APPLIED
MATH
203

ADVANCED
METHODS

APPLIED MATHEMATICS 203

ADVANCED METHODS

INSTRUCTOR: CARRIER ROOM P209: MWF 9

LECTURE 1: 9-25-61

Integral Equations:

An integral equation is defined as an equality with an unknown under an integral sign. If there is a derivative in it, it is known as an integro-differential equation. Usually, we can only handle linear integral equations. It is possible to use perturbation methods to convert non-linear equations into linear.

Consider an example of an integral equation:

$$a\, u(x) + b\, f(x) = \lambda \int_{A(x)}^{B(x)} K(x,t)\, u(t)\, dt$$

$u(x)$ is the unknown. $K(x,t)$ is called the kernal, usually a Green's function. If A and B are constants, we have a Friedholm equation. If $b=0$, we have homogeneous Friedholm equation. If $a=0$, Friedholm equation of the first kind. If $B(x)$ is non-constant, we have a Volterra equation.

We treat an integral equation by considering the homogeneous case first, then use results to find inhomogeneous solution. Some methods of partial differential equations are useful.

Fredholm's equation is also known as a linear integral equation of the second kind.

How do integral equations arise?

1. Conversion from ordinary differential equation. Consider:

$L(u) = -\lambda u$; $u(A) = u(B) = 0$, but any homogeneous boundary condition will do. Do by the Green's function method, find $K(x,t)$ and form:

$$u(x) = -\lambda \int K(x,t)\, u(t)\, dt$$

The reason we make this transformation is that the integral is easier to handle by approximate methods than the differential equation.

2. May arise from partial differential equation.

$$u(x,y) = \lambda \int K(x,y,x')\, u(x')\, dx'$$

This becomes an integral equation when we attempt to find $u(x,0)$:

$$u(x,0) = \lambda \int K(x,0,x')\, u(x')\, dx'$$

where the integration is carried over the boundary condition. This has the advantage of finding one variable over a limited domain rather than many variables over an infinite domain.

3. Integral equations may arise directly from a physical problem, for example, neutron diffusion.

We now examine the homogeneous Friedholm equation:

$$u(x) = \lambda \int_a^b K(x,t)\, u(t)\, dt$$

We will now write down some proofs or facts about homogeneous integral equations (linear), without actually doing the proofs at this time.

1. $K(x,t) = K^*(t,x)$ (Hermitean)
2. There is at least one value of λ and u such that the equality is obeyed.
3. We can form a new kernel of the form:

$$K(x,t) - \frac{u_1(x)\, u_1(t)}{\lambda_1}$$

4. The u_n's form a complete set, namely:

$$\int_a^b \left[F(x) - \sum a_n u_n(x) \right]^2 dx = 0$$

or else the u's terminate and hence no complete set. What do we do? Suppose we find that u_n breaks off after u_{17}. Then we can complete the set by finding V_{18} and on by making them orthogonal to u_1 thru u_{17} with the rule:

$$V_{27}(x) = \lambda \int_a^b \sum_{n=1}^{17} \frac{u_n(x)\, u_n(t)}{\lambda_n} V_{27}(t)\, dt = 0$$

If we admit this kind of extension, then we have completed the set where some of the eigenvalues are infinite because it is $\lambda^{-1} = 0$ that makes the integral zero. Is this too artificial? Turns out not to be.

Treatment of the Inhomogeneous Case:

We will want to use some of the properties of a complete set since we use expansions of the homogeneous solutions. Consider:

$$f(x) + \lambda^{-1} u(x) = \int_a^b K(x,t)\, u(t)\, dt$$

with the homogeneous solutions $u_n(x)$, $(\lambda^{-1})_n$

We anticipate:

$$f(x) = \sum a_n u_n(x); \quad u(x) = \sum b_n u_n(x)$$

Recall that λ^{-1} is specified in advance in an inhomogeneous problem. Substituting we get:

$$\sum a_n u_n(x) + (\lambda^{-1}) \sum b_n u_n(x) = \underbrace{\int_a^b \sum K(x,t)\left[b_n u_n(t)\right] dt}_{\sum b_n \lambda_n^{-1} u_n(x)}$$

from original homogeneous equation.

The u_n's are of course orthogonal so that we can operate with $\int_a^b u_{n'}(x)\,dx$ and get:

$$a_n \int u_n^2\,dx + (\lambda^{-1})\, b_n \int u_n^2\,dx = \frac{b_n}{\lambda_n} \int u_n^2\,dx$$

which leads to: $\quad b_n \text{ times } \dfrac{\lambda - \lambda_n}{\lambda\,\lambda_n} = a_n$

or $\quad b_n = \dfrac{\lambda\,\lambda_n}{\lambda - \lambda_n}\, a_n$

Hence the inhomogeneous solution is:

$$u(x) = \lambda \sum \frac{\lambda_n a_n}{\lambda - \lambda_n} u_n(x)$$

or we can rewrite this as:

$$u(x) = -\lambda f(x) + \sum_{n=1}^{m_0} \frac{\lambda / \lambda_n}{\lambda/\lambda_n - 1} a_n u_n(x)$$

where m_0 is the upper limit on the number of solutions of the homogeneous case. Note that $\lambda_n^{-1} = 0$ for $n >$ terminating number.

Solution of the Homogeneous Problem:

How do we find the eigenfunctions of the homogeneous equation? We usually don't. Most of the problems where it is possible to find them are artificial physical problems. In reality, we use an iterative method. At this point, note that an integral equation is really the limit of a matrix equation.

What is the iterative procedure? Assume a homogeneous Friedholm equation:

$$\lambda^{-1} u(x) = \int_a^b K(x,t)\, u(t)\, dt$$

Now suppose:

$$g(x) = \sum a_n u_n(x)$$

We now define the operation known as the <u>scalar</u> product or <u>inner</u> product of two functions:

$$\int_a^b K(x,t)\, g(t)\, dt \equiv K \cdot g$$

Do this operation on definition of g and get:

$$g_n = K \cdot g_{n-1} \quad , \text{ which comes from:}$$

$$K \cdot g = \int_a^b K(x,t) \sum a_n u_n(t)\, dt = \sum \frac{a_n u_n(x)}{\lambda_n}$$

Define: $\quad g_1 \equiv \sum \frac{a_n u_n(x)}{\lambda_n} \quad ; \quad g_n = \sum \frac{a_n u_n(x)}{(\lambda_n)^n}$

Then:

$$K \cdot g_{n-1} = \int_a^b K(x,t) \sum \frac{a_n u_n(t)}{(\lambda_n)^{n-1}}\, dt$$

$$= \sum \frac{a_n u_n(x)}{(\lambda_n)^n} = g_n$$

We form the inner product of the g's:

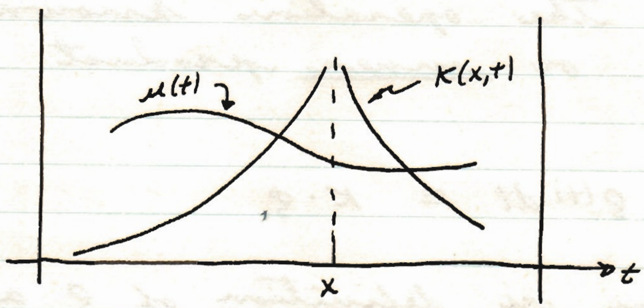
$$\frac{g_{n-1} \cdot g_n}{g_n \cdot g_n} = \frac{\sum a_n^2 / \lambda_n^{2n-1}}{\sum a_n^2 / \lambda_n^{2n}}$$

LECTURE 2 : 9-27-61

Recall: Homogeneous Fredholm equation:

$$u(x) = \lambda \int_a^b K(x,t)\, u(t)\, dt$$

The real meaning of an integral equation is that we are making the area under the curve of a product of two functions $K(x,t)\, u(t)$ equal to $\frac{u(x)}{\lambda}$.



We now consider integral equations defined in the infinite domain. For example:

$$u(x) = \lambda \int_{-\infty}^{\infty} \frac{\beta}{2} e^{-\beta|x-t|}\, u(t)\, dt$$

This is one of the few cases where we can anticipate a solution. We anticipate $u(x) = e^{sx}$ where s is a parameter to be determined. We will find that s will be in terms of λ so that solutions exist for all values of λ or we have continuous eigenvalues. Form:

$$\left[\int_a^b K(x-t)\, e^{s(t-x)}\, dt \right] e^{sx}$$

hence, from the equation:

$$\lambda \int_{-\infty}^{\infty} K(x-t)\, e^{s(t-x)}\, d(t-x) = 1$$

Make the substitutions: $\beta x = x' \to x \quad : \quad st \leftrightarrows s't'$

$\beta t = t' \to t \quad : \quad sx \leftrightarrows s'x'$

This removes the parameter β from the problem completely.

Then: $$\frac{\lambda}{2}\int_{-\infty}^{\infty} e^{-|x-t|}\, e^{s(t-x)}\, dt = 1$$

It is obvious that $s \leq 1$ or the integral will not converge. Integrating, we find:

$$\int_{-\infty}^{\infty} e^{-|x-t|}\, e^{s(t-x)}\, d(t-x) = \int_{0}^{\infty} e^{-(1-s)z}\, dz + \int_{-\infty}^{0} e^{(1+s)z}\, dz$$

$$= \frac{1}{1-s} + \frac{1}{1+s} = \frac{2}{1-s^2}$$

Hence: $$\frac{\lambda}{1-s^2} = 1 \quad \text{or} \quad \lambda = 1-s^2$$

It turns out that any linear combination of e^{sx}, e^{-sx} constitutes a solution. Interesting cases arise when s is complex. We have noted that eigenvalues in this problem are continuous, however, infinite domain problems do not arise in practice.
The solutions for imaginary s are:
e^{ikx}, e^{-ikx}, or $\sin[k(x+a)]$ which includes both exponentials.

Suppose we assume a solution of e^{ikx} but this time work only in the semi-infinite domain.
Integrate in parts:

$$\int_0^x e^{-x+t+ikt}\, dt = e^{-x} \left.\frac{e^{t(1+ik)}}{1+ik}\right]_0^x \text{, go to } e^{ik(x+a)}:$$

Do same for $\int_x^\infty$

Then: $$\frac{\lambda}{2} K \cdot e^{ik(x+a)} \longrightarrow \lambda \frac{e^{ik(x+a)}}{1+k^2} - \frac{\lambda}{2}\frac{e^{-x+ika}}{1+ik}$$

since: choose sign that cancels x.

$$\frac{\lambda}{2} K \cdot e^{ik(x+a)} = \frac{\lambda e^{\pm x+ika}}{2}\left\{ \left.\frac{e^{-t(1-ik)}}{-(1-ik)}\right]_x^\infty + \left.\frac{e^{t(1+ik)}}{1+ik}\right]_0^x \right\}$$

This will not be a solution as we don't get the original solution because of e^{-x+ika}. How about $-k$?

$$\frac{\lambda}{2} K \cdot e^{-ik(x+a)} \longrightarrow \lambda \frac{e^{-ik(x+a)}}{1+k^2} - \frac{\lambda}{2}\frac{e^{-x-ika}}{1-ik}$$

now choose a solution formed by the difference between k and $-k$: This will truly be a solution if:

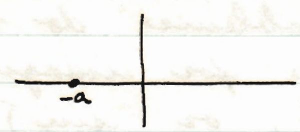
$$\frac{e^{ika}}{1+ik} = \frac{e^{-ika}}{1-ik}$$

or $$e^{ika - i\tan^{-1}k} = e^{-ika + i\tan^{-1}k}$$

Thus: $ka - \tan^{-1}k = n\pi$ gives the formula for k. However, multiplicity in n is unnecessary and we can take $n=0$:

Hence: $a = \dfrac{\tan^{-1}k}{k}$; $k = \sqrt{\lambda - 1}$

and the solution is $\sin k(x+a)$. This means origin of solution is at $-a$.



Cutting down to the semi-infinite domain does not destroy the continuous nature of the eigenvalues, but does restrict the form of the solution.

Consider a new problem in the finite domain:

$$u(x) = \lambda \int_{-a}^{a} \frac{\beta}{2} e^{-\beta|x-t|} u(t)\, dt$$

Transform to the unit interval: $x' = \frac{x}{a}$, $\beta' = \beta a \to \beta$

$\therefore$
$$u(x) = \lambda \int_{-1}^{1} \frac{\beta}{2} e^{-\beta|x-t|} u(t)\, dt$$

β plays a very important role in the finite domain. Anticipate the solution $\sin k(x+a)$, discreet eigenvalues, and only even-odd solutions (because $\int_{-a}^{a}$). Try the solution:

$$u(x) = \cos k(x+a) = \mathrm{Re}\, e^{ik(x+a)}$$

Substitute and calculate integral. It turns out that one need only try $u(x) = e^{ikx}$ when working in the finite domain.

We are concerned with the integral:

$$\int_{-1}^{1} e^{-\beta|x-t|} e^{ikt} dt$$

$$= \underbrace{\int_{-1}^{x} e^{-\beta x + \beta t + ikt} dt}_{e^{-\beta x}\left[\frac{e^{(\beta+ik)x} - e^{-(\beta+ik)}}{\beta+ik}\right]} + \underbrace{\int_{x}^{1} e^{-\beta t + \beta x + ikt} dt}_{e^{\beta x}\left[\frac{e^{-(\beta-ik)x} - e^{-(\beta-ik)}}{\beta-ik}\right]}$$

For $+k$: $\dfrac{2\beta e^{ikx}}{\beta^2+k^2} - \left[\dfrac{e^{-(x+1)\beta - ik}}{\beta+ik} + \dfrac{e^{(x-1)\beta+ik}}{\beta-ik}\right]$

For $-k$: $\dfrac{2\beta e^{-ikx}}{\beta^2+k^2} - \left[\dfrac{e^{-(x+1)\beta + ik}}{\beta-ik} + \dfrac{e^{(x-1)\beta-ik}}{\beta+ik}\right]$

Now for the solution $u(x) = \cos kx$, we must have:

$$\left[e^{-(x+1)\beta} + e^{(x-1)\beta}\right]\left[e^{i\{k+\tan^{-1}\frac{k}{\beta}\}} + e^{-i\{k+\tan^{-1}k/\beta\}}\right] = 0$$

or $\tan^{-1}\frac{k}{\beta} + k = \frac{n\pi}{2}$; n odd

Clearly, for:

$u(x) = \sin kx$, n is even.

Now: $\dfrac{\lambda\beta^2}{\beta^2+k^2} = 1$

$\therefore\ k = \beta\sqrt{\lambda-1}$; $\lambda > 1$

which is the rule for the eigenvalues.

LECTURE 3 : 9-29-61

We have examined an integral equation of the form:

$$u(x) = \lambda \int_a^b e^{-\beta|x-t|} \frac{\beta}{2} u(t)\, dt$$

and have found the following:

Infinite Domain: $u(x) = \sin k(x+a)$, with a taking on any value.

Semi-infinite Domain: $u(x) = \sin k(x+a)$, with a being governed by solution to transcendental equation but still continuous.

Finite Domain: $u(x) = \begin{cases} \cos kx \\ \sin kx \end{cases}$, with discreet eigenvalues being given by roots of transcendental equation in k.

In all forms: $k = \beta\sqrt{\lambda - 1}$, $\lambda > 1$

We have shown that the finite domain leads to a set of eigenfunctions which we assume complete.

Now we usually cannot guess the solution to an integral equation, and there is no formal method.
However, the methods of solution in the semi-infinite domain contain generalities. Hence we could solve a problem by formal methods in the semi-infinite domain and then use approximations to get the finite domain solution.
Consider then as an example, the Wiener-Hopf integral equation:

$$u(x) = \lambda \int_0^\infty \frac{\beta}{2} e^{-\beta|x-t|} u(t)\, dt$$

In the following, we may have to do some things without motivation.

Note, however, that this equation is close to the convolution integral:

$$G = \int_{-\infty}^{\infty} F_1(t) F_2(\xi - t)\, dt$$

and under Fourier transformation:

$$\bar{G} = \bar{F_1}\,\bar{F_2}$$

This motivates making the integral equation over into this form. We do this by defining the region of the solution by:

$$\mu(x) = \begin{cases} \mu'(x), & x>0 \\ 0, & x<0 \end{cases}$$

Using this will bring us closer to the definition of the convolution integral, however, the fact is that the integral will not give $\mu(x) = 0,\ x<0$ by inspection. We thus define another function to take care of this:

$$h(x) = \begin{cases} 0, & x>0 \\ h(x), & x<0 \end{cases}$$

We are then able to write, <u>using</u> the above definitions:

$$h(x) + \mu'(x) = \lambda \int_{-\infty}^{\infty} \frac{\beta}{2} e^{-\beta|x-t|} \mu'(t)\, dt$$

where: $$\mu(x) = \begin{cases} \lambda \int_0^{\infty} \frac{\beta}{2} e^{-\beta|x-t|} \mu'(t)\, dt, & x>0 \\ 0, & x<0 \end{cases}$$

$$h(x) = \begin{cases} \lambda \int_0^{\infty} \frac{\beta}{2} e^{-\beta|x-t|} \mu'(t)\, dt, & x<0 \\ 0, & x>0 \end{cases}$$

Now we have an equation which is consistent over the infinite domain.

We define the Fourier transform:

$$\bar{u}(\xi) = \int_{-\infty}^{\infty} e^{-i\xi x} u(x)\, dx \qquad (\text{now } u' \to u)$$

in the hope of obtaining an algebraic equation, and get using the convolution theorem:

$$\bar{h}(\xi) + \bar{u}(\xi) = \lambda \bar{K}(\xi)\, \bar{u}(\xi)$$

$$\bar{K}(\xi) = \int_{-\infty}^{\infty} \frac{\beta}{2} e^{-\beta|x|} e^{-i\xi x}\, dx = \frac{\beta^2}{\beta^2 + \xi^2}$$

$$\bar{h}(\xi) + \bar{u}(\xi) = \frac{\lambda \beta^2}{\beta^2 + \xi^2} \bar{u}(\xi)$$

We now have two "half" unknowns $\bar{h}(\xi)$, $\bar{u}(\xi)$.
Note the form of the Fourier transform of $u(x)$:

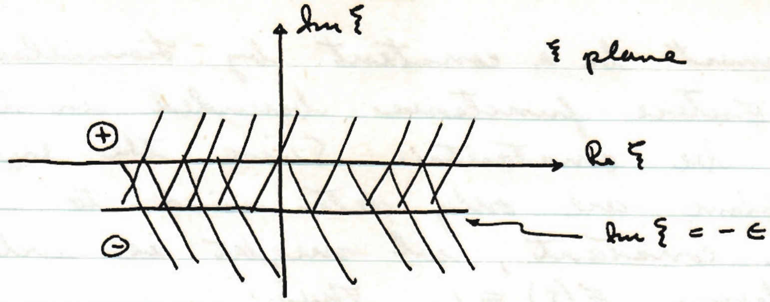
$$\bar{u}(\xi) = \int_{0}^{\infty} u(x)\, e^{-i\xi x}\, dx$$

It is apparent that this integral exists and converges for $\operatorname{Im} \xi < 0$ if we <u>assume</u> $u(x)$ does not approach ∞ any faster than an algebraic rate. Hence $\bar{u}(\xi)$ is analytic in the LHP of ξ.

What about $\bar{h}(\xi)$? It must converge also, but now we do not restrict it as much as $u(x)$ (this is not absolutely necessary, see M-F, 979).

$$\bar{h}(\xi) = \int_{-\infty}^{0} h(x)\, e^{-i\xi x}\, dx \quad \text{exists and converges if}$$

$\operatorname{Im} \xi > -\epsilon$ which means that $h(x)$ must go to infinity at a rate less than $Ae^{+\epsilon x}$, $x < 0$. We see then that $\bar{h}(\xi)$ is analytic in the half-plane above $\operatorname{Im} \xi = -\epsilon$. We hence have a "band of analyticity, $-\epsilon < \operatorname{Im} \xi < 0$, in which both $\bar{u}(\xi)$ and $\bar{h}(\xi)$ are regular. We now denote the two regions of analyticity as $\oplus$ for $\operatorname{Im} \xi > -\epsilon$ and $\ominus$ for $\operatorname{Im} \xi < 0$. We will see that $\epsilon = \beta$.



