos\frac{3\vartheta}{2} \quad ; \quad -8K = -\frac{1}{2}$$

Thus: $$g(\vartheta) = \cos\frac{\vartheta}{2} + \frac{\epsilon}{16}\cos\frac{3\vartheta}{2} + \dots + \alpha_1 \epsilon \cos\frac{\vartheta}{2} + \alpha_2 \epsilon \sin\frac{\vartheta}{2}$$

$$k^2 = 4 + 2\epsilon + \dots$$

Parameters chosen give only one of many possible solutions.



Recall that complete form of tension in string is given by:

$$T = T_0 + T_1 \cos \omega t + \frac{EA}{L} \int_0^L \frac{u'^2}{2}\, dx$$

which gives:

$$f''(\eta) + \left(1 + \epsilon \cos k\eta + \frac{f^2}{4\pi}\right) f = 0$$

This limits hieghts of waves as non-linear term becomes comparable to linear term.

LECTURE XXVI   4-28-61

Driven Oscillator:

$$u'' - \epsilon u'(1-u^2) + u = A \cos \omega t$$

If driving frequency is not far from natural frequency, we get oscillation at driving frequency. If ω is three times natural frequency, get subharmonic at $\omega/3$, straight synchronization:

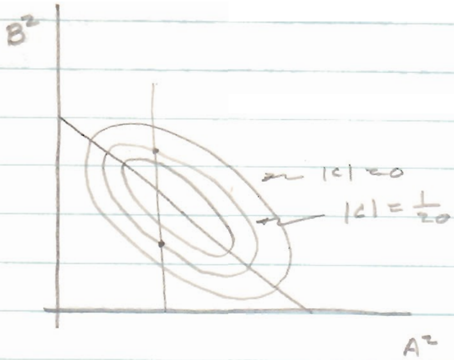
Substitute: $t' = \omega t$, $\frac{\epsilon}{\omega^2} = \epsilon'$ and drop primes:

$$u'' - \epsilon u'(1-u^2) + (1 + c\,\epsilon) u = A \cos t$$

of order 1, since $\omega \sim 1$, which is the natural frequency.

Solution of form: $B \cos t + O(\epsilon)$

Subharmonic: $B \cos t + C \cos 3t + O(\epsilon)$



Sub-harmonic Solutions

Can see if we drive too hard or too soft, there are no solutions.

Reference: Cohen, Proc. of Colloq. on Non-Linear Vib. (1951).

Use perturbations:

$$u = u_0(t) + \epsilon u_1(t) + \cdots$$

$$u_0'' + \epsilon u_1' + \cdots - \epsilon(u_0' + \epsilon u_1')(1 - u_1^2 - 2\epsilon u_0 u_1 + \cdots)$$

$$+ (1 + c\epsilon)(u_0 + \epsilon u_1 + \cdots) = A \cos 3t$$

$$u_0'' + u_0 = A \cos 3t$$

$$u_0'' = B \cos t + B' \sin t - \frac{A}{8} \cos 3t$$

If we go back and take the driver to be $A \cos(3t - \alpha)$, we can choose α to eliminate B' and get:

$$u_0 = B \cos t - \frac{A}{8} \cos(3t - \alpha)$$

$$u_1'' + u_1 + c u_0 - u_0'(1 - u_0^2) = 0$$

We see that it is probably not possible to get a periodic solution. What is trouble? May be in strength of driver amplitude A. Try weaker: ϵA

set:

$$u_0'' + u_0 = 0 \quad , \quad u_0 = B \cos t$$

$$u_1'' + u_1 + \underbrace{c u_0}_{cB \cos t} - \underbrace{u_0'(1 - u_0^2)}_{+B \sin t (1 - B^2 \cos^2 t)} = A \cos(3t - \alpha)$$

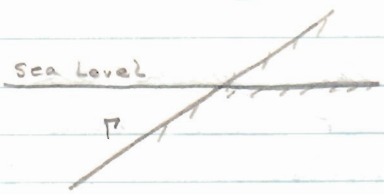
$$u_1'' + u_1 = -CB\cos t - B\left(1-\frac{B^2}{4}\right)\sin t + \frac{B^2}{4}\sin 3t$$

$$+ A\cos(3t-\alpha)$$

Carrier made mistake: A must be of order 1 instead of ϵ so first attempt was the correct one.

LECTURE XXVII 5-1-61

Flow Through Porous Media (Volcanic Oahu):



we replace conservation of momentum by $\vec{q} \sim -\text{grad}(p-p_0)$
$\delta\rho \sim \delta(p-p_0)$ (D'Arcy's Law)

For conservation of mass:

$$\text{div}(\rho A\vec{q}) + (\rho A)_t = 0$$ where A is the porosity.

Defining $\vec{q} = \text{grad}\,\varphi$: $\boxed{\nabla^2\varphi = \epsilon\varphi_t}$

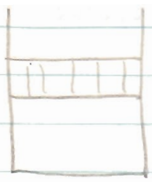
φ is essentially $p - p_0$.

Boundary Conditions: $\varphi(\Gamma) = e^{i\omega t}$

At the free surface:

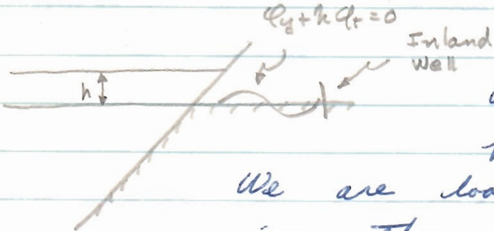
$$\varphi = -C\{\underbrace{p(x,y,t)}_{0} - \underbrace{p_0(x,y,t)}_{\rho g\eta}\} \quad \text{or} \quad \varphi(x,0,t) = C\rho g\eta$$

The flow equation is actually: $\vec{q} = \frac{a^2}{\mu}\,\text{grad}(p-p_0)$ where a is an average pore radius. Also relating density and pressure: $(\rho-\rho_0) = \gamma(p-p_0)$
μ is viscosity.



The mass flow rate is:

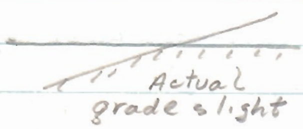
$$\dot{m} = \rho v A \quad \text{or} \quad \vec{v} = \frac{\vec{\dot{m}}}{\rho A}$$



Waves occur at a few cycles per day.

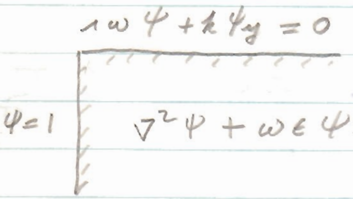
We are looking for the tidal fluctuations in the inland water table. We do not differentiate between fresh and salt water. Particle remains in surface giving $\eta_t = \varphi_y$.

Inland Boundary Conditions:



If we take the grade as flat, we must use the Wiener-Hopf method.

Other choice is to take vertical boundary condition.



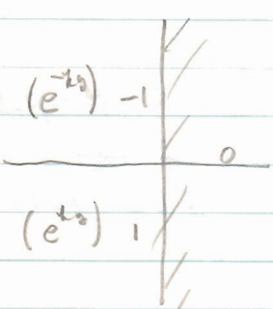
Taking for solution $\varphi = \psi(x,y)\,e^{i\omega t}$:

$$\nabla^2\psi + \omega\epsilon\psi = 0 \quad \text{(Helmholtz Equation)}$$

Choose the variable $\chi = k\psi_y + i\omega\psi$ and form an new problem:

$$\nabla^2\chi + \alpha^2\chi = 0\,; \quad \begin{array}{ll} \chi = 1 & \text{on } x=0,\ y<0 \\ \chi = 0 & \text{on } x>0,\ y=0 \end{array}$$

We now use symmetry to get a half plane problem instead of a quarter plane problem; We can do this as equation and BC are symmetric.



Now take F-transform:

$$\bar{\chi}_{xx} + (\alpha^2 - \xi^2)\bar{\chi} = 0$$

$$\text{BC;}\ \int_{\infty}^{0} + e^{-i\xi y + ky}\,dy - \int_{0}^{\infty} e^{-i\xi y - ky}\,dy$$

We get for $\bar{\chi}$:

$$\bar{\chi} = B(\xi)\, e^{-\sqrt{\xi^2-\alpha^2}\,x} = \frac{2\lambda\xi}{\lambda^2+\xi^2}\, e^{-\sqrt{\xi^2-\alpha^2}\,x}$$

This is in Campbell and Foster. Usually can neglect α which involves ϵ and this simplifies problem. Reason for using χ instead of ϕ is to make use of symmetry properties of boundary conditions in order to use F-transforms. We have rendered a non-symmetric boundary condition into a symmetric boundary condition.

LECTURE XXVIII 5-3-61

Approximation Methods:

1) Linearization
2) Idealization of Geometry

When a coordinate system can be chosen such that a coordinate = constant for the boundary conditions, usually separation of variables will work.

Review of Separation of Variables:

$L(u) = F(x,y)$

Consider $L(u) = 0$

Choose $u_n(x,y) = p(x)\, q(y)$, which should give a family of ordinary differential equations, each including the separation constant.

Completeness Theorem: If $[l(x)\, p']' + q(x)\, p - \lambda\, r(x)\, p = 0$

and $\lim_{N\to\infty} \int_a^b \left[f(x) - \sum_{n=0}^{N} a_n\, p_n(x)\right]^2 r(x)\, dx = 0$

then $p_n(x)$ form a complete set of functions.

If boundary conditions are homogeneous, have eigenvalue problem.

BC not homogeneous:

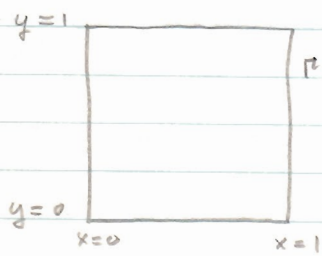
1) $u = \sum a_n u_n(x,y)$

Equation not homogeneous:

Expand: $F = b_n(y)\, p_n(x)$

and solution is $u = \sum c_n(y)\, p_n(x)$

Example: Membrane:



Consider forced vibrations of system:

$$\nabla^2 u - \frac{1}{c^2} u_{tt} = e^{i\omega t}$$

Choose for solution: $u = w(x,y)\, e^{i\omega t}$ and get:

$\nabla^2 w + k^2 w = 1$; we take for boundary condition $w(\Gamma) = 0$

Solve homogeneous case: $w = X(x)\, Y(y)$

Get:

$$-\frac{X''}{X} = \frac{Y''}{Y} + k^2 = \lambda$$

Consider: $X'' + \lambda X = 0$ subject to $X(0) = X(1) = 0$

$$X = \alpha \sin(x\sqrt{\lambda}) + \beta \cos(x\sqrt{\lambda}) \quad ; \quad \beta = 0, \quad \lambda_n = n^2\pi^2$$
$$= \sin n\pi x$$

Expand final solution in terms of these eigenfunctions:

$$W(x,y) = \sum_{n=1}^{\infty} a_n(y) \sin n\pi x$$

and resubstitute:

$$\sum \left\{ a_n'' + (k^2 - n^2\pi^2) a_n \right\} \sin n\pi x = 1$$

$$1 = \sum_{n=1}^{\infty} b_n(y) \sin n\pi x$$

or $\int_0^1 \sin n\pi x \, dx = \sum \int_0^1 b_n(y) \sin n\pi x \sin m\pi x \, dx$

which gives $b_n(y) = \frac{4}{n\pi}$ for n odd

Get $\sum_{n=1}^{\infty} \left\{ a_n'' + (h^2 - n^2\pi^2) a_n \right\} \sin n\pi x = \sum_{1,3,5}^{\infty} \frac{4}{n\pi} \sin n\pi x$

or $a_n'' + (h^2 - n^2\pi^2) a_n = \frac{4}{n\pi}$

subject to $a_n(0) = a_n(1) = 0$. This equation can presumably be solved and this completes the problem. When the BC on one of the separated variables is homogeneous, expand total solution as a series of these eigenfunctions.

Separation of variables is the easiest way to solve many partial differential equation problems.

Proof of Completeness:

$L[w(x)] + \lambda w(x) = 0$, $w(a) = w(b) = 0$

Convert to time problem:

$L(u) - u_t = 0$, $u(a,t) = u(b,t) = 0$; $u(x,0) = f(x)$

Take Laplace Transform:

$L(\bar{u}) - s\bar{u} = -f(x)$

We assume the homogeneous solution known; and we take the Green's function form for total solution:

$$\bar{u}(x,s) = -\int_a^b K(s,x,x') f(x') \, dx'$$

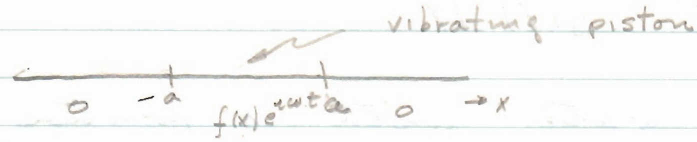
We now invert and change orders of integration:

$u(x,t) = \sum q_n(x,t)$ where the q_n's come from the residues.

Since $f(x) = u(x,0)$ can show that solutions form a complete set for solutions that approach BC continuously.

LECTURE XXX 5-8-61

Acoustic Potential Problem



$$\nabla^2 \varphi - \frac{1}{c^2}\varphi_{tt} = 0 \quad ; \quad \nabla^2 \psi + k^2 \psi = 0 \quad \text{if } \varphi = \psi(x,y)e^{i\omega t}$$

with $\psi_y(x,0) = \begin{cases} f(x) & , |x| < a \\ 0 & , |x| > a \end{cases}$

Using: $\bar{\Psi}(\xi, y) = \int_{-\infty}^{\infty} \psi(x,y)\, e^{-i\xi x}\, dx$

$\int_{-\infty}^{\infty} \psi_{xx}\, e^{-i\xi x}\, dx \to -\xi^2 \bar{\Psi}$ plus contributions at ∞ which we always tacitly assume vanish.

Usually proceed on basis of this assumption and then check results.

We now get:

$$\bar{\Psi}_{yy} + (k^2 - \xi^2)\bar{\Psi} = 0 \quad \text{or :} \quad \bar{\Psi} = A(\xi)\, e^{-\sqrt{\xi^2 - k^2}\, y} + B(\xi)\, e^{+\sqrt{\xi^2 - k^2}\, y}$$

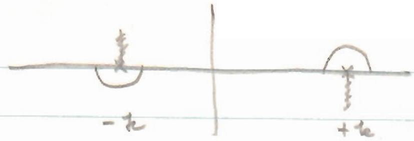
Since the last term is a growing exponential so $B(\xi)$ must vanish, even for $\xi < k$, since the analytic continuation of $B(\xi)$ into $\xi < k$ must still be zero.

Now:

$$\bar{\Psi}_y(\xi, 0) = \bar{f}(\xi) \quad \text{and} \quad \bar{\Psi}_y(\xi,0) = -\sqrt{\xi^2 - k^2}\, A(\xi)$$

Then: $$\bar{\Psi} = \frac{-\bar{f}(\xi)}{\sqrt{\xi^2 - k^2}}\, e^{-\sqrt{\xi^2 - k^2}\, y}$$

What is the path of integration for outgoing waves?

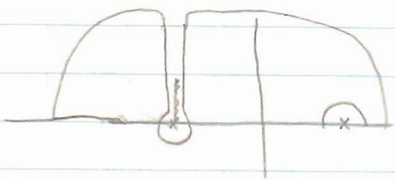


Either this path or the opposite. Chosen to give proper sign of exponential in y.

Our inversion formula is of the form:

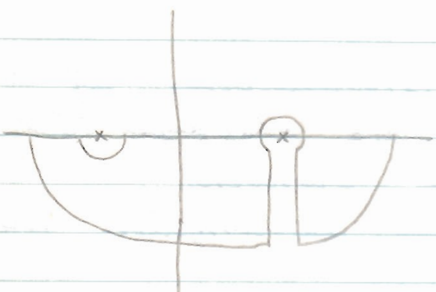
$$e^{i\omega t} \int e^{i\xi x} e^{-\sqrt{\xi^2 - k^2}\, y}\, g(\xi)\, d\xi$$

We try the above path with x positive in the integrand above. Using Cauchy's integral Theorem, we change contours to:



We have: $e^{i\omega t} \int e^{-ikx - bx}\, g(\xi)\, d\xi$

which gives an outgoing wave to the right. What about waves to left?



This contour corresponds to negative x and an outgoing wave to the left.

Use of the Convolution Theorem:

$$\varphi(x,y) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi x} \frac{e^{-\sqrt{\xi^2 - k^2}\, y}}{\sqrt{\xi^2 - k^2}}\, \bar{f}(\xi)\, d\xi$$

The Convolution Theorem states: For the integrand above in the form $\bar{f}(\xi)\, \bar{g}(\xi)$, we can solve in the form:

$$\int_{-\infty}^{\infty} f(\tau)\, g(x-\tau)\, d\tau$$

We find inverting $g(x) = H_0^{(2)}\left(k\sqrt{x^2+y^2}\right)$, $H_0^{(1)}\left(k\sqrt{x^2+y^2}\right)$

Then the convolution gives:

$$\phi = e^{i\omega t}\varphi = e^{i\omega t} \int_{-a}^{a} H_0^{(2)}\left(k\sqrt{(x-\tau)^2+y^2}\right) f(\tau)\, d\tau$$

The above type of problem uses Fourier Transforms because its domain is infinite. Problems in vibrating strings, etc., are better solved with Laplace Transforms. For example:

$$u_{xx} - u_{tt} = 0 \quad ; \quad \bar{u}_{xx} - s^2 \bar{u} = -s\, u(x,0) - u_t(x,0).$$

If the coefficients are not constant, can try to choose a special transform function:

$$\frac{1}{x}(x u_x)_x - \frac{9}{x^2} u + u_{yy} = 0$$

A proper choice is $x J_3(\alpha x)$

$$\bar{u}(\alpha) = \int u\, x J_3(\alpha x)\, dx \qquad \int u \frac{9}{x} J_3(\alpha x)\, dx \quad 9\alpha\, \bar{u}$$

$$\int J_3 (x u_x)_x \, dx \rightarrow J_3\, x u_x \Big]_0^\infty - \int x u_x J_3' \, dx$$

$$- x J_3' u + \int (x J_3')' u$$

Use:

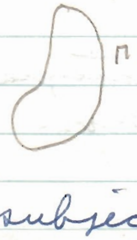
$$(x J_3')' - \frac{9}{x} J_3 + \alpha^2 x J_3 = 0$$

Mellin transforms can be made into Laplace Transforms by $x = \ln r$ from $r^{-\beta}$

LECTURE XXXI 5-10-31

$$L(u) = f(x,y,z)$$

$$u(x,y,z) = \int G(x,x',y,y',z,z')\, f(x',y',z')\, dx'dy'dz'$$



(u) $\left[\operatorname{div}[A \operatorname{grad} G] + qG\right] = g(r,r')$

(G) $\operatorname{div}\left\{A(\vec{r}) \operatorname{grad} u\right\} + qu = F(\vec{r})$

subject to $\beta u + u_n = 0$ on Γ or BC,

$$\int \left[u \operatorname{div} A \operatorname{grad} G - G \operatorname{div} A \operatorname{grad} u \right] = \int (ug - GF)$$

$$= \int \operatorname{div}\left[uA \operatorname{grad} G - GA \operatorname{grad} u \right] = 0$$

since $\quad u\, A G_n - G A u_n \;+\; u A B G \;-\; u A B G$

Then: $\quad u(\vec{r}') = \int F(\vec{r})\, G(\vec{r}, \vec{r}')\, d\vec{r}$

Consider Helmholtz equation:

$$\nabla^2 \varphi + k^2 \varphi = 0$$

or $\quad \frac{1}{r^2}(r^2 \varphi')' + k^2 \varphi = 0$

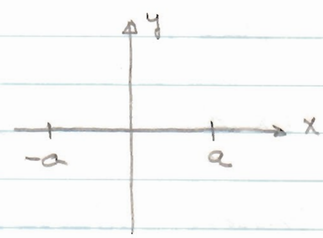
whose Green's function is $\quad \frac{e^{ikr}}{r}$

Many times transforms will better find the Green's function.

Consider: $\quad \nabla^2 u - k^2 u = 0$

$\qquad\qquad\quad \nabla^2 G - k^2 G = g$

$u_y = f(x) \quad$ on boundary $\quad |x| < a$



The line of integration is a large circle with a cut in to $-a \to a$ and around it over which $[-a \to a]$ the normal derivative vanishes. Using the above equations, we get

$$u(x,y) = \int_{-a}^{a} G_y(x, x', y, 0)\, h(x')\, dx'$$

$$h(x') = u(x', a^+) - u(x', a^-)$$

now: $\quad u_y(x,0) = f(x) = \int_{-a}^{a} G_{yy}(x, x', 0, 0')\, h(x')\, dx'$

which is an integral equation. An integral equation is one where the unknown is under the integral sign.

Example: $\nabla^2\nabla^2\psi - \nabla^2\psi_x = 0$

$y=0$

$\psi = -1$

$[\psi_{yy}] = f(x)$

$[\psi_{yyy}] = 0$

The principle object is to find $f(x)$.

We redefine the Fourier Transform in y into two pieces: $\int_{-\infty}^{0^+}$ and $\int_{0^+}^{\infty}$

plug into equation and use boundary conditions. That is, transform of ψ^{IV} gives ψ^{III} but BC are continuous at $0^-, 0^+$. Not so for transform of ψ^{III}. Get:

$$(\xi^2+\eta^2)(\xi^2+\eta^2+i\xi)\,\bar{\Psi} = i\eta\,\bar{f}(\xi)$$

Use convolution theorem:

$$\psi = \int_0^a f(x')\,F(x-x', y)\,dx'$$

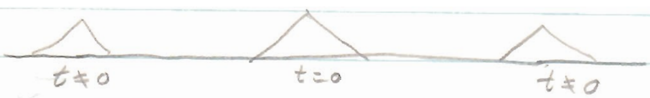
is form where F is Green's function which is composed of various wierd functions. Not responsible for new things covered in this lecture.

LECTURE XXXII 5-12-61

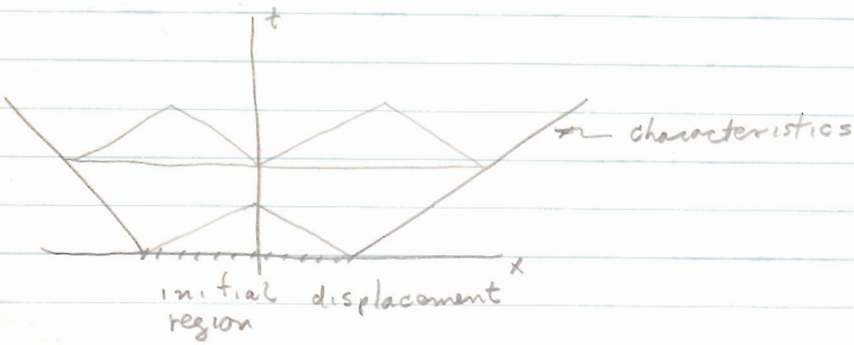
String Vibration:

$$u(x,t) = \frac{f(x-ct) + f(x+ct)}{2}$$

Which is
solution to hyperbolic wave equation.



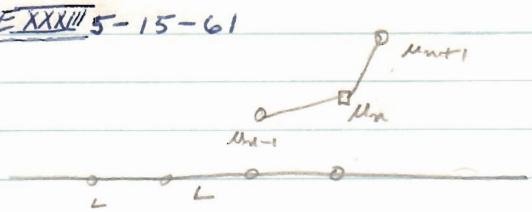
In x-t plane:



This shows propagation of wave down string. Must stay between characteristics. Shows that subsequent shape is determined by initial conditions

In n-dimension equations, if there are two families, hyperbolic; degenerate, parabolic; none, elliptic. Usually for higher orders, carrier use hyperbolic if characteristics exist, elliptic if they don't.

LECTURE XXXIII 5-15-61



$$m\ddot{u}_n = T\,\frac{u_{n+1}-u_n}{L} - \frac{T}{L}\,(u_n - u_{n-1})$$

$$f(x,t) = \sum_{n=-\infty}^{\infty} x^{2n}\,u_n(t)$$, introducing a generating function.

$$\tau = \frac{t}{\sqrt{\frac{mL}{T}}}$$

$$\therefore\quad \frac{mL}{T} f_{tt} = x^{-2} f - 2f + x^2 f = \left(x - \frac{1}{x}\right)^2 f$$

$$u_{n-1}\, x^{2(n-1)}$$

$$f_{\tau\tau} - \left(x-\frac{1}{x}\right)^2 f = 0$$

subject to $u_0 = 1,\ u_n = 0 \quad n \neq 0$
$\dot{u}_n = 0$

Thus $f = A_1 e^{(x+\frac{1}{x})t} + A_2 e^{-(x+\frac{1}{x})t} = \sum u_n x^{2n}$

$$\left.\begin{array}{l} A_1 + A_2 = 1 \\ A_1 - A_2 = 0 \end{array}\right\} \therefore\ f = \frac{1}{2} e^{(x-\frac{1}{x})t} + \frac{1}{2} e^{-(x-\frac{1}{x})t} = \sum u_n x^{2n}$$

or: $f = \sum u_n(t)\,\frac{1}{2\pi i}\oint \frac{x^{2n}}{x^{2m+1}}\,dx = u_m(t)$

but also:

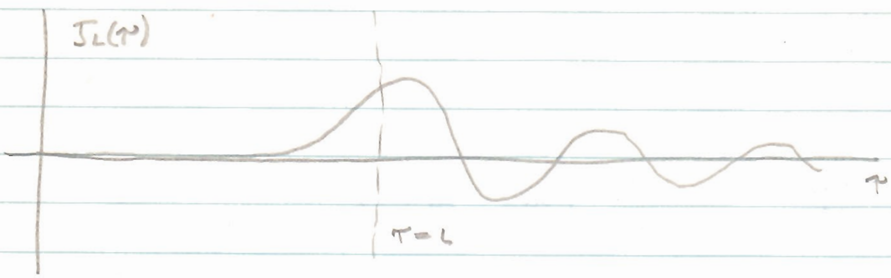
$$f = \frac{1}{2\pi i}\,\frac{1}{2}\oint e^{(x-\frac{1}{x})t}\,x^{-(2m+1)}\,dx$$

Choose unit circle as contour $x = e^{i\theta}$

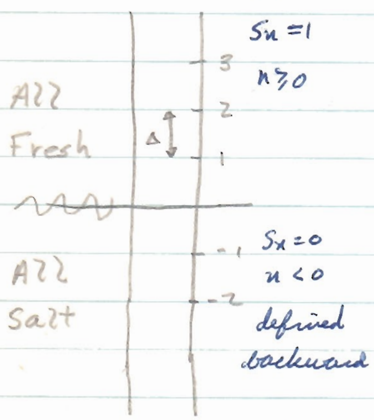
$$f = \frac{1}{4\pi i}\int_0^{2\pi} e^{i[2t\sin\theta - 2m\theta]}\,i\,d\theta = \frac{1}{2} J_{2m}(2t)$$

Then $u_m(t) = J_{2m}(2t)$

Forms of Bessel Functions:



Salinity Problem: Tidal motion causes mixing
Problem in volcanic ash in Hawaii



Recall:

$$S_{n,t} + W \frac{S_{n+1} - S_{n-1}}{2\Delta} - \frac{|W|}{2\Delta}\left(S_{n+1} + S_{n-1} - 2S_n\right) = 0$$

where W is the velocity.