Therefore:

$$\bar{h}_{\oplus}(\xi) + \bar{\mu}_{\ominus}(\xi) = \frac{\lambda\beta^2}{\beta^2+\xi^2}\,\bar{\mu}_{\ominus}(\xi)$$

or $$\bar{h}_{\oplus} = \underbrace{\frac{\xi^2-(\lambda-1)\beta^2}{\xi^2+\beta^2}}\,\bar{\mu}_{\ominus}$$

It may now be possible to factor this into the $\oplus, \ominus$ regions of analyticity, that is, one factor for $\oplus$, the other $\ominus$. Note $\lambda > 1$ by postulate.

$$\bar{h}_{\oplus} = -\frac{\xi^2-(\lambda-1)\beta^2}{(\xi-i\beta)(\xi+i\beta)}\,\bar{\mu}_{\ominus}$$

Denote as $\oplus$ that factor with neither zeroes nor singularities in the $\oplus$ region assuming $\beta > \epsilon$.

$$-(\xi+i\beta)_{\oplus}\,\bar{h}_{\oplus} = \left[\frac{\xi^2-(\lambda-1)\beta^2}{\xi-i\beta}\right]_{\ominus}\bar{\mu}_{\ominus}$$

In the overlap region the equality holds. What we then have above is the analytic continuation of one region into the other, with which we can extend the region of analyticity over the whole plane to give some <u>entire</u> function. When we find this entire function we have the solution.

$$-(\xi+i\beta)_{\oplus}\,\bar{h}_{\oplus} = \left[\frac{\xi^2-(\lambda-1)\beta^2}{\xi-i\beta}\right]_{\ominus}\bar{\mu}_{\ominus} = E(\xi)$$

Now $\bar{\mu}(\xi) \to 0$ as $\xi \to \infty$ in the LHP since $\mu(x)$ is algebraic and the same happens with $\bar{h}(\xi)$. This must be so since $\mu(x)$ is to be integrable at the origin. From the above equation, then, we see $E(\xi)$ is bounded at $\xi \to \infty$, Hence $E(\xi)$ must be a constant by Liouville's Theorem which says that Entire functions bounded in the whole complex plane must be constants. Since the eigenfunctions of the problem are only determined to within a multiplicative constant, we might as well in this problem choose $E(\xi) = 1$. Thus:

$$\bar{u}(\xi) = \frac{\xi - i\beta}{\xi^2 - (\lambda-1)\beta^2}$$

We define the Fourier inversion and invert:

$$u(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{u}(\xi)\, e^{i\xi x}\, d\xi$$

$$u(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\xi x}(\xi - i\beta)}{\xi^2 - (\lambda-1)\beta^2}\, d\xi$$

Define: $k = \beta\sqrt{\lambda-1}$

$$\therefore\ u(x) = i\left\{\frac{e^{ikx}\left[k - i\beta\right]}{2\underbrace{\sqrt{\lambda-1}\,\beta}_{k}} - \frac{e^{-ikx}\left[-k - i\beta\right]}{2k}\right\}$$

$$= i\left[\frac{e^{ikx} + e^{-ikx}}{2}\right] + i\frac{\beta}{k}\left[\frac{e^{ikx} - e^{-ikx}}{2i}\right]$$

$$\rightarrow \cos kx + \frac{\beta}{k}\sin kx \Rightarrow \sin\left(kx + \tan^{-1}\frac{k}{\beta}\right)$$

which is the same as we had before.

Next we do the Wiener-Hopf equation for a general kernel.

LECTURE 4 : 10-2-61

General Wiener-Hopf method:

Consider:

$$u(x) + f(x) = \lambda \int_0^\infty K(x-t)\, u(t)\, dt$$

Note that λ is a <u>known</u> number in the inhomogeneous case while it is an eigenvalue in the homogeneous case. We hope to use the Fourier transform method by converting the problem into a convolution integral. Hence we define:

$$u(x) = 0 \;,\quad x<0$$
$$f(x) = 0 \;,\quad x<0$$
$$H(x) = 0 \;,\quad x>0$$

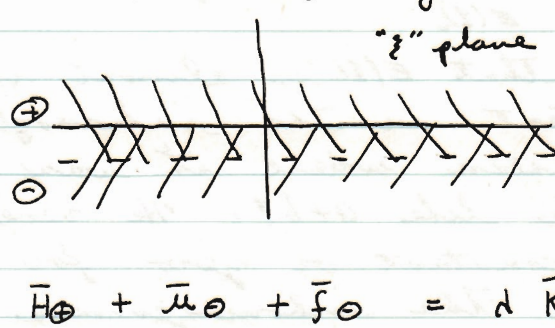
Then:

$$H(x) + f(x) + u(x) = \lambda \int_{-\infty}^{\infty} K(x-t)\, u(t)\, dt$$

We will talk only about kernels that do not diverge more rapidly than exponentially and take them properly normalized. Doing the Fourier transformation:

$$\bar{H}(\xi) + \bar{u}(\xi) + \bar{f}(\xi) = \lambda\, \bar{K}(\xi)\, \bar{u}(\xi)$$

Recall from last lecture the definitions of the ⊕ and ⊖ domains of analyticity:



Hence:

$$\bar{H}_\oplus + \bar{u}_\ominus + \bar{f}_\ominus = \lambda\, \bar{K}\, u_\ominus$$

$$\bar{u}_\ominus (1-\lambda \bar{K}) + \bar{f}_\ominus = -\bar{H}_\oplus$$

The principle problem now is to factor $1-\lambda\bar{K}$ into the two regions of analyticity.

We assume that this can be done and write:

$$1 - \lambda \bar{K} = \bar{G} = \bar{G}_{\oplus} \bar{G}_{\ominus}$$

Each factor must have no zeros or singularities in its region of analyticity and they must overlap and be analytic at every point in the overlap region or else there is no equality in the following equation:

$$\bar{u}_{\ominus} \bar{G}_{\ominus} + \frac{\bar{f}_{\ominus}}{\bar{G}_{\oplus}} = -\frac{\bar{H}_{\oplus}}{\bar{G}_{\oplus}}$$

However, we still have a mixed term because of the inhomogeneous nature. We will show later that this term can be represented as the sum of two properly analytic functions, viz:

$$\frac{\bar{f}_{\ominus}}{\bar{G}_{\oplus}} = \bar{F}_{\ominus} + \bar{F}_{\oplus}$$

Finally:

$$\bar{u}_{\ominus} \bar{G}_{\ominus} + \bar{F}_{\ominus} = -\left(\frac{\bar{H}_{\oplus}}{\bar{G}_{\oplus}} + \bar{F}_{\oplus} \right) = E(\xi)$$

which we know by analytic continuation is equal to an entire function. If we demand certain conditions on the behaviour of $\bar{u}(\xi)$ at the origin we can determine $E(\xi)$.

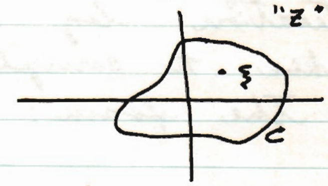
Carrier says that $E(\xi)$ is almost never anything other than a constant in the inhomogeneous case and has only once seen where it was not and then it grew at a rate like $a+b\xi$. He never saw anything but a constant or zero for the homogeneous case. Note that in the inhomogeneous case we must find the constant if $E(\xi)$ is such, compared to the homogeneous case where we can use unity because of the undetermined nature of the amplitude of eigenfunctions. In principle, we have now done the problem.

How do we perform the splitting and factoring required? Consider first the case of the mixed term:

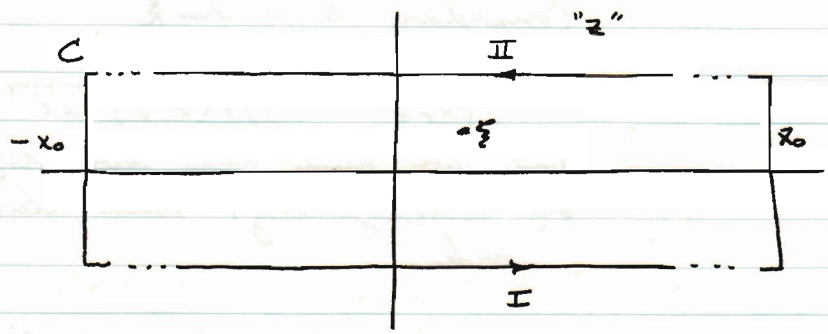
$$J(\xi) = \frac{\bar{f}_\ominus}{\bar{G}_\oplus} = \bar{F}_\ominus + \bar{F}_\oplus$$

Recall the Cauchy integral formula:

$$J(\xi) = \frac{1}{2\pi i} \oint_C \frac{J(z)}{z - \xi}\, dz$$



Deform the contour so that we have:



We hope that $J(\xi) \to 0$ as $x_0 \to \infty$. If it doesn't, subtract off enough so that it does and then add it back on at the end.

Now, if $J(x_0) \to 0$ as $x_0 \to \infty$, we have the sum of two functions:

$$\int_{I} + \int_{II}$$

Since the point ξ is arbitrary in the z plane, we can move it such that:

$\int_I$ gives analyticity in $\oplus \to \bar{F}_\oplus$

$\int_{II}$ gives analyticity in $\ominus \to \bar{F}_\ominus$

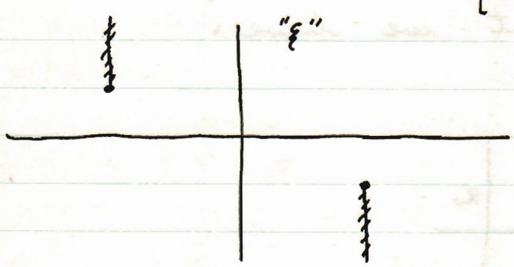
What about $1 + \lambda \bar{K} = \bar{G} = \bar{G}_\oplus \bar{G}_\ominus$?

Take $\ln \bar{G} = \ln \bar{G}_\oplus + \ln \bar{G}_\ominus$ and identify factors by inspection.

Note that integrals involved need not be trivial.

Consider a kernel with poles on the real axis. Then the regions of analyticity overlap except at two points. However, missing two points is just as bad as having no overlap at all. These kernels arise in considering the sound field emanating from a vibrating plate. Physically, we can improve the situation by considering a little damping which makes k complex. The kernel in this problem is of the form:

$$J_0\left[a\sqrt{\xi^2-k^2}\right] H_0^{(1)}\left[a\sqrt{\xi^2-k^2}\right]$$



Consider $\epsilon = \operatorname{Im} k$

then:

$$\sqrt{\xi^2+\epsilon^2} = \sqrt{\xi+i\epsilon}\,\sqrt{\xi-i\epsilon}$$

and we have the two regions of analyticity. This always works.

We will consider an example of a kernel which typifies many physical problems, the Hankel function of imaginary argument:

$$\frac{\beta}{\pi} K_0\left(\beta|x-t|\right)$$

and take for the inhomogeneous term a unit step function: $f(x) = S(x)$.

Consider the two separate problems:

① $u(x)$ not there: non-homogeneous

② $f(x)=0$, homogeneous

$$H(x) + u(x) + S(x) = \lambda\int_{-\infty}^{\infty} \frac{\beta}{\pi} K_0\left(\beta|x-t|\right) u(t)\,dt$$

$$\bar{H} + \bar{u} + \frac{1}{i\xi} = \lambda\frac{\beta}{\sqrt{\beta^2+\xi^2}}\,\bar{u}$$

$$\bar{u}\left[1-\frac{\lambda\beta}{\sqrt{\beta^2+\xi^2}}\right] = -\bar{H} - \frac{1}{i\xi}$$

$\frac{1}{i\xi}$ has analyticity in the $\ominus$ region because of the definition of the Fourier transform, viz:

$$\bar{f}(\xi) = \int_{-\infty}^{\infty} e^{-i\xi x} u(x)\,dx \;; \quad u(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi x} f(\xi)\,d\xi$$

The countour is taken around the UHP and the singularity indented around thru the LHP.
Then:

$$\bar{u}_{\ominus}\left[1 - \frac{\lambda\beta}{\sqrt{\beta^2+k^2}}\right] = -\bar{H}_{\oplus} - \frac{1}{i\xi_{\ominus}}$$

Thus we associate $\frac{1}{i\xi}$ with the UHP to get strip of analyticity.

For the non-homogeneous problem, 1 is not there. We can factor the kernel by inspection:

$$\frac{\lambda\beta}{\sqrt{\beta+i\xi}_{\ominus}}\;\frac{1}{\sqrt{\beta-i\xi}_{\oplus}}$$

We find after some juggling (adding and subtracting $\sqrt{\beta}$ from one term),

$$\bar{u}_{\ominus}\frac{\lambda\beta}{\sqrt{\beta+i\xi}_{\ominus}} = \bar{H}_{\oplus}\sqrt{\beta-i\xi}_{\oplus} + \left[\frac{\sqrt{\beta-i\xi} - \sqrt{\beta}}{i\xi_{\ominus}}\right]_{\oplus} + \frac{\sqrt{\beta}}{i\xi_{\ominus}}$$

$= 0$ (found by arguments on $\bar{u}(\xi)$ at origin)

$$\therefore\quad \bar{u}_{\ominus} = \frac{\sqrt{\beta}}{\lambda\beta}\;\frac{\sqrt{\beta+i\xi}}{i\xi}$$

We get, using the Tables of Campbell and Foster, a error-function type result.

LECTURE 5: 10-4-61

We now consider the homogeneous problem, case (E) of the last lecture:

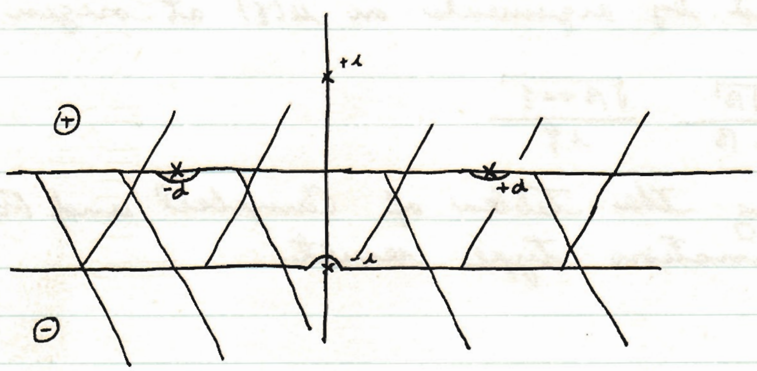
$$H(x) + \lambda\, \mu(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} K_0(|x-t|)\, \mu(t)\, dt$$

after suitable substitutions. $\bar{K}_0 = \frac{1}{\sqrt{1+\xi^2}}$ from before.
Now:

$$\bar{H}_\oplus + \lambda\, \bar{\mu}_\ominus = \frac{\bar{\mu}_\ominus}{\sqrt{1+\xi^2}}$$

or:
$$\left[\frac{\lambda\sqrt{1+\xi^2} - 1}{\sqrt{1+\xi^2}}\right] \bar{\mu}_\ominus = -\bar{H}_\oplus$$

Define as the zeroes of the numerator $d = \frac{\sqrt{1-\lambda^2}}{\lambda}$ which is analogous to k we used before.
We make these zeroes on the real axis associate with the UHP because we want no common zeroes or singularities in our strip of analyticity. However, these could be changed if wrong. We also have two branch points: $\pm i$



Juggle into more convenient form:

$$\frac{\lambda^2(1+\xi^2) - 1}{\sqrt{1+\xi^2}\,(\lambda\sqrt{1+\xi^2} + 1)}$$

The clearly not factorable quantity is $\frac{1}{\lambda\sqrt{1-\xi^2} + 1} = K$

At least not factorable by inspection.

The factorable quantity can be written:

$$\frac{\lambda^2(\xi^2-d^2)}{\sqrt{1+\xi^2}} = \left[\frac{\lambda^2(\xi^2-d^2)}{\sqrt{1+i\xi}}\right]_{\ominus} \frac{1}{\sqrt{1-i\xi}}_{\oplus}$$

Returning, take the logarithm of the not easily factorable term, and then take the derivative:

$$\frac{\partial \ln K}{\partial \xi} = -\left\{\frac{\lambda\xi}{\sqrt{1+\xi^2}\,(1+\lambda\sqrt{1+\xi^2})}\right\}$$

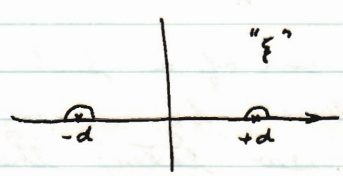
Now call the factors of $(1+\lambda\sqrt{1+\xi^2})$ which are to be found $L_\oplus(\xi)\,L_\ominus(\xi)$. We carry on assuming we have found $L_\oplus(\xi)\,L_\ominus(\xi)$ knowing that they cannot possess any other singularities other than those they had originally. Then:

$$\frac{\bar{u}_\ominus\,\lambda^2(\xi^2-d^2)}{\sqrt{1+i\xi}\,L_\ominus(\xi)} = -\bar{H}_\oplus(\xi)\,L_\oplus(\xi)\sqrt{1-i\xi} = 1$$

We cannot really evaluate the entire function because we don't know $L_\oplus$ or $L_\ominus$ but in the Carrier or applied mathematics spirit, we assume to be a constant as it almost always is, and proceed in search of a solution to be verified later. We now invert:

$$u(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\sqrt{1+i\xi}\,L_\ominus(\xi)}{\xi^2-d^2}\,e^{i\xi x}\,d\xi$$

Evaluating by choice of the proper contour:



we obtain:

$$u(x) = i\left[\frac{\sqrt{1+id}}{2d}\,L_\ominus(d)\,e^{idx} - \frac{\sqrt{1-id}}{2d}\,L_\ominus(-d)\,e^{-idx}\right]$$

We anticipate the $L_\ominus(d)$ to be symmetric in d and $-d$. We see now, however, that to get a solution, we only need to know the value of $L_\ominus$ at d.

Multiply and divide $\frac{\partial \ln K}{\partial \xi}$ by $1 - \lambda\sqrt{1+\xi^2}$ and get:

$$\frac{\lambda \xi (1 - \lambda\sqrt{1+\xi^2})}{\sqrt{1+\xi^2}\, d^2 (\xi^2 - d^2)} = -\frac{\xi}{\xi^2 - d^2} + \frac{\xi}{\lambda\sqrt{1+\xi^2}(\xi^2-d^2)}$$

We can see that the apparent poles at $\pm d$ cancel when we plug $\xi \to d$ in the equation. However, we cannot forget the branch points. We want to eventually get this into a sum of two regions of analyticity, $P_\oplus$ and $P_\ominus$. Recall

$$\frac{\partial \ln K}{\partial \xi} = -\frac{\partial}{\partial \xi} \ln L_\ominus L_\oplus = -\frac{\partial}{\partial \xi}\left[\ln L_\ominus + \ln L_\oplus\right]$$

$$= P_\oplus + P_\ominus$$

Then:

$$L_\ominus(\xi) = e^{-\int_0^\xi P_\ominus(\alpha)\, d\alpha} \quad ; \quad L_\oplus(\xi) = e^{-\int_0^\xi P_\oplus(\alpha)\, d\alpha}$$

α being a dummy variable. Now we would like to $P_\ominus + P_\oplus$ of the form:

$$\frac{1}{2}\left[\frac{1+f}{\sqrt{}} + \frac{1-f}{\sqrt{}}\right]$$

where f must cancel the respective branch points that would destroy the respective regions of analyticity. We also have to split up $\frac{1}{\xi^2 - d^2}$ by partial fractions.

We will have to pick f out of a hat. Carrier originally did this problem by using the Cauchy Integral Formula and saw later that he could have chosen the function f using the criteria that it cancel the respective branch points. We now write this down without further discussion.

$$-\frac{1}{2d}\left\{\frac{\pi/2+i\ln(\alpha+\sqrt{\alpha^2+1})}{\pi\sqrt{\alpha^2+1}}-\frac{\pi/2+i\ln(d+\frac{1}{d})}{\pi/d}\right\}_{\oplus}\frac{1}{\alpha-d}$$

$$-\frac{1}{2d}\left\{\frac{\pi/2+i\ln(\alpha+\sqrt{\alpha^2+1})}{\pi\sqrt{\alpha^2+1}}-\frac{\pi/2+i\ln(d+1/d)}{\pi/d}\right\}_{\oplus}\frac{1}{\alpha+d}$$

$P_{\oplus}(\alpha)$

$$-\frac{1}{2d}\left\{\frac{\pi/2-i\ln(\alpha+\sqrt{\alpha^2+1})}{\pi\sqrt{\alpha^2+1}}-\frac{\pi/2-i\ln(d+1/d)}{\pi/d}\right\}_{\ominus}\frac{1}{\alpha-d}$$

$$-\frac{1}{2d}\left\{\frac{\pi/2-i\ln(\alpha+\sqrt{\alpha^2+1})}{\pi\sqrt{\alpha^2+1}}-\frac{\pi/2-i\ln(d+1/d)}{\pi/d}\right\}_{\ominus}\frac{1}{\alpha+d}$$

$P_{\ominus}(\alpha)$

where we have used the dummy variable α for ξ. We see that the analyticity requirements are satisfied when we let $\alpha \to \pm i$ respective branch points dropping out. Also analyticity is satisfied when $\alpha \to \pm d$.

Now the greatest contribution to $P_{\ominus}(\alpha)$ comes from around d so we can expand $P_{\ominus}(\alpha)$ in power series. Evaluation then gives a solution of the form:

$$\operatorname{Im} e^{ikx + ik/\pi + \frac{1}{2}\tan^{-1}k} = \sin k\left(\beta x + \frac{\pi+2}{2\pi}\right)$$

for small k. k now is d ($k \equiv d$). The extrapolated end point is about $5/6\,k$. What about the structure at $k \sim 0$? This is an important question in stellar radiation. Now we must calculate the total integral:

$$\int_0^{\xi} P_{\ominus}(\alpha)\,d\alpha$$

Use $\int_0^{\infty} - \int_{\xi}^{\infty}$: Watch out for singularities at ∞. Bypass these by not splitting $\int$. We get something like: not splitting gives 3 parts

$$\int \frac{v\,du}{\sinh u - d}$$

Do for problem.

LECTURE 6 : 10-6-61

Lecturer: Wu: Theory of Linear Integral Equations

$$f(x) - \lambda \int_a^b K(x,y)\, f(y)\, dy = g(x) \qquad (1)$$

$K(x,y)$ and $g(x)$ are given quantities.

Postulates: Existence Requirements:

$$\int |f(x)|^2 dx < \infty \quad ; \quad \int |g(x)|^2 dx < \infty$$

$$\iint |K(x,y)|^2 dx\, dy < \infty \quad ; \quad K(x,y) \not\equiv 0$$

Theorems:

1. If $K(x,y) = K^*(y,x)$ then there is a λ and $f \not\equiv 0$ such that the homogeneous case exists:

$$f(x) - \lambda \int_a^b K(x,y)\, f(y)\, dy = 0$$

Even if this requirement is not satisfied, a homogeneous case still usually exists.

2. Either (1) has a solution for each g or:

$$\bar{f}(x) - \lambda \int_a^b K^*(y,x)\, \bar{f}(y)\, dy = 0$$ (- indicates another solution different than f)

has a non-trivial solution.

We will prove the above theorems. In doing so, we will take all statements outside the domain of linear integral equations as being true without proof.
Example: If a_n (Σ implied) $< A$, then there exists A_0 such that $a_n \leq B \Rightarrow B \geq A_0$, A_0 being the least upper bound on a_n. For example, if:

$$a_n = -2^{-\frac{1}{n}} \quad , \quad n = 0, 1, 2, \dots$$

Then obviously $A_0 = 0$ since this is the limit in $n = 0$.

We shall consider the integral equation in the limit of the sum:

$$f(x_j) - \lambda \lim_{\Delta \to \infty} \Delta \sum_i K(x_j, y_i) f(y_i) = g(x_j), \quad K \neq 0$$

We can now use matrix methods for operations with linear integral equations.

Review of Matrices:

$$M = \{M_{ij}\}, \quad x = \{x_i\} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}; \quad x_i \text{ may be complex.}$$

$$M^\dagger = \{M_{ji}^*\}; \quad M^* = \{M_{ij}^*\}; \quad M^T = \{M_{ji}\}$$

$$x^\dagger = (x_1^* \cdots x_n^*); \quad x^T = (x_1 \cdots x_n); \quad x^* = \{x_i^*\}$$

$$(M_1 M_2)^\dagger = M_2^\dagger M_1^\dagger$$

$$(M_1 M_2)^* = M_1^* M_2^*$$

$$(M_1 M_2)^T = M_2^T M_1^T$$

Scalar Product: $(x_1, x_2) = x_1^\dagger x_2$

Length of Vector $\|x\| = \sqrt{(x, x)}$

Hermitean Property: $M = M^\dagger$

Unitary Property: $\left. \begin{matrix} M M^\dagger = 1 \\ M^\dagger M = 1 \end{matrix} \right\} \quad 1 = \{\delta_{ij}\}$

The inverse of M, M^{-1}, exists if $\det M \neq 0$. Then the solution of $Mx = y$ is $x = M^{-1} y$.

Consider the equation $Mx = y$ and then the adjoint problem $M^\dagger x' = y'$ or:

$$x'^\dagger M = y'^\dagger$$

The reason for looking at the adjoint problem is that if x_i' satisfies $M^+ x_i' = 0$; $x_i'^+ M = 0$, and we have $Mx = y$, we can form:

$$\underbrace{x_i'^+ M}_{0} x = x_i'^+ y$$

then $x_i'^+ y = 0$ or $(x_i', y) = 0$ or x_i', y are orthogonal.

Statements about Matrix Equations:

1. Given M, either $Mx = y$ has a unique solution for each y or $Mx = 0$ has a number ρ of linearly independent non-trivial solutions. In the latter case, $M^+ x' = 0$ has the same number of solutions x_i', and $Mx = y$ has a solution if and only if $(x_i', y) = 0$

2. Canonical Form of Matrices:
Given M, there is an A such that:

$$A^{-1} M A = \left(\begin{array}{ccc|ccc} \mu_1 & & 0 & & & \\ & \mu_2 & & & 0 & \\ 0 & & \mu_3 & & & \\ \hline & & & \ddots & & 0 \\ & 0 & & & \ddots & \\ & & & 0 & & \ddots \end{array} \right)$$

Note that if M is not Hermitean, A may not diagonalize it but give some more general matrix.

3. If M is Hermitean, then there is U, $UU^+ = 1$ such that $U^+ M U$ is diagonal.

We now deviate from the usual matrix notation in that the μ's are not eigenvalues but related to inverse because of the nature of the construction of the integral equation.

$$x - \lambda M x = 0 \quad \text{has eigenvalues} \quad \lambda_i = \frac{1}{\mu_i}$$

We must be careful with this definition. For example: Consider:

$$M = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Then for the integral equation we have:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \lambda \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

gives:

$$\left.\begin{aligned} x_1 - \lambda x_2 &= 0 \\ x_2 &= 0 \end{aligned}\right\} \quad x_1 = x_2 = 0$$

or a trivial solution, therefore M has no eigenvalues as far as integral equations are concerned. Actually any eigenvalue at all satisfies the equation, so there are no eigenvalues particularly for this problem. In that sense, there are no eigenvalues. A more general M where there are no eigenvalues is where we have the form:

$$\begin{pmatrix} 0 & \times & & & & 0 \\ & 0 & \times & & & \\ & & 0 & \times & & \\ & & & 0 & \times & \\ & & & & 0 & \times \\ 0 & & & & & 0 \end{pmatrix}$$

A Hermitean matrix always has eigenvalues, hence the statement of Theorem 1 about Hermitean kernels.

Definition of Trace: $\quad \operatorname{Tr} M = \sum_i M_{ii}$

$$\operatorname{Tr} MN = \operatorname{Tr}\left\{ \sum_k M_{ik} N_{kj} \right\} = \sum_{ik} M_{ik} N_{ki}$$

$$= \sum_{ik} N_{ki} M_{ik} = \operatorname{Tr} NM$$

Consider then:

$\operatorname{Tr} A^{-1} M A = \operatorname{Tr} M A A^{-1} = \operatorname{Tr} M$ so a similitude transformation leaves the trace invariant.

Therefore:

$$\operatorname{Tr} M = \sum_i \mu_i = \sum_i \frac{1}{\lambda_i}$$

and in general:

$$\operatorname{Tr} M^n = \sum_i \frac{1}{\lambda_i^n}$$

LECTURE 7: 10-9-61

Lecturer: Wu : Theory of Integral Equations

$$f(x) - \lambda \int_a^b K(x,y)\, f(y)\, dy = g(x)$$

We will adopt the notation of matrix algebra to integral equations. Correspondence is as follows:

$$M \leftrightarrow K$$

$$M_{xy} \leftrightarrow K(x,y)$$

$$M^{+} \leftrightarrow K^{+}$$

$$M^{+}_{yx} \leftrightarrow K^{*}(y,x)$$

$$x \leftrightarrow f$$

$$x_1^{+} x_2 = (x_1, x_2) \leftrightarrow (f_1, f_2) = \int f_1^{+}(x)\, f_2(x)\, dx$$

$$\sqrt{x^{+}x} = \sqrt{(x,x)} = \|x\| \leftrightarrow \|f\| = \sqrt{(f,f)} = \sqrt{\int |f(x)|^2 dx}$$

always taking + value of $\sqrt{\ }$.

An $\int$ sign without limits always refers to the interval of the integral equation. All integrals are taken in the Lebesgue sense. All things of measure zero are considered identical.

Further Theorems and Statements about Complex Numbers:

1. A Cauchy series has a limit. If for $\{a_m\}$, $|a_m - a_n| \to 0$, then there is an a such that $|a_m - a| \to 0$.
2. If there is $\{f_n\}$ such that $\|f_n\|$ exists and $\|f_m - f_n\| \to 0$, then there is an f such that $\|f_m - f\| \to 0$ (Riesz-Fisher theorem, not true for Riemann integrals).
3. Given $\{a_n\}$, $|a_n| < 1$, then there is a subsequence $\{a_{n_i}\}$ such that $|a_{n_i} - a_{n_j}| \to 0$.

In the above, $\{\}$ means sequence of numbers.

Statements of Inequalities:

1. $|(f_1, f_2)| \leq \|f_1\|\,\|f_2\|$ (Schwartz Inequality)

Proof: Consider: $\| f_1 - \lambda e^{i\varphi} f_2 \| \geq 0$

or: $(f_1 - \lambda e^{i\varphi} f_2 , f_1 - \lambda e^{i\varphi} f_2) \geq 0$

or: $(f_1, f_1) + \lambda^2 (f_2, f_2) - 2 \,\mathrm{Re}\, \lambda e^{i\varphi} (f_1, f_2) \geq 0$

Choose φ to make the last quantity real, then, knowing λ real:

$$\| f_1 \|^2 - 2\lambda |(f_1, f_2)| + \lambda^2 \| f_2 \|^2 \geq 0$$

Hence the discriminant must be:

$$|(f_1, f_2)|^2 - \| f_1 \|^2 \| f_2 \|^2 \leq 0 \qquad \text{QED}$$

A consequence of this inequality is:

$$\| f_1 + f_2 \| = \sqrt{(f_1 + f_2 , f_1 + f_2)}$$

$$= \sqrt{\| f_1 \|^2 + \| f_2 \|^2 + 2 \underbrace{\mathrm{Re}\, (f_1, f_2)}_{\leq |(f_1, f_2)| \leq \| f_1 \| \| f_2 \|}}$$

$$\leq \| f_1 \| + \| f_2 \|$$

2. $\| f \|^2 \geq |(f, \varphi_i)|^2$

where φ_i is one of an orthonormal set $(\varphi_i , \varphi_j) = \delta_{ij}$

Why are matrices relevant? For degenerate kernels, integral equations go over directly into matrix equations. Definition of a degenerate kernel:

$$K(x, y) = \sum_{\substack{\text{finite} \\ \text{sum}}} \psi_i^*(y)\, \varphi_i(x)$$

as long as ψ, φ exist and are quadratically integrable. It is always possible to make an orthonormal set out of ψ_i, φ_i.

Reduction of the kernel to a matrix:

Make an orthonormal set out of ψ_i, φ_i. Call this set α_i, $i = 1 \ldots m$, $m \leq 2n$ where n is the order of ψ_i, φ_i. Then we can write:

$$\psi_i = \sum a_{ij}\,\alpha_j$$
$$\varphi_i = \sum b_{ij}\,\alpha_j$$

and: $$K(x,y) = \sum_{jk} \underbrace{\left[\sum_i a^*_{ij}\, b_{ik}\right]}_{C_{jk}} \alpha^*_j(y)\,\alpha_k(x)$$

$$= \sum_{j,k=1}^{m} C_{jk}\,\alpha^*_j(y)\,\alpha_k(x)$$

To solve $f(x) - \lambda\int K(x,y) f(y)\,dy = g(x)$, expand each term in terms of the α's:

$$g(x) = \sum_{i=1}^{m} G_i\,\alpha_i(x) + g_1(x) \quad ; \quad (\alpha_i, g_1) = 0$$

$$f(x) = \sum_{i=1}^{m} F_i\,\alpha_i(x) + f_1(x) \quad ; \quad (\alpha_i, f_1) = 0$$

Then:

$$\int K(x,y) f(y)\,dy = \sum_i \sum_{jk} C_{jk}\,F_i\,\delta_{ij}\,\alpha_k(x)$$

$$= \sum_i \left[\sum_k F_k\,C_{ki}\right]\alpha_i = \sum_i (FC)_i\,\alpha_i$$

Then the integral equation becomes, in the reduced form:

$$\sum_{i=1}^{m} (F - \lambda FC)_i\,\alpha_i + f_1 = \sum_{i=1}^{m} G_i\,\alpha_i + g_1$$

or equating coefficients:

$$f_1 = g_1$$
$$F(1 - \lambda C) = G$$

An integral equation with a degenerate kernel can always be reduced to matrix form. If $K = K^+$ and K is degenerate, there is at least one eigenvalue.

From now on, we always assume $K = K^+$. However, this is true only in about 10% of the physical problems. $K = K^+$ assures eigenvalues but is not a necessary condition.

In general, if:

$$\|K\| = \sqrt{\int |K(x,y)|^2 dx\, dy}$$ exists, then we can find a sequence of K_i, $\{K_i\}$, all degenerate, such that:

$$\|K - K_i\| \to 0.$$

There may be more than one non-trivial solution of the homogeneous equation:

$$f(x) - \lambda \int K(x,y)\, f(y)\, dy = 0 \quad , \quad \lambda \neq 0$$

Suppose there are N such solutions:
$f_1(x), f_2(x), \ldots\ldots$; all orthonormal. Then:

$$\|K\|^2 = \int dx\, dy\, |K(x,y)|^2 \geq \int dx \sum_i \Big| \underbrace{\int K(x,y)\, f_i(y)\, dy}_{\frac{f_i(x)}{\lambda}} \Big|^2 = \frac{N}{\lambda^2}$$

using inequality 2. Therefore:

$$N \leq \lambda^2 \|K\|^2$$

So far we have used only the existence of K. We now sketch some developments when we take $K = K^+$.
Consider:

$$I(\varphi) = (\varphi, \underbrace{K\varphi}_{\int K(x,y)\varphi(y)\, dy}) = \int \varphi^*(x)\, K(x,y)\, \varphi(y)\, dx\, dy$$

$$I^*(\varphi) = \int \varphi(x) \underbrace{K^*(x,y)}_{K(y\,x)} \varphi^*(y)\, dx\, dy = I(\varphi)$$

Then $I(\varphi)$ is real and there is a lowest upper bound to it.

LECTURE 8: 10-11-61

Lecturer: Wu

Recapitulation:

$$K(xy) = \sum_{jn} C_{jn}\, \varphi_n(x)\, \varphi_j^*(y)$$

If $\|K\|^2 = \int |K(xy)|^2\, dx\, dy$ exists then there is a set of degenerate kernels $\{K_i\}$ such that in the limit: $\|K - K_i\| \to 0$. Further, if $K = K^+$, then there is a set $\{K_i\}$ such that:

$$K_i'(xy) = \tfrac{1}{2}\left[K_i(xy) + K_i^*(yx) \right]$$

Consider for a moment: suppose $Kf = \int K(xy)\, f(y)\, dy$ with:

$$\|Kf\|^2 = \int dx \left| \int K(xy)\, f(y)\, dy \right|^2$$

$$\leq \int dx \left[\int |K(xy)|^2 dy \right]\left[\int |f(y)|^2 dy \right] = \|K\|^2\, \|f\|^2$$

by the Schwartz inequality. Then:

$$\|Kf\| \leq \|K\|\,\|f\|$$

Continuing from the last lecture with $K = K^+$:

$$I(\varphi) = (\varphi, K\varphi)$$

$$|I(\varphi)| \leq \|\varphi\|\,\|K\varphi\| \leq \|K\|\,\|\varphi\|^2$$

If $\|\varphi\| = 1$, then $|I(\varphi)| \leq \|K\|$

Suppose the lowest upper bound on $I(\varphi)$ is A with A' the greatest lower bound, and choose $A \neq 0$.

Problem: If $K \neq 0$, show that $I(\varphi) \neq 0$.

The Theorem we want now to prove is that there is a φ_1 such that:

$$\|\varphi_1\| = 1 \quad , \quad I(\varphi_1) = A \quad ; \quad A^{-1} K \varphi_1 = \varphi_1$$

Proof: Consider K and assume it exists. At this point $K = K^+$ is not needed.

Definition: The linear operator K is completely continuous if $\|f_i\| \le 1$, Kf_i containing a convergent subsequence. If $\{K_i\}$ is completely continuous and $\|K - K_i\| \to 0$, then K is completely continuous.

Proof: pick f_i ; $\|f_i\| \le 1$ that is a convergent subsequence. Now form $K_1 f_i$ which say forms a convergent subsequence of some set f_i^1, Then we say $K_1 f_i^1$ is convergent subsequence of some other set f_i^2 and so on.

$$f_i^1 = f_{\alpha_1(i)} \quad , \quad \|f_i^1\| \le 1$$

Form $K_2 f_i^1$ giving $f_i^2 = f'_{\alpha_2(i)}$, $\|f_i^2\| \le 1$, $K_2 f_i^2$ being a convergent subsequence. We can repeat this for each K_i and form complete subsequences:

$$\begin{array}{cccc} f_1 & f_2 & f_3 & \cdots \\ f_1^1 & f_2^1 & f_3^1 & \cdots \\ f_1^2 & f_2^2 & f_3^2 & \cdots \\ \vdots & \vdots & \vdots & \end{array}$$

Each is a subsequence of a subsequence.

Now choose the subsequence formed by the "diagonal":

$$h_i = f_i^{i-1}$$

For the sequence $\{h_i\}$, if $i > n$ $\{h_i\}$ is a subsequence of $\{f_i^n\}$.

Now $K_n f_i^n$ converges, therefore $K_n h_i$ converges. The purpose of using the "diagonal" was to find a convergent subsequence independent of n. Now form for all n:

$$\|Kh_a - Kh_b\| \leq \|Kh_a - K_n h_a\| + \|K_n h_a - K_n h_b\| + \|K_n h_b - K h_b\|$$

$$\|Kh_a - K_n h_a\| = \|(K-K_n) h_a\| \leq \|K - K_n\| \to 0$$

$$\|K_n h_b - K h_b\| = \|(K_n - K) h_b\| \leq \|K - K_n\| \to 0$$

$$\|K_n h_a - K_n h_b\| = \|K_n (h_a - h_b)\| \leq \|K_n\| \|h_a - h_b\| \to 0$$

Thus $K h_i$ converges.

> Problem: If K_1, K_2 are completely continuous, show that $K_1 + K_2$ is.