We take a generating function:

This is same as if pail of sand were half filled with salt water and then fresh water poured carefully on.

$$f(x,t) = \sum S_n(t)\, x^{2n}$$

and get:

$$f_t + \frac{W}{2\Delta}\left(\frac{1}{x^2} - x^2\right) f - \frac{|W|}{2\Delta} \underbrace{\left(\frac{1}{x^2} + x^2 - 2\right)}_{\left(x - \frac{1}{x}\right)^2} f = 0$$

Then: $$f = A\, e^{-\frac{\left(\frac{1}{x^2} - x^2\right)}{2\Delta} \int_0^t W\,dt + \frac{\left(\frac{1}{x} - x\right)^2}{2\Delta} \int_0^t |W|\,dt}$$

Now: $$\sum_0^\infty x^{2n} = \sum_{-\infty}^{\infty} S_n(0)\, x^{2n} = \frac{1}{1 - x^2}$$

so any contour integration must be within the unit circle. Then:

$$S_n = \oint_\Gamma \frac{1}{1-x^2} \frac{e^{-\frac{\left(\frac{1}{x^2} - x^2\right)}{2\Delta} \int_0^t W\,dt + \frac{\left(\frac{1}{x} - x\right)^2}{2\Delta} \int_0^t |W|\,dt}}{x^{2n+1}}\, dx^2$$

Use steepest descents. What is large?
n is large, and $\int_0^t |w|\,dt$ is large.

Get: $e^{\frac{(\frac{t}{x}-x)^2}{2\sigma}\int |w|\,dt \;-\; 2n \ln x}$

Finally: S_n is of form:

$$S_n = \frac{1 + \operatorname{erf}\left\{ \dfrac{n - \int_0^t w\,dt}{c\int |w|\,dt + \cdots} \right\}}{2}$$

5-17-61

Van der Poel Oscillator:

$$u'' - \epsilon u'(1-u^2) + (1+\epsilon c)u = \cos 3t$$

When ϵ is zero, problem is trivial.
Assume:

$$u(t,\epsilon) = u_0(t) + \epsilon u_1(t) + \epsilon^2 u_2(t) + \cdots$$

Plug in d.e. and collect terms. Solve each equation in turn in terms of boundary conditions.

Asymptotic Expansions and semiconvergent series:

Semi-convergent: $f(x) = \sum_{n=0}^{N} a_n x^{-n} + R_N$
if $x^N R_N \to 0$ as $x \to \infty$
then semiconvergent and
asymptotically: $f(\infty) \sim \sum_{n=0}^{\infty} a_n x^{-n}$

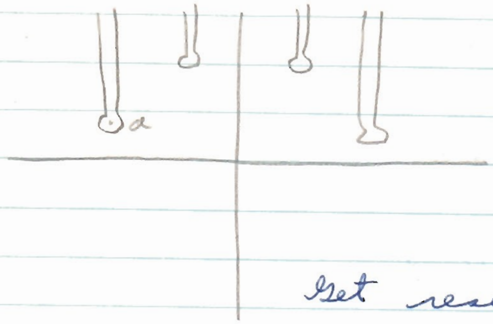
Example: asymptotic form of $K_0(z)$:

$$c\sqrt{z}\,K_0(z)\,e^{z} \sim \left(1 + \frac{a_1}{z} + \frac{a_2}{z} + \cdots\right)$$

Very good for z about 5 and above, no good for $z = 1$ or 0.

Asymptotics from contour integrals:

Integral of form: $\int e^{i\xi x} f(\xi)\, d\xi$



Expand in ascending powers of x as if the domain extended to infinity.

Get result like $\frac{e^{iax}}{\sqrt{x}}\left[1+\frac{a_1}{x}+\frac{b_1}{x^2}+\dots\right]$

Can do for each singularity. If a is closest to real axis, it is dominant over the others.

We expand:

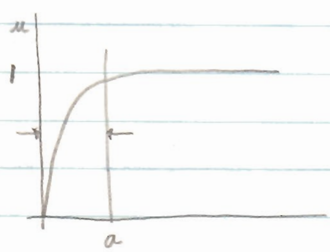
$$f(\xi) = \sum_i d_n (\xi - a)^{-n+\beta}$$

Back to Perturbation theory:

Consider:

$$\epsilon u'' - u = -1 \;;\quad u(0) = u(1) = 0$$

If try usual approach, find $u = 1$, cannot satisfy BC. Reason is that u is not an analytic function of ϵ. We hope that ϵ becomes important only at the boundaries.



Take: $u = u_0 + \psi(\xi) + \chi(\xi)$.

$(x-a)\epsilon^k = \xi$

$$u_n'' + \lambda u_n = f(t) \;;\quad v_n'' + \lambda v_n = 0$$

with solution λ_n, v_n

Expand $f(t) = \sum_n b_n v_n(t)$

$$u_n = \sum_n v_n(t) \frac{b_n}{1-\lambda_n}$$

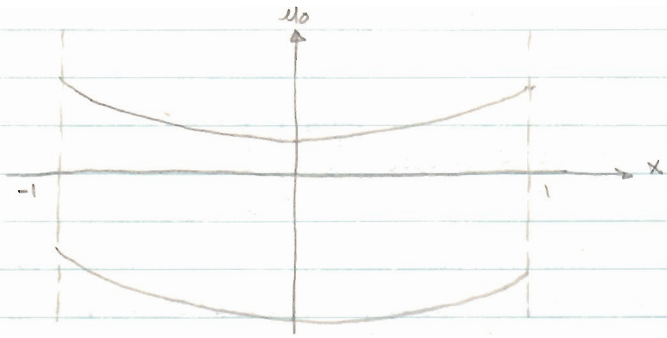
LECTURE XXXIV 5-19-61

If high order derivative is multiplied by ϵ will have trouble due to boundary layer.

$$\epsilon u'' + (1-x^2)u + u^2 = 1 \quad ; \quad u(-1) = u(1) = 0$$

$$0 < \epsilon << 1 \quad ; \quad u = u^{(0)} + \epsilon u^{(1)} + \dots$$

$$u^{(0)} = \frac{-(1-x^2) \pm \sqrt{(1-x^2)^2 + 4}}{2}$$



However, we know that we must have steep slopes near the boundaries

$$u = u_0 + u_1(\eta) + u_2(\xi) + \dots$$

$$\eta = (1+x)\epsilon^{\nu} \quad , \epsilon^{\nu} \text{ gives scale near boundary.}$$

$$\epsilon\left[u_0'' + \epsilon^{2\nu} u_1''\right] + \eta\epsilon^{-\nu}(2 - \eta\epsilon^{-\nu})u_1 + 2u_1 u_0 + u_1^2 = 0$$

In order to have u_1 same order of magnitude we choose $\nu = -1/2$. Get:

$$u_1'' + 2u_1 u_0 + u_1^2 = 0$$

Can expand u_0 in series about boundary point and get:

$$u_0 = 1 + O(\epsilon^{1/2}) \approx 1$$

Thus $u_1'' + 2u_1 + u_1^2 = 0$

$$u_1 = +1 \quad , \eta = 0$$
$$u_1 \to 0 \quad , \eta \to \infty$$

$$\frac{(u_1')^2}{2} + u_1^2 + \frac{u_1^3}{3} = 0$$

$$\frac{(u_1')^2}{2} + 1 - \frac{1}{3} = 0$$

Found by multiplying u_1' and integrating
no solution

Must use other root of u_0:

$$\frac{(u_1')^2}{2} - u_1^2 + \frac{u_1^3}{3} = 0$$

$$d\eta = \frac{\pm\, du_1}{2\sqrt{u_1^2 - \frac{u_1^3}{3}}} \quad , \quad \int_1^{u_1} \cdots du_1 = \int_0^{\eta} d\eta$$

Get:

$$u_1 = 3\left[1 - \tanh^2\left(\frac{\eta}{\sqrt{2}} + \operatorname{arc\,tanh}\sqrt{\frac{2}{3}}\right)\right]$$

η will give width of boundary conditions of layer

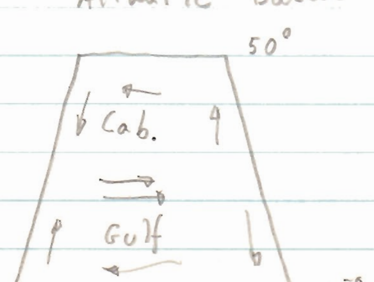
Suppose the numbers lumped into ϵ cause it to run from very small to very large values. When ϵ is very large, divide thru by it and get ordinary perturbation term. Problem goes from boundary layer to small perturbation. In between, must use numerical calculation.

Example:

$$k\, \nabla^2\nabla^2\psi - \epsilon\left[\psi_y \nabla^2\psi_x - \psi_x \nabla^2\psi_y\right] - \psi_x f'(y) + g(y) = 0$$

which is the curl of the momentum equation. This describes ocean motion, $g(y)$ is basin, $\psi_x f'(y)$ is due to Coriolis force.

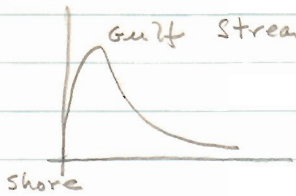
Altantic Basin



ϵ is Reynolds number

k is a viscosity

Velocity Profile:



NOTES ON LECTURES

LECTURE V: Prove that $\bar{u} = \int_0^a v(x,x';s)\, f(x')\, dx'$ is a solution of $\bar{u}_{xx} - s\bar{u} = f(x)$ given that $v_{xx} - sv = \delta(x-x')$:

$$\bar{u}_{xx} = \int_0^a v_{xx}(x,x';s)\, f(x')\, dx'$$

$$v_{xx}(x,x';s) = \delta(x-x') + s\,v(x,x';s)$$

$$\therefore\ \bar{u}_{xx} = \int_0^a \{\delta(x-x')\} f(x')\, dx' + s\int_0^a v(x,x';s)\, f(x')\, dx'$$

Then: $f(x) + s\bar{u} - s\bar{u} \equiv f(x)$, QED

We have:

$$v(x,x';s) = \begin{cases} A \sinh \sqrt{s}\, x & x < x' \\ B \sinh \{(a-x)\sqrt{s}\} & x > x' \end{cases}$$

Forming the Wronskian:

$$\begin{vmatrix} A \sinh \sqrt{s}\, x & B \sinh \{(a-x)\sqrt{s}\} \\ \sqrt{s}\, A \cosh \sqrt{s}\, x & -\sqrt{s}\, B \cosh \{(a-x)\sqrt{s}\} \end{vmatrix}$$

$$= -\sqrt{s}\, AB \left\{ \sinh \sqrt{s}\, x \cosh \{\ \} + \sinh \{\ \} \cosh \sqrt{s}\, x \right\}$$

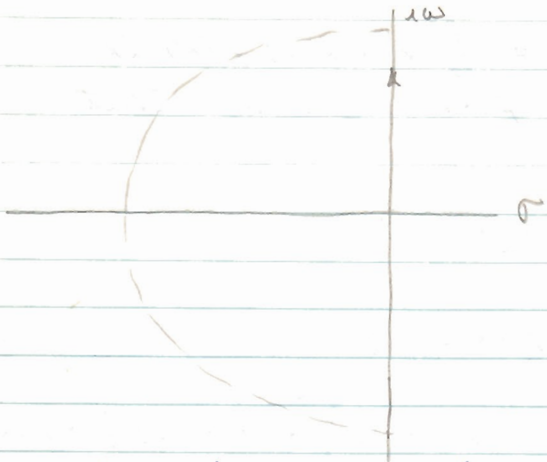
$$= -\sqrt{s}\, AB \sinh \sqrt{s}\, a$$

Now at the poles of the integrands in the inversion integral, the Wronskian vanishes and the two solutions are linearly dependent and the integral over dx' can be closed.

The inversion integral is:

$$u(x,t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \bar{u}(x,s)\, e^{st}\, ds$$

We choose the path such that;



Taking the poles of $\bar{u}(x,s)$ to lie in the left-half plane which is usually the case for physical problems. Then in general:

$$u(x,t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \bar{u}(x,s)\, e^{st}\, ds$$

$$= \sum_{n=1}^{\infty} R_n$$

Usually, all the poles of $\bar{u}(x,s)$ will be simple and we have:

$$R_n = e^{s_n t} \lim_{s \to s_n} (s - s_n)\, \bar{u}(x,s)$$

Further, if $\bar{u}(x,s)$ is of the form $\bar{u}(x,s) = \frac{P(s)}{Q(s)}$

where P and Q are analytic at s_n and $P(s_n) \neq 0$ and $Q(s_n) = 0$ is a simple zero, then

$$R_n = e^{s_n t} \lim_{s \to s_n} \frac{P(s)}{\left\{\frac{Q(s)}{s - s_n}\right\}} = e^{s_n t}\, \frac{P(s_n)}{Q'(s_n)}$$

and:

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \frac{P(s_n)}{Q'(s_n)}\, e^{s_n t}}$$

We assume that $\bar{u}(x,s)$ satisfies the vanishing criteria on the circular boundary.

LECTURE VII: The problem is to show that:

$$\bar{u} = \int_0^x \frac{w_1(x',s)\, w_2(x,s)\, f(x')\, h(x')\, dx'}{Q(x')}$$

$$+ \int_x^a \frac{w_2(x',s)\, w_1(x,s)\, f(x')\, h(x')\, dx'}{Q(x')}$$

is a solution of: $[p(x)\,\bar{u}']' + \{q(x) - s\,h(x)\}\,\bar{u} = -h(x)\,f(x)$

$$\bar{u}_x = \int_x^\infty \frac{w_{1x}(x,s)\; w_2(x',s)\, f(x')\, h(x')\, dx'}{Q(x')} - \frac{w_1 w_2 h f}{Q}$$

$$+ \int_0^x \frac{w_{2x}(x,s)\; w_1 f h \, dx'}{Q(x')} + \frac{w_1 w_2 h f}{Q}$$

$$\bar{u}_{xx} = \int_x^\infty \frac{w_{1xx}(x,s)\; w_2 f h \, dx'}{Q} - \frac{w_1' w_2 f h}{Q}$$

$$+ \int_0^x \frac{w_{2xx}(x,s)\; w_1 f h\, dx'}{Q} + \frac{w_2' w_1 f h}{Q}$$

Now:

$$p' w_1' + p w_1'' + q w_1 - s h w_1 = 0$$
$$p' w_2' + p w_2'' + q w_2 - s h w_2 = 0$$

Then:

$$p \left\{ \int_0^x \frac{w_{2xx}\; w_1 f h \, dx'}{Q} + \int_x^a \frac{w_{1xx}\; w_2 f h\, dx'}{Q} + \frac{fh}{Q}(-w_1' w_2 + w_2' w_1) \right\} + p' \left\{ \int_0^x \frac{w_{2x}\; w_1 f h\, dx'}{Q} + \int_x^a \frac{w_{1x}\; w_2 f h\, dx'}{Q} \right\} + (q - sh) \left\{ \int_0^x \frac{w_2(x)\; w_1 f h\, dx'}{Q} + \int_x^a \frac{w_1(x)\; w_2 f h\, dx'}{Q} \right\} = -hf$$

We get:

$$\frac{p}{Q}\left\{ -w_1' w_2 + w_2' w_1 \right\} = -1$$

or $\dfrac{p}{Q} W = -1$, $\quad Q = -pW$

Now: $W = \begin{vmatrix} w_1 & w_2 \\ w_1' & w_2' \end{vmatrix} = (w_1 w_2' - w_2 w_1')$

We have:

$$p' w_1' w_2 + p w_1'' w_2 + (q - sh) w_1 w_2 = 0$$
$$p' w_2' w_1 + p w_2'' w_1 + (q - sh) w_1 w_2 = 0$$

Then: $p'(w_2' w_1 - w_2 w_1') + p(w_2'' w_1 - w_2 w_1'') = 0$

or $\{ p W \}' = 0$

Therefore: $pW = \text{constant}$, $W = \frac{\text{constant}}{p}$

Thus $Q = \text{constant}$

This constant is independent of x but may be dependent on s.

Sturm-Liouville Theory: Green's Functions

(1) $$\boxed{\begin{aligned} &L[u(x,t)] + h(x)\, u_t(x,t) = 0 \\ &L[u] = [p(x)\, u'(x,t)]' + q(x)\, u(x,t) \end{aligned}}$$ Defined in $a < x < b$

(2) Boundary conditions: Most general - homogeneous:

$$\left.\begin{aligned} A u'(a,t) + B u(a,t) &= 0 \\ C u'(b,t) + D u(b,t) &= 0 \end{aligned}\right\} \quad u(x,0) = f(x)$$

(3) Transform:

$$\boxed{\begin{aligned} \bar{u}(x,s) &= \int_0^\infty u(x,t)\, e^{-sx}\, dx \\ u(x,t) &= \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \bar{u}(x,s)\, e^{sx}\, ds \end{aligned}}$$

(4) $$L[\bar{u}(x,s)] + s\, h(x)\, \bar{u}(x,s) = h(x)\, f(x)$$

(5) The solution can be written in the form:

$$\boxed{\bar{u}(x,s) = \int_a^b G(x,x',s)\, h(x')\, f(x')\, dx'}$$

where $G(x,x',s)$ is the Green's function and is the solution of:

(6) $$L\left[G(x,x',s)\right] + s\, h(x)\, G(x,x',s) = \delta(x-x')$$

Get: $G_1(x,\xi,s)$: satisfies BC at $x = a$
$G_2(x,\xi,s)$, satisfies BC at $x = b$

(7) Now: $$\boxed{G(x,x',s) = -\begin{cases} \dfrac{G_1(x',s)\, G_2(x,s)}{p(x')\, W_G(x',s)} & x > x' \\[2ex] \dfrac{G_1(x,s)\, G_2(x',s)}{p(x')\, W_G(x',s)} & x < x' \end{cases}}$$

(8) Now : $u(x,t) = \int_a^b h(x') f(x') dx' \left\{ \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} G(x,x';s) e^{st} ds \right\}$

(9) The poles and eigenvalues of the equation are determined by the vanishing of the Wronskian:

$W_G(s_n) = 0$

At this point, the two sides of the Green's function are dependent and equal since the Wronskian vanishes, and we can combine into one. Further, if they are simple poles, we can write:

(10) $$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} G(x,x';s) e^{st} ds = \sum_{n=1}^{\infty} \frac{G_1(x';s_n) G_2(x,s_n)}{p(x') W_G'(s_n)} e^{s_n t}$$

(11) $\therefore$ $$\boxed{u(x,t) = \int_a^b \frac{h(x') f(x') dx'}{p(x')} \left\{ \sum_{n=1}^{\infty} \frac{G_1(x';s_n) G_2(x,s_n)}{W_G'(s_n)} e^{s_n t} \right\}}$$

If : $L[\bar{u}(x,s)] + s^2 h(x) \bar{u}(x,s) = h(x)[g(x) + s f(x)]$

from $L[u(x,t)] + h(x) u_{tt} = 0$; $u(x,0) = f(x)$
$u_t(x,0) = g(x)$

Then: $$u(x,t) = \int_a^b \frac{h(x')\, dx'}{p(x')} \sum_{n=1}^{\infty} \frac{G_1(x',s_n) G_2(x,s_n) e^{s_n t}}{W_G'(s_n)} \left\{ s_n f(x') + g(x') \right\}$$

In general: for linear, non-homogeneous equations:

$L[u(x,y,z)] = f(x,y,z)$; L denotes linear now.

$$u(x,y,z) = \iiint_{\text{boundaries}} G(x,x',y,y',z,z') f(x',y',z') dx' dy' dz'$$

Sturm - Liouville Theory: Separation of Variables

(1) $L[w(x,t)] + h(x)\, w_t(x,t) = 0$: Defined in $a < x < b$

with homogeneous boundary conditions in the spatial coordinates and $w(x,0) = f(x)$

(2) Assume
$$\boxed{\begin{aligned} w_n(x,t) &= u_n(x)\, v_n(t) \\ w(x,t) &= \sum_n a_n w_n(x,t) \end{aligned}}$$

(3) $$\frac{L[u(x)]}{h(x)\, u(x)} = -\lambda = -\frac{v'(t)}{v(t)}$$

(4) $L[u(x)] + \lambda h(x) u(x) = 0 \quad ; \quad v(t) = e^{\lambda t}$

(5) λ_n is found from the boundary conditions. The solutions are then $u_n(x)$ and

$$w(x,t) = \sum_n a_n u_n(x)\, e^{\lambda_n t}$$

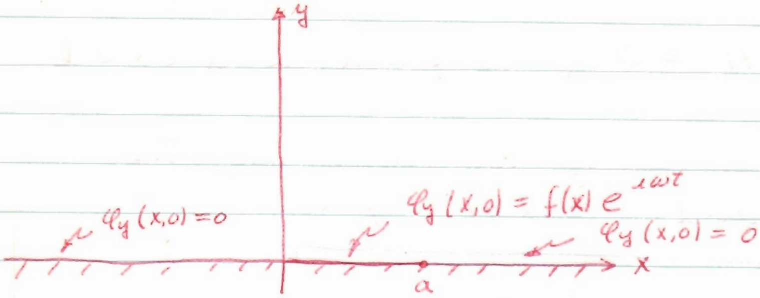
(6) Since the u_n's from a complete set of orthogonal functions, we have

$$a_n = \frac{\int_a^b h(x') f(x')\, u_n(x')\, dx'}{\int_a^b [u_n(x')]^2 [h(x')]^2\, dx'}$$

(7) $\therefore$ $$\boxed{w(x,t) = \int_a^b h(x')\, f(x')\, dx' \left\{ \sum_{n=1}^{\infty} \frac{u_n(x')\, u_n(x)\, e^{\lambda_n t}}{\int_a^b [h(x')\, u_n(x')]^2\, dx'} \right\}}$$

Separation of variables gives essentially the same result as the Green's function method for Sturm - Liouville problems; however, it can also be used in multi-dimensional problems where boundaries are coordinates equal to a constant.

Acoustic Potential Problem: Use of Fourier Transforms



$$\boxed{\varphi_{xx}+\varphi_{yy}-\frac{1}{c^2}\varphi_{tt}=0}$$

Must be outgoing wave in result and φ vanishes at $\pm\infty$

(1) Assume $\varphi=\psi e^{i\omega t}$; $\psi_{xx}+\psi_{yy}+k^2\psi=0$; $k=\frac{\omega}{c}$

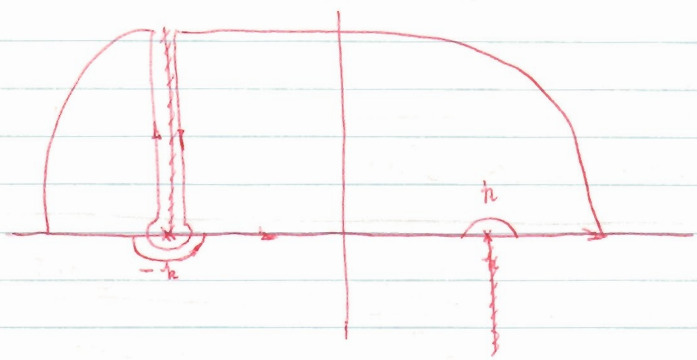
(2) Transform along x: $\bar{\Psi}(\xi,y)=A(\xi)\,e^{-\sqrt{\xi^2-k^2}\,y}$

(3) $\bar{\Psi}(\xi,y)=\dfrac{-\bar{f}(\xi)}{\sqrt{\xi^2-k^2}}\,e^{-\sqrt{\xi^2-k^2}\,y}$

(4) Then:

$$\boxed{\psi(x,y)=\int_0^a f(x')\,dx'\cdot\frac{-1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-\sqrt{\xi^2-k^2}\,y}}{\sqrt{\xi^2-k^2}}\,e^{i\xi(x-x')}\,d\xi}$$

(5) Proper Contour; Want Outgoing Wave; $\varphi=\psi e^{i\omega t}$



Since we need $\varphi\sim e^{i\omega t-ikx}$ for outgoing to the right, we must include $-k$ as a residue.

Can make an asymptotic expansion about one of the real axis poles.

Sturm - Liouville Theory : Perturbation Theory

(1) Consider : $\boxed{u'' + \lambda u + \epsilon f(x) u = 0}$

subject to a set of homogeneous boundary conditions

(2) $u(x, \epsilon) = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \ldots$

$\lambda(\epsilon) = \lambda_0 + \epsilon \lambda_1 + \epsilon^2 \lambda_2 + \ldots$

(3) $u_0'' + \lambda_0 u_0 = 0$

$u_1'' + \lambda_0 u_1 = -f(x) u_0 - \lambda_1 u_0$

$u_2'' + \lambda_0 u_2 = -f(x) u_1 - \lambda_1 u_1 - \lambda_2 u_0$

$\vdots$

$u_m'' + \lambda_0 u_m = -f(x) u_{m-1} - \sum_{i=1}^{m} \lambda_i u_{m-i}$

(4) Assume $u_m = \sum_n a_{mn} u_{0n}$

where the u_{0n}'s are chosen to be orthonormal by satisfying the boundary conditions and a normality condition.

(5) Operate with $\int u_{0n'} \, dx$; and get :

$$\lambda_{1n} = -\int f(x) \left[u_{0n}\right]^2 dx$$

$$a_{mn} = \frac{\sum_{n'} a_{m-1,n'} \int f(x) u_{0n'} u_{0n} \, dx}{\lambda_{0n} - \lambda_{01}}$$

$$u_m = \sum_n \left\{ \frac{\sum_{n'} a_{m-1,n'} \int f(x) u_{0n'} u_{0n} \, dx}{\lambda_{0n} - \lambda_{01}} \right\} u_{n0}$$

Some Useful Relations

Conservation of Momentum: $\rho u_{,t} + \rho u_j u_{,j} + p_{,i} = 0$

Conservation of Mass: $(\rho u_i)_{,i} + \rho_{,t} = 0$

Heat: $\operatorname{div}\left\{-k \operatorname{grad} T + \rho c T \vec{v}\right\} + \frac{\partial}{\partial t}(c\rho T) = 0$

Waves: $\frac{\partial^2 \eta}{\partial t^2} = g\left\{\frac{\partial}{\partial x}\left(h \frac{\partial \eta}{\partial x}\right) + \frac{\partial}{\partial y}\left(h \frac{\partial \eta}{\partial y}\right)\right\}$

Bessel's Equation: $R'' + \frac{1}{r}R' + \left(\lambda - \frac{\alpha^2}{r^2}\right)R = 0$

with a finite solution: $R = J_\alpha(\sqrt{\lambda}\, r)$

$u'' - s x u = 0$; $\quad u_1 = x^{1/2} J_{1/3}\left\{\frac{2}{3} s^{1/2} x^{3/2}\right\}$

$u_2 = x^{1/2} J_{-1/3}\left\{\frac{2}{3} s^{1/2} x^{3/2}\right\}$

Week from Monday

APPLIED MATHEMATICS 202

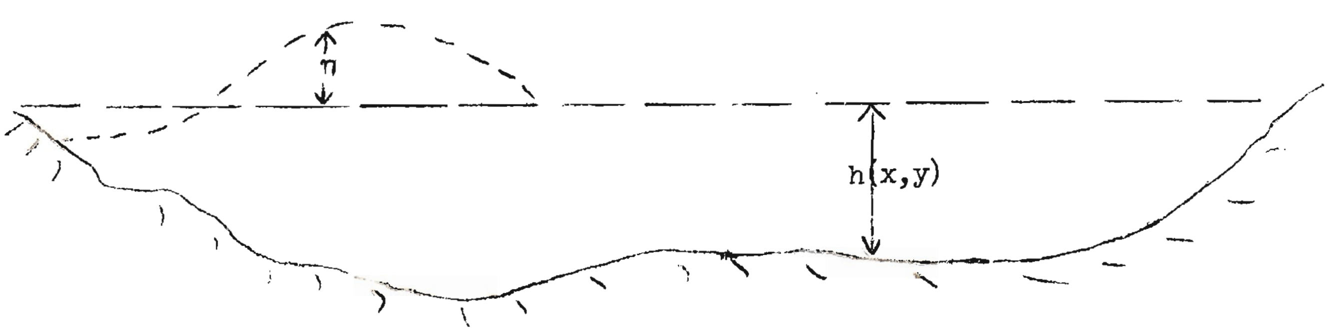
Problem Set 1 Spring Term 1961

1) An inextensible cord of length L and mass per unit length ρ is suspended from a fixed point. Deduce the equations governing the lateral oscillations of the cord.