If K is degenerate, then K is completely continuous:

$$K(xy) = \varphi_x(x)\, \varphi_y(y)$$

Take a sequence $\{f_n\}$: $\|f_n\| \leq 1$; $K f_n = \varphi_x(x) \underbrace{(\varphi_y, f_n)}_{\text{number} = a_n}$

Then $|a_n| \leq \|\varphi_y\| \|f_n\| \leq 1$. Suppose b_n is a subsequence of a_i, then $K f_{g(n)}$ converges.

Now we use $K = K^+$: Approximate K by K_n degenerate kernels. Define:

$$I_n(\varphi) = (\varphi, K_n \varphi)$$

We can verify $A_n \to A$ in limit. Now stipulate $K_n \not\equiv 0$, $K_n = K_n^+$. Choose $A_n > 0$, then $A_n^{-1} \to A^{-1}$. Since the K_n are degenerate, they are equivalent to matrices and since $K_n = K_n^+$ we can write:

$$K_n \varphi_n = \underset{\substack{\downarrow \\ A_n}}{\lambda_n} \varphi_n$$

$\|\varphi_n\| = 1$. $K \varphi_n'$ converges; $K \varphi_n' \to g$ or $K \varphi_{\alpha(n)} \to g$

now: $(K - K_{\alpha(n)})\,\varphi_{\alpha(n)} \to 0$ in limit, $K_{\alpha(n)}\varphi_{\alpha(n)} \to g$,
$A_{\alpha(n)}\varphi_{\alpha(n)} \to g$, $A\varphi_{\alpha(n)} \to g$, $\varphi_{\alpha(n)} \to \frac{g}{A}$.

Then: $K\frac{g}{A} = g$ and $A^{-1}Kg = g$, $\|g\| = A$ so we must renormalize to say Φ. Then:

$$A^{-1}K\Phi = \Phi, \quad I(\Phi) = (\Phi, K\Phi) = (\Phi, A\Phi) = A$$

LECTURE 9: 10-13-61

If $K = K^{+} \not\equiv 0$, $I(\varphi) = (\varphi, K\varphi)$, A = lower upper bound $I(\varphi) > 0$, $\|\varphi\| \le 1$, then φ_1 such that $I(\varphi_1) = A$, $\|\varphi_1\| = 1$, and $A^{-1}K\varphi_1 = \varphi_1$, all from last time.

Statements:

1. $I(\varphi) = (\varphi, K\varphi)$: $\delta I(\varphi) = I(\varphi + \delta\varphi) - I(\varphi)$

$$= \underbrace{(\delta\varphi, K\varphi)}_{(\varphi,\delta\varphi)=0} + \underbrace{(\varphi, K\delta\varphi)}_{(K\varphi,\delta\varphi) = (\delta\varphi^{*}, (K\varphi)^{*})} \quad ; \|\varphi\| = 1 \; ; \|\varphi + \delta\varphi\| = 1$$

Then $K\varphi = C\varphi$ and $A = (\varphi, K\varphi) = C(\varphi, \varphi) = C$

so $K\phi = A\phi$ and $A^{-1}K\phi = \phi$

2. Expansion Theorem:

Plancherel's Theorem (Fourier transforms)

If $f(x)$ exists, $\|f\|$ exists, from $-\infty$ to ∞ then we can define the Fourier transform $F(k)$ with the following properties:

$$\lim_{B\to\infty} \int_{-\infty}^{\infty} dk \left| F(k) - \frac{1}{\sqrt{2\pi}} \int_{-B}^{B} f(x)\, e^{-ikx}\, dx \right|^2 = 0$$

This is often written:

$$F(k) = \underset{B\to\infty}{\mathrm{l.i.m.}} \frac{1}{\sqrt{2\pi}} \int_{-B}^{B} f(x)\, e^{-ikx}\, dx$$

which says that in the mean limit, $\|F - F_B\| \to 0$.
The result we are aiming for is of the form:

$$\sum \frac{\varphi\varphi^{*}}{\lambda} \to K$$

Consider: φ_1, λ_1 such that $\lambda_1 K \varphi_1 = \varphi_1$, $\|\varphi_1\| = 1$ and:

$$\lambda_1 = \begin{cases} A^{-1} & \text{if } A \geq -A' \\ A'^{-1} & \text{if } A' < -A \end{cases}$$

Write: $K^{(1)}(xy) = K(xy) - \dfrac{\varphi_1(x)\,\varphi_1^*(y)}{\lambda_1}$

Then: $K^{(1)}\varphi_1 = K\varphi_1 - \dfrac{1}{\lambda_1}\varphi_1 = 0$

Consider $K^{(1)}$ operating on a general φ:

$$I^{(1)}(\varphi) = (\varphi, K^{(1)}\varphi)$$

If $\varphi = \alpha\varphi_1 + \varphi_1'$, $(\varphi_1, \varphi_1') = 0$, then $(\varphi_1, K^{(1)}\varphi_1') = (K^{(1)}\varphi_1, \varphi_1') = 0$

Therefore: $I^{(1)}(\varphi) = I^{(1)}(\varphi_1')$

$$\downarrow$$

if $A^{(1)} \leq A$; $A^{(1)\prime} \geq A'$

If we take φ_2, λ_2, $\|\varphi_2\| = 1$, $(\varphi_2, \varphi_1) = 0$, $|\lambda_2| \geq |\lambda_1|$ results, and we can get the sequence: $\{\varphi_\alpha, \lambda_i\}$,

$$0 < |\lambda_1| \leq |\lambda_2| \leq |\lambda_3| \leq \dots$$

Consider for a moment the computation of $\|K\|^2$ in terms of $K^{(1)}$:

$$\|K\|^2 = \iint dx\,dy\,|K(xy)|^2 = \iint dx\,dy \left(K^{(1)}(xy) + \frac{\varphi_1(x)\,\varphi_1^*(y)}{\lambda_1}\right)$$

$$\cdot \left(K^{(1)*}(yx) + \frac{\varphi_1^*(x)\,\varphi_1(y)}{\lambda_1}\right) = \iint dx\,dy\,|K^{(1)}(xy)|^2 + \frac{1}{\lambda_1^2}\iint dx\,dy\,|\varphi_1(x)\,\varphi_1^*(y)|^2$$

$$= \|K^{(1)}\|^2 + \frac{1}{\lambda^2}$$

Upon repetition: $\|K\|^2 = \|K^{(n)}\|^2 + \displaystyle\sum_{\alpha=1}^{n} \frac{1}{\lambda_\alpha^2}$

Then we see that $\displaystyle\sum_{\alpha=1}^{n} \frac{1}{\lambda_\alpha^2}$ converges in limit.

Returning, form: $K^{(m)}(xy) - K^{(n)}(xy) = \sum_{\lambda=n+1}^{m} \frac{\varphi_\lambda(x)\,\varphi_\lambda^*(y)}{\lambda_\lambda}$

with $m > n$. Then:

$$\| K^{(m)}(xy) - K^{(n)}(xy) \|^2 = \sum_{\lambda=n+1}^{m} \frac{1}{\lambda_\lambda^2}$$

Now we define the partial sum: $S^{(m)} = \sum_{\lambda=1}^{m} \frac{1}{\lambda_\lambda^2}$
Then:

$$\sum_{\lambda=n+1}^{m} \frac{1}{\lambda_\lambda^2} = | S^{(m)} - S^{(n)} |$$

Then: $\| K^{(m)} - K^{(n)} \| \to 0$ and $K^{(m)} \to K'$ in some limit.
We want to show: $K' = 0$. Note: $|(\varphi_1, K^{(m)}\varphi)| \leq \frac{1}{\lambda_{m+1}}$.
Then in the limit:

$$\left. \begin{matrix} (\varphi, K'\varphi) = 0 \\ K' = 0 \end{matrix} \right\} \text{ Also, } \|K\|^2 = \sum_{\lambda=1}^{\infty} \frac{1}{\lambda_\lambda^2}$$

so: $K - \sum \frac{\varphi\varphi^*}{\lambda} \to 0$ as $n \to \infty$, or:

$$\lim_{n\to\infty} \iint dx\,dy \left| K(xy) - \sum_{\lambda=1}^{n} \frac{\varphi_\lambda(x)\,\varphi_\lambda^*(y)}{\lambda_\lambda} \right|^2 = 0$$

so the kernel can be expanded in eigenfunctions.

3. Results: $K^2 \leftrightarrow \iint dz\; K(xz)\,K(zy) = K^2(xy)$
$\|K^2\| \leq \|K\|^2$

Now: $\operatorname{Tr} K^2 = \int dx\; K^2(xx) = \int dx\,dz\; K(xz)\underbrace{K(zx)}_{K^*(xz)} = \|K\|^2$

Then:

$$\operatorname{Tr} K^2 = \sum_{\lambda=1}^{\infty} \frac{1}{\lambda_\lambda^2}$$

Recall from matrix theory: $M = M^\dagger$, then $\operatorname{Tr} M^2 = \sum_\lambda \frac{1}{\lambda_\lambda^2}$

4. If we can write: $\varphi = \sum_{\lambda=1}^{\infty} a_\lambda \varphi_\lambda + \varphi'$, $(\varphi_\lambda, \varphi') = 0$
Then:

$$K\varphi = \sum_{\lambda=1}^{\infty} \frac{a_\lambda}{\lambda_\lambda}\,\varphi_\lambda$$

or: $$\lim_{n\to\infty} \int dx \left| \int dy\, K(xy)\, \varphi(y) - \sum_{i=1}^{n} \frac{a_i}{\lambda_i} \varphi_i(x) \right|^2 = 0$$

Mercer's Theorem:

Assume: $K(xy)$ is continuous in both variables and of λ_i, all except a finite number are positive.

Then:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\varphi_i(x)\, \varphi_i^*(y)}{\lambda_i} = K(xy)$$ <u>pointwise</u> and not just in mean.

<u>LECTURE 10 : 10-16-61</u>

We now discuss non-Hermitean kernels and the non-homogeneous integral equation:

$$f(x) - \lambda \int K(xy)\, f(y)\, dy = g(x) \quad ; \quad K(xy) \neq K^*(yx)$$

and $\|f\|$, $\|g\|$ exist.

$$\|K\|^2 = \int |K(xy)|^2 dx\, dy = \int |K^*(yx)|^2 dx\, dy = \|K^+\|^2$$

<u>Statements</u>:

1. K, K^+ are completely continuous.

2. Take for the general case:

$$M = \begin{pmatrix} \mu & 1 & 0 & 0 \\ 0 & \mu & 1 & 0 \\ 0 & 0 & \mu & 1 \\ 0 & 0 & 0 & \mu \end{pmatrix} \quad ; \quad \mu \neq 0$$

Define $N = 1 - \frac{1}{\mu} M$ and $Nx = 0$ as a matrix equation.

$$Nx = \begin{pmatrix} 0 & -\frac{1}{\mu} & 0 & 0 \\ 0 & 0 & -\frac{1}{\mu} & 0 \\ 0 & 0 & 0 & -\frac{1}{\mu} \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \quad ; \quad \text{Then: } x_2 = x_3 = x_4 = 0$$

Then $x = \begin{pmatrix} x_1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ or only one eigenvector. We see this result tells us nothing about the dimensionality of the original matrix.

To find the dimensionality, form:

$$N^2 x = \begin{pmatrix} 0 & 0 & \frac{1}{\mu^2} & 0 \\ 0 & 0 & 0 & \frac{1}{\mu^2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \quad \text{gives} \quad x_3 = x_4 = 0$$

Continuing this, we find:

$$\begin{array}{ccccccccccc} N^0 = 1 & & N & & N^2 & & N^3 & & N^4 & \cdots & \text{same hereafter} \\ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & & \begin{pmatrix} x_1 \\ 0 \\ 0 \\ 0 \end{pmatrix} & & \begin{pmatrix} x_1 \\ x_2 \\ 0 \\ 0 \end{pmatrix} & & \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ 0 \end{pmatrix} & & \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} & & \\ \eta_0 & < & \eta_1 & < & \eta_2 & < & \eta_3 & < & \eta_4 & = & \eta_5 \; \cdots \end{array}$$

η is the notation for a space. After the 4th term we have a <u>multiplicity</u> of subspaces. These spaces do not have to increase their dimension by one each time necessarily.

We now return to a consideration of integral equations. Take K with λ as a possible eigenvalue. Define the operator:

$$T = 1 - \lambda K$$

along with the spaces given by:

$$\begin{array}{ccccccccccc} T^0 = 1 & & T & & T^2 & & T^3 & & T^4 & & \cdots \\ \eta_0 = \{0\} & \leq & \eta_1 & \leq & \eta_2 & \leq & \eta_3 & \leq & \eta_4 & \leq & \cdots \end{array}$$

Let us examine $T^2 = 1 - 2\lambda K + \lambda^2 K^2$. We define:

$T^n = 1 - K_n \quad ; \quad \|K_n\|$ exists.

These all form finite dimension spaces.

To show this, we must make some statements:

1. If $\mathcal{N}_n \neq \mathcal{N}_{n+1}$, then $\mathcal{N}_i \neq \mathcal{N}_{i+1}$ for $i \leq n$

Therefore: $\left.\begin{array}{lcl} T^{n+1} f = 0 & ; & T^{i+1} f' = 0 \\ T^{n} f \neq 0 & ; & T^{i} f' \neq 0 \end{array}\right\}$ can be stipulated.

and: $f' = T^{n-i} f \neq 0$. Now:

2. (a) Either $\mathcal{N}_0 < \mathcal{N}_1 < \mathcal{N}_2 < \cdots$, or:

(b) $\mathcal{N}_0 < \mathcal{N}_1 < \mathcal{N}_2 < \cdots < \mathcal{N}_n = \mathcal{N}_{n+1}$
with dimension of $\mathcal{N}_n$ being the multiplicity.

(a): Must be f_n in $\mathcal{N}_n$; $\|f_n\| = 1$; $(f_n, \mathcal{N}_{n-1}) = 0$
if (a) is true. Compute: $m > n$:

$$K f_m - K f_n = \frac{1}{\lambda}\Big[\underset{\mathcal{N}_m}{\underset{\downarrow}{f_m}} + \underset{\mathcal{N}_{n-1}}{\underset{\downarrow}{f_n}} - \underset{\mathcal{N}_{n-1}}{\underset{\downarrow}{T_m}} - \underset{\mathcal{N}_{n-1}}{\underset{\downarrow}{T f_n}} \Big]$$

$$= \frac{1}{\lambda}\Big[f_m - \varphi \Big] \quad , \quad (f_n, \varphi) = 0$$

$$\| K f_m - K f_n \|^2 = \left| \frac{1}{\lambda^2} \right| \Big\{ \|f_m\|^2 + \|\varphi\|^2 \Big\} \geq \left| \frac{1}{\lambda^2} \right|$$

This is case of Hermitean kernels. ?

(b): If (b) true and if $\mathcal{N}_1 = \{0\}$, $\mathcal{N}_n = \{0\}$

3. If the integral equation is written $Tf = g$ and can be written for every g, then λ is not an eigenvalue.

Proof: Assume λ is an eigenvalue or $\mathcal{N}_0 < \mathcal{N}_1$ which is an equivalent statement, Then $\mathcal{N}_0$ is included in $\mathcal{N}_1$. Write down f_n's up to $n < m$. $\left.\begin{array}{l} T^n f_n = 0 , \quad T^{n-1} f_n \neq 0 \\ \text{and; putting } f_n = g , \ Tf = f_n , \ f \neq 0 \end{array}\right\}$ $\begin{array}{l} T^{n+1} f = 0 \\ T^n f \neq 0 \end{array}$

Therefore f is included in N_{m+1} but not in N_m. But this is a contradiction.

4. Define $T^{+} = 1 - \lambda^{*} K^{+}$. Then if λ is not an eigenvalue of K, $T^{+}f = g$ can be solved, and λ^{*} is not an eigenvalue of T^{+}.

Proof: Choose g_0 such that $(g_0, S) = 0$ where S is some linear space orthogonal to g_0. Then take $S' = TS$. Now there is an f_0 such that: $(f_0, S') = 0$.

Pick f_0 so that $(f_0, Tg_0) = 1$

Put $g_1 = T^{+} f_0$

$$(g_1, g_0) = (T^{+}f_0, g_0) = (f_0, Tg_0) = 1$$

$$(g_1, g_2) = (T^{+}f_0, g_2) = (f_0, Tg_2) = 0$$

The g's are taken from the space S.

Also: $(g_1 - g_0, g_0) = 0$

$(g_1 - g_0, g_2) = 0$

$\vdots$ for all g's in S

So we conclude $g_1 = g_0$ and $T^{+} f_0 = g_0$ QED.

(1)

AM 203 (11)
10-18-61

Recap. K $\qquad \eta_0 < \eta_1 < \cdots \eta_m = \eta_{m+1}$

3. If $Tf = g$ $(T = 1 - \lambda K)$ can be solved for every g then λ is not an eigenvalue of K

4. If λ is not an eigenvalue of K, then $T^{\dagger} f = g$ $(T^{\dagger} = 1 - \lambda^{*} K^{\dagger})$ can be solved for every g.

* Pf. g_0 ; $\|g_0\| = 1$

S ; $(g_0, S) = 0$

$S' = TS$

Find: f_0 : $(f_0, S') = 0$, $(f_0, Tg_0) = 1$

Tg_0 cannot be element of S' because λ is not an eigenvalue.

If $Tg_0 = Tg$, g in S

$T(g_0 - g) = 0$

$\|g_0 - g\| \geq 1$ or $g_0 - g \neq 0$

but this is contradiction

Now $T^{\dagger} f_0 = g_1$

$(g_1 - g_0, g_0) = 0$

$(g_1 - g_0, g) = 0$ for g in S

$g_1 - g_0 = 0$ $\quad \therefore T^{\dagger} f_0 = g_0$

Hence adjoint eq can be solved.

5. A. If λ is not an eigenvalue of K, then λ^* is not an eigenvalue of K^+.

B. $K^{++} = K$

C. If λ^* is not an eigenvalue of K^+, then λ is not an eigenvalue of K

Either $(1-\lambda K)f = 0$ and $(1-\lambda^* K^+)\bar{f} = 0$ both have non-trivial solution or both $(1-\lambda K)f = g$ and $(1-\lambda^* K^+)\bar{f} = \bar{g}$ can be solved for every $g, \bar{g}$ (Fredholm alternative)

Suppose we have $(1-\lambda K)f_1 = 0$, $f_1 \neq 0$

$$T f_1 = 0$$

$$(1-\lambda^* K^+)f = g \; ; \; T^+ f = g$$

Form $(g, f_1) = (T^+ f, f_1) = (f, T f_1) = (f, 0) = 0$

Then it is necessary that g be orth. to f_1 to have solution

4'. If $T f_i = 0$ and $Tf = 0$ implies $f = \sum_i a_i f_i$

then $T^+ f = g$ can be solved if and only if $(g, f_i) = 0$.

** Go back to ** and add following: $(g_0, f_i) = 0$

$g_0 - g = \sum_i a_i f_i$, $(g_0, g - g) = 0$ $\therefore \|g_0\|^2 = 0$

but this is contradiction

$Tf = g \quad : \quad (1 - \lambda K) f = g$

$(g, \bar{f_i}) = 0 \qquad \bar{f_i}$ satisfies $T^+ \bar{f_i} = 0$

$(1 - \lambda^* K^+) \bar{f_i} = 0$

From K^+ we can form sequence like for K:

$K^+: \quad \mathcal{N}_0' < \mathcal{N}_1' < \cdots < \mathcal{N}_{m'}' = \mathcal{N}_{m'+1}' = \cdots$

$\mathcal{N}_1' \neq \mathcal{N}_1$, etc.

Now $m = m'$

$\dim \mathcal{N}_m = \dim \mathcal{N}_{m'}'$ all the way to $\mathcal{N}_0, \mathcal{N}_0'$

References:

Application: W. Hunziker, thesis ETH Zurich

Examples: R. Courant & D. Hilbert

suppl. to Ch. ~~6~~ 3

Theory: F. Riesz and B. Sz.-Nagy: Functional Analysis.