2. Derive the differential equation which implies the conservation of mass of a compressible fluid whose density is $\rho(x_i, t)$ and whose velocity field is $u_i(x_i, t)$.

3. In certain situations heat is transferred in a fluid by conduction and by convection. If the conductive heat flux vector (heat flow through the fluid per unit area per unit time) is given by $\vec{q} = -k \operatorname{grad} T$ and if the heat capacity per unit mass of the fluid is C, find the equation implying the conservation of heat in three-dimensional time dependent phenomena.

4. Let the depth of a basin of an incompressible liquid at rest be given by $h(x,y)$; see Figure. Assume that vertical accelerations are negligible in any motion to be considered and that the x and y components of velocity are independent of z, the vertical coordinate. Denoting the change in surface elevation by $\eta(x,y,t)$, deduce the equations which imply the conservation of mass and momentum for wave motion in the basin.



5) Starting from the equations derived in the first lecture and considering only motions for which u_x , v_x are very small compared to unity, derive a single partial differential equation governing the small amplitude lateral oscillations of an elastic string.

6) Starting from the momentum conservation equation derived in the second lecture, the results of problem (2), and the thermodynamic rule $p/p_o = (\rho/\rho_o)^\gamma$, and considering motions for which the changes in ρ , the changes in p , and the velocity components are very small compared to ρ_o , p_o , and the "speed of sound," $\gamma\, p_o/\rho_o$, respectively, deduce a single differential equation for the changes in p associated with small amplitude motions in a compressible fluid. In the foregoing, ρ_o and p_o are the constant values of ρ and p which characterize the motionless condition of the fluid.

7) Using the results of Problem (4), deduce the equation governing $\eta(x,y,t)$ for motions wherein η_x and η_y are each very small compared to unity.

APPLIED MATHEMATICS 202

PROBLEM SET NO. 2 Spring 1961

(1) Find the eigenfunctions, $u_n(x)$, associated with the problem

$u_{xx} - u_t = 0$, in $0 < x < a$; $t > 0$.

$$u(0,t) = u_x(a,t) + \alpha\, u(a,t) = 0$$

$$u(x,0) = f(x)$$

and establish the completeness and orthogonality of these $u_n(x)$.

(2) Repeat question (1) for the problem

$$u_{xx} - x\, u_t = 0 \quad \text{in } 0 < x < L,\ 0 < t,$$

$$u(0,t) = u(L,t) = 0$$

$$u(x,0) = f(x).$$

(3) Let

$$u_{xx} + u_{yy} = e^{-y^2} \sin x$$

in $0 < x < \pi$, $-L < y < L$

with $u(0,y) = u(\pi,y) = 0$

$u(x,-L) = u(x,L) = 0$

(a) Find $u(x,y)$ in the form, $u = \sum g_n(y)\, u_n(x)$ where $u_n(x)$ are suitably chosen eigenfunctions.

(b) Why is this choice of representation superior to that in which the $g_n(y)$ are determined by eigenfunction considerations?

(3) Cont'd

(c) What modification occurs if $e^{-y^2} \sin x$ is replaced by e^{-y^2} ?

(d) What modification occurs if the boundary condition at $x = \pi$ is replaced by

$$u_x(\pi,y) + \alpha\, u(\pi,y) = 0 \quad ?$$

(e) What is $u(x,t)$ if the boundary conditions at $y = -L, L$ are replaced by

$$u(x,-L) = u_y(x,-L) = 0 \quad ?$$

Work out the detailed answers for (a), (c), (e) but not for (d).

(4) A solid cylinder of length L, radius a, and thermal diffusivity ν, is insulated at its ends and is initially at the uniform temperature T_0. At its surface heat is lost to an adjacent medium in such a way that $\partial T/\partial r + k\,T = 0$ at $r = a$. Find the relevant eigenfunctions, the criterion for the determination of the eigenvalues, the formula for any needed coefficients, and the product series which represents $T(r,t)$.

(5) Find the eigenmode of oscillation for small displacements of the hanging cord problem of Problem Set No.1. Sketch the spatial dependence of the first three such modes.

(6) A membrane, with tension T, has edges which lie on $\theta = \pi/10$, $\theta = 3\pi/10$, $r = a$. Find the frequency of the two lowest frequency eigenmodes. Indicate the shape of the membrane in the lowest eigenmode.

(7) An elastic bar of length L is so attached to a rigid frame that its ends can undergo no displacement. Since the ends are free to turn, the curvature of the bar's center line is zero at the end points. The bar is under a compressive force, P, and its small lateral displacement, u, is governed by the equation

$$K\, u_{xxxx} + P\, u_{xx} + R\, u_{tt} = 0 ,$$

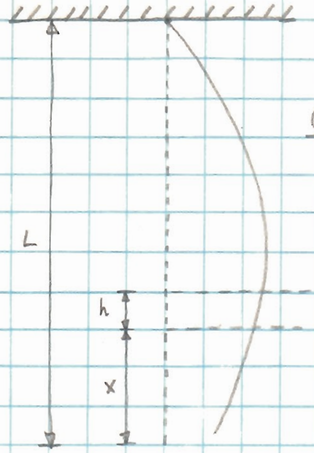
where the positive quantities K and R depend on the cross section and properties of the rod. For a given rod, what is the criterion that the straight configuration be not unstable?

Assignment #1
Problems 1- 7

35
30

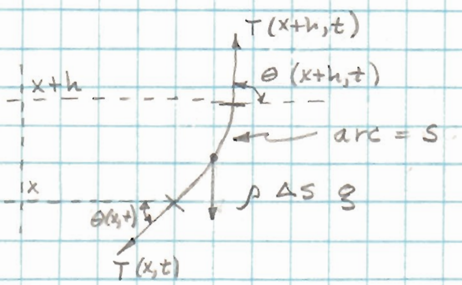
Paul Grant
DEAP - AP - 16
AM 202
2 - 10 - 61

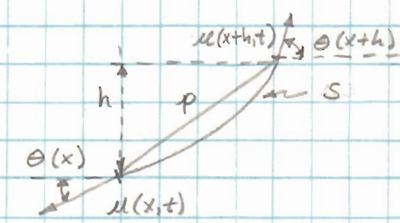
1.



Density of cord = ρ gms/cm and is constant

The cord is inextensible, hence there are no longitudinal displacments.





(1) Newton's Law equations:

$$T(x+h,t)\cos\theta(x+h,t) - T(x,t)\cos\theta(x,t) = \rho\Delta s\,\ddot{u}_c$$

$$T(x+h,t)\sin\theta(x+h,t) - T(x,t)\sin\theta(x,t) - \rho\Delta s\, g = \cancel{0}\ \rho\Delta s\,\ddot{v}$$

u_c is the position of the centroid of the element being considered of the cord. There is no $\ddot{v}$ component as the cord is inextensible. Altho a particular point on the string has vertical and horizontal components of acceleration, we are considering the net displacement of the whole cord at a particular time, a displacement which is only transverse. The displacement u at a fixed x for various times does not continuously refer to the same material particle on the cord.

(2) Dividing by h and taking $\cancel{\lim_{h\to 0}}$:

$$(T\cos\theta)_x = \rho\, s_x\, \ddot{u}$$

$$(T\sin\theta)_x = \rho\, s_x\, g$$

(3) We have four unknowns: T, θ, s_x and $\ddot{u}$.

From geometry:

$$\frac{\Delta s}{h} = \frac{p}{h} = \frac{\left\{ h^2 + [u(x+h) - u(x)]^2 \right\}^{1/2}}{h}$$

$$\lim_{h \to 0} \frac{\Delta s}{h} = s_x = \left\{ 1 + u_x^2 \right\}^{1/2}$$

(4) Using the same limit process:

$$\sin\theta = \frac{1}{\sqrt{1 + u_x^2}} \; ; \quad \cos\theta = \frac{u_x}{\sqrt{1+u_x^2}}$$

We now have enough equations to solve for all the unknowns; and the problem is in principle finished.

(5)
$$\left(\frac{T u_x}{\sqrt{1+u_x^2}} \right)_x = \rho \left\{ 1 + u_x^2 \right\}^{1/2} \ddot{u}$$

$$\left(\frac{T}{\sqrt{1+u_x^2}} \right)_x = \rho \left\{ 1 + u_x^2 \right\}^{1/2} g$$

T may be in principle eliminated to give a single partial differential equation.

(6) If the assumption of small displacements is made, $u_x \ll 1$, we have:

$$(T u_x)_x = \rho \ddot{u} \; ; \quad T_x = \rho g \quad \text{or} \quad T = \rho g x$$

if ρ is constant along the cord.

$$\therefore \quad g \frac{\partial}{\partial x} \left(x \frac{\partial u}{\partial x} \right) = \frac{\partial^2 u}{\partial t^2}$$

$$\text{or} \quad x \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} = \frac{1}{g} \frac{\partial^2 u}{\partial t^2}$$

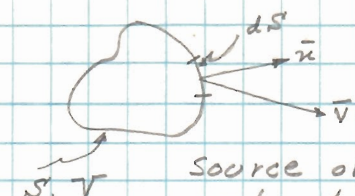
which upon separation of variables would give Bessel's equation for the x solution. The rest of the problem is essentially a boundary value problem which is beyond the intent of the problem so far.

7

Assignment # 1
Continued

Paul Grant
DEAP-AP-1G
AM 202
2-10-61

2. Consider the following element of fluid volume:



Source or sink strength $= \sigma$

Consider the density of sources within V to be σ. That is, at any point within V there is σ grams per second per unit volume flowing into V.

Thus the total amount of fluid being created within V is:

(1) $\displaystyle\int_V \sigma \, dV$ grams/sec.

Now the outflow in grams/sec thru an area dS is the velocity of the fluid $\perp$ to dS times the density at dS, say ρ; that is:

(2) $-\rho \bar{V}\cdot\bar{n}\, dS$

and the net outflow is: $\displaystyle -\int_S \rho\bar{V}\cdot\bar{n}\, dS$

The total mass at any time within V is clearly $\displaystyle\int_V \rho \, dV$ and the net rate of change is:

$$\frac{\partial}{\partial t}\int_V \rho\, dV = \int_V \frac{\partial \rho}{\partial t}\, dV$$ assuming no discontinuities in ρ. Therefore:

(3) $$\int_V \sigma\, dV - \int_S \rho\bar{V}\cdot\bar{n}\, dS = \int_V \frac{\partial\rho}{\partial t}\, dV$$

Involking the Divergence theorem: $\displaystyle\int_S \rho\bar{V}\cdot\bar{n}\, dS = \int_V \operatorname{div}(\rho\bar{V})\, dV$

we have:

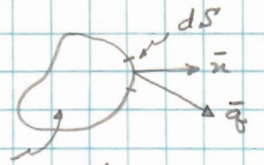
(4) $$\int_V \left\{ \operatorname{div}(\rho\bar{V}) + \rho_t - \sigma \right\} dV = 0$$, or, since this holds for any volume, and, we consider the integrand continuous:

$$\operatorname{div}(\rho\bar{V}) + \rho_t - \sigma = 0 \quad \text{, or,} \quad \left\{\rho(x_i,t)\, u_j(x_i,t)\right\}_{,j} + \rho_t(x_i,t) - \sigma(x_i,t) = 0$$

10

which is the well-known equation of continuity. With $\sigma = 0$, the equation implies conservation of mass.

3. Consider the following element of fluid volume:



Assume no sources or sinks of heat.

(1) The amount of heat in a given volume dV is given by the well-known calorimetric relation:

$dQ = c\rho T\, dV$ where T is the temperature of the element, c its heat capacity and ρ the fluid density.

(2) $\therefore$ the total heat in V $= \int_V c\rho T\, dV = Q$

(3) Now the rate of increase with time of Q within V is:

$$\frac{\partial Q}{\partial t} = \frac{\partial}{\partial t}\int_V c\rho T\, dV = \int_V c\,\frac{\partial}{\partial t}(\rho T)\, dV,$$ differentiation under the integral sign assuming no change in the volume limits and a well-behaved integrand with c constant in time, ρ may be time dependent because of convection.

(4) Now the heat flow into V thru an element of surface dS is: $-\bar{q}\cdot\bar{n}\, dS$, $\bar{q}$ being the heat flow vector given by $\bar{q} = -k \operatorname{grad} T$ where k is the termal conductivity and $\bar{q}$ has units of heat flow / unit area · unit time. Thus the total heat flow into V is:

$$\frac{\partial Q}{\partial t} = -\int_{S'} \bar{q}\cdot\bar{n}\, dS = -\int_V \operatorname{div}\bar{q}\, dV$$ by the divergence or Gauss' Theorem.

(5) Since (4) and (3) are equal and hold for each volume element of the fluid regardless of size, we have:

$$\operatorname{div}\bar{q} + c\,\frac{\partial}{\partial t}(\rho T) = 0$$

which implies from the derivation that when no sources or sinks are present, heat is conserved.

(6) If we assume heat transfer due to conduction only, and the conductivity k to be space independent, we have:

$$\operatorname{div}(k \operatorname{grad} T) = c\,\frac{\partial}{\partial t}(\rho T)$$ should include convection

8

or $$\nabla^2 T = \frac{c\rho}{k}\,\frac{\partial T}{\partial t}$$ which is the well-known diffusion equation.

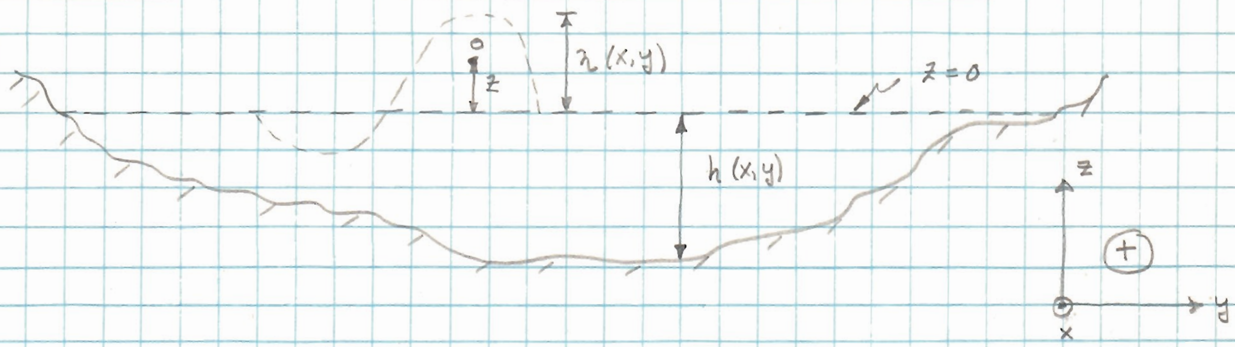
(7) Altho it is usual to assume c time independent, perhaps for the sake of generality we should write for the continuity equation:

$$\operatorname{div}\bar{q} = \frac{\partial}{\partial t}(c\rho T)$$

Assignment #1
Continued

Paul Grant
DEAP - AP - 1G
AM 202
2 - 10 - 61

4. We have the following model for the basin:

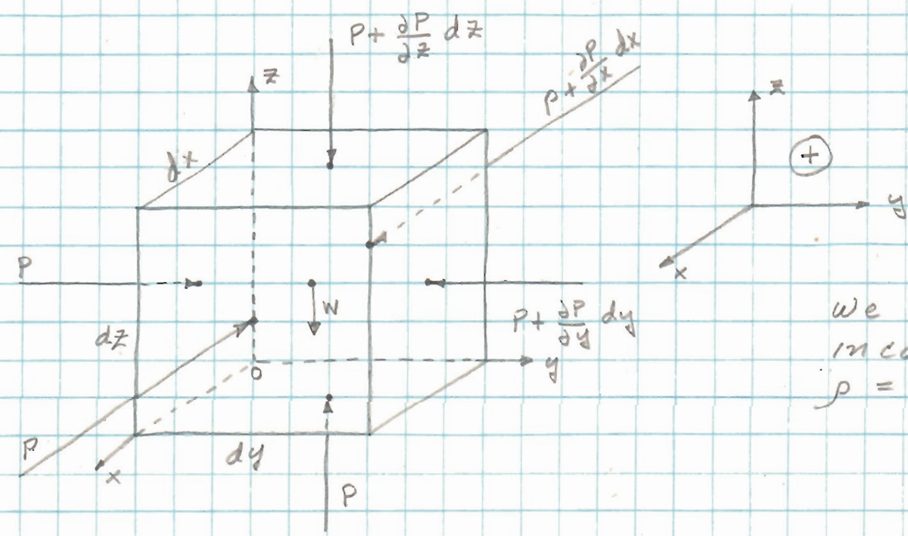


(1) If we make the assumption of no vertical acceleration, the pressure at point o will be given merely by the hydrostatic pressure:

$$p = p_0 + \rho g (\eta - z)$$

where p_0 is the atmospheric pressure.

(2) We now examine a differential volume of fluid whose origin is at o:



We assume incompressible, ρ = constant

(3) Consider the dynamical equation derived in lecture using conservation of momentum:

$$\rho \frac{\partial \bar{v}}{\partial t} + \rho (\bar{v} \cdot \text{grad}) \bar{v} + \text{grad } P = \rho \bar{F}$$

Each one of the terms is a force per unit volume which means that if we form equations using Newton's second law, those equations will imply conservation of momentum.

(4) We denote the velocity of the fluid as:

$$\bar{V} = u\,\bar{i} + v\,\bar{j} + w\,\bar{k}$$

(5) $$\rho \frac{du}{dt}\,dx\,dy\,dz = P\,dy\,dz - \left(P + \frac{\partial P}{\partial x}dx\right)dy\,dz)$$
$$= -\frac{\partial P}{\partial x}\,dx\,dy\,dz$$

or $$\rho\frac{du}{dt} = -\frac{\partial P}{\partial x}$$

Similarly:
$$\rho\frac{dv}{dt} = -\frac{\partial P}{\partial y} \quad ; \quad \rho\frac{dw}{dt} = -\frac{\partial P}{\partial z} - \rho g = 0$$

since the vertical accelation is zero.

(6) Using (1): $$\frac{du}{dt} = -g\frac{\partial \eta}{\partial x} \quad ; \quad \frac{dv}{dt} = -g\frac{\partial \eta}{\partial y}$$

(7) By the definition of the total derivative:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{dx}{dt}\frac{\partial}{\partial x} + \frac{dy}{dt}\frac{\partial}{\partial y} + \frac{dz}{dt}\frac{\partial}{\partial z}$$

$$= \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

(8) Thus we have for (6):

$$\left.\begin{aligned}\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= -g\frac{\partial \eta}{\partial x}\\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} &= -g\frac{\partial \eta}{\partial y}\end{aligned}\right\}$$

Using the fact given that u, v are independent of z.

which imply conservation of momentum.

(9) To show that (8) implies conservation of momentum, use (3) directly:

$$\rho\left(\bar{i}\frac{\partial u}{\partial t} + \bar{j}\frac{\partial v}{\partial t} + \bar{k}\frac{\partial w}{\partial t}\right) + \rho\left\{u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right\}(u\bar{i} + v\bar{j} + w\bar{k})$$

$$+ \rho g\left\{\bar{i}\frac{\partial \eta}{\partial x} + \bar{j}\frac{\partial \eta}{\partial y} - \bar{k}\right\} = -\rho g\,\bar{k}$$

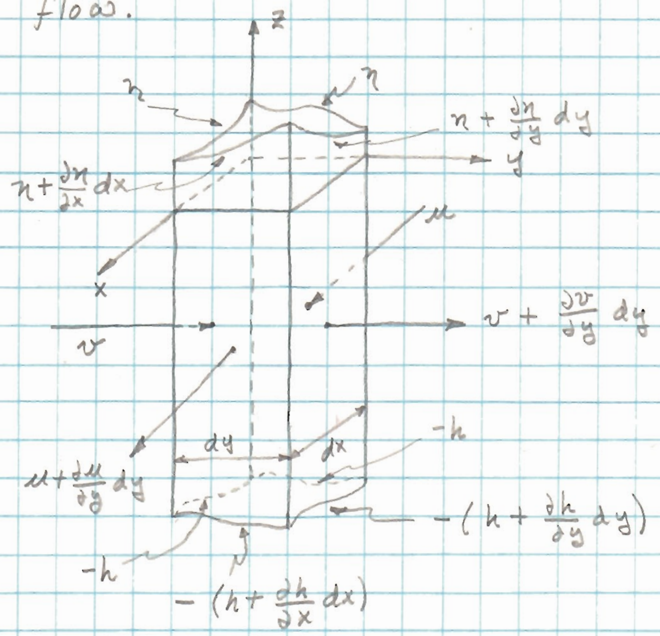
Equating components, recalling $\frac{dw}{dt} = \frac{\partial w}{\partial t} + w\frac{\partial w}{\partial z} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} = 0$ and that u, v are independent of z, equations (8) result immediately.

Assignment #1
Continued

Paul Grant
DEAP - AP-16
AM 202
2 - 10 - 61

4. Continued

(10) We now calculate the equation of continuity. Since ρ is constant, we need only consider volume flow.



Consider the columnar element shown. It is implicitly assumed that u and v are independent of z.

(11) The net volume flow in:

$$v\,dx\,(h+\eta) + u\,dy\,(h+\eta)$$

The net flow out:

$$\left(v + \frac{\partial v}{\partial y} dy\right) dx \left\{ h + \frac{\partial h}{\partial y} dy + \eta + \frac{\partial \eta}{\partial y} dy \right\}$$

$$+ \left(u + \frac{\partial u}{\partial x} dx\right) dy \left\{ h + \frac{\partial h}{\partial x} dx + \eta + \frac{\partial \eta}{\partial x} dx \right\}$$

(12) Flow in − Flow out = net increase in volume

$$= \frac{\partial}{\partial t} \left\{ (h+\eta)\,dx\,dy \right\}$$

(13) Flow in − Flow out $= - \left\{ h \frac{\partial v}{\partial y} dy + \eta \frac{\partial v}{\partial y} dy + v \frac{\partial h}{\partial y} dy + v \frac{\partial \eta}{\partial y} dy \right.$

$$\left. + \frac{\partial v}{\partial y}\frac{\partial h}{\partial y}(dy)^2 + \frac{\partial v}{\partial y}\frac{\partial \eta}{\partial y}(dy)^2 \right\} dx - \left\{ h \frac{\partial u}{\partial x} dx + \eta \frac{\partial u}{\partial x} dx + u \frac{\partial h}{\partial x} dx \right.$$

$$\left. + u \frac{\partial \eta}{\partial x} dx + \frac{\partial u}{\partial x}\frac{\partial h}{\partial x}(dx)^2 + \frac{\partial u}{\partial x}\frac{\partial \eta}{\partial x}(dx)^2 \right\} dy$$

$= \left[-\frac{\partial}{\partial y}\{v(h+\eta)\} - \frac{\partial}{\partial x}\{u(h+\eta)\} \right] dx\,dy$, neglecting terms of second order as usual.

(14) Unless an earthquake occurs, h is independent of time, therefore, the equation of continuity becomes:

$$\frac{\partial}{\partial x}\left\{u(h+\eta)\right\} + \frac{\partial}{\partial y}\left\{v(h+\eta)\right\} = -\frac{\partial \eta}{\partial t}$$

(15) Usually, $\eta \ll h$, so that it can be neglected in comparison to h and we have:

$$\frac{\partial}{\partial x}(uh) + \frac{\partial}{\partial y}(vh) = -\frac{\partial \eta}{\partial t}$$

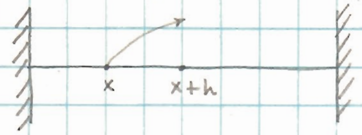
This is saying that the hieght of the waves is much less than the depth of the basin, which is generally true except possibly at the edges where we assume it has a negligible effect on the total wave motion.

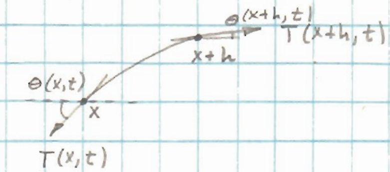
10

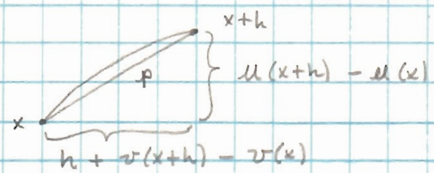
Assignment # 1
Continued

Paul Grant
DEAP - AP - 1G
AM 202
2 - 10 - 61

5.







It was shown in lecture that in the limit of $h \to 0$, the following equations result:

(1) $(T\cos\theta)_x = \rho A \ddot{v}$, $\cos\theta = \dfrac{1+v_x}{[(1+v_x)^2 + u_x^2]^{1/2}}$

$(T\sin\theta)_x = \rho A \ddot{u}$, $\sin\theta = \dfrac{u_x}{[(1+v_x)^2 + u_x^2]^{1/2}}$

$$T = T_0 + EA\left[\left\{(1+v_x)^2 + u_x^2\right\}^{1/2} - 1\right]$$

(2) Clearly, for $u_x, v_x \ll 1$

$\sin\theta \simeq u_x$, $\cos\theta \simeq 1$

$T \simeq T_0$

(3) $(T_0)_x = 0$, $(T_0 u_x)_x = \rho A \ddot{u}$

(4) $\therefore \quad u_{xx} = \dfrac{\rho A}{T_0} \ddot{u}$

which is the usual equation for undamped motion of a string.