W. V. Lovitt: Lin. Int. Eq. (sym. kernel)

Example:

Usually non-hermitian K has no eigenvalues:

$M = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ If $\lambda M x = x$ then $x = 0$

C-H:

$\sum_{n=1} \frac{\sin nx \sin(n+1)y}{n^2}$ Kowalewski:

solve: $f(x) - \lambda \int_0^\pi \sin x \cos y \, f(y) \, dy = 0$

~~$K = \sin x \sin y$~~

$f(x) = C \sin x$

$K = \sin x \cos y$
non-symmetric kernel

$C = \lambda \int_0^\pi \cos y \, f(y) \, dy$

$= C\lambda \int_0^\pi \cos y \sin y \, dy = -\frac{C\lambda}{4} \cos 2y \Big]_0^\pi = 0 \therefore$ no eigenvalue, $f(x) = 0$

AM 203 (12) ①

10-20-61

Lecturer: Carrier

Limited domains and difference kernels

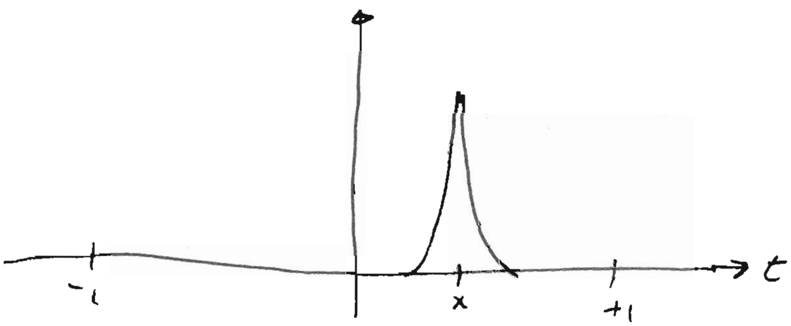
Two cases: narrow kernel with wide domain
wide kernel " narrow domain

Ref: Carrier, Jahre Grenzschicht forschung
Also: Carleman ?

~~Wide~~ Narrow kernel case:

Non-Homogeneous.

$$1 = \int_{-1}^{1} \frac{\beta}{\pi} K_0\left(\beta|x-t|\right) u(t)\, dt$$

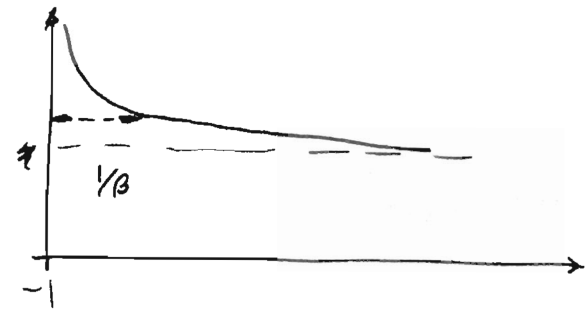


kernel is so narrow that we stipulate that we can use infinite domain if x not near +1, -1

Near end points we slop over. Then define two semi-infinite ~~step~~ domain probs, $-1 \to \infty$, $1 \to -\infty$ since we can translate domain.

$$\therefore \quad 1 = \int_0^\infty \frac{\beta}{\pi} K_0\, \beta|x-t|\, u_0(t)\, dt$$

Sol: $u_0(x) = (\pi\beta x)^{-1/2} e^{-\beta x} + \operatorname{erf}(\sqrt{\beta x})$



So we have some hope of using this method if $\frac{1}{\beta} \ll 1$.

For inf domain.

$$u_0(x) = 1 + |\pi\beta x|^{-1/2} e^{-\beta x} - \operatorname{erfc}\sqrt{\beta x}$$

$$= 1 + \chi(\beta x)$$

Total solution: $1 + \chi[\beta(x+1)] + \chi(\beta(1-x))$

$$= \left\{ 1 + [\pi\beta(x+1)]^{-1/2} e^{-\beta(x+1)} - \operatorname{erfc}\sqrt{\beta(x+1)} \right.$$

$$\left\{ + [\pi\beta(1-x)]^{1/2} e^{-\beta(1-x)} - \operatorname{erfc}\sqrt{\beta(1-x)} \right.$$

Subst. this in int. eq. to see how good:

$$\underbrace{-\int_{-1}^{\infty} \frac{\beta}{\pi} K_0\, u(RH) + \int_{-\infty}^{1} u(LH) - 1 + \frac{\beta}{4}\int_{1}^{\infty} \chi\{\beta(x+1)\}}_{K_0(\beta|x-t|)}$$

$$\underbrace{-\frac{\beta}{4}\int_{1}^{\infty} \chi(\beta(x-1))\, K_0(\beta|x-t|)\, dx}$$

extra term

These term contribute $\frac{e^{-2\beta}}{2}$ asymptotically as x is inside $-1 \to +1$.

This must be small compared to 1 which mean

$\beta \geq 2$

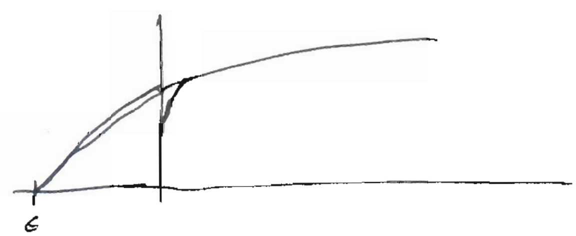
Homo. Eigen. Prob.:

~~$u(x) =$~~

$$u(x) = \lambda \int_{-1}^{1} \frac{\beta}{\pi} K_0(\beta|x-t|)\, u(t)\, dt$$

$$u_0(x) = \lambda \int_0^{\infty} K \quad u_0(t)\, dt$$

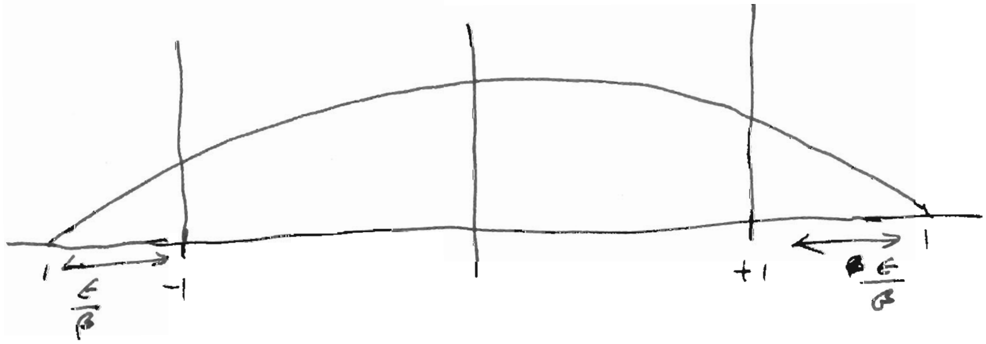
recall soln of form: $\sinh(\beta x + \epsilon) + \chi(\beta x)$



conjecture:

$$u(x) = \cos(\lambda\beta x) + \chi[\beta(x+1)] + \chi[\beta(x-1)]$$

best educated guess



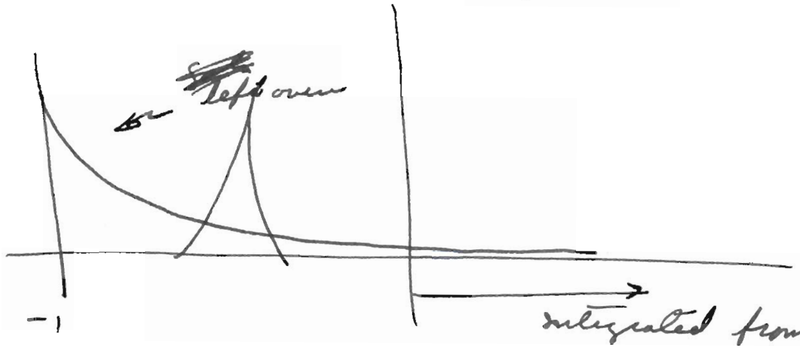
must have $\cos(\lambda\beta x) = \sinh(\beta(x+1) + \epsilon)$

means $\cos(\lambda\beta + \epsilon) = 0$ — set similar correction for other end

gives eigenvalue.

Recall $\epsilon(\lambda)$; $\lambda(\lambda)$

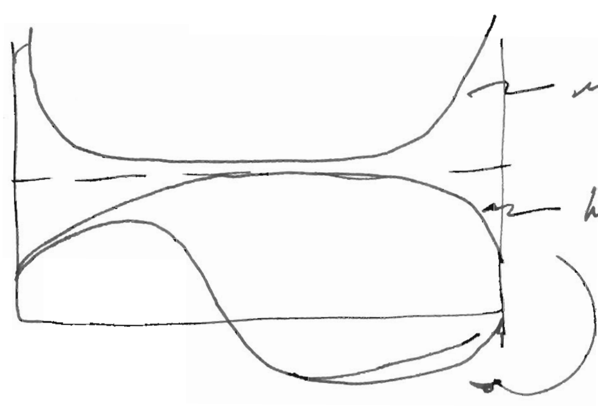
Check by substituting back into int. eq. as before.
Left-overs are given by int.



Write down asy. form of X : $\chi \sim \dfrac{e^{-\beta(x+1)}}{\beta(x+1)}$

so we overestimate by $\dfrac{e^{-2\beta}}{2\beta}$ again.

Recall expansion of inhomo in terms of homo.



Would take tremendous amount of homo solution to get approx inhomo. sol'n.

AN

AM203

10-23-61

(13)

(1)

Short interval Problem

Consider: $\int_{-1}^{1} \ln|x-t|\, f(t)\, dt = g(x)$

Take derivative: definition of int in Cauchy Princ. Value.

$$\lim_{\epsilon \to 0} \int_{-1}^{x-\epsilon} + \int_{x+\epsilon}^{1}$$

$$\therefore \quad \int_{-1}^{1} \frac{f(t)}{x-t}\, dt = g'(x)$$

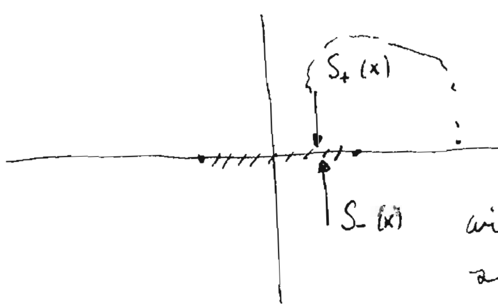
Now define (very poor motivation):

$$S(z) = \frac{\sqrt{z^2-1}}{\pi i} \int_{-1}^{1} \frac{f(t)}{z-t}\, dt \qquad \sqrt{z^2-1} = h(z)$$

$$h(\sqrt{z}) = 1$$

Single valued everywhere except along line ± 1

Take limit as $z \to x$ from above: $(S_+(x))$



will have different values on each side of BL.

For $S_+(x)$:

$$S_+(x) = \frac{i\sqrt{1-x^2}}{\pi i}\left[g'(x) - \pi i f(x)\right]$$

For $S_-(x)$

$$S_-(x) = \frac{-i\sqrt{1-x^2}}{\pi i}\left[g'(x) + \pi i f(x)\right]$$

Then:

$$S_+(x) - S_-(x) = \frac{2}{\pi}\sqrt{1-x^2}\, g'(x)$$

Shows discontinuity across Branch line:

Use Cauchy int. formula to choose most general fn. ~~of~~ $S(z)$

$$S(z) = \frac{2}{\pi(\pi i)}\int_{-1}^{1} \frac{\sqrt{1-u^2}\, g'(u)}{u-z}\, du + C(z)$$

Claim that this int defines single-valued function except along ± line, also resolves to S_+ and S_-. $C(z)$ can be at most a constant from the growth of the original $S(z)$.

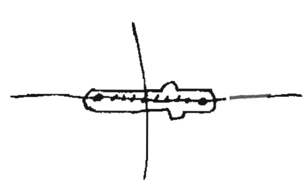
First take: $S_+ + S_- = -2i\sqrt{1-x^2}\, f(x)$

$$\cancel{2S(x)} = \frac{2\cdot 2}{\pi(2\pi i)}\int_{-1}^{1} \frac{\sqrt{1-u^2}\, g'(u)}{u-x}\, du$$

$$\therefore\ f(x) = \frac{1}{\pi^2\sqrt{1-x^2}}\int_{-1}^{1}\frac{\sqrt{1-u^2}\, g'(u)}{u-x}\, du + \frac{iC}{\sqrt{1-x^2}}$$

What is C? Look at: (out of hat!)

$$\int_{-1}^{1}\frac{g(x)\,dx}{\sqrt{1-x^2}} = \int_{-1}^{1} f(t)\,dt\ \underbrace{\int_{-1}^{1}\frac{dx}{\sqrt{1-x^2}}\ \ln|x-t|}_{\text{take derivative}}\ \cancel{dx}$$



$$\int_{-1}^{1}\frac{dx}{\sqrt{1-x^2}\,(x-t)} = 0$$

Then $\int_{-1}^{1} \frac{g(x)\,dx}{\sqrt{1-x^2}} = \int_{-1}^{1} f(t)\,dt \; K$

↑ constant

That is: $K = \int_{-1}^{1} \frac{\ln|x|}{\sqrt{1-x^2}}\,dx = 2\int_{0}^{1} \frac{\ln x \, dx}{\sqrt{1-x^2}}$

take $t=0$ since derivative $=0$ shows ind. of t and can take anyplace.

Let $x = \cos\theta$ and get:

$$K = \int_{0}^{\pi/2} \ln(\cos\theta)\,d\theta = \pi \log 2$$

$$K\int_{-1}^{1} f(t)\,dt = \ell C K \underbrace{\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}}}_{\pi}$$

Then: $\ell C = \frac{1}{\pi^2 \ln 2} \int_{-1}^{1} \frac{g(x)\,dx}{\sqrt{1-x^2}}$

when integrating $f(x)$, it goes to zero because

$$\int \frac{dx}{\sqrt{1-x^2}\,(x-t)} = 0$$

$g'(x)$ has to be integrable for a solution to exist.

There is no operational method on these problems.

In many cases, can approximate the kernel logarithmically over short intervals.

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Approx. Method: Wide kernel; short interval

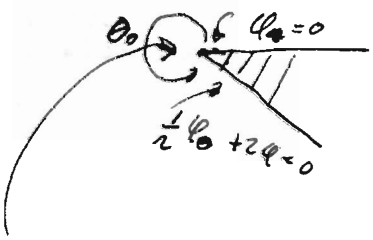
Consider:

$$f(x) = \int_{-a}^{a} K_0|x-t|\, g(t)\, dt \quad , \; f(x) \text{ known.}$$

Comes from a certain differential equation. If we consider semi-∞ domain get singularities in $g(t)$ are of form:

$$g(t) \sim (t+a)^{-1/2}$$
$$(a-t)^{-1/2}$$

Suppose as a BC problem: Radiation or Diffraction



$$\nabla^2 \varphi + \varphi = 0$$

Is there any $r^n \varphi_n(\theta)$ that satisfies d.e. and BC? Yes, get from Laplacian term.

Want to investigate possible singularity here. Can be characterized by leading terms of series solution.

$\boxed{\begin{matrix} n(n+1) \\ n^2-n \end{matrix}}$

$$r^{n-2}\left[n^2 \varphi_n + \varphi_n''\right] = 0 \; ; \quad \varphi_n = \sin(n\theta + \underset{0}{\overset{\shortparallel}{c}})$$

from top BC

$\cos n\theta_0 = 0$

$$\varphi_\theta = n \cos n\theta_0 = 0 \text{ on } \theta = \theta_0$$

$$n = \frac{(2m+1)\,\pi/2}{\theta_0}$$: In int. eq. $\theta_0 = \pi$; m are always integers

gives strongest type of singularity

If we expand around sing. can observe type of sing and pick worse.

Hence we assume that character of sing. is no worse than $(a \pm t)^{-1/2}$. Write, then:

$$g(t) = \frac{h(t)}{(a^2 - t^2)^{1/2}}$$

Make Trig. substitution: $t = a \sin\alpha$, $dt = a \cos\alpha \, d\alpha$

Then:

$$f(x) = \int_{-\pi/2}^{\pi/2} K_0 |x - a \sin\alpha| \, H(\alpha) \, d\alpha \quad ; \quad h(t) = H(\alpha)$$

Make a polynomial substitution for $H(\alpha)$, series for K_0, first few terms. If use 5 term polynomial, have 5 coef. parameters to match RHS to $f(x)$. Can do by:

$$0 = \frac{\partial}{\partial a_i} \left[f(x) - \sum_i a_i \, g_i(x) \right]^2 \quad \text{where } a_i \text{ are the}$$

minimizing the difference squared

Coefficients of the polynomial rep. for g.

Wiener-Hopf : Differential Equations:

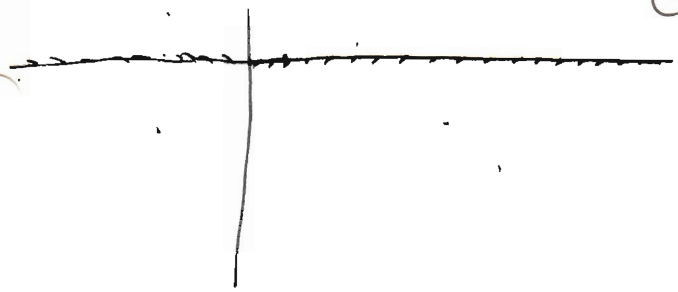
Redefine F trans.:

$$\bar{f}(\eta) = \int^{*} f(y) \, e^{-i\eta y} \, dy \quad \text{where} \quad \int^{*} = \int_{-\infty}^{y_0^-} + \int_{y_0^+}^{\infty}$$

y_0 is a left-out point. Reason for this is:

Consider:

$$\nabla^2 \nabla^2 \psi - a \nabla^2 \psi_x = 0$$



$\psi = 0$ on cut, $\psi_y = -1$ on Γ

ψ is odd in y

$\psi \to 0$ as $\sqrt{x^2+y^2} \to \infty$

have discontinuity in y direction, hence y_0.

nothing in x so use ordinary F trans.