✓

Assignment #1
Continued

Paul Grant
DEAP-AP-1G
AM 202
2-10-61

6. (1) We carry over the following results from lecture and problem 2:

Conservation of Momentum: Free Fluid:

$$\rho \frac{\partial \bar{V}}{\partial t} + \rho (\bar{V} \cdot \text{grad}) \bar{V} + \text{grad } P = 0$$

Conservation of mass:

$$\text{div}(\rho \bar{V}) + \frac{\partial \rho}{\partial t} = 0$$

Also given, the equation of state of the fluid:

$$\frac{P}{P_0} = \left(\frac{\rho}{\rho_0}\right)^\gamma \quad \text{or} \quad \rho = \rho_0 \left(\frac{P}{P_0}\right)^{1/\gamma}$$

(2) It will prove advantageous to make the substitution for $\frac{\partial \rho}{\partial t}$ in terms of P:

$$\frac{\partial \rho}{\partial t} = \frac{\rho_0}{P_0 \gamma} \left(\frac{P}{P_0}\right)^{\frac{1-\gamma}{\gamma}} \frac{\partial P}{\partial t}$$

$$\text{div}(\rho \bar{V}) + \frac{\rho_0}{P_0 \gamma} \left(\frac{P}{P_0}\right)^{\frac{1-\gamma}{\gamma}} \frac{\partial P}{\partial t} = 0$$

(3) We consider ρ, P and $|\bar{V}|$ to be small compared to ρ_0, P_0 and $c^2 = \gamma P_0/\rho_0$, that is:

$$\rho = \rho_0 + \epsilon \rho', \quad \epsilon \rho' \ll \rho_0$$

$$P = P_0 + \epsilon P', \quad \epsilon P' \ll P_0$$

$$\bar{V} = \epsilon \bar{V}', \quad |\epsilon \bar{V}'| \ll c = \sqrt{\frac{\gamma P_0}{\rho_0}}$$

ϵ is a dimensionaless quantity which displays the magnitude of the perturbation and is of the order of the velocity over the speed of sound.

(4) We can then form:

$$\frac{(\rho_0 + \epsilon \rho') \frac{\partial}{\partial t} (\epsilon V') + (\rho_0 + \epsilon \rho')(\epsilon \bar{V}' \cdot \text{grad}) \epsilon \bar{V}' + \text{grad}(P_0 + \epsilon P')}{\epsilon} = 0$$

$$\frac{\text{div}\{(\rho_0 + \epsilon \rho') \epsilon \bar{V}'\} + \frac{\rho_0}{\gamma P_0} \left(\frac{P_0 + \epsilon P'}{P_0}\right)^{\frac{1-\gamma}{\gamma}} \frac{\partial}{\partial t}(\epsilon P')}{\epsilon} = 0$$

(5) Take $\lim\limits_{\epsilon \to 0}$ and get:

$$\rho_0 \frac{\partial}{\partial t} \bar{V}' + \text{grad } P' = 0$$

$$\rho_0 \text{ div } \bar{V}' + \frac{\rho_0}{\gamma P_0} \frac{\partial P'}{\partial t} = 0$$

(6) Now form:

$$\rho_0 \text{ div } \frac{\partial}{\partial t} \bar{V}' + \text{div grad } P' = 0$$

$$\rho_0 \text{ div } \frac{\partial}{\partial t} \bar{V}' + \frac{\rho_0}{\gamma P_0} \frac{\partial^2 P'}{\partial t^2} = 0$$

(7) $\therefore \quad \nabla^2 P' = \dfrac{\rho_0}{\gamma P_0} \dfrac{\partial^2 P'}{\partial t^2}$

(8) We now make the change in notation $P' \to P$ with the implicit assumption that P represents small deviations from P_0:

$$\nabla^2 P = \frac{\rho_0}{\gamma P_0} \frac{\partial^2 P}{\partial t^2}$$

We see that this is the usual wave equation of the form:

$$\nabla^2 \psi = \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2}$$

Assignment #1
Continued

Paul Grant
DEAP-AP-1G
AM 202
2-10-61

7. (1) We have for the momentum equations:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -g\frac{\partial \eta}{\partial x}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -g\frac{\partial \eta}{\partial y}$$

and for continuity:

$$\frac{\partial}{\partial x}\{u(h+\eta)\} + \frac{\partial}{\partial y}\{v(h+\eta)\} = -\frac{\partial \eta}{\partial t}$$

or $\frac{\partial}{\partial x}(uh) + \frac{\partial}{\partial y}(vh) = -\frac{\partial \eta}{\partial t}$ if $h \gg \eta$.

(2) If the oscillations are small, that is, $\frac{\partial \eta}{\partial x}, \frac{\partial \eta}{\partial y} \ll 1$, then terms of $u\frac{\partial u}{\partial x}$, $v\frac{\partial u}{\partial y}$, etc., are of the second order and can be neglected.

$$\therefore \quad \frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x}\;; \quad \frac{\partial v}{\partial t} = -g\frac{\partial \eta}{\partial y}$$

(3) For purposes of manipulation, it is easier to put these equations in vector form, keeping always in mind that the z component does not exist:

$$\left.\begin{array}{ll}\text{Momentum: } h & \frac{\partial}{\partial t}\bar{v} = -g \operatorname{grad}\eta \\ \text{Continuity:} & \frac{\partial \eta}{\partial t} = -\operatorname{div}(h\bar{v})\end{array}\right\} \begin{array}{l}\cdot \operatorname{div} \\ \cdot \frac{\partial}{\partial t}\end{array}$$

$$\operatorname{div}\left\{h\frac{\partial}{\partial t}\bar{v}\right\} = -g\operatorname{div}\{h \operatorname{grad}\eta\}$$

$$\frac{\partial^2 \eta}{\partial t^2} = -\operatorname{div}\left\{h\frac{\partial}{\partial t}\bar{v}\right\}$$

$$\text{or} \quad \frac{\partial^2 \eta}{\partial t^2} = g\operatorname{div}\{h \operatorname{grad}\eta\}$$

(4) Finally:

$$\frac{\partial^2 \eta}{\partial t^2} = g\left\{\frac{\partial}{\partial x}\left(h\frac{\partial \eta}{\partial x}\right) + \frac{\partial}{\partial y}\left(h\frac{\partial \eta}{\partial y}\right)\right\}$$

which is the wave equation of the surface of the basin under the assumptions made above.

(OVER)

(5) The criteria $\frac{\partial \eta}{\partial x}, \frac{\partial \eta}{\partial y} \ll 1$ means that the surface waves are long. For the total derivative of the wave velocity (x component):

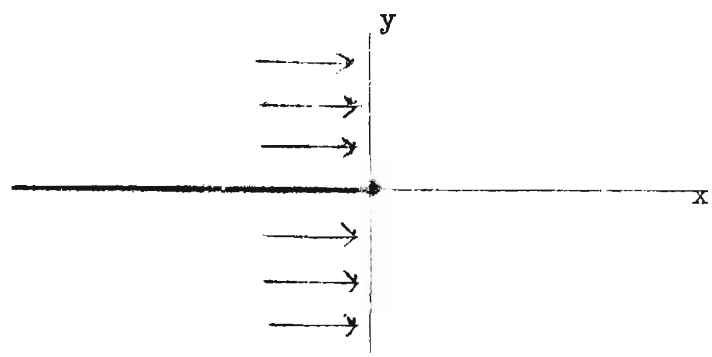
$$\frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}$$

where $\frac{du}{dt}$ is the rate of change of velocity as one moves in the fluid. $\frac{\partial u}{\partial t}$ is the velocity at a point. If the wave is long, the velocity of the wave is essentially that of a point in the wave. This is the meaning of $\frac{\partial \eta}{\partial x}, \frac{\partial \eta}{\partial y} \ll 1$.

APPLIED MATHEMATICS 202

Problem Set No. 3 Spring 1961

1. Fluid issues at speed u_o from either side of an insulating plate as shown in Figure 1.



The thermal diffusivity of the fluid is ν, the temperature of the upper fluid at $x = 0$ is T_1, and that of the lower fluid at $x = 0$ is T_2.

Assuming that the density and velocities stay constant, find $T(x,y)$ in $x > 0$.

2. Find the solution of

$$\phi_{xx} + \phi_{yy} - k^2\phi = G(x,y)$$

such that $\phi \longrightarrow 0$ as $x^2 + y^2 \longrightarrow 0$

Put your result in the form $\phi(x,y) = \iint K(x,x',y,y')\, G(x',y')\,dx'dy'$.

$$\theta(x,y) = \theta_0 \int_{-\infty}^{\infty} e^{-\xi^2 x + i\xi y} \int_0^{\infty} e^{-i\xi y} \, dy$$

$$\theta(x,y) \qquad \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_0^{\infty} \theta_0 \, e^{-i\xi y} \, e^{i\xi y'} \, e^{-\xi^2 x} \, dy \, d\xi$$

$$\theta(x,y) = \frac{\theta_0}{2} \left(1 + \operatorname{erf} \left(y \sqrt{\frac{c}{x}} \right) \right) \qquad y > 0$$

$$= \frac{\theta_0}{2} \left(1 - \operatorname{erf} \left(y \sqrt{\frac{c}{x}} \right) \right) \qquad y < 0$$

$$\bar{q} = -k \operatorname{grad} T$$

$$\int \bar{q} \cdot \bar{n} \, ds + \int \rho c T \, \bar{v} \cdot \bar{n} \, ds$$

$$\bar{v} = \operatorname{grad} \varphi$$

ST 2-216a

3. A gas, whose sound speed is c moves horizontally at speed U past the boundary $y = 0$. The strip nominally lying in $0 < x < a,\ y = 0$ moves vertically with the rigid body motion $y = y_0\, e^{i\omega t}$, where $\omega\, y_0 << c$.

Show that the small disturbances propagated into the stream obey the equation

$$\Delta \phi - \frac{1}{c^2}\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)^2 \phi = 0$$

where $u_i = \phi,i$

(a) Find ϕ for the experiment described above, for the case $u/c < 1$, and discuss the phenomenon.

(b) Find ϕ for the case $u/c > 1$, and discuss the phenomenon.

Assignment #2
Problem Set #2

Paul Grant
DEAP-AP-1G
AM 202
3-4-61

40
25 — $\frac{65}{70}$

1. (1) $u_{xx} - u_t = 0$
subject to: $u(0,t) = u_x(a,t) + \alpha u(a,t) = 0$
$u(x,0) = f(x)$

(2) Take the Laplace transform:

$\bar{u}_{xx} - s\bar{u} = -f(x)$, $\bar{u}(0,s) = \bar{u}_x(a,s) + \alpha\bar{u}(a,s) = 0$
or: $\bar{u}_{xx} - s\bar{u} = f(x)$; then $u(x,0) = -f(x)$

(3) Form the equation to find the Green's function:

$\bar{V}_{xx} - s\bar{V} = \delta(x-x')$
with $\bar{V}(x'+) = \bar{V}(x'-)$
and $\bar{V}_x(x'+) - \bar{V}_x(x'-) = 1$

(4) Consider the homogeneous equation:

$\bar{V}_{xx} - s\bar{V} = 0$; then: $\bar{V} = A\cosh\sqrt{s}\,x + B\sinh\sqrt{s}\,x$

Since $\bar{u}(0,s) \Rightarrow \bar{V}(0,s) = 0$, $\bar{V} = B\sinh\sqrt{s}\,x$, $x < x'$

At $\bar{u}(a,s) \Rightarrow \bar{V}(a,s)$, $\bar{V}_x(a,s) + \alpha\bar{V}(a,s) = 0$

By a suitable modification of the constants A and B above, the solution could also have been written:

$\bar{V} = C\cosh\sqrt{s}(a-x) + D\sinh\sqrt{s}(a-x)$

(5) $\bar{V}_x(a,s) = \left\{-\sqrt{s}\,C\sinh\sqrt{s}(a-x) - \sqrt{s}\,D\cosh\sqrt{s}(a-x)\right\}_a$

$= -\sqrt{s}\,D$

$\bar{V}(a,s) = C$, $\therefore -\sqrt{s}\,D + \alpha C = 0$, $C = \frac{\sqrt{s}}{\alpha}D$

$\therefore \bar{V} = D\left\{\frac{\sqrt{s}}{\alpha}\cosh\sqrt{s}(a-x) + \sinh\sqrt{s}(a-x)\right\}$ $x > x'$

(6) Consider: $A\cosh x + B\sinh x$

$= C\sinh\gamma\cosh x + C\cosh\gamma\sinh x = C\sinh(x+\gamma)$

$= \sqrt{B^2-A^2}\,\sinh(x+\gamma)$; $\gamma = \tanh^{-1}\frac{A}{B}$

$\therefore \bar{V} = D\sinh\left\{\sqrt{s}(a-x) + \gamma\right\}$, $x > x'$

$\gamma = \tanh^{-1}\frac{\sqrt{s}}{\alpha}$

(7) $\therefore \quad \bar{v} = \begin{cases} A \sinh \sqrt{s}\, x & x < x' \\ B \sinh \{\sqrt{s}(a-x) + \gamma\} & x > x' \end{cases}$

The notation A, B is not same as before.

(8) Using conditions (3):

$$\begin{vmatrix} \sinh \sqrt{s}\, x' & -\sinh\{\sqrt{s}(a-x') + \gamma\} \\ \sqrt{s} \cosh \sqrt{s}\, x' & \sqrt{s}\cosh\{\sqrt{s}(a-x')+\gamma\} \end{vmatrix} \begin{vmatrix} A \\ B \end{vmatrix} = \begin{vmatrix} 0 \\ -1 \end{vmatrix}$$

$$\bar{v} = \bar{v}(x, x', s) = -\begin{cases} \dfrac{\sinh\{\sqrt{s}(a-x')+\gamma\}}{\sqrt{s}\sinh\{\sqrt{s}a+\gamma\}} \sinh \sqrt{s}\, x, & x < x' \\ \dfrac{\sinh \sqrt{s}\, x'}{\sqrt{s}\sinh\{\sqrt{s}a+\gamma\}} \sinh\{\sqrt{s}(a-x)+\gamma\}, & x > x' \end{cases}$$

(9) $u(x,t) = \dfrac{1}{2\pi i}\displaystyle\int_{-i\infty}^{i\infty} \bar{u}(x,s)\, e^{st}\, ds = \dfrac{1}{2\pi i}\int_0^a f(x')\, dx' \int_{-i\infty}^{i\infty} \bar{v}(x,x';s)\, e^{st} ds$

if we take $\bar{u}(x,s) = \displaystyle\int_0^a \bar{v}(x,x';s)\, f(x')\, dx'$ as a solution.

(10) $\displaystyle\int_0^a \bar{v}(x,x',s)\, f(x')\, dx' = -\left\{\int_0^x [x > x']\, f(x')\, dx' + \int_x^a [x < x']\, f(x')\, dx'\right\}$

(11) Consider $\dfrac{1}{2\pi i}\displaystyle\int_{-i\infty}^{i\infty} [x<x']\, e^{st}\, ds$:

$$I(s) = \frac{\sinh\{\sqrt{s}(a-x')+\gamma\}\sinh\sqrt{s}\,x\; e^{st}}{\sqrt{s}\sinh\{\sqrt{s}a+\gamma\}}$$

Poles at $\sqrt{s}a + \gamma = \sqrt{s}a + \tanh^{-1}\dfrac{\sqrt{s}}{\alpha} = i n\pi$

$$\frac{\sqrt{s}}{\sigma} = \tanh\{in\pi - \sqrt{s}a\} = -\tanh\sqrt{s}\,a$$

No pole at $s = 0$ as: $\displaystyle\lim_{\sqrt{s}\to 0} \frac{\sinh \sqrt{s}\, x}{\sqrt{s}} = x \neq 0$

(12) It will prove nice to make the substitution: $\boxed{\sqrt{s} = i\beta}$

Then: $\dfrac{1}{2\pi i}\displaystyle\int_{-i\infty}^{i\infty} [x<x']\, e^{st}\, ds = -\frac{2}{2\pi i}\int_{\Gamma} [x<x']\, e^{-\beta^2 t}\, \beta\, d\beta$

$$I(s) = I(\beta) = \frac{\beta \sin\{\beta(a-x')+\delta\}\sin\beta x\; e^{-\beta^2 t}}{\beta\sin\{\beta a+\delta\}}$$

with $\delta = \tan^{-1}\dfrac{\beta}{\alpha}$

Assignment #2
Continued

Paul Grant
DEAP-AP-16
AM 202
3-4-61

1. Continued

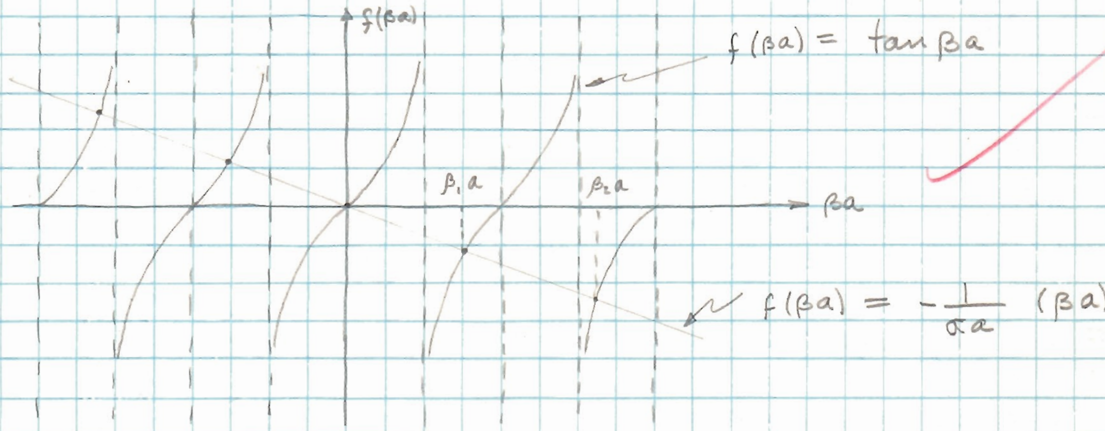
(13) The poles are at $\beta a - \tan^{-1}\frac{\beta}{\alpha} = n\pi$

$$\frac{\beta}{\alpha} = -\tan(\beta a - n\pi) = -\tan\beta a$$

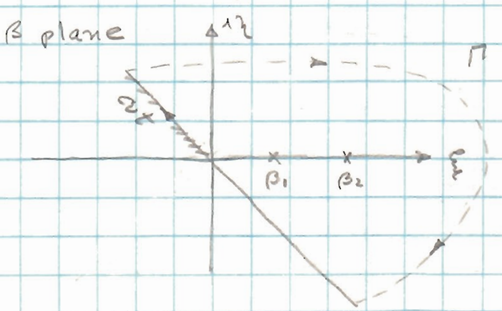
or $\beta = -\alpha \tan\beta a$

the poles of $I(\beta)$ are at the roots of this trigonometric equation β_n found graphically. Consider:

$$\beta a = -\alpha a \tan\beta a$$



We take the β_n to be positive and real.



now: $\sqrt{s} = i\beta$
let, $s = ic$ where c is + and real

$$\sqrt{c}\, e^{i\pi/4} = i\beta = i\xi - \eta$$

$$= \sqrt{c}\,\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\sqrt{c}$$

$$\xi = \frac{\sqrt{2}}{2}\sqrt{c}, \quad \eta = -\frac{\sqrt{2}}{2}\sqrt{c}$$

$\therefore\ \xi = -\eta$ which determines Γ

Also, $\lim_{\beta\to\infty} I(\beta) = 0$ so $I(\beta)$ vanishes on the part of Γ that is not $\xi = -\eta$

(14) $$R(\beta_n) = \lim_{\beta\to\beta_n} \frac{N(\beta)}{D'(\beta)} = \lim_{\beta\to\beta_n}\left\{ \frac{\sin\{\beta(a-x')+\delta\}\sin\beta x\,\beta e^{-\beta^2 t}}{\sin\{\beta a+\delta\} + \beta(a+\delta')\cos\{\beta a+\delta\}} \right\}$$

$$\delta' = \frac{d}{d\beta}\tan^{-1}\frac{\beta}{\alpha} = \frac{1}{1+\frac{\beta^2}{\alpha^2}}\cdot\frac{1}{\alpha} = \frac{\alpha}{\alpha^2+\beta^2}$$

$$R(\beta_n) = -\frac{e^{-\beta_n^2 t}}{\left\{a + \frac{\alpha}{\alpha^2+\beta_n^2}\right\}}\ \frac{\sin\{\beta_n(a-x')+\delta_n\}\sin\beta_n x}{\cos\{\beta_n a+\delta_n\}}$$

(15) $\sin\{\beta_n(a-x')+\delta_n\} = \sin(\beta_n a+\delta_n)\cos\beta_n x' - \cos(\beta_n a+\delta_n)\sin\beta_n x'$

$$\therefore\ R(\beta_n) = \frac{-e^{-\beta_n^2 t}}{\left\{a+\frac{\alpha}{\alpha+\beta_n^2}\right\}}\sin\beta_n x' \sin\beta_n x$$

(16) Now: $\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}[x<x']\,e^{st}\,ds = 2\sum_{n=1}^{\infty} R(\beta_n)$

A casual inspection of $[x>x']$ shows the same pole points as $[x<x']$ and the yielding of the same residues at these poles as $[x<x']$. Thus the two solutions are linearly dependent at their poles, such that (10) can be closed.

(17) $\therefore\ \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\bar{V}(x,x',s)\,e^{st}\,ds = -2\sum_{n=1}^{\infty}R(\beta_n)$

(18) From (9):

$$u(x,t) = -\sum_{n=1}^{\infty}\frac{2e^{-\beta_n t}}{\left\{a+\frac{\alpha}{\alpha^2+\beta_n^2}\right\}}\sin\beta_n x\int_0^a f(x')\sin\beta_n x'\,dx'$$

(19) From (1) and (2):

$$f(x) \equiv -u(x,0) = \sum_{n=1}^{\infty}\frac{2}{\left\{a+\frac{\alpha}{\alpha^2+\beta_n^2}\right\}}\sin\beta_n x\int_0^a f(x')\sin\beta_n x'\,dx'$$

where the β_n are roots of $\beta = -\alpha\tan\beta a$. (19) shows completeness in the fact of the <u>identity</u> between $f(x)$ and the expansion in eigenfunctions of the original equation. The eigenfunctions are clearly:

$$V_n(x) = \left\{\frac{2}{\left\{a+\frac{\alpha}{\alpha^2+\beta_n^2}\right\}}\right\}^{1/2}\sin\beta_n x$$

or more simply: $V_n(x) = \sin\beta_n x$, lumping the first term into the coefficient. At any rate, consider (4):

$\bar{V}_{xx} - s\bar{V} = 0$; $\bar{V}(0) = 0$, $\bar{V}_x(a) + \alpha\bar{V}(a) = 0$
or $\bar{V}_{xx} + \beta^2\bar{V} = 0$ which is of the sturm-louiville type with $p=1$, $q=0$, $\lambda=\beta^2$, $h=1$.

$$\begin{array}{l} V_m\{\ (V_n)_{xx} + \beta_n^2 V_n = 0\ \} \\ V_n\{\ (V_m)_{xx} + \beta_m^2 V_m = 0\ \} \end{array} \quad (\beta_n^2-\beta_m^2)\int_0^a V_n V_m\,dx = \int_0^a\{V_m V_{n_{xx}} - V_n V_{m_{xx}}\}\,dx$$

$$= \{V_m V_{n_x} - V_n V_{m_x}\}\Big|_0^a = V_m(a)V_{n_x}(a) - V_n(a)V_{m_x}(a)$$

Now: $V_m(a)V_{n_x}(a) = -\alpha V_m(a)V_n(a)$; $V_n(a)V_{m_x}(a) = -\alpha V_n(a)V_m(a)$

$\therefore (\beta_n^2-\beta_m^2)\int_0^a V_n V_m\,dx = 0$, $n\neq m$ thus establishing orthogonality.

10

Assignment #2
Continued

Paul Grant
DEAP-AP-1G
AM 202
3-4-61

2. (1) Given: $u_{xx} - x u_t = 0$, $0 < x < L$, $0 < t$

subject to:

$$u(0,t) = u(L,t) = 0, \quad u(x,0) = f(x)$$

(2) Transforming: $\bar{u}_{xx} - s x \bar{u} = -x f(x)$

$$\bar{u}(0,s) = \bar{u}(L,s) = 0$$

(3) From lecture: $[p(x)\bar{u}']' + q(x)\bar{u} - s h(x)\bar{u} = -h(x) f(x)$

The homogeneous solutions are: $w_1(x,s)$ $x < x'$

$w_2(x,s)$ $x > x'$

$$\bar{u}(x,s) = \int_0^x \frac{w_1(x',s)\, w_2(x,s)\, f(x')\, h(x')\, dx'}{[-pW]}$$

$$+ \int_x^a \frac{w_2(x',s)\, w_1(x,s)\, f(x')\, h(x')\, dx'}{[-pW]}$$

(4) $w_{xx} - s x w = 0$, $w(0) = w(L) = 0$

$$w = A x^{1/2} J_{1/3}\left\{\tfrac{2}{3}(-s)^{1/2} x^{3/2}\right\} + B x^{1/2} J_{-1/3}\left\{\tfrac{2}{3}(-s)^{1/2} x^{3/2}\right\}$$

(5) $$J_\nu(z) = \left(\tfrac{1}{2}z\right)^\nu \sum_{n=0}^{\infty} \frac{(-1)^n \left(\tfrac{1}{2}z\right)^{2n}}{n!\,\Gamma(\nu+n+1)}$$

If $z \ll 1$, $J_\nu(z) \sim z^\nu$

$\therefore$ $w \sim A x + B$ for x near zero

Then for $x = 0$, $B = 0$, $A \neq 0$

At L: $w(L) = A L^{1/2} J_{1/3}\left\{\tfrac{2}{3}(-s)^{1/2} L^{3/2}\right\} = 0$

or the eigenvalues are at the roots s_n of

$$J_{1/3}\left\{\tfrac{2}{3}(-s)^{1/2} L^{3/2}\right\} = 0.$$

$\therefore$ $w_1 = x^{1/2} J_{1/3}\left\{\tfrac{2}{3}(-s)^{1/2} x^{3/2}\right\}$, $x < x'$

(6) At the other boundary:

$(w_2)_{xx} - s x w_2 = 0$

$$w_2 = A x^{1/2} J_{1/3}\{\cdots x^{3/2}\} + B x^{1/2} J_{-1/3}\{\cdots x^{3/2}\}$$

$$\frac{A}{B} = -\frac{J_{-1/3}\{\cdots L^{3/2}\}}{J_{1/3}\{\cdots L^{3/2}\}}$$

(7) $\therefore\ w_2 = x^{1/2}\left\{ J_{1/3}\{\cdots x^{3/2}\} J_{-1/3}\{\cdots L^{3/2}\} - J_{-1/3}\{\cdots x^{3/2}\} J_{1/3}\{\cdots L^{3/2}\}\right\},\ x > x'$

(8) $W = \begin{vmatrix} w_1 & w_2 \\ w_1' & w_2' \end{vmatrix} : \quad w_1' = \frac{1}{2} x^{-1/2} J_{1/3}\{\cdots x^{3/2}\} + (-s)^{1/2} x\, J_{1/3}'\{\cdots x^{3/2}\}$

$$w_2' = 1/2\, x^{-1/2}\left\{ J_{1/3}(x) J_{-1/3}(L) - J_{-1/3}(x) J_{1/3}(L)\right\} + (-s)^{1/2} x \left\{ J_{1/3}'(x) J_{-1/3}(L) - J_{-1/3}'(x) J_{1/3}(L)\right\}$$

$$w_1 w_2' = \frac{1}{2}\left\{ J_{1/3}^2(x) J_{-1/3}(L) - J_{1/3}(x) J_{-1/3}(x) J_{1/3}(L)\right\} + (-s)^{1/2} x^{3/2}\left\{ J_{1/3}(x) J_{1/3}'(x) \cdot J_{-1/3}(L) - J_{1/3}(x) J_{-1/3}'(x) J_{1/3}(L)\right\}$$

$$w_1' w_2 = \frac{1}{2}\left\{ J_+(x) J_+(x) J_-(L) - J_+(x) J_-(x) J_+(L)\right\} + (-s)^{1/2} x^{3/2}\left\{ J_+'(x) J_+(x) J_-(L) - J_+'(x) J_-(x) J_+(L)\right\}$$

$$W = (-s)^{1/2} x^{3/2}\left[J_+(L)\left\{ J_+' J_- - J_+ J_-'\right\}\right]$$

(9) $$J_\nu(z) J_{-\nu}'(z) - J_\nu'(z) J_{-\nu}(z) = -\frac{2 \sin \nu\pi}{\pi z} \qquad \text{(Copson)}$$

$$\therefore\ J_+' J_- - J_+ J_-' = \frac{2 \sin \pi/3}{\pi\left(\frac{2}{3}(-s)^{1/2} x^{3/2}\right)}$$

$$W = \frac{3}{\pi} \sin \pi/3\ J_+(L) = \frac{3\sqrt{3}}{2\pi} J_+(L)$$

We now have the Green's function.