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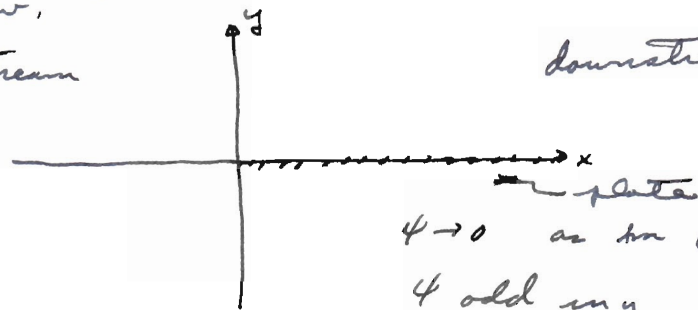
Stream Problem: $\bar{v} = \operatorname{curl} \hat{z}\psi$, $\psi =$ stream function.
viscous fluid flow,

upstream

downstream



$\psi \to 0$ as Im $(x+iy)^{1/2} \to \infty$

ψ odd in y ; $\psi_y = 1$ on $y=0,\ x>0$, $\psi(x,0) = 0$ no flow thru plate

Note: because ψ odd it can be discontinuous at $y=0$, however ψ_y being even cannot be. ψ_{yy} is odd again. We take the discontinuity defined by:

$$\psi_{yy}(x,0^+) - \psi_{yy}(x,0^-) = f(x)$$

Define:

$*$ single FT in y
$-$ double FT
$+$ single FT in x

$$\left.\begin{aligned}\psi^*(x,\eta) &= \left(\int_{-\infty}^{0^-} + \int_{0^+}^{\infty}\right) \psi(x,y)\, e^{-i\eta y}\, dy = \rho^*(\quad) \\ \bar{\Psi}(\xi,\eta) &= \int\int^{+*} \psi(x,y)\, e^{-i\eta y - i\xi x}\, dx\, dy\end{aligned}\right\}$$

allows one to use discontinuous derivative

The equation is:

$$\nabla^2\nabla^2\psi - a\nabla^2\psi_x = 0$$

Operate with FT,

$$\int \psi_{yyyy}\, e^{-i\eta y}\, dy = \underbrace{\psi_{yyy}\, e^{-i\eta y}\Big]_{-\infty}^{0^-}\ {}_{0^+}^{\infty}}_{\text{vanishes because continuous}} + i\eta \underbrace{\int \psi_{yyy}\, e^{-i\eta y}\, dy}$$

$$\hookrightarrow \underbrace{i\eta\, \psi_{yy} e^{-i\eta y} \Big]_{-\infty}^{0^-} \;\; {}_{0^+}^{\infty}}_{-i\eta f(x)} - \eta^2 \int \psi_{yy}\, e^{-i\eta y}\, dy$$

So the double ∇^2 term gives for ψ_{yyyy}:

$$-i\eta\, f(x) + \eta^4 \psi^*$$

get no extra term from $2\psi_{yyxx}$ and ψ_{xxxx}

Finally get:

$$\left[(\xi^2+\eta^2)^2 + ia\xi(\xi^2+\eta^2)\right] \bar{\Psi}(\xi,\eta) = i\eta\, \bar{f}(\xi)$$

$$\bar{\psi} = \bar{f}(\xi) \frac{i\eta}{(\xi^2+\eta^2)(\xi^2+\eta^2+ia\xi)}$$

Transforming back over η:

$$\psi^+(\xi,y) = \frac{\bar{F}(\xi)}{2\pi} \int_{-\infty}^{\infty} e^{i\eta y} \frac{i\eta\, d\eta}{(\xi^2+\eta^2)(\xi^2+\eta^2+ia\xi)}$$

This int. is easy to do since we have simple poles: Use:

$$\int \frac{f(\eta)}{g(\eta)}\, d\eta = \frac{f(\eta_0)}{g'(\eta_0)}\, 2\pi i$$

Poles are at: $\eta = i|\xi|$, $i\sqrt{\xi^2+ia\xi}$ (UHP)

$$\psi^+(\xi,y) = i\bar{f}(\xi)\left[\underbrace{\frac{i}{2}\frac{e^{-|\xi|y}}{ia\xi}}_{\text{res at } i|\xi|} + \underbrace{\frac{i}{2}\frac{e^{-\sqrt{(\xi^2+ia\xi)}\,y}}{-ia\xi}}_{\text{res at } i\sqrt{\xi^2+ia\xi}}\right]$$

We now use BC on ψ_y: Compute:

$$\psi_y^+(\xi,y) = i\bar{f}(\xi)\left[\frac{i}{2}\frac{-|\xi|e^{-|\xi||y|}}{ia\xi} - \frac{i}{2}\frac{-\sqrt{\xi^2+ia\xi}}{ia\xi}e^{-\sqrt{}|y|}\right]$$

If we had taken contour around LHP, would get reversal of sign in exp and out front. Can combine

$$\psi^+(\xi y) = i\bar{f}(\xi)\left[\frac{i}{2}\frac{e^{-|\xi||y|}}{ia\xi} + \frac{i}{2}\frac{e^{-\sqrt{}|y|}}{-ia\xi}\right]\underbrace{\frac{|y|}{y}}_{\text{sign change}}$$

so get one formula from two. Use this to find ψ_y^+ at the boundary:

$$\psi_y^+(\xi 0) = \frac{i}{2}\bar{f}(\xi)\frac{\sqrt{\xi^2+ia\xi} - |\xi|}{a\xi}$$

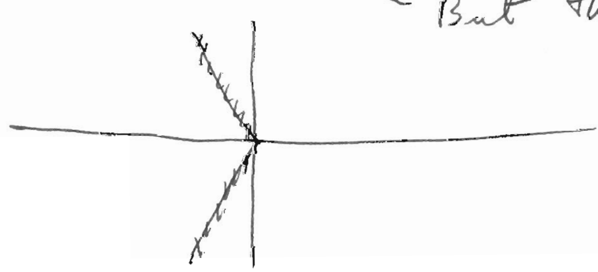
Now define: $\psi_y(x0) = u(x) + v(x)$: $u(x) = \begin{cases} 1 & x>0 \\ 0 & x<0 \end{cases}$

Then: $v(x) = \begin{cases} 0 & x>0 \\ \psi_y & x<0 \end{cases}$

$\therefore$ $\psi_y^+(\xi,0) = \bar{u}(\xi) + \bar{v}(\xi)$

$= \left(\frac{1}{i\xi}\right)_{\ominus} + \bar{v}(\xi)_{\oplus}$ now: $\bar{f}(\xi)$ is $\ominus$ fn.

$\frac{\sqrt{\xi^2+ia\xi} - |\xi|}{a\xi}$ is mixed and we have Wiener Hopf. But this requires branch lines:



No common region of analyticity.

We have to doctor mixed term: go back to $\bar\Psi$

$$\bar\Psi = \bar f(\xi)\,\frac{i\eta}{(\xi^2+\eta^2+\epsilon^2)\left(\eta^2+(\xi+ia)(\xi+i\epsilon)\right)}$$

because we would like to write for mixed term:

$$\sqrt{(\xi+ia)(\xi+i\epsilon)} - \sqrt{(\xi+i\epsilon)(\xi-i\epsilon)}$$

Then get:

$$\bar V_\oplus + \left(\frac{1}{i\xi}\right)_\ominus = -\frac{\bar f(\xi)_\ominus}{2}\cdot\frac{1}{\left(\sqrt{\xi-i\epsilon}\right)_\ominus\left[\sqrt{\xi+ia}+\sqrt{\xi+i\epsilon}\right]_\oplus}$$

$$\bar V_\oplus\left[\;\;\right]_\oplus + \underbrace{\frac{\left[\sqrt{\xi+ia}+\sqrt{\xi+i\epsilon}\right]_\oplus}{(i\xi)_\ominus}} = \frac{-\bar f(\xi)}{2\sqrt{\xi-i\epsilon}}$$

$$\left(\frac{\sqrt{\xi+ia}+\sqrt{\xi+i\epsilon}-\left(\sqrt{ia}+\sqrt{i\epsilon}\right)}{i\xi}\right)_\oplus$$

$$+\left[\frac{\sqrt{ia}+\sqrt{i\epsilon}}{i\xi}\right]_\ominus$$

Use fact that Ψ_{yy} is integrable over finite plate, $f(x)$ cannot be less than $\frac{1}{\sqrt{x}}$. Leads to Entire function $=0$. Then by usual anal. cont., etc:

$$\bar f(\xi) = \frac{-2\sqrt{a}}{\sqrt{i\xi}}$$

$$f(x) = \frac{-2\sqrt{a}}{\sqrt{\pi x}}$$ This is negative of viscous friction or shear stress.

Note that we haven't found ψ yet. once we find $\bar{f}(\xi)$, we just invert,

$$\psi(xy) \cong \int \frac{e^{i\xi x}\, e^{-\sqrt{\xi^2+ib\xi}\; y}}{\xi\,\sqrt{\xi+ib}}\, d\xi$$

get something like: $\psi = y - 2r\, p(s)$

$$p(s) = \int_0^s \operatorname{erf}\left(2\sqrt{a}\, s\right) ds$$

$$r + is = \sqrt{x + iy}$$

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Recall: $\bar{u}(\xi) + \bar{V}_+(\xi) = \dfrac{\bar{f}(\xi)}{2\left[\sqrt{\xi^2+\epsilon^2} + \sqrt{(\xi+ia)(\xi-i\epsilon)}\right]}$

$$1 = \int_0^\infty \frac{f(t)}{2} K(x-t)\,dt$$

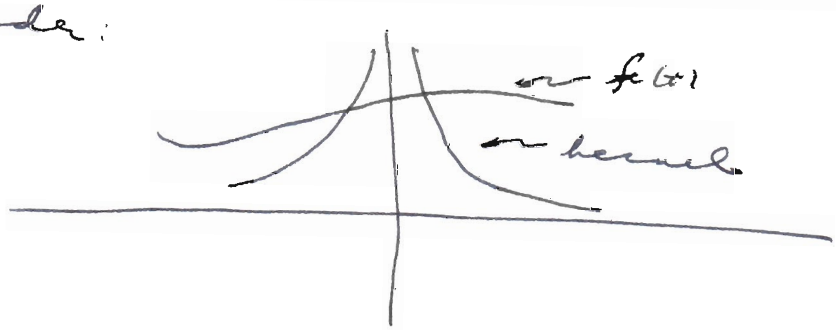
~~For~~ For exercise, try inverting above

More on Weiner - Hopf:

To difficult to factor: what can we say?

Problem came from MHD viscous flow over plate.

Consider:



$$u(x) = \int_0^\infty K(x-t)\,f(t)\,dt$$

$$\bar{u}_\ominus + \bar{V}_\oplus = \bar{K}\,\bar{f}_\ominus$$

We look for another kernel with properties of above but analytically simpler. Same area, width, lopsidedness, etc.

The area is. $\int_{-\infty}^{\infty} K(x)\,dx = \bar{K}(0)$

The first moment is: $\int_{-\infty}^{\infty} x\,K(x)\,e^{-i\xi x}\,dx : \bar{K}'(0)$

What about width? Demand exactly the same type of singularity. $\bar{K}(\infty)$ (behavior in limit)

Could also use second moment: $\bar{K}''(0)$, but hard to set.

Consider
$$\frac{1}{1+\sqrt{\xi^2+1}}$$

Can be matched by:
$$\frac{1}{\sqrt{\xi^2+4}}$$

A better one is:
$$\left.\frac{1}{\sqrt{\xi^2+c^2}}\;\frac{\xi^2+a^2}{\xi^2+b^2}\;\right\}$$ this gives 2nd and 4th moment by choice of a, b, c.

must use more ~~$\mathcal{S}$~~ judgement: if kernel originally smooth, substitute must be smooth.

Another Example:
$$1 = \underbrace{\int_0^\infty E_1(|x-t|)}_{\frac{1}{1+|\xi|}}\, u(t)\, dt$$

$\bar{K}'(0) = 0$

Area $= 1$

$\bar{K}(\infty) \sim \frac{1}{\xi}$

$\bar{K}''(0)$ doesn't exist

Try substitute: $\frac{1}{\sqrt{1+\xi^2}}$ } easier to factor $\frac{1}{\sqrt{(1+\xi)(1-\xi)}}$

First try factoring $\frac{1}{1+|\xi|}$:

$$\left.\begin{array}{l} \frac{1}{\sqrt{\xi+1}}\, e^{-\frac{1}{2\pi}\int_0^{\xi}\frac{\ln s}{1-s^2}\,ds} \\ \frac{1}{\sqrt{\xi+1}}\, e^{\frac{1}{2\pi}\int_0^{\xi}\frac{\ln s}{1-s^2}\,ds} \end{array}\right\}$$ show that product is above.

Problem: Find which of above is UHP or LHP fn. ~~the~~ log branch taken to be ~~the~~ that for which argument $\to 0$ and not 2π. In this way, find $\overline{K}_{\oplus}$ and $\overline{K}_{\ominus}$

$-\pi/2 <$ angle of $s < 3\pi/2$

$-3\pi/2 <$ " $< \pi/2$

Results:

Original:

$u(x) \sim 1 + \frac{c}{x}$ ~~small~~ large x $\simeq \frac{1}{\sqrt{\pi x}} + O(x \ln x)$ ~~large~~ small x

Subst.

$u^*(x) \sim 1 + O\, e^{-x}$ large x ← difference here is because of lack of 2nd moments.

$\simeq \frac{1}{\sqrt{\pi x}} + O(x^{3/2})$ small x

Problem: Fluid flowing by plate:

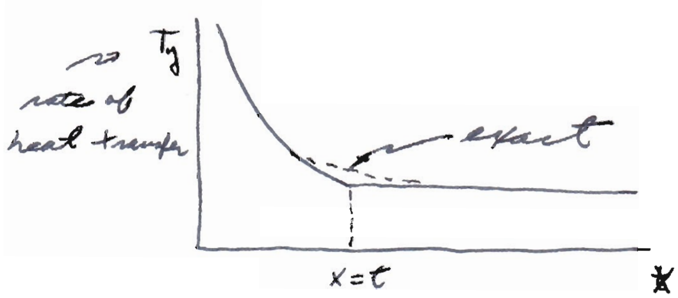
$T(x,0) = \delta(t-0)$ for $x>0$

Okaya $\nabla^2 T - T_x - T_t = 0$

and $T(\infty,t) = 0$

F tran in x and y, split in y, L trans. in t. need to consider s in L tran. as real + number, OK. Then anal. cont. into s plane. Also use fake kernel. Can use ~~steepest~~ descent and Wiener-Hopf.

Result:



Method of subst. kernel gives poor answer for homo. case.

$$\left.\begin{aligned} u(x) &= \lambda \int k u \, dt \\ \bar{Q} + \bar{u} &= \lambda \bar{K} \bar{u} \\ -\bar{Q} &= (1-\lambda\bar{K})\bar{u} \end{aligned}\right\}$$

Just about through with int. eq.

Volterra Eq:

$$u(x) = f(x) + \int_0^x K(x,t)\, u(t)\, dt$$

$$u_0(x) = f(x)$$

$$u_1(x) = f(x) + K \cdot u_0(x)$$

$$u_n(x) = f(x) + K \cdot u_{n-1}(x)$$

If $K(x,t) = K(x-t)$, can use L trans. and conv. int.

$$\bar{u}(s) = \bar{f}(s) + \bar{K}(s)\, \bar{u}(s)$$

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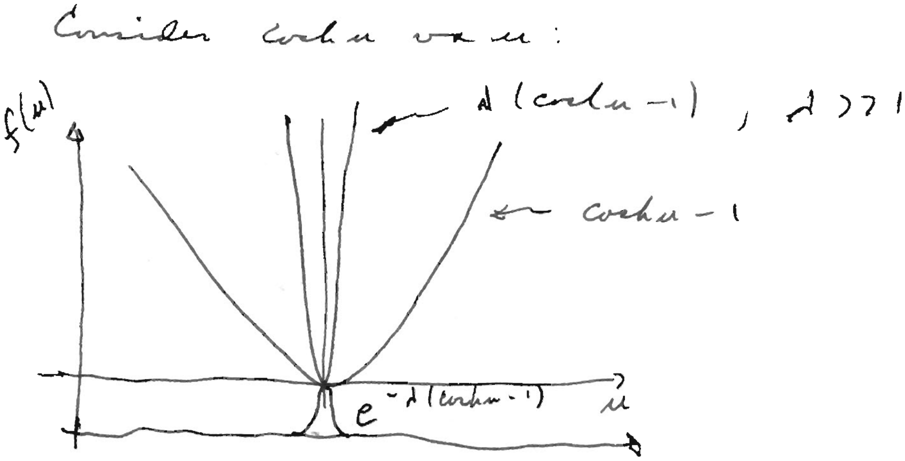
Method of Steepest Descent:

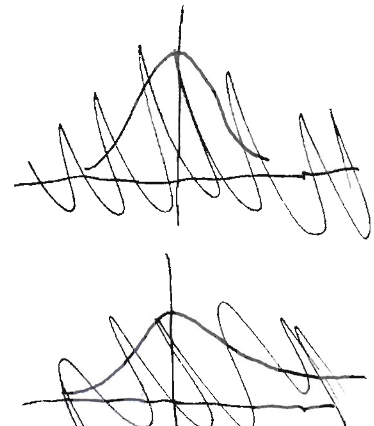
Consider:

$$\int_{-1}^{1} e^{-\lambda \cosh u}\, du = e^{-\lambda}\int_{-1}^{1} e^{-\lambda(\cosh u - 1)}\, du \quad ; \quad \lambda \text{ real}, \gg 1$$

In this example, λ real implies $\cosh u$ real.

Consider $\cosh u$ vs u:





This behavior implies that we can replace $\cosh u - 1$ by $\frac{u^2}{2}$ because λ is large. Then we have:

$$\int_{-1}^{1} e^{-\lambda \cosh u}\, du \sim e^{-\lambda}\int_{-1}^{1} e^{-\lambda \frac{u^2}{2}}\, du$$

$$\sim \sqrt{\frac{2}{\lambda}}\, e^{-\lambda}\int_{-\infty}^{\infty} e^{-\frac{\lambda u^2}{2}}\, du\sqrt{\tfrac{\lambda}{2}} \simeq \sqrt{\frac{2\pi}{\lambda}}\, e^{-\lambda}$$

error involved is $O\left(\frac{e^{-\lambda/2}}{\lambda}\right)$

Another Approach:

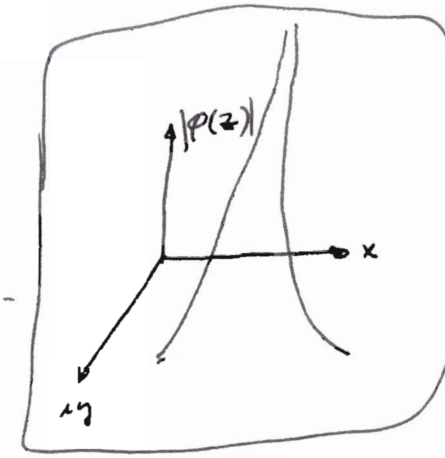
Take $t^2 = \cosh u - 1$

Can find $u(t)$ explicitly or in series rep.

Then can write:

$\boxed{\text{remainder}}$

$$\int e^{-\lambda t^2} \underbrace{\left[\sum_{n=0}^{N} a_n t^n + R_N(t) \right] dt}_{du}$$



Consider the result:

$$\lambda^{-1/2} \left[\sum \frac{\alpha_n}{\lambda^n} + R_N \right]$$

We stipulate $\lambda^N R_N(\lambda) \to 0$ as $\lambda \to \infty$

Poincaré's definition of asymptotic expansion.

Cutoff just before minimum. Taking more will make more inaccurate. Way of finding is to write general term and cutoff just before smallest term, error is no greater than smallest term.

In many cases, one term is best.

The above is a way of finding asymptotic expansion.

More General Problem:

$$\int_a^b g(z)\, e^{-\lambda \varphi(z)}\, dz$$

$g(z)$ and $\varphi(z)$ are analytic in the usual sense.

$\varphi(z)$, $g(z)$ do not take on extremum in complex plane, but they can have saddle points.

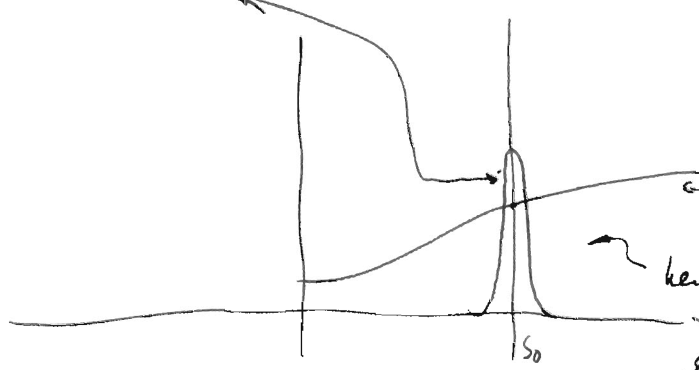
z^2 has saddle point at origin.

If path a – b climbs over saddle point, method of steepest descent may work.

If there is path a – b such that Im z is fixed and Re z goes thru s.p. can use m.s.d. Call this path s, s being arc length. s_0 is s.p.

$$e^{-i\lambda\psi - \lambda\Phi(s_0)} \int_{-\infty}^{\infty} \underbrace{e^{-\lambda\left(\Phi(s) - \Phi(s_0)\right)}}$$

What about $g(z)$?



hence can replace $g(z)$ by value at s_0 or z_0

$$g(z_0)\, e^{-i\lambda\psi - \lambda\Phi(s_0)} \int_{-\infty}^{\infty} \underbrace{e^{-\lambda\left(\Phi(s) - \Phi(s_0)\right)}} \underbrace{d\,z(s)}_{ds}$$

$$= \int_a^b g(z)\, e^{-\lambda\varphi(z)}\, dz$$

at s.p. power series expansion must start with squared term

$c(s - s_0)^2$

$$c = \frac{\Phi''(s_0)}{2!}$$

This gives:

$$g(z_0)\sqrt{\frac{2\pi}{c\lambda}}\; e^{-i\lambda\psi - \lambda\Phi(s_0)}$$

If we have form:

$$\int_a^b g(z) e^{-i\varphi(z,\alpha)}\, dz$$

may have special values of α where sp's coalesce. Then squared term is zero and first term is cubic.

What is method of stationary phase?

$$\varphi_r,\ \varphi_i : \quad \left.\begin{aligned} \frac{\partial \varphi_r}{\partial x_1} &= \frac{\partial \varphi_i}{\partial x_2} \\ \frac{\partial \varphi_r}{\partial x_2} &= -\frac{\partial \varphi_i}{\partial x_1} \end{aligned}\right\}\ \text{CR relation}$$

We want max. ~~s~~directional derivative which is gradient direction:

$$\frac{\partial \varphi_r}{\partial x_1},\quad \frac{\partial \varphi_r}{\partial x_2}$$

choose x_1 so that it is in direction: means $\frac{\partial \varphi_r}{\partial x_2} = 0$

hence: $\frac{\partial \varphi_i}{\partial x_1} = 0$ so φ_i is constant along path.

Thus stationary phase

First Example: Wave along beach:

Wave amplitude

$$\eta(x, y, t, \alpha, v)$$

~~αα~~ $\ell = 2a$

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1-body

$$[-\nabla^2 + V(r)]\psi = E\psi \qquad V(r) = \begin{cases} \infty & r<a \\ 0 & r>a \end{cases} \rightarrow \frac{V(r)}{2} = \begin{cases} \infty & r<a \\ 0 & r>a \end{cases}$$

$$[\nabla^2 + 4\pi a\,\delta(r)\frac{\partial}{\partial r} r]\psi' = E\psi'$$

2-body

$$[-\nabla_1^2 - \nabla_2^2 + V(r_{12})]\psi = E\psi$$

$$R = \frac{r_1 + r_2}{2}$$

$$r = r_1 - r_2$$

$$r_{12} = |r|$$

$$\{-\tfrac{1}{2}\nabla_R^2 - 2\nabla_r^2 + V(\underset{|r|}{r})\}\psi = E\psi$$

$$\{-\tfrac{1}{2}\nabla_R^2 + \{-\nabla_r^2 + \frac{V(r)}{2}\}\}\psi = E\psi$$

$$\{-\tfrac{1}{2}\nabla_R^2 + 2\{-\nabla_r^2 + 4\pi a\,\delta(r)\frac{\partial}{\partial r} r\}\}\psi' = E\psi'$$

$$\{-\nabla_1^2 - \nabla_R^2 + 8\pi a\,\delta(r_1 - r_2)\frac{\partial}{\partial r_{12}} r_{12}\}\psi' = E\psi'$$

3 body:

$$[-\nabla_1^2 - \nabla_2^2 - \nabla_3^2 + V(r_{12}) + V(r_{31}) + V(r_{23})]\psi = E\psi$$

$$[-\nabla_1^2 - \nabla_2^2 - \nabla_3^2 + 8\pi a\,\delta(r_3 - r_2)\frac{\partial}{r_{23}} r_{23} + 8\pi a\,\delta(r_2 - r_1)\frac{\partial}{r_{21}} r_{21}$$

$$+ 8\pi a\,\delta(r_1 - r_3)\frac{\partial}{\partial r_{31}} r_{31}]\psi' = E\psi'$$

must be taken to 5th term, never been done.

Poincaré - Lighthill method

Consider: $(x+\epsilon u)\dfrac{du}{dx} + u = 0$, $u(1) = 1$

$u = u_0(\xi) + \epsilon u_1(\xi) + \cdots$ $\quad u(\xi=1)=1 \begin{cases} u_0(\xi=1) = 1 \\ u_1(\xi=1) = 0, \text{etc.} \end{cases}$

$x = \underbrace{x_0(\xi)}_{\xi} + \epsilon x_1(\xi) + \cdots$ $\quad x(\xi=1)=1 \begin{cases} x_0(\xi=1) = 1 \\ x_1(\xi=1) = 0, \text{etc.} \end{cases}$

The usual pert. procedure would have $x_1, x_2, \cdots = 0$ for all ξ.

Here we get:

$$\xi \frac{du_0}{d\xi} + u_0 = 0$$

$$\xi \frac{du_1}{d\xi} + u_1 = \left(x_1 + \frac{1}{\xi}\right)\frac{1}{\xi^2} - \frac{1}{\xi}\frac{dx_1}{d\xi}$$

In ~~usual~~ usual pert. Th.:

$$\xi \frac{du_0}{d\xi} + u_0 = 0 \quad ; \quad \xi \frac{du_1}{d\xi} + u_1 = \frac{1}{\xi^3}$$

get: $u = \dfrac{1}{\xi} + \epsilon\left(\dfrac{1}{2\xi} - \dfrac{1}{2\xi^3}\right) + \cdots$; $x = \xi$

However, we get increasing reciprocal powers.
Use P-L method to choose x_1 to cancel $\frac{1}{\xi^3}$

Here we find : $u_1 = 0$, $x_1 = \dfrac{\xi^2 - 1}{2\xi}$

$\left.\begin{matrix} u_i = 0 \\ x_i = 0 \end{matrix}\right\}$ $i \geq 2$, however, this is not general.

We now consider a non-trivial problem due to Carrier. Comm. of Pure and Applied Mathematics 7, 11 (1954)

$$(x^2+\epsilon w)\frac{dw}{dx}+w-(2x^3+x^2)=0 \quad , \; w(1)=A\neq 1 \text{ over interval } (0,1)$$

$$w = w_0 + \epsilon w_1 + \cdots$$

$$x = \underbrace{x_0(\xi)}_{\xi} + \epsilon X_1(\xi) + \cdots$$

$$w(\xi=1)=A \rightarrow w_0(\xi=1)=A, \quad w_1(\xi=1)=0$$

$$X(\xi=1)=0 \rightarrow x_0(\xi=1)=0, \quad X_1(\xi=1)=0$$

We examine first under usual procedure:

$$w = w_0 + \epsilon w_1 + \cdots$$

$$x = \xi$$

or: $X_1(\xi)=0$ so we can see at each stage what will happen in the usual procedure. Now go to P-L:

$$X_0^2\frac{dw_0}{dX_0}+w_0-(2X_0^3+x_0^2)=0$$

$$e^{\int \frac{dx}{x^2}} = e^{-\frac{1}{x}}$$

$$x^2\frac{d}{dx_0}\left(w_0 e^{-\frac{1}{x_0}}\right) = e^{-\frac{1}{x_0}}(2x_0^3+x_0^2)$$

$$\frac{d}{dx_0}\left(w_0e^{-\frac{1}{x_0}}\right) = e^{-\frac{1}{x_0}}(2x_0+1) = \frac{d}{dx_0}\left(x_0^2e^{-\frac{1}{x_0}}\right)$$

or $w_0 = \xi^2 + \underbrace{Ke^{\frac{1}{\xi}}}_{\overline{w}_0}$ where from BC: $eK = A-1$

The worse sing. $e^{\frac{1}{\xi}}$ comes from x^2 term in front of $\frac{dw}{dx}$.

second order:

$$\xi^2 \frac{d}{d\xi}\omega_1 + \omega_1 + (2\xi x_1 + \omega_0)\frac{d\omega_0}{d\xi} + (6\xi^2 x_1 + 2\xi x_1)$$

$$+ (\omega_0 + 2\xi^3 + \xi^2)\frac{dx_1}{d\xi} = 0$$

in usual pert. Th.

$$\xi^2 \frac{d\omega_1}{d\xi} + \omega_1 + \underbrace{\omega_0 \frac{d\omega_0}{d\xi}}_{e^{2/\xi}} = 0$$

We have choice of x_1 to cancel. Choose $X_1 \sim e^{1/\xi}$

and set worst terms $= 0$.

$$(2\xi x_1 + \bar{\omega}_0)\frac{d\bar{\omega}_0}{d\xi} + \bar{\omega}_0 \frac{dx_1}{d\xi} = 0 \qquad \bar{\omega}_0 = Ke^{1/\xi}$$

Find That:

$$x_1(\xi) = -Ke^{\frac{1}{\xi}}(1 - 2\xi + 2\xi^2) + eK_1\xi^2$$

Check:

$$\underline{\xi^2 \frac{d\omega_1}{d\xi} + \omega_1} + (6\xi^2 + 2\xi)x_1 + (2\xi^3 + 2\xi^2)\frac{dx_1}{d\xi}$$

$$\underbrace{+ 2\xi \frac{-K}{\xi^2} e^{1/\xi} + (2\xi x_1 + Ke^{1/\xi})2\xi}_{\text{contain all terms with } e^{1/\xi}} + \dots$$

$$e^{1/\xi}(\xi^2)\frac{d}{d\xi}(\omega_1 e^{-\frac{1}{\xi}}) \quad \text{use } x_1 : 0\, e^{1/\xi}$$

$$O\xi e^{\frac{1}{\xi}}$$

must divide out

behavior near $x = 0$:

$\xi - \epsilon K e^{1/\xi} \sim 0$

$\omega \sim Ke^{1/\xi}$

$\sim \frac{\xi}{\epsilon}$

Set: $\omega = \xi^2 + Ke^{1/\xi} + \epsilon(e^{1/\xi}(\text{bounded fcn.})) + \epsilon^2 \dots$

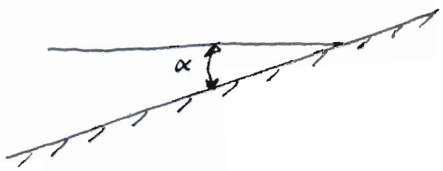
$X = \xi + \epsilon[Ke^{1/\xi}(1 - 2\xi + 2\xi^2) + K_1\xi^2] + \dots$

(18)

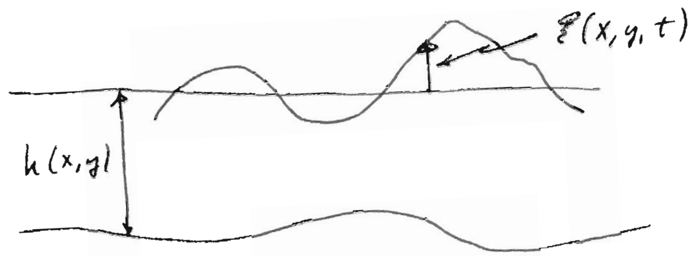
(1)

AM 203
11-6-61

Hurricane: edge waves or pressure concentration in fluid of geometry: Greenspan, 1956



We use shallow water theory derived heuristically from hydrodynamics. Our waves will be about 3 ft in height with wavelength of 100 miles.



velocities are:

$u(x,y,t)$

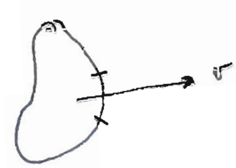
$v(x,y,t)$

$w(x,y,z,t)$ (will not have anything dependent on w)

That is, $\frac{dw}{dt}$ is much less than g, gravity, and pressure is always hydrostatic.

Cons. of mass: Take cyl. contour:

Top View:



rate of efflux of fluid $= \int \rho(h+\eta)\,\vec{v}\cdot\vec{n}\,ds$

S is length

The fluid inside $= \int_A \rho(h+\eta)\,dA$ A is area.

Hence: $\int_S \rho(h+\eta)\,\vec{v}\cdot\vec{n}\,ds = -\frac{\partial}{\partial t}\int_A \rho(h+\eta)\,dA$

(2)

Use divergence theorem:

$$[\rho(h+\eta)\, v_\alpha]_{,\alpha} + [\rho(h+\eta)]_{,t} = 0$$

We take ρ constant and $\underbrace{h >> \eta}$ over most of range,
and $h_{,t} = 0$ — very crude because not true at shore.

$$\therefore \quad (hU)_{,x} + (hV)_{,y} + \eta_{,t} = 0$$

Conservation of Momentum:

$$\rho(v_{\alpha,t} + v_\beta v_{\alpha,\beta}) + p_{,\alpha} = 0 \quad \text{(horizontal)}$$

we ignore viscosity

(vertical) $$(p + \rho g z)_{,z} = 0$$

$$\text{or:} \quad p + \rho g z = f(x,y,t) = \underbrace{P(x,y,t) + \rho g \eta}$$

Now: at free surface $$\underbrace{p(x,y,\eta)}_{\text{forcing term, not constant. Take independent of } \eta \text{. mean}} + \rho g \eta = \underbrace{P(x,y,t)}_{\text{gauge pressure}} + \rho g \eta$$

$p \sim p(x,y,0) + \cancel{\eta\, p_z(x,y,0)}$

Then: $p = P(x,y,t) + \rho g(\eta - z)$

Some figures: $\eta \sim 3$ ft, $d \sim 200$ miles, period ~ 6 hrs.

We throw out $V_j V_{i,j}$ subject to further verification and set:

$$\rho V_{i,t} + P_{,i} + \rho g \eta_{,i} = 0 \qquad \text{(horizontal)}$$

We now want to combine this with mass eq. Get:

$$[h(-g\eta_x)]_x + [h(-g\eta_y)]_y + \eta_{tt} = \frac{(hP_x)_x + (hP_y)_y}{\rho}$$

Note: hyperbolic and inhomo.

What are BC? $\quad \eta(x,y,t) \to 0$ as $x^2+y^2 \to \infty$

Boundedness at origin $\eta(0,y,t) < M$

because $h = 0$ at $x=0$.

What about initial conditions? can take storm starting infinitely long ago.

Now choose P, smooth, approximates physical problem, and is mathematically simple. Carrier chose:

$$\frac{P_0\, a\,(x+a)}{(y-vt)^2 + (x+a)^2} \quad \underbrace{H(t)}_{\text{step fn.}} \qquad v \sim 40 \text{ mph}$$

Pressure moving north and with center a distance a inshore; a also gives size of pressure. P_0 gives amplitude.

What kind of waves can arise?

Trapped waves:

$v \approx \sqrt{gh}$

high velocity turns waves back in.

no waves escape.
energy trapped.
no attenuation

Others: $\eta = e^{ik(y-ct)} f(x)$

solution usually form orthonormal set so all waves can be represented in term of this set.

11-8-61 (20)

AM 203 (20)

①

Recall: $\left[(h\eta_x)_x + (h\eta_y)_y\right] - \frac{1}{g}\eta_{tt} = g_1(x,y,t)$

$$= \frac{P_0 a}{\rho g} \frac{(y-vt)^2 - (x+a)^2}{\left[(y-vt)^2 + (x+a)^2\right]^2} H(t) \quad ; \quad \text{take } h = \alpha x$$

Take F transform in y:

$$\bar{\eta} = \int_{-\infty}^{\infty} e^{-ihy} \eta(x,y,t)\,dy$$

$$(x\bar{\eta}_x)_x - k^2 x\bar{\eta} - \frac{1}{\alpha g}\bar{\eta}_{tt} = \bar{q}_1$$

When we take $\bar{q}_1$ actually take with respect to $y-vt$:

$$\bar{F}(k) = e^{-ikvt}\int_{-\infty}^{\infty} F(y-vt)\, e^{-ik(y-vt)}\, d(y-vt) \quad H(t)$$

and taking L. tran. get $\frac{1}{s+ikv}$

Now take L. trans.

$$\bar{\bar{\eta}} = \int e^{-st}\bar{\eta}(x,k,t)\,dt$$

$$(x\bar{\bar{\eta}}_x)_x - k^2 x\bar{\bar{\eta}} - \frac{s^2}{\alpha g}\bar{\bar{\eta}} = \bar{\bar{q}}_1$$

$$\bar{\bar{q}}_1 = \frac{f(k,x)}{s + ikv}$$

Now ask what elementary fns satisfy eq?

$\bar{\bar{\eta}} \sim e^{-kx}$ would do it if LHS $\sim e^{-kx}$ $f(k,x)$

note that it cancels out linear term.

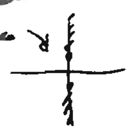
Now the pressure distribution was chosen to give this simple problem. ~~initially~~ This is hardest of the problem and is way it was done by Greenspan chronologically. Now $\bar{\bar{\eta}}$ turns out to be:

$$\bar{\bar{\eta}} = \frac{\gamma k_* e^{-k_*(x+a)}}{(s+ikv)(s^2+\alpha g k_*)} \quad ; \quad \gamma = \frac{-\alpha\pi P_0 a}{\rho}$$

where $k_* = \lim_{\epsilon \to 0} \sqrt{k^2+\epsilon^2}$ (multiple valued) $k_*^2 \equiv k^2$

chosen because k will not satisfy BC at $\pm\infty$

something like $\times$



Now invert $\bar{\bar{\eta}}$, first with respect to s.

$$\bar{\eta} = \frac{\gamma k_*}{2\pi i} e^{-k_*(x+a)} \int_{-i\infty}^{i\infty} \frac{e^{st}\, ds}{(s+ikv)(s^2+\alpha g k_*)}$$

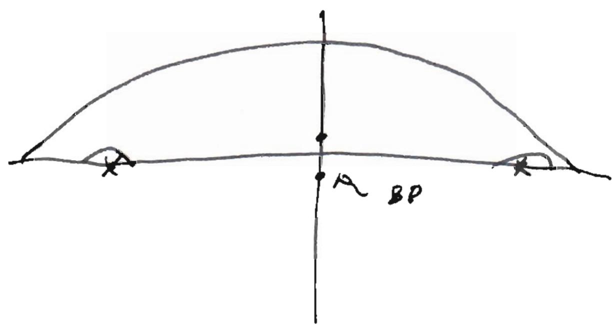
The poles are at: $-ikv$, $s = \pm i\sqrt{\alpha g k_*}$

$$= \gamma k_* e^{-k_*(x+a)} \left[\frac{e^{-ikvt}}{(\alpha g k_* - k^2 v^2)} + \frac{e^{+i\sqrt{\alpha g k_*}\,t}}{(i\sqrt{\alpha g k_*} + ikv)\,2i\sqrt{\alpha g k_*}} + \frac{e^{-i\sqrt{\alpha g k_*}\,t}}{(-i\sqrt{\alpha g k_*} + ikv)(-2i)\sqrt{\alpha g k_*}} \right]$$

We now do inversion over k: First term comes out to be,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\gamma\, e^{-ikvt + iky - k_*(x+a)}}{\alpha g - k^* v^2}\, dk$$

What about $\lambda^2 = \dfrac{(\alpha g)^2}{v^4}$

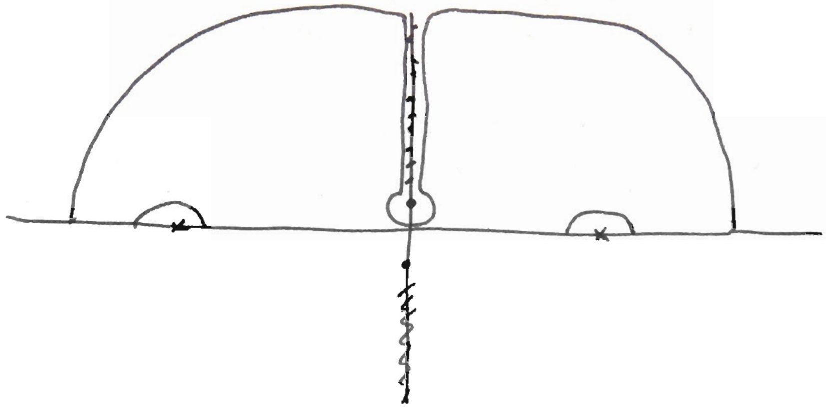


choose side of sing. that gives strong answer for $-y$ (where storm has been) and wave for $+y$ (ahead of storm) Then choose UHP

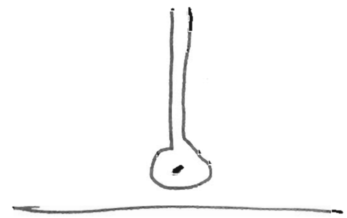
This does not effect the total ans but allows interpretation of above term as behavior at $t \sim -\infty$.

Now neglect waves far off shore so x will not be large parameter. Also y and t when small give only transients. Thus we consider as large parameters y and t.