(10) Consider: $$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{(x')^{1/2} J_+(x')\, x^{1/2}\left\{ J_+(x) J_-(L) - J_-(x) J_+(L)\right\} e^{st}\, ds}{J_+(L)}$$

The poles are clearly at the zeroes of $J_{1/3}\left\{\frac{2}{3}(-s)^{1/2} x^{3/2}\right\} = 0$

Note that $J_-(x,s) J_+(x',s)$ is an entire function. Thus the integral reduces to:

$$(x')^{1/2} x^{1/2}\left\{ \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{J_+(x',s) J_+(x,s) J_-(L,s)\, e^{st}\, ds}{J_+(L,s)}\right.$$

(11) Make the substitution: $s = \alpha^2 e^{i\pi}$

If $s = ic = c e^{i\pi/2}$; $\alpha = c^{1/2} e^{-i\pi/4} = \xi + i\eta$

$= \frac{\sqrt{2}\, c^{1/2}}{2}(1 - i)$, $\therefore\ \xi = -\eta$

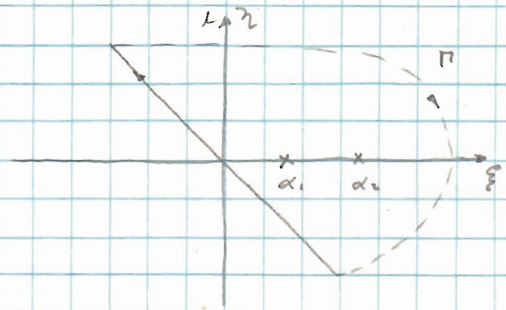
Paul Grant
DEAP-AP-16
AM 202
3-6-61

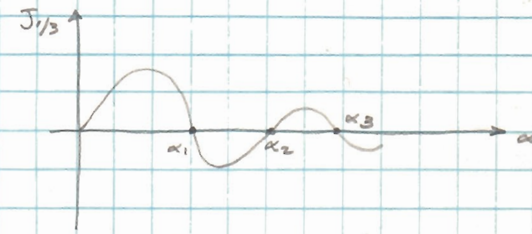
Problem 2 continued:

(12) $\therefore$ we have $I(s) \to \dfrac{-2}{2\pi i} \displaystyle\int_{\Gamma} \dfrac{\alpha\, J_+(x',\alpha)\, J_+(x,\alpha)\, e^{-\alpha^2 t}\, d\alpha\; J_-(L,\alpha)}{J_+(L,\alpha)}$

where the poles are the roots of $J_{1/3}\left\{\frac{2}{3} L^{3/2} \alpha\right\} = 0$

and are positive real.





No pole at $\alpha = 0$ since $J_{1/3}\{\cdot\cdot\alpha\} \sim \alpha^{1/3}$.

(13) $R_n = -\dfrac{e^{-\alpha_n^2 t}\, \alpha_n\, J_+(x',\alpha_n)\, J_+(x,\alpha_n)\, J_-(L,\alpha)}{\frac{2}{3} L^{3/2}\, J_+'(L,\alpha_n)} = \dfrac{e^{-\alpha_n^2 t}\, \alpha_n\, J_+(x',\alpha_n)\, J_+(x,\alpha_n)\, J_-(L,\alpha_n)}{\frac{2}{3} L^{3/2}\, J_{4/3}(L,\alpha_n)}$

$$\frac{d}{dz} J_\nu(az) = \frac{\nu}{z} J_\nu(az) - a J_{\nu+1}(az)$$

(14) Consider: $\dfrac{1}{2\pi i} \displaystyle\int_{-i\infty}^{i\infty} \dfrac{(x)^{1/2} J_+(x)\, (x')^{1/2} \left\{ J_+(x') J_-(L) - J_-(x') J_+(L) \right\}}{J_+(L)} e^{st}\, ds$

$$= \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{(x)^{1/2} J_+(x)\, (x')^{1/2}\, J_+(x')\, J_-(L)\, e^{st}\, ds}{J_+(L)}$$

which gives the same result as before.

(15) $\therefore$ $u(x,t) = \dfrac{2\pi}{\sqrt{3}\, L^{3/2}} \displaystyle\sum_{n=1}^{\infty} \dfrac{J_{-1/3}\left\{\frac{2}{3} L^{3/2} \alpha_n\right\}}{J_{4/3}\left\{\frac{2}{3} L^{3/2} \alpha_n\right\}} e^{-\alpha_n^2 t} \alpha_n x^{1/2} J_{1/3}\left\{\frac{2}{3} \alpha_n x^{3/2}\right\}$

$$\cdot \int_0^L (x')^{1/2} J_{1/3}\left\{\tfrac{2}{3} \alpha_n x'^{3/2}\right\} h(x') f(x')\, dx'$$

(16) $u(x,0) = f(x) = \displaystyle\sum_{n=1}^{\infty} x^{1/2} J_{1/3}\left\{\frac{2}{3} \alpha_n x^{3/2}\right\} \left\{ \frac{2\pi}{\sqrt{3}\, L^{3/2}} \int_0^L \frac{J_{-1/3}\left\{\frac{2}{3} L^{3/2} \alpha_n\right\}}{J_{4/3}\left\{\frac{2}{3} L^{3/2} \alpha_n\right\}} \alpha_n (x')^{1/2} \right.$

$$\left. \cdot\, J_{1/3}\left\{\tfrac{2}{3} \alpha_n x'^{3/2}\right\} h(x') f(x')\, dx' \right\}$$

which establishes the completeness of the $u_n(x) = x^{1/2} J_{1/3}\left\{\frac{2}{3} \alpha_n x^{3/2}\right\}$

(17) The equation of the $u_n(x)$ is:

$$u_{xx} + \beta^2 x\, u = 0$$

so that Sturm-Liouville theory gives:

$$(\beta_m^2 - \beta_n^2)\int_0^L x\, u_n(x)\, u_m(x)\, dx = \left(u_n' u_m - u_m' u_n\right)\Big|_0^L$$

(18) Now $u_n(x) = x^{1/2}\, J_{1/3}\left\{\frac{2}{3}\alpha_n x^{3/2}\right\}$

$u_n(0) = 0$; $u_n(L) = L^{1/2} J_{1/3}\left\{\frac{2}{3}\alpha_n L^{3/2}\right\} = 0$
since this defines the zeroes of $u_n(x)$.

$\therefore$ $(\beta_m^2 - \beta_n^2)\int_0^L x\, u_n(x)\, u_m(x)\, dx = 0$, $m \neq n$

and the u_n's form a complete orthogonal set.

10 ✓

Paul Grant
DEAP-AP-1G
AM 202
3-13-61

3. a. (1) Given: $u_{xx} + u_{yy} = e^{-y^2} \sin x$; $0 < x < \pi$, $-L < y < L$

subject to: $u(0,y) = u(\pi,y) = 0$
$u(x,-L) = u(x,L) = 0$

(2) Consider the homogeneous equation:

$$V_{xx} + V_{yy} = 0 \quad , \quad \left.\begin{array}{l} V(0,y) = V(\pi,y) = 0 \\ V(x,-L) = V(x,L) = 0 \end{array}\right\} \quad V(x,y) = X(x)\,Y(y)$$

$$\therefore \quad \frac{X''}{X} = -\frac{Y''}{Y} = \lambda^2$$

(3) $X'' + \lambda^2 X = 0$; $X = A \sin \lambda x + B \cos \lambda x$

$Y'' - \lambda^2 Y = 0$; $Y = C \sinh \lambda y + D \cosh \lambda y$

$V(0,y) = 0$: $X = 0 = B$
$V(\pi,y) = 0$: $X = 0 = \sin \lambda\pi$, $\therefore$ $\lambda = n$, $n = 0, 1, 2, 3, \dots$

$\therefore$ $X_n = \sin nx$

We choose these as the eigenfunctions: $u_n(x) = \sin nx$

(4) We assert that the solution can be written in terms of the eigenfunctions, viz:

$$u(x,y) = \sum_n l_n(y) \sin nx$$

with $g(x,y) = \sum_n g_n(y) \sin nx$

where $g(x,y) = e^{-y^2} \sin x$

(5) $u_x = \sum_n l_n(y)\, n \cos nx$, $u_{xx} = -\sum_n l_n(y)\, n^2 \sin nx$

$u_{yy} = \sum_n l_n''(y) \sin nx$

$$\therefore \quad \sum_n \left\{ -n^2 l_n(y) + l_n''(y) \right\} \sin nx = \sum_n g_n(y) \sin nx$$

(6) $\therefore$ $l_n''(y) - n^2 l_n(y) = g_n(y)$

(7) $e^{-y^2} \sin x = \sum_{n'} g_{n'}(y) \sin n'x$

(8) $$e^{-y^2}\int_0^{\pi} \sin x \, \sin nx \, dx = \sum_{n'} g_{n'}(y) \int_0^{\pi} \sin n'x \, \sin nx \, dx$$

(9) $$\int_0^{\pi} \sin n'x \, \sin nx \, dx = \frac{\sin (n'-n)\pi}{2(n'-n)}$$

$$\left.\begin{aligned} &= 0, \quad n' \neq n \\ &= \frac{1}{2} \lim_{n' \to n} \frac{\sin(n'-n)\pi}{(n'-n)} = \pi/2, \quad n' = n \end{aligned}\right\} = \pi/2 \; \delta_{n'n}$$

$$\therefore \quad g_n(y) = e^{-y^2} \delta_{n1}$$

(10) $$\therefore \quad l_n'' - n^2 l_n = e^{-y^2} \delta_{n1}$$

$$l_n'' - n^2 l_n = 0, \quad n \neq 1 \quad ; \quad l_1'' - l_1 = e^{-y^2}, \quad n = 1$$

$$l_n = A_n \cosh ny + B_n \sinh ny, \quad n \neq 1$$

(11) $$l_1'' - l_1 = e^{-y^2} \; ; \quad l_1(x,-L) = l_1(x,L) = 0$$

(12) Consider $w'' - w = 0$: $$w_g = A \cosh y + B \sinh y = A e^{y} + B e^{-y}$$

$$w_1 = e^{y}, \quad w_2 = e^{-y}$$

$$W(w_1, w_2) = \begin{vmatrix} w_1 & w_2 \\ w_1' & w_2' \end{vmatrix} = \begin{vmatrix} e^{y} & e^{-y} \\ e^{y} & -e^{-y} \end{vmatrix} = -1 - 1 = -2$$

$$w_p = w_2 \int \frac{f w_1}{W} dy - w_1 \int \frac{f w_2}{W} dy$$

$$= e^{-y} \int \frac{e^{-y^2} e^{y} \, dy}{-2} - e^{y} \int \frac{e^{-y^2} e^{-y}}{-2} dy$$

$$= -\frac{e^{+1/4}}{2} \left\{ e^{-y} \int e^{-(y-1/2)^2} d(y - 1/2) - e^{y} \int e^{-(y+1/2)^2} d(y+1/2) \right\}$$

(13) Now, $$\operatorname{erf}(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x e^{-r^2} dr$$

$$\therefore \quad \frac{\sqrt{\pi}\, e^{+1/4}}{4} \left\{ e^{y} \operatorname{erf}(y + 1/2) - e^{-y} \operatorname{erf}(y - 1/2) \right\}$$

(14) $$\therefore \quad l_1 = A e^{y} + B e^{-y} + \frac{\sqrt{\pi}\, e^{+1/4}}{4} \left\{ e^{y} \operatorname{erf}(y+1/2) - e^{-y} \operatorname{erf}(y - 1/2) \right\}$$

$$0 = A e^{-L} + B e^{L} + \frac{\sqrt{\pi}\, e^{+1/4}}{4} \left\{ e^{-L} \operatorname{erf}(-L + 1/2) - e^{L} \operatorname{erf}(-L - 1/2) \right\}$$

$$0 = A e^{L} + B e^{-L} + \frac{\sqrt{\pi}}{4} e^{+1/4} \left\{ e^{L} \operatorname{erf}(L + 1/2) - e^{-L} \operatorname{erf}(L - 1/2) \right\}$$

Assignment #2
Continued

Paul Grant
DEAP-AP-1G
AM 202
3-13-61

3 continued:

(15)
$$\begin{vmatrix} e^{-L} & e^{L} \\ e^{L} & e^{-L} \end{vmatrix} \begin{vmatrix} A \\ B \end{vmatrix} = \begin{vmatrix} \left[\frac{\sqrt{\pi}\, e^{+1/4}}{4} \left\{ e^{L} \operatorname{erf}(L+1/2) - e^{-L} \operatorname{erf}(L-1/2) \right\}\right] \\ \left[\frac{\sqrt{\pi}}{4}\, e^{+1/4} \left\{ e^{L} \operatorname{erf}(L+1/2) - e^{-L} \operatorname{erf}(L-1/2) \right\}\right] \end{vmatrix}$$

$$A = e^{-2L} - e^{2L} = -2 \sinh 2L$$

$$A = \frac{\sinh L}{\sinh 2L} [\quad] = \frac{1}{2 \cosh L} [\quad]$$

$$B = \frac{\sinh L}{\sinh 2L} [\quad] = \frac{1}{2 \cosh L} [\quad]$$

(16) For A_n, B_n, $n \neq 1$; $\sinh 2nL = 0$ which is impossible since $2nL$ is real, $\therefore$ $A_n = B_n = 0$, $n \neq 1$.

(17)
$$u(x,y) = \frac{\sqrt{\pi}\, e^{+1/4}}{4} \left[- \frac{\left\{ e^{L} \operatorname{erf}(L+1/2) - e^{-L} \operatorname{erf}(L-1/2) \right\}}{\cosh L} \cosh y \right.$$
$$\left. + \left\{ e^{y} \operatorname{erf}(y+1/2) - e^{-y} \operatorname{erf}(y-1/2) \right\} \right] \sin x$$

b.(1) The RHS of a.(1) contains an eigenfunction of the X equation, thus reducing the series a.(4) to one term for $u(x,y)$. This makes for greater convenience if the result can be expressed in closed form. This would not work if the solutions of a.(3) in Y were taken as eigenfunctions because e^{-y^2} is not an eigenfunction of this equation.

c. (1) $u_{xx} + u_{yy} = e^{-y^2}$; $u(0,y) = u(\pi,y) = 0$; $u(x,-L) = u(x,L) = 0$

$u_{xx} + u_{yy} = 0$; Take $u_n(x,y) = X_n(x)\,Y_n(y)$

(2) $\frac{Y''}{Y} = -\frac{X''}{X} = \lambda^2$; $Y'' + \lambda^2 Y = 0$, $Y = A\cos\lambda y + B\sin\lambda y$

$B = 0$; $Y_n = \cos\frac{(2n-1)\pi y}{2L}$; $\lambda_n = \frac{(2n-1)\pi}{2L}$, $n = 1, 2, 3, \ldots$

(3) Assume: $u(x,y) = \sum_n h_n(x)\cos\lambda_n y$; $g(x,y) = \sum_n g_n(x)\cos\lambda_n y$

(4) $\therefore\ h_n'' - \lambda_n^2 h_n = g_n(x)$; because $g(x,y) = e^{-y^2}$, $g_n(x) = g_n$, a constant.

(5) $(h_n)_c = A_n e^{\lambda_n x} + B_n e^{-\lambda_n x}$

Take $(h_n)_p = k$: $-\lambda_n^2 k = g_n$

$\therefore\ h_n = A_n e^{\lambda_n x} + B_n e^{-\lambda_n x} - g_n/\lambda_n^2$

(6) $\therefore\ u(x,y) = \sum_n \left\{ A_n e^{\lambda_n x} + B_n e^{-\lambda_n x} - g_n/\lambda_n^2 \right\}\cos\lambda_n y$

(7) $\begin{vmatrix} 1 & 1 \\ e^{\lambda_n\pi} & e^{-\lambda_n\pi} \end{vmatrix} \begin{vmatrix} A_n \\ B_n \end{vmatrix} = \begin{vmatrix} g_n/\lambda_n^2 \\ g_n/\lambda_n^2 \end{vmatrix}$; $A_n = \frac{g_n e^{-\lambda_n \pi/2}}{2\lambda_n^2 \cosh \lambda_n \pi/2}$

$B_n = \frac{g_n e^{\lambda_n \pi/2}}{2\lambda_n^2 \cosh \lambda \pi/2}$

(8) Now: $e^{-y^2} = \sum_{n'} g_{n'} \cos\lambda_{n'} y$

$\int_{-L}^{L} e^{-y^2}\cos\lambda_n y\, dy = \sum_{n'} g_{n'} \int_{-L}^{L} \cos\lambda_n y \cos\lambda_{n'} y\, dy$

$= L g_n = \int_0^L e^{-y^2}\left(e^{i\lambda_n y} + e^{-i\lambda_n y}\right) dy$

$= e^{-\lambda_n^2/4}\left\{ \int_0^L e^{-(y-i\lambda_n/2)^2} dy + \int_0^L e^{-(y+i\lambda_n/2)^2} dy \right\}$

(9) $g_n = \frac{e^{-\lambda_n^2/4}}{L}\left\{ \mathrm{erf}(L - i\lambda_n/2) + \mathrm{erf}(L + i\lambda_n/2) \right\}$

if you expand e^{-y^2} in a Fourier series in $(-L,+L)$ you get

$e^{-y^2} = \sum g_n \cos\left(\frac{n\pi y}{L}\right)$

where $g_n = \frac{1}{L}\int_{-L}^{+L} dy\, e^{-y^2}\cos\left(\frac{n\pi y}{L}\right)$.

Assignment #2
Continued

Paul Grant
DEAP - AP - 16
AM 202
3-13-61

3 Continued:

d. (1) The X solution is now $X_n = \sin \lambda_n x$ where λ_n is a root of:

$$\lambda \cos \lambda\pi + \alpha \sin \lambda\pi = 0$$

appropriate changes must be made throughout the calculation. Note that $\sin x$ is no longer one of the eigenfunctions.

e. (1) We have from a.:

$$l_n = A_n e^{ny} + B_n e^{-ny}, \quad n \neq 1$$

$$l_1 = A_1 e^{y} + B_1 e^{-y} + \frac{\sqrt{\pi}\, e^{+1/4}}{4}\left\{e^{y}\,\mathrm{erf}\,(y+1/2) - e^{-y}\,\mathrm{erf}\,(y-\tfrac{1}{2})\right\}$$

$$l_n(-L) = \{l_n(L)\}_y = 0$$

(2) $$\{l_1(y)\}_y = A_1 e^{y} - B_1 e^{-y} + \frac{\sqrt{\pi}\, e^{+1/4}}{4}\left\{e^{y}\,\mathrm{erf}\,(y+1/2) + e^{-y}\,\mathrm{erf}\,(y-1/2)\right.$$

$$\left. + \underbrace{e^{y}\frac{2}{\sqrt{\pi}}e^{-(y+1/2)^2} - e^{-y}\frac{2}{\sqrt{\pi}}e^{-(y-1/2)^2}}_{0}\right\}$$

$$\{l_n(y)\}_y = nA_n e^{ny} - nB_n e^{-ny}$$

$$\begin{vmatrix} e^{-nL} & e^{nL} \\ e^{nL} & -e^{-nL} \end{vmatrix} \begin{vmatrix} A_n \\ B_n \end{vmatrix} = 0 \; ; \qquad \cosh 2nL = 0, \text{ impossible,} \quad \therefore A_n = B_n = 0, \; n \neq 1$$

(3) $$\begin{vmatrix} e^{-L} & e^{L} \\ e^{L} & -e^{-L} \end{vmatrix} \begin{vmatrix} A_1 \\ B_1 \end{vmatrix} = \begin{vmatrix} \frac{\sqrt{\pi}\, e^{+1/4}}{4}\left\{e^{L}\,\mathrm{erf}\,(L+1/2) - e^{-L}\,\mathrm{erf}\,(L-1/2)\right\} \\ \frac{\sqrt{\pi}\, e^{+1/4}}{4}\left\{e^{L}\,\mathrm{erf}\,(L+1/2) + e^{-L}\,\mathrm{erf}\,(L-1/2)\right\} \end{vmatrix}$$

(4) $$\Delta = -2\cosh 2L$$

$$A_1 = \frac{\sqrt{\pi}\, e^{+1/4}}{-8\cosh 2L}\left\{-\mathrm{erf}\,(L+1/2) + e^{-2L}\,\mathrm{erf}\,(L-1/2) - e^{2L}\,\mathrm{erf}\,(L+1/2) - \mathrm{erf}\,(L-1/2)\right\}$$

$$B_1 = \frac{\sqrt{\pi}\, e^{+1/4}}{-8\cosh 2L}\left\{\mathrm{erf}\,(L+1/2) + e^{-2L}\,\mathrm{erf}\,(L-1/2) - e^{2L}\,\mathrm{erf}\,(L+1/2) + \mathrm{erf}\,(L-1/2)\right\}$$

(5) $$A_1 e^{y} + B_1 e^{-y} = \frac{\sqrt{\pi}\, e^{+1/4}}{-8 \cosh 2L} \Big\{ -2 \sinh y \,\mathrm{erf}(L + 1/2) - 2 \sinh y \,\mathrm{erf}(L - 1/2)$$

$$+ 2 \cosh y \left[e^{-2L} \mathrm{erf}(L - 1/2) + e^{2L} \mathrm{erf}(L + 1/2) \right] \Big\}$$

(6) $$u(x, y) = \frac{\sqrt{\pi}\, e^{+1/4}}{4} \Big[\frac{1}{\cosh 2L} \left\{ \mathrm{erf}(L + 1/2) + \mathrm{erf}(L - 1/2) \right\} \sinh y$$

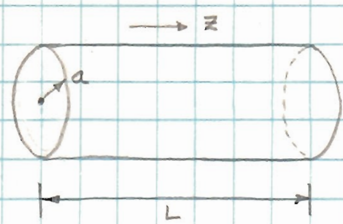
$$- \frac{1}{\cosh 2L} \left\{ e^{-2L} \mathrm{erf}(L - 1/2) + e^{2L} \mathrm{erf}(L + 1/2) \right\}$$

$$+ \left\{ e^{y} \mathrm{erf}(y + 1/2) - e^{-y} \mathrm{erf}(y - 1/2) \right\} \Big] \sin x$$

7

Assignment # 2
Continued

Paul Grant
DEAP-AP-16
AM 202
3-8-61

4. 

(1) The appropriate equation and boundary conditions are:

$$\frac{\partial T}{\partial t} = \nu \nabla^2 T \quad , \quad T = T(r, \varphi, z, t)$$

$$T(r, \varphi, z, 0) = T_0$$

$$\frac{\partial T}{\partial r} + kT = 0 \quad \text{at } r = a$$

$T(0, \varphi, z, t)$ must be finite

$$T_z(r, \varphi, 0, t) = T_z(r, \varphi, L, t) = 0$$

(2) $$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}$$

(3) Assume a solution: $T = R(r)\,\Phi(\varphi)\,Z(z)\,\Pi(t)$

$$\frac{1}{Rr}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \frac{1}{r^2 \Phi}\frac{\partial^2 \Phi}{\partial \varphi^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} - \frac{1}{\nu}\frac{1}{\Pi}\frac{\partial \Pi}{\partial t} = 0$$

$$\therefore \frac{d^2 Z}{dz^2} + \lambda_z^2 Z = 0 \quad ; \quad Z = A\cos\lambda_z z + B\sin\lambda_z z$$

$$Z_z(0) = \left\{-\lambda_z A\sin\lambda_z z + \lambda_z B\cos\lambda_z z\right\}_0 = \lambda_z B = 0$$

$\therefore B = 0$

$$Z_z(L) = 0 = -\lambda_z A\sin\lambda_z L \quad ; \quad \lambda_z = \frac{2n\pi}{L}, \ n = 0, 1, 2, \cdots$$

$$\therefore Z(z) = A\cos\frac{2n\pi z}{L} \quad ; \ n = 1, 2, \cdots$$

(4) $$\frac{\partial^2 \Phi}{\partial \varphi^2} + m^2 \Phi = 0 \ , \quad \Phi = e^{\pm i m \varphi} \ , \quad m = 0, 1, 2, 3 \ldots$$

m is an integer if we require that $T(r, \varphi, z, t)$ be single-valued.

(5) $$\frac{1}{R}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) - \frac{m^2}{r} - \lambda_z^2 r - \frac{r}{\nu}\frac{1}{\Pi}\frac{\partial \Pi}{\partial t} = 0$$

(6) Take as solution: $T(r, \varphi, z, t) = \sum_{mnp} a_{mnp} R_{mn}(r)\,\Pi_{np}(t)\,Z_p(z)\,e^{im\varphi}$

(7) $$\frac{1}{\nu \Pi}\frac{\partial \Pi}{\partial t} = -\lambda^2 \quad ; \quad \frac{1}{Rr}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) - \frac{m^2}{r^2} - \lambda_z^2 + \lambda^2 = 0$$

(8) $T_{np} = e^{-\nu \lambda_{np}^2 t}$; $\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left\{(\lambda^2 - \lambda_z^2) - \frac{m^2}{r^2}\right\} R = 0$

(9) Calling $\beta^2 = \lambda^2 - \lambda_z^2$, we have:

$$R = A\, J_m(\beta r) + B\, Y_m(\beta r)$$

For the temperature to be finite at $r=0$, we must have:

$$R = A\, J_m(\beta r)$$

(10) At $r = a$: $\beta J_m'(\beta a) + k\, J_m(\beta a) = 0$; this equation determines the eigenvalues β_n.

$$\therefore\ R_n = J_m(\beta_{np} r)$$

(11) $$T(r,\varphi,z,t) = \sum_{mnp} a_{mnp}\, e^{im\varphi} \cos\frac{2p\pi z}{L}\, e^{-\nu\lambda_{np}^2 t}\, J_m(\beta_{np} r)$$

(12) Now: $$T_0 = \sum_{mnp} a_{mnp}\, e^{im\varphi} \cos\frac{2p\pi z}{L}\, J_m(\beta_{np} r)$$

$$T_0 \int_0^a\int_0^{2\pi}\int_0^L e^{-im'\varphi} \cos\frac{2p'\pi z}{L}\, J_{m'}(\beta_{n'p'} r)\, r\, dr\, d\varphi\, dz$$

$$= a_{m'n'p'} \int_0^a\int_0^{2\pi}\int_0^L \cos^2\frac{2p'\pi z}{L} \left\{J_{m'}(\beta_{n'p'} r)\right\}^2 r\, dr\, d\varphi\, dz$$

$$= \pi L\, a_{m'n'p'} \int_0^a r \left\{J_{m'}(\beta_{n'p'} r)\right\}^2 dr$$

It becomes clear at this point that no solutions will exist unless $m' = p' = 0$. This is reasonable since the problem is cylindrically symmetry.

$$\therefore\ T_0 \int_0^a\int_0^{2\pi}\int_0^L r\, J_0(\beta_n r)\, dr\, d\varphi\, dz = \pi L\, a_n \int_0^a r\left\{J_0(\beta_n r)\right\}^2 dr$$

$$= 2\pi L\, T_0 \int_0^a r\, J_0(\beta_n r)\, dr$$

(13) $$\int_0^a r\, J_0(\beta_n r)\, dr = \frac{1}{\beta_n^2}\int_0^{\beta_n a} u\, J_0(u)\, du = u\, J_1(u)\Big]_0^{\beta_n a} \frac{1}{\beta_n^2}$$

$$= \frac{a}{\beta_n} J_1(\beta_n a)\ ;\quad \int_0^a r\left\{J_0(\beta_n r)\right\}^2 dr = \frac{a^2}{2}\left\{J_1(\beta_n a)\right\}^2$$

Only if $J_0(\beta_n a) = 0$

$$a_n \frac{a^2}{2}\left\{J_1(\beta_n a)\right\}^2 = 2\, T_0\, \frac{a}{\beta_n} J_1(\beta_n a)$$

(14) $$\therefore\ T(r,t) = \sum_n a_n\, e^{-\nu\beta_n^2 t}\, J_0(\beta_n r)\ ;\quad a_n = \frac{4T_0}{J_1(\beta_n a)}$$

and the eigenvalues are determined by the roots of:

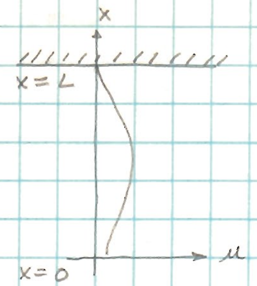
$$\beta J_0'(\beta a) + k\, J_0(\beta a) = 0$$

8

Assignment #2
Continued

Paul Grant
DEAP-AP-1G
AM 202
3-11-61

5.



(1) The differential equation as derived in the first problem set is:

$$(x u_x)_x = \frac{1}{g} u_{tt}$$

subject to: $u(L,t) = 0$

$u_t(x,0) = g(x)$, $u(x,0) = f(x)$

(2) $$x u_{xx} + u_x = \frac{1}{g} u_{tt}$$

Let $u = XT$: $$\frac{x}{X} X_{xx} + \frac{X_x}{X} = \frac{1}{g}\frac{T_{tt}}{T} = -\lambda^2$$

(3) $$\frac{d^2 X}{dx^2} + \frac{1}{x}\frac{dX}{dx} + \frac{\lambda^2}{x} X = 0$$

(4) Let $\xi = \frac{1}{2} x^{1/2}$: $$\frac{d\xi}{dx} = \frac{1}{4} x^{-1/2} = \frac{1}{8\xi}$$

$x = 4\xi^2$ $$\frac{d^2\xi}{dx^2} = -\frac{1}{8} x^{-3/2} = \frac{-1}{64\xi^3}$$

$$\frac{dX}{dx} = \frac{d\xi}{dx}\frac{dX}{d\xi} = \frac{1}{8\xi}\frac{dX}{d\xi}$$

$$\frac{d^2X}{dx^2} = \frac{d^2\xi}{dx^2}\frac{dX}{d\xi} + \left(\frac{d\xi}{dx}\right)^2 \frac{d^2X}{d\xi^2}$$

$$= -\frac{1}{64\xi^3}\frac{dX}{d\xi} + \frac{1}{64\xi^2}\frac{d^2X}{d\xi^2}$$

$$\therefore \frac{1}{64\xi^2}\frac{d^2X}{d\xi^2} - \frac{1}{64\xi^3}\frac{dX}{d\xi} + \frac{1}{4\xi^2}\cdot\frac{1}{8\xi}\frac{dX}{d\xi} + \frac{\lambda^2 X}{4\xi^2} = 0$$

$$\frac{d^2X}{d\xi^2} + \frac{1}{\xi}\frac{dX}{d\xi} + 16\lambda^2 X = 0$$

(5) $$\therefore X = A J_0(2\lambda x^{1/2}) + B Y_0(2\lambda x^{1/2})$$

(6) The solution that is finite at $x = 0$ is:

$$X = A J_0(2\lambda x^{1/2})$$

(7) The eigenvalues are determined by the zeroes of

$$J_0(2\lambda L^{1/2}) = 0$$

(8) $$\therefore X_n = J_0(2\lambda_n x^{1/2})$$

(9) $T_n = a_n \sin g\lambda_n t + b_n \cos g\lambda_n t$

(10) $\therefore \quad u(x,t) = \sum_{n=0}^{\infty} (a_n \sin g\lambda_n t + b_n \cos g\lambda_n t)\, J_0(2\lambda_n x^{1/2})$

(11) $u(x,0) = f(x) = \sum_{n=0}^{\infty} b_n\, J_0(2\lambda_n x^{1/2})$

$$b_n = \frac{\int_0^{L^{1/2}} f(x)\, J_0(2\lambda_n x^{1/2})\, x^{1/2}\, d(x^{1/2})}{\int_0^{L^{1/2}} \{J_0(2\lambda_n x^{1/2})\}^2\, x^{1/2}\, d(x^{1/2})}$$

(12) $u_t(x,0) = g(x) = \sum_{n=0}^{\infty} g\lambda_n a_n\, J_0(2\lambda_n x^{1/2})$

$$a_n = \frac{1}{g\lambda_n}\, \frac{\int_0^{L^{1/2}} g(x)\, J_0(2\lambda_n x^{1/2})\, x^{1/2}\, d(x^{1/2})}{\int_0^{L^{1/2}} \{J_0(2\lambda_n x^{1/2})\}^2\, x^{1/2}\, d(x^{1/2})}$$

(13) Consider: $J_0(\alpha_n) = 0$: $\left.\begin{array}{l}\alpha_1 = 2.40\\ \alpha_2 = 5.52\\ \alpha_3 = 8.65\end{array}\right\}$ $X_n = J_0(2\lambda_n x^{1/2})$, $J_0(2\lambda_n L^{1/2}) = 0$

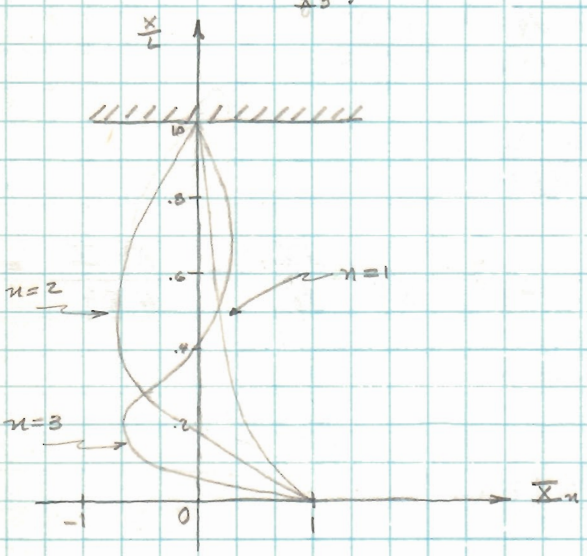
$\lambda_n = \alpha_n / 2L^{1/2}$

$\therefore \quad X_n = J_0\left\{\alpha_n \left(\frac{x}{L}\right)^{1/2}\right\}$

Zeroes of: X_1 : $\left(\frac{x}{L}\right)^{1/2} = 1$: $\frac{x}{L} = 1$

X_2 : " $= 2.40/5.52,\ 1$: " $= .189,\ 1$

X_3 : " $= 2.40/8.65,\ 5.52/8.65,\ 1$: " $= .077,\ .41,\ 1$



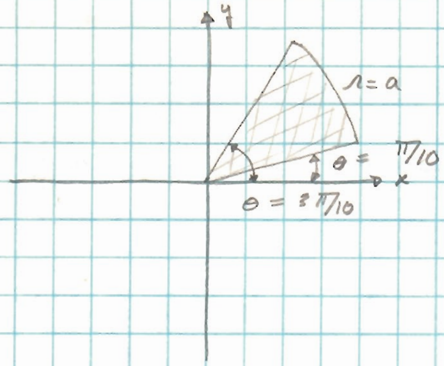
For convenience, the vertical scale is normalized with respect to the string length

10

Assignment #2
Continued

Paul Grant
DEAP-AP-16
AM 202
3-13-61

6.



(1) The equation of a freely vibrating membrane with small oscillations is:

$$u_{tt} = \gamma^2 (u_{xx} - u_{yy}) \; ; \quad \gamma^2 = \frac{T}{\rho}$$

(2) In polar co-ordinates; $\frac{1}{r}(r u_r)_r + \frac{1}{r^2} u_{\vartheta\vartheta} - \frac{1}{\gamma^2} u_{tt} = 0$

subject to: $u(a,\theta,t) = 0$, $u(r, \pi/10, t) = u(r, 3\pi/10, t) = 0$

and $u(r,\vartheta,0) = f(r,\vartheta)$; $u_t(r,\vartheta,0) = g(r,\vartheta)$

(3) In the interest of simplicity, as we want only to examine the eigenvalues, we shall take the steady state solution, $u(r,\vartheta,t) = V(r,\vartheta)e^{i\omega t}$, Then:

$$\frac{1}{r}(r V_r)_r + \frac{1}{r^2} V_{\vartheta\vartheta} + \frac{\omega^2}{\gamma^2} V = 0$$

(4) If $V(r,\vartheta) = R(r)\,\Theta(\vartheta)$:

$$\frac{1}{rR}(rR')' + \frac{1}{r^2\Theta}\Theta'' + \frac{\omega^2}{\gamma^2} = 0$$

or $$\frac{r}{R}(rR')' + \frac{\omega^2}{\gamma^2} r^2 = -\frac{\Theta''}{\Theta} = m^2$$

or $$\Theta'' + m^2\Theta = 0 \; ; \quad r^2R'' + rR' + \left\{\frac{\omega^2}{\gamma^2} r^2 - m^2\right\} R = 0$$

(5) $\Theta = A\cos m\vartheta + B\sin m\vartheta$

$$A\cos m\,\pi/10 + B\sin m\,\pi/10 = 0$$
$$A\cos m\,3\pi/10 + B\sin m\,3\pi/10 = 0$$

$$\cos m\pi/10 \,\sin m\,3\pi/10 - \sin m\,\pi/10\,\cos m\,3\pi/10 = 0$$

or $$\sin\left\{m\,3\pi/10 - m\,\pi/10\right\} = \sin\frac{m\pi}{5} = 0$$

The eigenvalues are given by $m = 5n$, $n = 0, 1, 2, 3, \ldots$

(6) ∴ We have two separate solutions for even and odd n:

ODD: $\Theta_n = A_n \cos 5(2n+1)\vartheta$; $n = 0, 1, 2, 3, \ldots$
EVEN: $\Theta_n = B_n \sin 10n\vartheta$; $n = 0, 1, 2, 3, \ldots$

(7) $R'' + \frac{1}{r}R' + \left\{ \frac{\omega^2}{\gamma^2} - \frac{m^2}{r^2} \right\} R = 0$

whose finite solution is:

$$R_m = J_m \left\{ \frac{\omega}{\gamma} r \right\}$$

subject to: $J_m \left\{ \frac{\omega}{\gamma} a \right\} = 0$

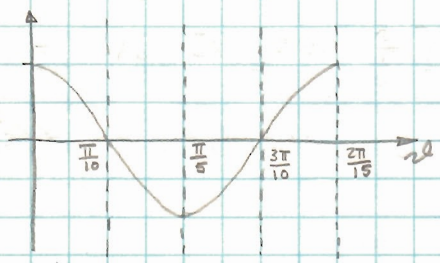
which gives the roots ω_{mp}, $R_{mp} = J_m \left\{ \frac{\omega_{mp}}{\gamma} r \right\}$

(8) $V_{np} = A_n \cos 5(2n+1)\vartheta \; J_{5(2n+1)} \left\{ \frac{\omega_{np}}{\gamma} r \right\}$: ODD

$V_{np} = B_n \sin 10n\vartheta \; J_{10n} \left\{ \frac{\omega_{np}}{\gamma} r \right\}$: EVEN

(9) Lowest ODD: $V_{0p} = A_0 \cos 5\vartheta \; J_5 \left\{ \frac{\omega_{0p}}{\gamma} r \right\}$

Lowest EVEN: $V_{1p} = B_1 \sin 10\vartheta \; J_{10} \left\{ \frac{\omega_{1p}}{\gamma} r \right\}$



From Janke & Emke:

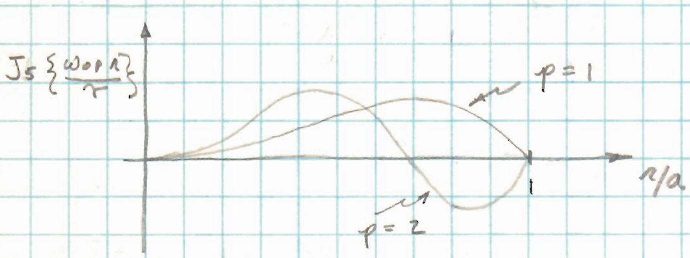
$J_5(x_n) = 0$; $x_1 = 8.78$, $x_2 = 12.34$

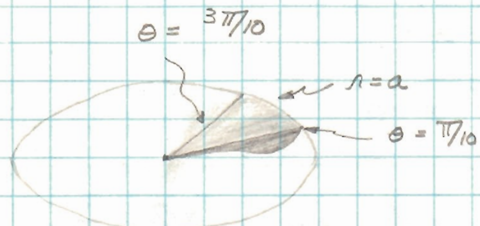
$\omega_{01} = 8.78 \frac{\gamma}{a}$; $\omega_{02} = 12.34 \frac{\gamma}{a}$

10

(10) ODD has two lowest eigen modes: $V_{01} = A_0 \cos 5\vartheta \; J_5 \left\{ 8.78 \frac{r}{a} \right\}$

$V_{02} = A_0 \cos 5\vartheta \; J_5 \left\{ 12.34 \frac{r}{a} \right\}$



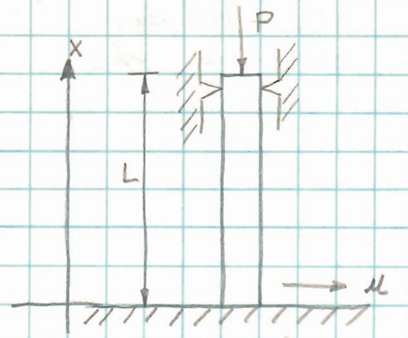


$V_{01} = A_0 \cos 5\vartheta \; J_5 \left\{ 8.78 \frac{r}{a} \right\}$

Assignment # 2
Continued

Paul Grant
DEAP-AP-16
AM 202
3-14-61

7.



(1) Given: $K u_{xxxx} + P u_{xx} + R u_{tt} = 0$

with: $u(0,t) = u(L,t) = 0$

$u_{xx}(0,t) = u_{xx}(L,t) = 0$

since $\kappa = u'' / (1+u'^2)^{3/2}$

Assume for completeness:

$u(x,0) = f(x)$

$u_t(x,0) = g(x)$

(2) Take $u = X(x)\,T(t)$:

$$K\frac{X''''}{X} + P\frac{X''}{X} = -R\frac{T''}{T} = \lambda^2$$

(3) $T'' + \frac{\lambda^2}{R} T = 0$; $T = a \cos\frac{\lambda}{\sqrt{R}}t + b \sin\frac{\lambda}{\sqrt{R}}t$

(4) $X'''' + \frac{P}{K}X'' - \frac{\lambda^2}{K}X = 0$

form auxiliarly eq:

$$m^4 + \frac{P}{K}m^2 - \frac{\lambda^2}{K} = 0$$

$$m^2 = \frac{-\frac{P}{K} \pm \left[\frac{P^2}{K^2} + 4\frac{\lambda^2}{K}\right]^{1/2}}{2}$$

$$m = \frac{\pm \lambda}{\sqrt{2K}}\left\{\left[P^2 + 4K\lambda^2\right]^{1/2} + P\right\}^{1/2} ; \quad \pm\frac{1}{\sqrt{2K}}\left\{\left[P^2+4K\lambda^2\right]^{1/2} - P\right\}^{1/2}$$

$$= \pm\alpha \qquad ; \quad \pm\beta$$

(5) $\therefore$ $X = Ae^{\alpha x} + Be^{-\alpha x} + Ce^{\beta x} + De^{-\beta x}$

$X'' = \alpha^2 A e^{\alpha x} + \alpha^2 B e^{-\alpha x} + \beta^2 C e^{\beta x} + \beta D e^{-\beta x}$

(6) Using the boundary conditions, we can form the following secular determinant:

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ e^{\alpha L} & e^{-\alpha L} & e^{\beta L} & e^{-\beta L} \\ \alpha^2 & \alpha^2 & \beta^2 & \beta^2 \\ \alpha^2 e^{\alpha L} & \alpha^2 e^{-\alpha L} & \beta^2 e^{\beta L} & \beta^2 e^{-\beta L} \end{vmatrix} = 0$$

OVER:

(7) If λ should be ever imaginary, the solution to (3) will be hyperbolic functions that increase without limit as time proceeds. These are physically unallowable and the constants K and P must be arranged so that the roots λ of the secular determinant are always real, thus, this is the criterion.

(8)
$$\begin{vmatrix} 0 & 0 & (\alpha^2-\beta^2) & (\alpha^2-\beta^2) \\ 0 & 0 & (\alpha^2-\beta^2)e^{\beta L} & (\alpha^2-\beta^2)e^{-\beta L} \\ \alpha^2 & \alpha^2 & \beta^2 & \beta^2 \\ \alpha^2 e^{\alpha L} & \alpha^2 e^{-\alpha L} & \beta^2 e^{\beta L} & \beta^2 e^{-\beta L} \end{vmatrix}$$

$$= \alpha^4(\alpha^2-\beta^2)^2 e^{\beta L}\begin{vmatrix} 1 & 1 \\ e^{\alpha L} & e^{-\alpha L}\end{vmatrix} - \alpha^4(\alpha^2-\beta^2)^2 e^{-\beta L}\begin{vmatrix} 1 & 1 \\ e^{\alpha L} & e^{-\alpha L}\end{vmatrix}$$

or $\alpha^4(\alpha^2-\beta^2)^2 \sinh \beta L \sinh \alpha L = 0$

(9) $\sinh \alpha L = 0$: $\sinh\left\{\frac{\lambda}{\sqrt{2K}}\left[(P^2+4K\lambda^2)^{1/2}+P\right]^{1/2}\right\}L = 0$

$$\frac{1}{2K}\left[(P^2+4K\lambda^2)^{1/2}+P\right]L^2 = n^2\pi^2 \quad , \quad n = 1, 2, \ldots$$

$$(P^2+4K\lambda^2) = \left\{\frac{2Kn^2\pi^2}{L^2} - P\right\}^2 = \frac{4K^2n^4\pi^4}{L^4} - \frac{4Kn^2\pi^2 P}{L^2} + P^2$$

$$\lambda_n = \frac{n^2\pi^2}{L^2}\left\{\frac{Kn^2\pi^2}{L^2} - P\right\}^{1/2}$$

(10) $\sinh \beta L = 0$: $\sinh\left\{\frac{1}{\sqrt{2K}}\left[(P^2+4K\lambda^2)^{1/2}-P\right]^{1/2}\right\}L = 0$

$$\frac{1}{2K}\left[(P^2+4K\lambda^2)^{1/2}-P\right]L^2 = -n^2\pi^2$$

$$(P^2+4K\lambda^2) = \left\{-\frac{2Kn^2\pi^2}{L^2} + P\right\}^2 = \frac{4K^2n^4\pi^4}{L^4} - \frac{4Kn^2\pi^2 P}{L^2} + P^2$$

16

$$\lambda_n = \frac{n^2\pi^2}{L^2}\left\{\frac{Kn^2\pi^2}{L^2} - P\right\}^{1/2}$$

These are the roots as $\alpha^4 = 0$ gives the first eigenvalue and $(\alpha^2-\beta^2)^2 = 0$ is an identity. For λ_n to be real, obviously:

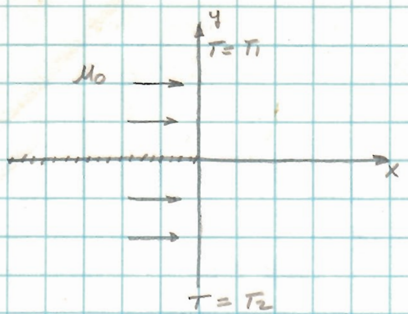
$$\frac{K\pi^2}{L^2} \geq P$$ in the first eigenmode and hence for all others.

Assignment #3

Paul Grant
DEAP-AP-16
AM 202
4-15-61

$\frac{24}{30}$

1.



(1) The equation of continuity of heat flow is:

$$\operatorname{div}\bar{q} + \frac{\partial}{\partial t}(c\rho T) = 0$$

$$\bar{q} = -k \operatorname{grad} T + \rho c T \bar{v}$$

(2) Under equilibrium conditions: $\operatorname{div}\bar{q} = 0$

or $-k \nabla^2 T + \rho c \operatorname{div} T\bar{v} = 0$

(3) In this problem, $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} - \frac{u_0}{\nu}\frac{\partial T}{\partial x} = 0$

subject to: $T(0,y) = \begin{cases} T_2, & y > 0 \\ T_1, & y < 0 \end{cases}$

(4) Choose the new variable $\theta = T - T_2$:

$$\left\{\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2} - \frac{u_0}{\nu}\frac{\partial}{\partial x}\right\}\theta(x,y) = 0 \quad ; \quad \theta(0,y) = \begin{cases} \theta_0 = T_1 - T_2, & y > 0 \\ 0, & y < 0 \end{cases}$$

(5) Use Fourier transforms in y and η:

$$\left.\begin{aligned} \bar{\theta}(x,\eta) &= \int_{-\infty}^{\infty} \theta(x,y)\, e^{-i\eta y}\, dy \\ \theta(x,y) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{\theta}(x,\eta)\, e^{i\eta y}\, d\eta \end{aligned}\right\} \quad \theta \text{ and } \theta_y \text{ vanish at } y = \pm\infty$$

(6) $\therefore \left\{\frac{\partial^2}{\partial x^2} - \frac{u_0}{\nu}\frac{\partial}{\partial x} - \eta^2\right\}\bar{\theta}(x,\eta) = 0$

The vanishing solution at $x = \infty$ is:

$$\bar{\theta}(x,\eta) = A(\eta)\exp\frac{1}{2}\left\{\frac{u_0}{\nu} - \left[\left(\frac{u_0}{\nu}\right)^2 + 4\eta^2\right]^{1/2}\right\}x$$

(7) $\bar{\theta}(0,\eta) = A(\eta) = \int_{-\infty}^{\infty}\theta(0,y)\, e^{-i\eta y}\, dy = \theta_0\int_0^{\infty} e^{-i\eta y}\, dy$

$= \frac{\theta_0}{i\eta}$

(8) $\therefore \quad \theta(x,y) = \frac{\theta_0}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{\frac{1}{2}\left\{\frac{u_0}{\nu} - \left[\left(\frac{u_0}{\nu}\right)^2 + 4\eta^2\right]^{1/2}\right\}x + i\eta y}}{\eta}\, d\eta$

(9) As suggested in lecture, we make the approximation that the temperature is slowly varying in x such that $\theta_{xx} = 0$.

(10) $\therefore \quad \left\{ \frac{\partial}{\partial x} + \frac{\nu \xi^2}{u_0} \right\} \bar{\Theta}(x,\xi) = 0$

and $\quad \bar{\theta}(x,\eta) = A(\xi)\, e^{-\frac{\nu \xi^2}{u_0} x}$

where $A(\xi)$ is the same as before.

(11) $$\theta(x,y) = \frac{\theta_0}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{-\frac{\nu}{u_0}\eta^2 x + i\eta y}}{\eta}\, d\eta$$

or: $$\theta(x,y) = \frac{\theta_0}{2\pi} \int_{-\infty}^{\infty}\int_{0}^{\infty} e^{-\frac{\nu}{u_0}\eta^2 x + i\eta(y-y')}\, d\eta\, dy'$$

(12) $$\int_{-\infty}^{\infty} e^{-\frac{\nu}{u_0}\eta^2 x + i\eta(y-y')}\, d\eta = e^{-\frac{u_0}{4\nu x}(y-y')^2} \int_{-\infty}^{\infty} e^{-\frac{\nu x}{u_0}\left\{\eta - \frac{i u_0}{2\nu}(y-y')\right\}^2} d\eta$$

$$= \sqrt{\frac{\pi u_0}{\nu x}}\; e^{-\frac{u_0}{4\nu x}(y-y')^2}$$

(13) $$\therefore \quad \theta(x,y) = \frac{\theta_0}{2}\sqrt{\frac{u_0}{\nu \pi x}} \int_{0}^{\infty} e^{-\frac{u_0}{4\nu x}(y-y')^2}\, dy'$$

(14) Let $\quad s = \sqrt{\frac{u_0}{4\nu x}}\,(y-y') \quad ; \quad dy' = -2\sqrt{\frac{\nu x}{u_0}}\, ds$

(15) $$\therefore \quad \theta(x,y) = -\frac{\theta_0}{\sqrt{\pi}} \int_{\frac{1}{2}\sqrt{\frac{u_0}{\nu x}}\,y}^{-\infty} e^{-s^2}\, ds = \frac{\theta_0}{\sqrt{\pi}} \int_{-\frac{1}{2}\sqrt{\frac{u_0}{\nu x}}\,y}^{\infty} e^{-s^2}\, ds$$

$$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-s^2}\, ds$$

(16) $$\theta(x,y) = \frac{\theta_0}{2}\operatorname{erfc}\left\{-\frac{1}{2}\sqrt{\frac{u_0}{\nu x}}\,y\right\} = \frac{\theta_0}{2}\left[1 + \operatorname{erf}\left\{\frac{1}{2}\sqrt{\frac{u_0}{\nu x}}\,y\right\}\right]$$

Now, as $x \to 0$, $y < 0$, $\operatorname{erfc}\left\{-\frac{1}{2}\sqrt{\frac{u_0}{\nu x}}\,y\right\} \to 0$, $\therefore \theta(0,y) = 0$, $y < 0$

as $x \to 0$, $y > 0$, $\operatorname{erfc}\left\{-\frac{1}{2}\sqrt{\frac{u_0}{\nu x}}\,y\right\} \to 2$, $\therefore \theta(0,y) = \theta_0$, $y > 0$

(17) $$\therefore \quad T(x,y) = \frac{T_1 - T_2}{2}\left[1 + \operatorname{erf}\left\{\frac{1}{2}\sqrt{\frac{u_0}{\nu x}}\,y\right\}\right] + T_2$$

Assignment #3
Continued

Paul Grant
DEAP - AP-1G
AM 202
3-31-61

2. (1) $\varphi_{xx} + \varphi_{yy} - k^2 \varphi = G(x,y)$; $\varphi \to 0$ as $x^2 + y^2 \to 0$

(2) The appropriate Fourier transform pair is:

$$\bar{\varphi}(\xi, \eta) = \iint e^{-i(\xi x + \eta y)} \varphi(x,y)\, dx\, dy$$

$$\varphi(x,y) = \frac{1}{4\pi^2} \iint e^{i(\xi x + \eta y)} \bar{\varphi}(\xi, \eta)\, d\xi\, d\eta$$

(3) $\therefore \{-\xi^2 - \eta^2 - k^2\} \bar{\varphi} = \bar{G}(\xi, \eta)$

or $\bar{\varphi} = \dfrac{-\bar{G}}{\xi^2 + \eta^2 + k^2}$

(4) $$\varphi(x,y) = \frac{-1}{4\pi^2} \iint e^{i(\xi x + \eta y)} \frac{\bar{G}}{\xi^2 + \eta^2 + k^2}\, d\xi\, d\eta$$

$$= -\frac{1}{4\pi^2} \iint \frac{e^{i(\xi x + \eta y)}}{\xi^2 + \eta^2 + k^2}\, d\xi\, d\eta \iint e^{-i(\xi x' + \eta y')} G(x', y')\, dx'\, dy'$$

(5) $$\therefore \varphi(x,y) = \frac{-1}{4\pi^2} \iint G(x', y')\, dx'\, dy' \iint \frac{e^{i\xi(x-x') + i\eta(y-y')}}{\xi^2 + \eta^2 + k^2}\, d\xi\, d\eta$$

(6) We wish to evaluate the Green's function:

$$-\frac{1}{4\pi^2} \iint_{-\infty}^{\infty} \frac{e^{i\xi(x-x') + i\eta(y-y')}}{\xi^2 + \eta^2 + k^2}\, d\xi\, d\eta = K(x, x'; y, y')$$

Put in polar coordinate form: $\bar{R} = \bar{i}\xi + \bar{j}\eta$
$\bar{r} = \bar{i}(x - x') + \bar{j}(y - y')$

$$K = -\frac{1}{4\pi^2} \int_0^{\infty}\int_0^{2\pi} \frac{e^{i\bar{R}\cdot\bar{r}}}{R^2 + k^2} R\, dR\, d\varphi$$

$$= -\frac{1}{4\pi^2} \int_0^{\infty}\int_0^{2\pi} \frac{e^{iRr\cos\varphi}}{R^2 + k^2} R\, dR\, d\varphi$$

(7) Consider: $\displaystyle\int_0^{2\pi} e^{iRr\cos\varphi}\, d\varphi$

From Jahnke & Emde: $J_n(z) = \dfrac{i^{-n}}{2\pi} \displaystyle\int_0^{2\pi} e^{iz\cos\varphi} e^{in\varphi}\, d\varphi$

$$\therefore \int_0^{2\pi} e^{iRr\cos\varphi}\, d\varphi = 2\pi J_0(Rr)$$

(8) $$\therefore K = -\frac{1}{2\pi} \int_0^{\infty} \frac{J_0(Rr)\, R\, dR}{R^2 + k^2}$$

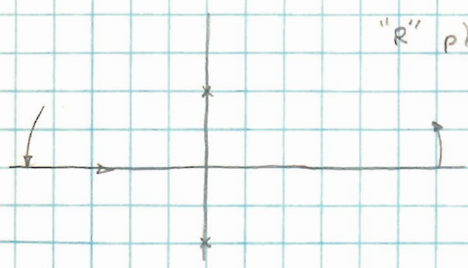
(9) Recall the theorem given in lecture:

$$\int_0^\infty \varphi(\alpha)\, J_0(\alpha r)\, d\alpha = \frac{1}{2}\int_{-\infty}^{\infty} \varphi(\alpha)\, H_0^{(1)}(\alpha r)\, d\alpha$$

or

$$\int_0^\infty \frac{R}{R^2+k^2}\, J_0(Rr)\, dR = \frac{1}{2}\int_{-\infty}^{\infty} \frac{R}{R^2+k^2}\, H_0^{(1)}(Rr)\, dR$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \frac{R\, H_0^{(1)}(Rr)\, dR}{(R+ik)(R-ik)}$$



"R" plane

$$\text{Residue} = \lim_{R\to ik} \frac{R\, H_0^{(1)}(Rr)}{R+ik}$$

$$= \frac{1}{2} H_0^{(1)}(ikr)$$

$$\therefore \int_0^\infty \frac{R}{R^2+k^2}\, J_0(Rr)\, dR = \frac{\pi i}{2} H_0^{(1)}(ikr)$$

$$H_\nu^{(1)}(z) = J_\nu(z) + i\, Y_\nu(z)$$

(10) There is probably a mistake in the boundary conditions as stated which should read $\varphi \to 0$ as $x^2+y^2 \to \infty$. The Hankel function satisfies this requirement. Using an identity given in Campbell & Foster, viz:

$$K_\nu(z) = \frac{1}{2}\pi i^{\nu+1} H_\nu^{(1)}(iz)$$

we have:

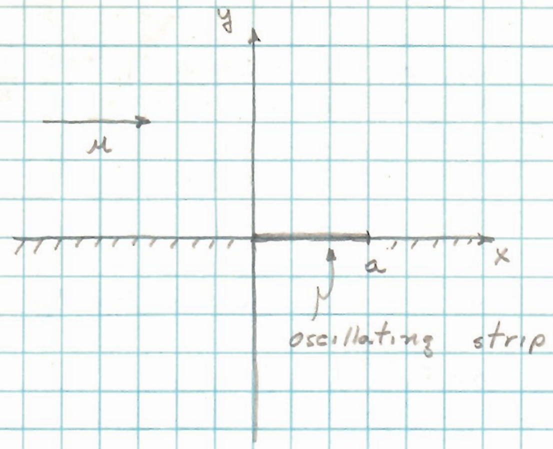
$$\varphi = -\frac{1}{2\pi}\iint_{-\infty}^{\infty} K_0\left\{k\sqrt{(x-x')^2+(y-y')^2}\right\} G(x',y')\, dx'dy'$$

10

Assignment #3
Continued

Paul Grant
DEAP - AP - 16
AM 202
4-15-61

3.



(1) The strip $0 < x < a$, $y = 0$ oscillates as:

$$y = y_0 e^{i\omega t}$$

where $\omega y_0 \ll c$

(2) We write the conservation equations:

Conservation of momentum: Free Fluid:

$$\rho \frac{\partial \bar{v}}{\partial t} + \rho (\bar{v} \cdot \text{grad}) \bar{v} + \text{grad } P = 0$$

Conservation of mass:

$$\text{div} (\rho \bar{v}) + \frac{\partial \rho}{\partial t} = 0$$

Equation of state: $\frac{P}{P_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma}$; $c^2 = \frac{\gamma P_0}{\rho_0}$; $\rho = \rho_0 \left(\frac{P}{P_0}\right)^{1/\gamma}$

(3) Apply the method of perturbations:

$$\rho = \rho_0 + \epsilon \rho' , \quad \epsilon \rho' \ll \rho_0$$
$$P = P_0 + \epsilon P' , \quad \epsilon P' \ll P_0$$
$$\bar{v} = u \bar{i} + \epsilon \bar{v}' , \quad |\epsilon \bar{v}'| \ll c$$

(4) $(\rho_0 + \epsilon\rho')\frac{\partial}{\partial t}(u\bar{i} + \epsilon \bar{v}') + (\rho_0 + \epsilon\rho')\left\{(u\bar{i} + \epsilon\bar{v}')\cdot \text{grad}\right\}(u\bar{i} + \epsilon\bar{v}') + \text{grad}(P_0 + \epsilon P') = 0$

or $(\rho_0 + \epsilon\rho')\frac{\partial}{\partial t}(\epsilon\bar{v}') + (\rho_0 + \epsilon\rho')\left\{u \frac{\partial}{\partial x}(\epsilon\bar{v}') + \epsilon\bar{v}' \cdot \text{grad}(\epsilon\bar{v}')\right\} + \text{grad } \epsilon P' = 0$

Divide by ϵ and take $\lim_{\epsilon \to 0}$:

$$\rho_0 \frac{\partial \bar{v}'}{\partial t} + \rho_0 u \frac{\partial \bar{v}'}{\partial x} + \text{grad } P' = 0$$

(5) Now: $\text{div}\left\{\rho_0 \left(\frac{P}{P_0}\right)^{1/\gamma} \bar{v}\right\} + \frac{1}{c^2}\left(\frac{P}{P_0}\right)^{\frac{1-\gamma}{\gamma}} \frac{\partial P}{\partial t} = 0$

$$\text{div}\left\{\rho_0 \left(\frac{P_0 + \epsilon P'}{P_0}\right)^{1/\gamma}(u\bar{i} + \epsilon\bar{v}')\right\} + \frac{1}{c^2}\left(\frac{P_0 + \epsilon P'}{P_0}\right)^{\frac{1-\gamma}{\gamma}} \frac{\partial}{\partial t}(P_0 + \epsilon P') = 0$$

(5) Dividing by ϵ, taking $\lim\limits_{\epsilon \to 0}$:

$$\rho_0 \operatorname{div} \bar{v}' + \frac{1}{c^2}\frac{\partial p'}{\partial t} + \lim_{\epsilon \to 0} \frac{u}{\epsilon}\frac{\partial}{\partial x}\left\{\rho_0\left(\frac{P_0+\epsilon p'}{P_0}\right)^{1/\gamma}\right\} = 0$$

$$\lim_{\epsilon \to 0} \frac{u}{\epsilon}\frac{\partial}{\partial x}\left\{\rho_0\left(\frac{P_0+\epsilon p'}{P_0}\right)^{1/\gamma}\right\} = \lim_{\epsilon \to 0}\frac{u}{\epsilon}\frac{\rho_0}{\gamma}\left(\frac{P_0+\epsilon p'}{P_0}\right)^{1/\gamma - 1}\cdot\frac{1}{P_0}\epsilon\frac{\partial p'}{\partial x}$$

$$= \frac{u}{c^2}\frac{\partial p'}{\partial x}$$

(6) we assume $\bar{v}' = \operatorname{grad}\varphi$ where φ is a velocity potential:

$$\rho_0 \operatorname{grad}\frac{\partial\varphi}{\partial t} + \rho_0 u \operatorname{grad}\frac{\partial\varphi}{\partial x} + \operatorname{grad} p' = 0$$

$$\rho_0 \nabla^2\varphi + \frac{1}{c^2}\left\{\frac{\partial p'}{\partial t} + u\frac{\partial p'}{\partial x}\right\} = 0$$

(7) Now: $\dfrac{d}{dt} = \dfrac{\partial}{\partial t} + u\bar{i}\cdot\operatorname{div} = \dfrac{\partial}{\partial t} + u\dfrac{\partial}{\partial x}$

$$\therefore\ \rho_0 \operatorname{grad}\frac{\partial^2\varphi}{\partial t^2} + 2\rho_0 u \operatorname{grad}\frac{\partial\varphi}{\partial x\partial t} + \rho_0 u^2 \operatorname{grad}\frac{\partial^2\varphi}{\partial x^2}$$
$$+ \operatorname{grad}\left\{\frac{\partial p'}{\partial t} + u\frac{\partial p'}{\partial x}\right\} = 0$$

and: $\rho_0 \operatorname{grad}\nabla^2\varphi + \dfrac{1}{c^2}\operatorname{grad}\left\{\dfrac{\partial p'}{\partial t} + u\dfrac{\partial p'}{\partial x}\right\} = 0$

(8) $\therefore\ \nabla^2\varphi - \dfrac{1}{c^2}\left\{\dfrac{\partial}{\partial t} + u\dfrac{\partial}{\partial x}\right\}^2\varphi = 0$,

solving equations (7) simultaneously and integrating.

a. (1) Since the problem is homogeneous in the z direction, we need only be concerned with the x-y plane.

Boundary Conditions: $0 < x < a$; $y = 0$; $y = y_0 e^{i\omega t}$
elsewhere: ; $y = 0$

$$\frac{\partial\varphi}{\partial y} = v_y' = \frac{\partial y}{\partial t} = i\omega y_0 e^{-i\omega t} = \varphi_y(x,0)$$

$$\varphi = \int\frac{\partial\varphi}{\partial y}dy = \int\left(\frac{\partial\varphi}{\partial y}\right)^2 dt = -\omega^2 y_0^2\int e^{i2\omega t}dt$$
$$= i\omega y_0^2 e^{i2\omega t} = \varphi(x,0)$$

(2) Anticipate solution $\varphi = \Phi(x,y)e^{i\omega t}$

which must be outward going and vanish at ∞.

Assignment #3
Continued

Paul Grant
DEAP - AP - 1G
AM 202
4-15-61

Problem 3
Continued

a. (3)
$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} - \frac{1}{c^2}\left\{ i\omega + u\frac{\partial}{\partial x}\right\}^2 \psi = 0$$

or:
$$\left\{ \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2} - \frac{u^2}{c^2}\frac{\partial^2}{\partial x^2} - \frac{2i\omega u}{c^2}\frac{\partial}{\partial x} + \frac{\omega^2}{c^2}\right\} \psi(x,y) = 0$$

where:
$$\begin{aligned} \psi_y(x,0) &= i\omega y_0 \quad ; \quad 0 < x < a \\ &= 0 \quad ; \quad \text{elsewhere} \end{aligned}$$

(4) Take Fourier transforms in x and ξ:
$$\bar{\Psi}(\xi, y) = \int_{-\infty}^{\infty} \psi(x,y)\, e^{-i\xi x}\, dx$$
$$\psi(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{\Psi}(\xi,y)\, e^{i\xi x}\, d\xi$$

$$\left\{ \frac{\partial^2}{\partial y^2} - \xi^2 + \frac{u^2}{c^2}\xi^2 + \frac{2\omega u}{c^2}\xi + \frac{\omega^2}{c^2}\right\} \bar{\Psi}(\xi, y) = 0$$

The solution that vanishes at $y = \infty$ is:
$$\bar{\Psi}(\xi, y) = A(\xi)\exp\left\{ -\left[\xi^2\left(1 - \frac{u^2}{c^2}\right) - \frac{2\omega u}{c^2}\xi - \frac{\omega^2}{c^2}\right]^{1/2} y \right\}$$

$$\bar{\Psi}_y(\xi, 0) = -A(\xi)\left[\xi^2\left(1 - \frac{u^2}{c^2}\right) - \frac{2\omega u}{c^2}\xi - \frac{\omega^2}{c^2}\right]^{1/2}$$

$$\bar{\Psi}_y(\xi, 0) = i\omega y_0 \int_0^a e^{-i\xi x'}\, dx' = \frac{\omega y_0}{\xi}\left[1 - e^{-i\xi a}\right]$$

(5)
$$\therefore \quad \psi(x,y) = \frac{-i\omega y_0}{2\pi}\int_{-\infty}^{\infty}\int_0^a \frac{\exp\left\{-\left[\xi^2\left(1-\frac{u^2}{c^2}\right) - \frac{2\omega u}{c^2}\xi - \frac{\omega^2}{c^2}\right]^{1/2} y + i\xi(x - x')\right\}}{\left[\xi^2\left(1 - \frac{u^2}{c^2}\right) - \frac{2\omega u}{c^2}\xi - \frac{\omega^2}{c^2}\right]^{1/2}}\, dx'\, d\xi$$

(6) We now make use of the relation given in lecture, viz:
$$\int_{-\infty}^{\infty} e^{-\sqrt{}\,|y|}\, e^{-i\eta y}\, dy = \frac{2\sqrt{}}{\sqrt{}^{\,2} + \eta^2}$$

$$\therefore \quad e^{-\sqrt{}\,|y|} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2\sqrt{}}{\sqrt{}^{\,2} + \eta^2}\, e^{i\eta y}\, dy$$

Suppose $\sqrt{}^{\,2} < 0$ so the poles in the η-plane are on the real axis. Then what? Must be very careful here.

(7) Substituting in (5):

$$\psi(x,y) = \frac{\omega y_0}{2\pi^2 i}\int_0^a dx' \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{i\xi(x-x')+i\eta y}\, d\xi\, d\eta}{\xi^2\left(1-\frac{u^2}{c^2}\right) - \frac{2\omega u}{c^2}\xi - \frac{\omega^2}{c^2}+\eta^2}$$

(8) Call $\gamma = \sqrt{1-\frac{u^2}{c^2}}$;

Consider:

$$\int_{-\infty}^{\infty} \frac{e^{i\xi(x-x')}\, d\xi}{\xi^2\gamma^2 - \frac{2\omega u}{c^2}\xi - \frac{\omega^2}{c^2}+\eta^2}$$

The roots of the denominator are:

$$\xi_0 = \frac{\omega\, u/c^2}{\gamma^2} \pm \frac{i}{\gamma}\left\{\eta^2 - \frac{\omega^2/c^2}{\gamma^2}\right\}^{1/2}$$

Then we have:

$$\int_{-\infty}^{\infty} \frac{e^{i\xi(x-x')}\, d\xi}{\gamma^2(\xi-\xi_0^-)(\xi-\xi_0^+)}$$

Integrating around the UHP and using Routh's Theorem:

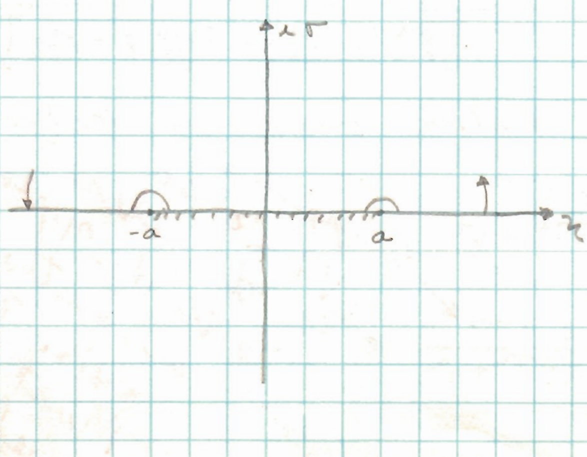
$$\text{Residue} = \lim_{\xi\to\xi_0^+} \frac{e^{i\xi(x-x')}}{\gamma^2(\xi-\xi_0^-)} = \frac{e^{i\xi_0^+(x-x')}}{\gamma^2(\xi_0^+-\xi_0^-)}$$

(9)
$$\int_{-\infty}^{\infty} \frac{e^{i\xi(x-x')}\, d\xi}{\gamma^2(\xi-\xi_0^-)(\xi-\xi_0^+)} = \frac{2\pi i}{\gamma^2}\, \frac{e^{i\frac{\omega u}{c^2\gamma^2}(x-x')}\; e^{-\frac{1}{\gamma}\left\{\eta^2-\frac{\omega^2}{\gamma^2c^2}\right\}^{1/2}(x-x')}}{-\frac{2i}{\gamma}\left\{\eta^2-\frac{\omega^2}{\gamma^2c^2}\right\}^{1/2}}$$

(10) $\therefore$
$$\psi(x,y) = \frac{-\omega y_0}{2 i\pi\gamma}\int_0^a e^{i\frac{\omega u}{c^2\gamma^2}(x-x')}\,dx' \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{\gamma}\left\{\eta^2-\frac{\omega^2}{\gamma^2c^2}\right\}^{1/2}(x-x')}\; e^{i\eta y}\, d\eta}{\left\{\eta^2-\frac{\omega^2}{\gamma^2c^2}\right\}^{1/2}}$$

(11) Take $\frac{u}{c}<1$; $\alpha = \frac{\omega}{\gamma c}$ and consider:

$$\int_{-\infty}^{\infty} \frac{e^{-\frac{(x-x')}{\gamma}\{z^2-\alpha^2\}^{1/2}}\; e^{izy}\, dz}{\{z^2-\alpha^2\}^{1/2}}$$ around the following contour:



Assignment #3
Continued

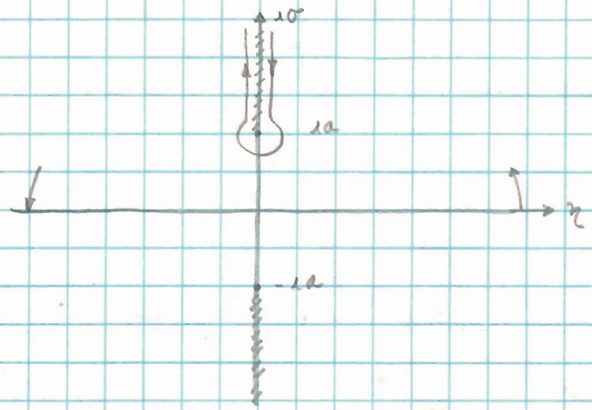
Paul Grant
DEAP-AP-1G
AM 202
4-15-61

Problem #3
Continued

(12) Take $\frac{u}{c} > 1$; $\gamma = -i\gamma$, $a = ia$ and consider:

$$\int_{-\infty}^{\infty} \frac{e^{\frac{i(x-x')}{\gamma}\{z^2+\alpha^2\}^{1/2}}\, e^{izy}\, dz}{\{z^2+\alpha^2\}^{1/2}}$$

around the following contour:



(13) Instead of using contour integration, we will use the tables of Campbell and Foster:

$$\int_{-\infty}^{\infty} F(f)\, e^{i2\pi f g}\, df = G(g)\ ; \quad \int_{-\infty}^{\infty} G(g)\, e^{-i2\pi f g}\, dg = F(f)$$

$$F = \frac{\exp\left[-ix\left(4\pi^2 f^2 - y^2\right)^{1/2}\right]}{i\left(4\pi^2 f^2 - y^2\right)^{1/2}}\ ; \quad G = J_0\left[y\left(g^2 - x^2\right)^{1/2}\right]$$

$$F = \frac{\exp\left[-\sigma\left(4\pi^2 f^2 + \rho^2\right)^{1/2}\right]}{\left(4\pi^2 f^2 + \rho^2\right)^{1/2}}\ ; \quad G = \frac{1}{\pi} K_0\left[\rho\left(g^2 + \sigma^2\right)^{1/2}\right]$$

(14) In (11), $\frac{u}{c} < 1$, take $\rho^2 = -\alpha^2$, $\sigma = \frac{(x-x')}{\gamma}$, $z = 2\pi f$, $g = y$ and get:

$$2K_0\left[-\alpha^2\left(y^2 + \frac{(x-x')^2}{\gamma^2}\right)^{1/2}\right] = 2K_0\left[\alpha^2\left(y^2 + \frac{(x-x')^2}{\gamma^2}\right)^{1/2}\right] - i\pi 2 I_0\left[\alpha^2\left(y^2 + \frac{(x-x')^2}{\gamma^2}\right)^{1/2}\right]$$

(15) We now have integrals of the type:

$$\int_0^a e^{i\frac{\omega u}{c^2\gamma^2}(x-x')}\, I_0\left[\alpha^2\left(y^2 + \frac{(x-x')^2}{\gamma^2}\right)^{1/2}\right] dx'\ ; \quad \frac{u}{c} < 1$$

(15) For $\frac{u}{c} > 1$, take $\gamma^{-} = -\alpha^2$, $z = 2\pi f$, $x = \frac{(x - x')}{\gamma}$, $\varsigma = y$ and get:

$$2\pi \; J_0 \left[\alpha^2 \left(y^2 - \frac{(x-x')^2}{\gamma^2}\right)^{1/2}\right]$$, which gives integrals of the type:

$$\int_0^a e^{i \frac{u\omega}{c^2\gamma^2}(x-x')} \; J_0\left[\alpha^2 \left(-y^2 + \frac{(x-x')^2}{\gamma^2}\right)^{1/2}\right] dx'$$

At this point the problem appears intractable or not worth the further effort. For $u < c$, φ will probably a smooth function because of the properties of K_0 and I_0. For $u < c$, the function J_0 will give oscillatory φ or shock waves. If $\gamma y < x - x'$, integral will vanish, thus limits on integral should be modified.

5

One must be very careful in using transform tables, particularly if one substitutions are made. Physical considerations play an important part in this problem for they determine the branch cut configuration. Different configurations give different answers!

APPLIED MATHEMATICS 202

Problem Set No. 4 April 1961

1. Classify each of the following systems of differential equations (i.e. is it elliptic, parabolic, ...?)

(a)
$$(\rho u)_x + \rho_t = 0$$
$$\rho u_t + \rho u u_x + (\rho^\gamma)_x = 0 \; ;$$

where ρ , u , are the unknown functions and the real constant γ is greater than unity.

(b)
$$(\rho u_i)_{,i} = 0$$
$$\rho \, u_j \, u_{i,j} + (\rho^\gamma)_{,i} = 0$$

where i and j take on the values 1, 2, and 3; ρ and the u_i are the unknown functions; and γ is again a real constant greater than unity.

(c)
$$\sigma_{ij,j} = u_{i,tt}$$
$$\sigma_{ij} = 2\mu \, \epsilon_{ij} + \lambda \, \epsilon_{kk} \, \delta_{ij}$$
$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

where μ and λ are positive constants.

2. Find the real characteristics of those foregoing equation systems which possess such real characteristics, and find for 1(a) the form these equations take when the characteristic variables replace the coordinates as independent variables.

AM 202 PROBLEM SET 4

Problem

1. $(\rho u)_x + \rho_t = 0$

$\rho u_t + \rho u u_x + (\rho^\gamma)_x = 0$

Let $c = \sqrt{\gamma}\, \rho^{\frac{\gamma-1}{2}}$, $m = \frac{2}{\gamma-1}$

New Equations: $c u_x + m c_x u + m c_t = 0$

$u u_x + u_t + m c c_x = 0$

Let $\alpha(x,t)$ & $\beta(x,t)$ be the two families of curves in the (x,t) plane. Then:

$$c u_\alpha \alpha_x + c u_\beta \beta_x + m u c_\alpha \alpha_x + m u c_\beta \beta_x + m c_\alpha \alpha_t + m c_\beta \beta_t = 0$$
$$u u_\alpha \alpha_x + u u_\beta \beta_x + u_\alpha \alpha_t + u_\beta \beta_t + m c c_\alpha \alpha_x + m c c_\beta \beta_t = 0$$

Now along a curve $\alpha(x,t) = $ constant, $\frac{\partial}{\partial \alpha}$ represent the normal derivative and $\frac{\partial}{\partial \beta}$ the tangential derivative. Along curve $\alpha =$ constant, $\frac{\partial}{\partial \beta}$ is continuous and only $\frac{\partial}{\partial \alpha}$ can be discontinuous. Thus we have on subtraction:

$$\alpha_x \{ c [u_\alpha] + m u [c_\alpha] \} + m [c_\alpha] \alpha_t = 0 \qquad \text{I}$$
$$\alpha_x \{ u [u_\alpha] + m c [c_\alpha] \} + [u_\alpha] \alpha_t = 0$$

The determinential solution gives:

$$u \alpha_x + \alpha_t = \pm c \alpha_x$$

If $c = u$, there is one family of curves and the system is ~~hyperbolic~~ parabolic. Otherwise, there are two families of curves α and β and the system is hyperbolic. These curves are:

$$(u+c)\alpha_x + \alpha_t = 0 \qquad \text{II}$$
$$(u-c)\beta_x + \beta_t = 0$$

(1) Continued

Eliminating α_t & β_t between I and II we can arrive at the slope:

$$\frac{dx}{dt} = (u \pm c)$$

Problem (2):

$(\rho u_i)_{,i} = 0 \; ; \; \rho u_j u_{i,j} + (p^{\gamma})_{,i} = 0$

Let $c^2 = \gamma \rho^{\gamma-1}$; $m = \frac{2}{\gamma - 1}$

Obtain finally: $c\left[u_i, \alpha_s\right] \alpha_{s,i} + m\, u_i \left[c, \alpha_s\right] \alpha_{s,i} = 0$ (I)

$u_i \left[u_j, \alpha_s\right] \alpha_{s,i} + m\, c \left[c, \alpha_s\right] \alpha_{s,j} = 0$ (II)

(1) $u_i \alpha_{s,i} = 0$

(2) $c^2 \left\{ (\alpha_{s,i})(\alpha_{s,i}) \right] + \left[u_i \alpha_{s,i} \right]^2 = 0$

where we do not sum over s where s is repeated.

If (1) is true, then from II we have:
$[c, \alpha_s]\, \alpha_{s,j} = 0$, so either $\alpha_{s,j} = 0$ or $[c, \alpha_s] = 0$.
Now $[c, \alpha_s] \neq 0$ by hypothesis and $\alpha_{s,j} = 0$ is of little use. Hence the equation giving the characteristic surface is:

$$c^2 \left[(\alpha_{s,i})(\alpha_{s,i}) \right] - \left[u_i \alpha_{s,i} \right]^2 = 0$$

or $c^2 \left| \nabla \alpha_s \right|^2 - u^2 \left| \nabla \alpha_s \right|^2 \cos^2(\vec{u}, \nabla \alpha_s) = 0$

Let $M = u/c$ where $u^2 = u_1^2 + u_2^2 + u_3^2$

$\therefore \quad \cos^2(\vec{u}, \nabla \alpha_s) = \frac{1}{M^2}$

Problem (2) Continued

So we have real characteristic surfaces for $M > 1$, hence hyperbolic. The characteristic surfaces are surfaces tangential to a circular cone whose axis is along $\vec{u}$ and whose semi-apex angle is such that $\cos\alpha = \frac{1}{M}$.

If $M = 1$, the characteristic surfaces are surfaces tangent to a plane which is normal to $\vec{u}$ and the system is parabolic.

For $M < 1$, there are no real characteristics and the equation is elliptic.

Problem (3):

$$\sigma_{ij,j} = u_{i,tt}$$
$$\sigma_{ij} = 2\mu\,\epsilon_{ij} + \lambda\,\delta_{ij}\,\epsilon_{kk}$$
$$\epsilon_{ij} = 1/2\,(u_{i,j} + u_{j,i})$$

We can reduce this system of equations into one set consisting only of u_i's:

$$u_{i,tt} = \mu\,(u_{i,jj} + u_{j,ij}) + \lambda\,\delta_{ij}\,u_{k,kj}$$

Assume a new set of independent variables $\alpha_p(x_1, x_2, x_3, x_4)$ where $p = 1,2,3,4$ and $x_4 = t$. The new equation is then:

$$u_{i,\alpha_p\alpha_q}\,\alpha_{p,4}\,\alpha_{q,4} = \mu\,(u_{i,\alpha_p\alpha_q}\,\alpha_{p,j}\,\alpha_{q,j} + u_{j,\alpha_p\alpha_q}\,\alpha_{p,i}\,\alpha_{q,j})$$
$$+\ \lambda\,\delta_{ij}\,u_{k,\alpha_p\alpha_q}\,\alpha_{p,k}\,\alpha_{q,j}$$

Now look at the hypersurface α_s such that $\frac{\partial^2}{\partial\alpha_s^2}$ of the u_i's are discontinuous across α_s while all other derivatives of the u_i's as well as the u_i's themselves remain continuous across α_s.

Problem 4 Continued:

Evaluating on each side of α_s and subtracting:

$$\alpha_{s,4}\,\alpha_{s,4}\,[u_i, \alpha_s \delta_s] - \Big\{ \mu\, \alpha_{s,j}\, \delta_{s,j}\, [u_i, \alpha_s \alpha_s]$$

$$+ \mu\, \alpha_{si}\, \delta_{s,j}\, [u_j, \alpha_s \delta_s] + \lambda\, \delta_{ij}\, \alpha_{s,k}\, \alpha_{s,j}\, [u_k, \alpha_s \delta_s] \Big\} = 0 \qquad \text{I}$$

The above can be considered as three homogeneous equations in one $[u_i, \alpha_s \alpha_s]$. From now on do not sum over when s repeats. Solve determinant and get:

$$\alpha_{s,4}^2 - \mu\,(\alpha_{s,j})\,(\alpha_{s,j}) = 0 \qquad \text{II}$$

$$\alpha_{s,4}^2 - (2\mu + \lambda)\,(\alpha_{s,j})\,(\alpha_{s,j}) = 0 \qquad \text{III}$$

Both equs. II and III give solutions that are compatible with one. Since both μ and λ are always positive, the equations are always hyperbolic as the eqns have real characteristic surfaces.

The characteristic hypersurfaces given by II are hypersurfaces tangent to the circular hypercone whose axis is parallel to the t axis and whose semi-apex angle is given by $\tan \varphi = \sqrt{\mu}$. The ones given by III are tangent to the circular hypercone whose axis is parallel to the t axis and whose semi-apex angle φ is given by $\tan \varphi = \sqrt{2\mu + \lambda}$.

ASSIGNMENT #4

Paul Grant
DEAP - AP - 16
AM 202
4 - 22 - 61

23/30

1. a. (1) Consider:

$$a_{11} u_{1,x} + a_{12} u_{1,y} + a_{21} u_{2,x} + a_{22} u_{2,y} + a_1 u_1 + a_2 u_2 + a = 0$$
$$b_{11} u_{1,x} + b_{12} u_{1,y} + b_{21} u_{2,x} + b_{22} u_{2,y} + b_1 u_1 + b_2 u_2 + b = 0$$

$$a_{ij} = a_{ij}(u_1, u_2, x, y) \; ; \; b_{ij} = b_{ij}(u_1, u_2, x, y)$$

(2) Choose two new independent variables via the curves: $\xi(x,y) = c$; $\eta(x,y) = c$, across which we assume the derivative of u is discontinuous and along which it is continuous. Taking:

$$u_{1,x} = u_{1,\xi}\xi_x + u_{1,\eta}\eta_x \text{ , etc.}$$

(3) We pick another point along η and subtract resulting equations. Pick point along ξ and repeat: Get:

$$\begin{vmatrix} a_{11}\xi_x + a_{12}\xi_y & a_{21}\xi_x + a_{22}\xi_y \\ b_{11}\xi_x + b_{12}\xi_y & b_{21}\xi_x + b_{22}\xi_y \end{vmatrix} \begin{vmatrix} [u_{1,\xi}] \\ [u_{2,\xi}] \end{vmatrix} = 0$$

$$\begin{vmatrix} a_{11}\eta_x + a_{12}\eta_y & a_{21}\eta_x + a_{22}\eta_y \\ b_{11}\eta_x + b_{12}\eta_y & b_{21}\eta_x + b_{22}\eta_y \end{vmatrix} \begin{vmatrix} [u_{1,\eta}] \\ [u_{2,\eta}] \end{vmatrix} = 0$$

$\frac{\eta_x}{\eta_y}$ or $\frac{\xi_x}{\xi_y}$: roots: real and distinct : hyperbolic
repeated real : parabolic
complex : elliptic

(4) Consider:

$$\rho_x u + \rho u_x + \rho_t = 0$$
$$\rho u_t + \rho u u_x + \gamma \rho^{\gamma-1} \rho_x = 0$$

9

Choose: $\begin{matrix} u_1 = u \\ u_2 = \rho \end{matrix} \Bigg\{ \begin{matrix} 1 \to x \; ; \; 2 \to t \\ x \to x \; ; \; y \to t \end{matrix} \Bigg\{ \begin{matrix} a_{11} = \rho ; \; a_{12} = 0 \; : \; a_{21} = u \; ; \; a_{22} = 1 \\ b_{11} = \rho u ; \; b_{12} = \rho \; : \; b_{21} = \gamma\rho^{\gamma-1} ; \; b_{22} = 0 \end{matrix}$

(5)
$$\begin{vmatrix} \rho \frac{\xi_x}{\xi_t} & u\frac{\xi_x}{\xi_t} + 1 \\ u\frac{\xi_x}{\xi_t} + 1 & \gamma\rho^{\gamma-2}\frac{\xi_x}{\xi_t} \end{vmatrix} = 0 \; ; \; \gamma\rho^{\gamma-1}\left(\frac{\xi_x}{\xi_t}\right)^2 - \left\{ u^2\left(\frac{\xi_x}{\xi_t}\right)^2 + 2u\left(\frac{\xi_x}{\xi_t}\right) + 1 \right\} = 0$$

$$\frac{\xi_x}{\xi_t} = \frac{u}{(\gamma\rho^{\gamma-1} - u^2)} \pm \frac{\left\{4u^2 + 4\gamma\rho^{\gamma-1} - 4u^2\right\}^{1/2}}{2(\gamma\rho^{\gamma-1} - u^2)}$$

$$= \frac{u \pm \gamma^{1/2}\rho^{\frac{\gamma-1}{2}}}{\gamma\rho^{\gamma-1} - u^2} \quad , \therefore \text{ hyperbolic}$$

(6) Choose + root for $\frac{\xi_x}{\xi_t}$ and − root for $\frac{\eta_x}{\eta_t}$:

$$\frac{\xi_x}{\xi_t} = \frac{1}{\gamma^{1/2}\rho^{\frac{\gamma-1}{2}} - u} \quad ; \quad \frac{\eta_x}{\eta_t} = \frac{-1}{\gamma^{1/2}\rho^{\frac{\gamma-1}{2}} + u}$$

or
$$\left.\begin{aligned} \left\{\gamma^{1/2}\rho^{\frac{\gamma-1}{2}} - u\right\}\xi_x - \xi_t = 0 \\ \left\{\gamma^{1/2}\rho^{\frac{\gamma-1}{2}} + u\right\}\eta_x + \eta_t = 0 \end{aligned}\right\} \text{Characteristic equations.}$$

(7) Take $c = \gamma^{1/2}\rho^{\frac{\gamma-1}{2}}$

$$\xi_x = \frac{t_\eta}{t_\eta x_\xi - t_\xi x_\eta = J} = \frac{t_\eta}{J} \quad ; \quad \xi_t = -\frac{x_\eta}{J}$$

$$\eta_x = -\frac{t_\xi}{J} \quad ; \quad \eta_t = \frac{x_\xi}{J}$$

Then:
$$(c - u)\, t_\eta + x_\eta = 0$$
$$(c + u)\, t_\xi - x_\xi = 0$$

These equations arise from the relations :

$$\xi_\eta = \xi_x x_\eta + \xi_t t_\eta \quad : \quad \xi_\xi = \xi_x x_\xi + \xi_t t_\xi$$

$$\eta_\xi = \eta_x x_\xi + \eta_t t_\xi \quad : \quad \eta_\eta = \eta_x x_\eta + \eta_t t_\eta$$

Assignment #4
Continued

Paul Grant
DEAP-AP-1G
AM 202
4-22-61

1\. b. (1) We have:

$$(\rho u_i)_{,i} = 0 \quad ; \quad (\rho u_x)_{,x} + (\rho u_y)_{,y} + (\rho u_z)_{,z} = 0$$

or: $\rho u_{x,x} + \rho u_{y,y} + \rho u_{z,z} + u_x \rho_{,x} + u_y \rho_{,y} + u_z \rho_{,z} = 0$

$$\rho u_x u_{x,x} + \rho u_y u_{x,y} + \rho u_z u_{x,z} + \gamma \rho^{\gamma-1} \rho_{,x} = 0$$
$$\rho u_x u_{y,x} + \rho u_y u_{y,y} + \rho u_z u_{y,z} + \gamma \rho^{\gamma-1} \rho_{,y} = 0$$
$$\rho u_x u_{z,x} + \rho u_y u_{z,x} + \rho u_z u_{z,z} + \gamma \rho^{\gamma-1} \rho_{,z} = 0$$

(2) We will have a determinant of the form:

$$\begin{bmatrix} a_{11}\xi_{,x} + a_{12}\xi_{,y} + a_{13}\xi_{,z} & \cdots \\ \vdots & \\ d_{11}\xi_{,x} + d_{12}\xi_{,y} + d_{13}\xi_{,z} & \cdots \end{bmatrix} \begin{bmatrix} [u_{1,\xi}] \\ [u_{2,\xi}] \\ [u_{3,\xi}] \\ [u_{4,\xi}] \end{bmatrix} = 0$$

since there are 4 unknowns in three independent variables. Also: Surfaces: $\xi(x,y,z)$, $\eta(x,y,z)$, $J(x,y,z)$

$$u_{1,x} = u_{1,\xi}\xi_{,x} + u_{1,\eta}\eta_{x} + u_{1,J} J_{x} \quad ; \text{ etc}$$

which gives the determinant upon substitution into:

$$a_{11} u_{1,x} + a_{12} u_{1,y} + a_{13} u_{1,z} + \cdots \quad ; \text{ etc.}$$

(3) Now: $u_1 = u_x$, $u_2 = u_y$, $u_3 = u_z$, $u_4 = \rho$

$a_{11} = \rho$, $a_{12} = a_{13} = 0$ | $b_{11} = \rho u_x$, $b_{12} = \rho u_y$, $b_{13} = \rho u_z$
$a_{21} = a_{23} = 0$, $a_{22} = \rho$ | $b_{21} = b_{22} = b_{23} = 0$
$a_{31} = a_{32} = 0$, $a_{33} = \rho$ | $b_{31} = b_{32} = b_{33} = 0$
$a_{41} = u_x$, $a_{42} = u_y$, $a_{43} = u_z$ | $b_{41} = \gamma\rho^{\gamma-1}$, $b_{42} = b_{43} = 0$

$c_{11} = c_{12} = c_{13} = 0$ | $d_{11} = d_{12} = d_{13} = 0$
$c_{21} = \rho u_x$, $c_{22} = \rho u_y$, $c_{23} = \rho u_z$ | $d_{21} = d_{22} = d_{23} = 0$
$c_{31} = c_{32} = c_{33} = 0$ | $d_{31} = \rho u_x$, $d_{32} = \rho u_y$, $d_{33} = \rho u_z$
$c_{41} = c_{43} = 0$, $c_{42} = \gamma\rho^{\gamma-1}$ | $d_{41} = d_{42} = 0$, $d_{43} = \gamma\rho^{\gamma-1}$

(4)
$$\begin{vmatrix} \rho\xi_{,x} & \rho\xi_{,y} & \rho\xi_{,z} & (u_x\xi_{,x} + u_y\xi_{,y} + u_z\xi_{,z}) \\ \rho(u_x\xi_{,x} + u_y\xi_{,y} + u_z\xi_{,z}) & 0 & 0 & \gamma\rho^{\gamma-1}\xi_{,x} \\ 0 & \rho(u_x\xi_{,x}\cdots) & 0 & \gamma\rho^{\gamma-1}\xi_{,y} \\ 0 & 0 & \rho(\quad) & \gamma\rho^{\gamma-1}\xi_{,z} \end{vmatrix} = 0$$

(5)

$$\rho\xi_{,x}\begin{vmatrix} 0 & 0 & \gamma\rho^{\gamma-1}\xi_{,x} \\ \rho(\ ) & 0 & \gamma\rho^{\gamma-1}\xi_{,y} \\ 0 & \rho(\ ) & \gamma\rho^{\gamma-1}\xi_{,z}\end{vmatrix} - \left[\rho(\ )\right]^4$$

$$-\rho\xi_{,y}\begin{vmatrix} \rho(\ ) & 0 & \gamma\rho^{\gamma-1}\xi_{,x} \\ 0 & 0 & \gamma\rho^{\gamma-1}\xi_{,y} \\ 0 & \rho(\ ) & \gamma\rho^{\gamma-1}\xi_{,z}\end{vmatrix} + \rho\xi_{,z}\begin{vmatrix} \rho(\ ) & 0 & \gamma\rho^{\gamma-1}\xi_{,x} \\ 0 & \rho(\ ) & \gamma\rho^{\gamma-1}\xi_{,y} \\ 0 & 0 & \gamma\rho^{\gamma-1}\xi_{,z}\end{vmatrix} = 0$$

(6) $$\rho\left[\rho(\ )\right]^2\gamma\rho^{\gamma-1}\left[(\xi_{,x})^2+(\xi_{,y})^2+(\xi_{,z})^2\right] - \left[\rho(\ )\right]^4 = 0$$

$$\left\{u_x\xi_{,x} + u_y\xi_{,y} + u_z\xi_{,z}\right\}^2 = 0$$

(7) $$\gamma\rho^{\gamma-2}\left[(\xi_{,x})^2+(\xi_{,y})^2+(\xi_{,z})^2\right] - \left[u_x\xi_{,x} + u_y\xi_{,y} + u_z\xi_{,z}\right]^2 = 0$$

(8) Since we have three solutions, we can arbitrarily assign one to each surface and have (possibly):

$$\left.\begin{aligned} u_x\xi_{,x} + u_y\xi_{,y} + u_z\xi_{,z} &= 0 \\ u_x\eta_{,x} + u_y\eta_{,y} + u_z\eta_{,z} &= 0 \end{aligned}\right\}$$

These two solns. are not physically valid if you check back to your original eqns.

$$\gamma\rho^{\gamma-2}\left[(\zeta_{,x})^2+(\zeta_{,y})^2+(\zeta_{,z})^2\right] - \left[u_x\zeta_{,x} + u_y\zeta_{,y} + u_z\zeta_{,z}\right]^2 = 0$$

The first two are planes (same plane) and the last is apparently the surface formed by the intersection of a surface and plane (probably degenerate into a line).

From the first two equations, the set appears to be parabolic, but the presence of the third seems to make the system semi-hyperbolic unless it is degenerate.

b

For Clarification refer to soln

Assignment #4
Continued

Paul Grant
DEAP - AP - 16
AM 202
4-22-61

1. c. (1) $\sigma_{ij,j} = u_{i,tt}$

$\sigma_{ij} = 2\mu \epsilon_{ij} + \lambda \epsilon_{kk} \delta_{ij}$

$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$

$\epsilon_{kk} = u_{k,k}$

(2) $\sigma_{ij,j} = u_{i,tt}$

$\sigma_{ij} = \mu(u_{i,j} + u_{j,i}) + \lambda u_{k,k} \delta_{ij}$

(3) $\therefore \quad \mu(u_{i,j} + u_{j,i})_{,j} + \lambda\{u_{k,k}\delta_{ij}\}_{,j} - u_{i,tt} = 0$

or: $\mu(u_{i,jj} + u_{j,ij}) + \lambda u_{i,ii} - u_{i,tt} = 0$

$\therefore \mu(2u_{x,xx} + u_{x,yy} + u_{x,zz} + u_{y,xy} + u_{z,xz}) + \lambda u_{x,xx} - u_{x,tt} = 0$

$\mu(u_{y,xx} + 2u_{y,yy} + u_{y,zz} + u_{x,yx} + u_{z,yz}) + \lambda u_{y,yy} - u_{y,tt} = 0$

$\mu(u_{z,xx} + u_{z,yy} + 2u_{z,zz} + u_{x,zx} + u_{y,zy}) + \lambda u_{z,zz} - u_{z,tt} = 0$

7

(4) $\lambda + 2\mu > 0$, $\therefore$ hyperbolic ✓

Incomplete. There are other solns also possible.
refer posted soln.

2\. a. (1) If $c = \gamma^{1/2}\rho^{\frac{\gamma-1}{2}}$;

$$(c-u)\,\xi_x = \xi_t \quad ; \quad (c+u)\,\eta_x = -\eta_t$$

$$d\xi = \xi_x\,dx + \xi_t\,dt \quad , \therefore \quad \frac{dx}{dt} = -\frac{\xi_t}{\xi_x} \quad , \text{etc}$$

$$\therefore \quad \frac{dx}{dt} = u - c \quad ; \quad \frac{dx}{dt} = u + c$$

$$x = \int (u-c)\,dt \quad ; \quad x = \int (u+c)\,dt$$

are the two characteristics.

(2) The characteristic differential equations with ξ and η as independent variables is :

$$(c-u)\,t_\eta + x_\eta = 0$$

$$(c+u)\,t_\xi - x_\xi = 0$$

b. (1) The characteristic differential equations are given in 1b.

HARVARD UNIVERSITY

APPLIED MATHEMATICS 202

Final Examination May 24, 1961

1. (a) Find the eigenvalues and eigenfunctions of:

$$u''(x) + \lambda\, u(x) = 0\,, \quad \text{in } 1 < x < 4\,;$$
$$u(1) = u(4) = 0$$

(b) Demonstrate the orthogonality of the eigenfunctions of:

$$[p(x)\, u'(x)]' + q(x)\, u(x) + \lambda\, \alpha(x)\, u(x) = 0$$
$$u(a) = u'(b) = 0$$

p, q, and α are continuous, one-signed functions of x.

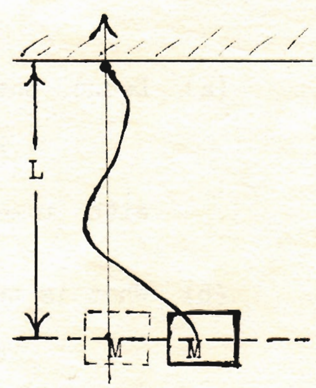
2. Let $u_y - (y^2 + 1)\, u_{xx} = 0 \quad \text{in} \quad 0 < y < \infty, \; -\infty < x < \infty$

and $u(x,0) = e^{-ax^2}$.

Find $u(x,y)$. Note that $\displaystyle\int_{-\infty}^{\infty} e^{-a\sigma^2 + b\sigma}\, d\sigma = \sqrt{\frac{\pi}{a}}\, e^{b^2/4a}$

3. One end of an inextensional cord of length L and mass per unit length m is connected to a rigid support. A mass M hangs at the other end.

(a) Formulate the boundary value problem associated with the small amplitude lateral oscillations of the cord and mass.



(Cont'd on next page)

3. (Cont'd)

(b) Give the quantitative description of the oscillation which ensues when the motion is initiated by giving an initial displacement

$$u(x,0) = 1/2 \;; \qquad 0 \le x \le L/2$$

$$u(x,0) = 1 - x/L \;; \qquad L/2 \le x \le L \,,$$

and an initial velocity

$$u_t(x,0) = 0 \qquad 0 \le x \le L \,.$$

Any number defined by messy integral or transcendental equations need not be evaluated beyond the specification of such definitions.

4. Classify the equation system

$$x^3 u_x = v_t$$

$$t\, v_x = u_t$$

and find any characteristic curves it possesses.

5. (a) Deduce the Green's function associated with the problem

$$\Delta u - k^2 u = f(x,y,z) \text{ in the infinite domain,}$$

with $u \longrightarrow 0$ as $x^2 + y^2 + z^2 \longrightarrow \infty$.

(b) What is the Green's function when the same equation holds in

$$0 < x < \infty, \quad 0 < y < \infty \;, \quad -\infty < z < \infty$$

and $u(0,y,z) = 0$, $u_y(x,0,z) = 0$?

6. Find one of the eigenfunctions, $u(x,\epsilon)$, and eigenvalues, $\lambda(\epsilon)$, of

$$u''(x) + \lambda (1 + \epsilon \cos 2x) \; u(x) = 0$$

$$u(0,\epsilon) = u(\pi,\epsilon) = 0$$

when $0 < \epsilon \ll 1$.

AM 202 - FINAL EXAMS

6-2-60

① Classify the system of equations:

$$u_x - (x-a) v_y = 0$$
$$u_y - v_x + v = 0$$

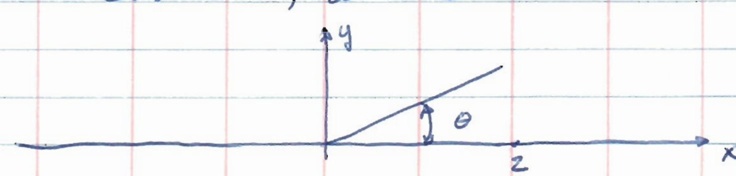
and locate its characteristics if it has any.

② A membrane occupies the annular space between two rigid rings, one at $r=a$, the other at $r=b>a$. The outer ring is fixed in position, but the inner one is free to translate in a direction perpendicular to the plane of the membrane. The membrane density is ρ gms/cm^2 and the ring density is σ gms/cm. Find the equations which govern the eigenmodes of oscillation and find some of the modes.

③ The potential φ of the acoustic waves in the gas in region R is governed by the equation:

$$\nabla^2 \varphi - \frac{1}{c^2} \varphi_{tt} = 0$$

R is the region $y>0$, $-\infty < x < \infty$, $-a < z < a$. The boundaries at $z = \pm a$ are rigid and unmoving as are those at $y=0$ with $-\infty < x < 0$, and $x > 2$. However, a rigid plate of length 2 is hinged at $x=0$ (see figure) and is oscillated so that $\theta(t) = \alpha \cos \omega t$, where $\alpha \ll 1$.



Find the potential φ in R. An integral suffices as the answer.

(4) With $u'' + \epsilon(1-x^2)u + \lambda u = 0$; $u(0) = 0$;
$u(1) + 2u'(1) = 0$ and $0 < \epsilon << 1$.
Find the eigenfunctions $u_n(x)$ and eigenvalues λ_n.

(5) It is given that:
$$u_{xx} + u_{yy} + k^2 u = f(x,y) \quad \text{in} \quad x>0,\ y>0,$$
$$u(0,y) = u_y(x,0) = 0 \text{ ; and :}$$
$$u \to 0 \quad \text{as} \quad x^2+y^2 \to \infty;$$
where $f(x,y)$ is to be regarded as a known function. Find the Green's function, G, associated with this problem and write $u(x,y)$ as an integral involving G.

(6) Deduce the asymptotic properties of the solution of problem 3.