Now: for above integral need not use msd. Take large y:

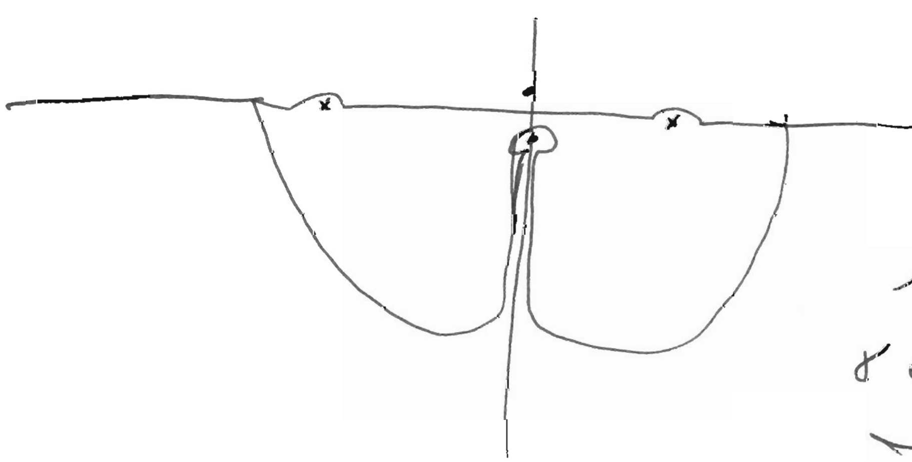


Replace this with:



Expand around BP, int. term by term, and get leading term giving asymptotic development of problem where storm stands at $y = vt$. Other terms of $O\left(\frac{1}{y-vt}\right)$.

What about contour:



Here pick up residues:

Get:

$$\frac{\gamma e^{-\frac{\alpha g}{v^2}(x+a)} \cos\left[\frac{\alpha g}{v^2}(y-vt)+q\right]}{v^2}$$

which we stipulate describes behavior from $y = -\infty$ to something less than $y = vt$. For any given speed have wave number. Have dispersion and hence group velocity.

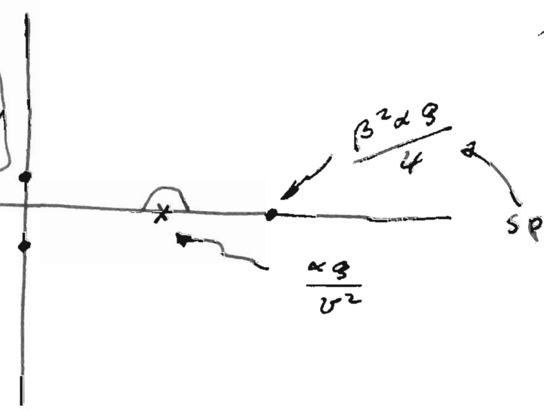
AM205 (21)

11-10-61 (1)

Recall one of the integrals to be performed is:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\gamma k_* \, e^{i\sqrt{\alpha g k_*}\,t}\, e^{iky}\, e^{-k_*(x+a)}}{\left(i\sqrt{\alpha g k_*}+ikv\right)2i\sqrt{\alpha g k_*}}\,dk$$

$$\boxed{\begin{aligned}&(nk|H_0|n'k')\\ &= E_n(0)\,\delta_{nn'} + \frac{\hbar^2 k^2}{2m^*}\,\delta_{k'k}\,\delta_{nn'}\end{aligned}}$$



roots of denom

$kv = \sqrt{\alpha g k}$

$k = \frac{\alpha g}{v^2}$

Assume t and y are the large parameters with $t = \beta y$ assumed. Then we get:

$$e^{iy\left(k - \sqrt{\alpha g k_*}\,\beta\right)}$$

What we are interested in is the point:

$$1 - \frac{\beta\sqrt{\alpha g}}{2\sqrt{k_*}}\left(\frac{k}{k_*}\right) = 0$$

or $2\sqrt{k_*} = \beta\sqrt{\alpha g}\,\frac{k}{k_*}$; $k = \frac{\beta^2\alpha g}{4}$

now sp can be either to right or left of pole

We agree that all paths along real axis go over singularities

(2)

Path of int. does pass thru s.p. Can we deform to get path of ~~least~~ steepest descent?

Consider: (to get path of steepest descent)

$$\text{Im}\ i\left(k - \sqrt{\alpha g k_*}\,\beta\right) = \text{Im}\ i\left(\frac{\beta^2\alpha g}{4} - \sqrt{\frac{\alpha g \beta^2 \alpha g}{4}}\,\beta\right)$$

Constant phase: same for any k, so get sp first.

$$= \text{Im}\left[-i\left(\frac{\beta^2\alpha g}{4}\right)\right] \text{ ; or: } \text{Im}\ i\left[\left(k - \sqrt{\alpha g k_*}\,\beta\right) + \left(\frac{\beta^2\alpha g}{4}\right)\right] = 0$$

or $\text{Re}\left[\left(k - \sqrt{\alpha g k_*}\,\beta\right) + \frac{\beta^2\alpha g}{4}\right] = 0$

In RHP, $k_* = k$, then we have: (see the perfect square)

$$\text{Re}\left(\sqrt{k} - \frac{\beta\sqrt{\alpha g}}{2}\right)^2 = 0$$

Which root do we use? pick one:

$$\sqrt{k} - \frac{\beta\sqrt{\alpha g}}{2} = A\sqrt{i} \quad \text{(must be imag to be} = 0\text{)}$$

Use parabolic coordinates: $\boxed{k = p + iq}$

then:

$$\text{Re}\sqrt{k} = \sqrt{\frac{\rho + p}{2}} \qquad p^2 + q^2 = \rho^2$$

$$\text{Im}\sqrt{k} = \sqrt{\frac{\rho - p}{2}}$$

Now: $k = A^2 i + \sqrt{i}\,A\beta\sqrt{\alpha g} + \frac{\beta^2\alpha g}{4}$

$$p = \frac{\beta^2\alpha g}{4} + \frac{A\beta\sqrt{\alpha g}}{\sqrt{2}} \ ; \quad q = A^2 + \frac{A\beta\sqrt{\alpha g}}{\sqrt{2}}$$

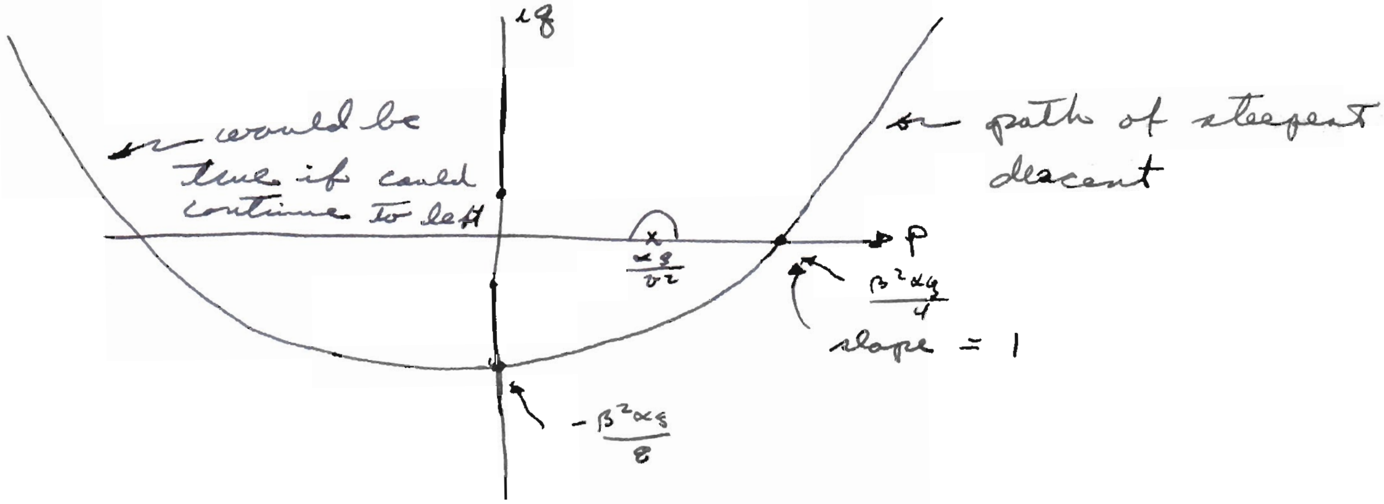
$$q = A\left[A + \frac{\beta\sqrt{\alpha g}}{\sqrt{z}}\right]$$

$$A = \frac{\sqrt{z}\,p}{\beta\sqrt{\alpha g}} - \frac{\beta\sqrt{\alpha g}}{2\sqrt{z}} = \frac{\sqrt{z}}{\beta\sqrt{\alpha g}}\left[p - \frac{\beta^2\alpha g}{4}\right]$$

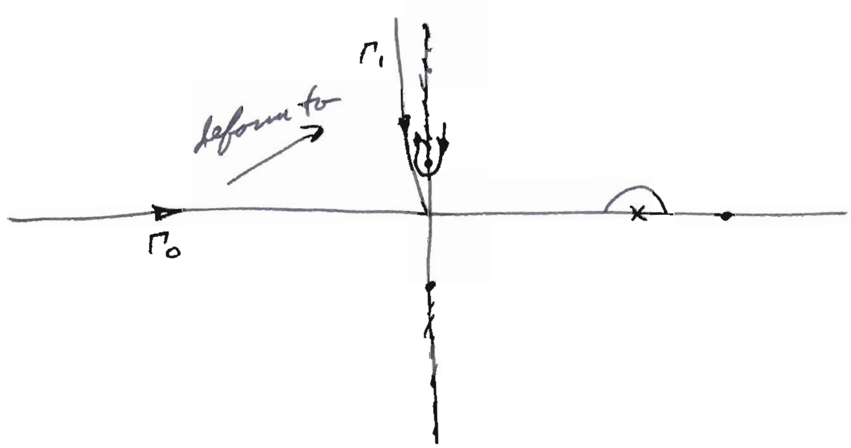
Then:

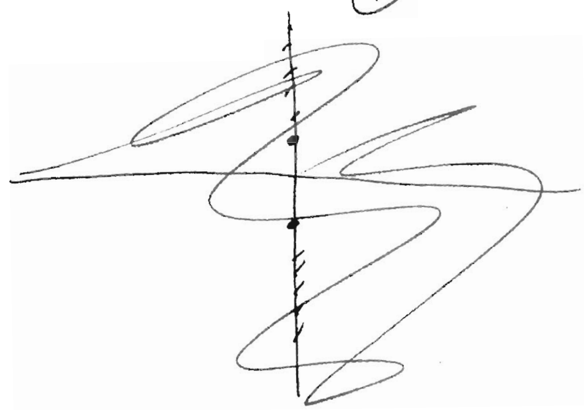
$$\boxed{\begin{aligned} q &= \left(\frac{\sqrt{z}}{\beta\sqrt{\alpha g}}\right)^2\left[p - \frac{\beta^2\alpha g}{4}\right]\left[p + \frac{\beta^2\alpha g}{4}\right] \\ &= \frac{z}{\beta^2\alpha g}\left(p^2 - \frac{\beta^4\alpha^2 g^2}{16}\right)\end{aligned}}$$

or we have parabola in p-q plane.



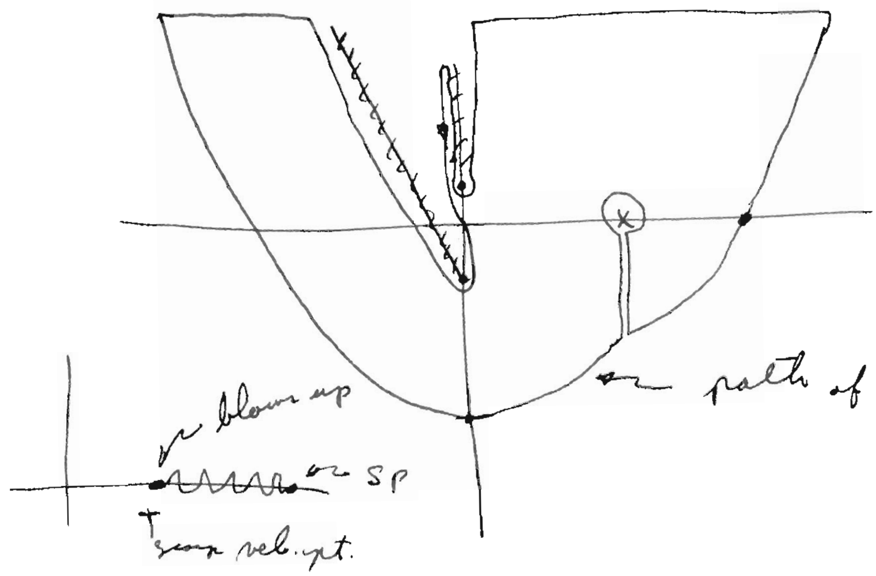
However, path cuts branch line





We can now deform to:

This is only for + y



on path of steepest descent

Conventional steepest descent can be used for SP far enough away from pole. We want to consider this special case, however, so we digress for a moment.

Consider:

$$\int f(z) e^{-\lambda g(z,\alpha)} dz$$

path of s.d. is real axis

$$\lim_{z \to z_1} f(z)(z - z_1) = 1 \quad , \quad g'(z_0) = 0$$

z_0 = SP
z_1 = pole

Write: $$\int \frac{(z-z_1)}{(z-z_1)} f(z) e^{-\lambda g(z,\alpha)} dz$$

erf ~ e^{-cx^2}

$$\to \left[(z_0 - z_1) f(z_0)\right] \int \frac{e^{-\lambda \frac{(z-z_0)^2}{2}}}{z - z_1} dz$$

which can be found in tables to be erf. type.

The way it goes for our problem is to take out $e^{-h\alpha(x+a)}$ as a smooth function and look up rest of integrand in Campbell and Foster, formula 809

Carrier suggests doing whole problem in detail and convince myself of all approximation.

Use of erf given

C & F formula 809:

$$\frac{\exp(-\sigma^{1/2} p^{1/2})}{1 + \beta^{3/2} p^{1/2}}$$

$$\rightarrow \frac{1}{\pi^{1/2}\beta^{3/2} g^{1/2}} \exp\left(\frac{-\sigma}{4g}\right)$$

$$- \frac{1}{\beta^3} \exp\left(\frac{\sigma^{1/2}}{\beta^{3/2}} + \frac{g}{\beta^3}\right) \operatorname{erfc}\left(\frac{\sigma^{1/2}}{2g^{1/2}} + \frac{g^{1/2}}{\beta^{3/2}}\right); \; g > 0$$

(22)

AM203
11-13-61

Method of Stationary Phase

We restrict to only linear terms.

Consider the following:

$$\int_{-\infty}^{\infty} e^{-\lambda(\cosh x - 1)}\,dx \sim \sqrt{\frac{2\pi}{\lambda}} \quad \text{for } \lambda \text{ large and } >0$$

$\int_{-\infty}^{\infty} e^{-\lambda\left(\frac{x^2}{2}\right)}\,dx$, etc.

what about?

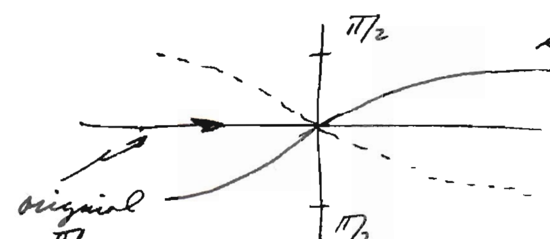
$$I = \int_{-\infty}^{\infty} e^{i\lambda(\cosh x - 1)}\,dx$$

Call: $f(z) = -i(\cosh z - 1)$; $z_0 = 0$

The path of steepest descent is therefore:

$\operatorname{Im} f(z) = 0$ since $\operatorname{Im} f(z_0) = 0$ and we want imaginary part constant. This means:

$\cosh x \cos y = 1$



or path of steepest descent

How can we deform path?

Locally: $\left(1+\frac{x^2}{2}\right)\left(1-\frac{y^2}{2}\right) \sim 1$

$x^2 - y^2 = 0$, $x \sim y$

.50000086 ~

By standard prescription:

$$f''(z) = -i \cosh z$$

$$I \sim \int dz\, e^{-(-i)\, z^2/2}$$

$$= \int_{-\infty}^{\infty} dz\; e^{-i \frac{z^2}{2}}$$

We can evaluate along original path

$$I = \int_{-\infty}^{\infty} e^{i\lambda(\cos z - 1)}\, dz \sim \int_{-\infty}^{\infty} e^{i\lambda \frac{z^2}{2}}\, dz = e^{i\pi/4}\sqrt{\frac{2\pi}{\lambda}}$$

For finite limits:

$$\int_{-\infty}^{a} e^{i\lambda(\cosh a + (z-a)\sinh a - 1)}\, dz$$

$$\simeq e^{i\lambda(\cosh a - a\sinh a - 1)} \left. \frac{e^{i\lambda z \sinh a}}{i\lambda \sinh a} \right]_{-\infty}^{a}$$

$$= \frac{e^{i\lambda(\cosh a - 1)}}{i\lambda \sinh a} = O\left(\frac{1}{\lambda}\right)$$

Suppose ~~we~~ we have pole near critical point

$$\int \frac{e^{i\lambda x^2}\, dx}{x - z_0} \longrightarrow \text{erf}$$

name of stationary phase comes from i in exponent and since f' vanishes phase is stationary.

$$\int dz\, e^{i\lambda z^2} \quad \begin{cases} 1 \to \dfrac{1}{\sqrt{\lambda}} \\ \dfrac{1}{z-z_0} \to \text{erf} \\ \dfrac{1}{\sqrt{z-z_0}} \to\ ? \end{cases}$$

Call $\left.\begin{array}{l} \sqrt{\lambda}\, z = t \\ z_0\sqrt{\lambda} = \mu \end{array}\right\} \quad \int_{-\infty}^{\infty} dt\, e^{it^2} \frac{1}{\sqrt{t-\mu}}$

If μ far from zero, exp can be taken out.

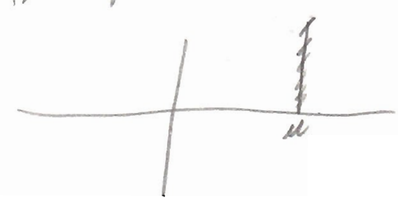
We are interested in pole close to critical pt.

Differentiate to set d.e.

$$I'(\mu) = 2i \int_{-\infty}^{\infty} dt\, e^{it^2} \sqrt{t-\mu} + 2i\mu I(\mu)$$

$$(I' + 2i\mu I)' = -iI$$

Take the branch



$$I'' - 2i\mu I' - iI = 0$$

whose solution is: $I = c\sqrt{\mu}\, e^{i\frac{\mu^2}{2}} \underbrace{Z_{1/4}\left(\frac{\mu^2}{2}\right)}_{AJ_{1/4} + BY_{1/4}}$

Now for $\mu \to \infty$:

$$I \approx i \frac{1}{\sqrt{\mu}} e^{i\pi/4} \sqrt{2\pi}$$

$$I \approx C\sqrt{\mu}\, e^{i\frac{\mu^2}{2}} H^{(2)}_{1/4}\left(\frac{\mu^2}{2}\right) \approx C\sqrt{\mu}\, e^{i\frac{\mu^2}{2}} \sqrt{\frac{4}{\pi\mu^2}}$$

$$\cdot\, e^{-i\left(\frac{\mu^2}{2} - \pi/8 - \pi/4\right)}$$

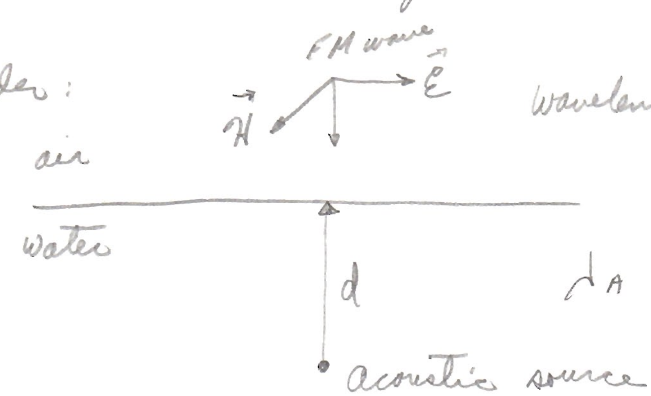
Therefore: $C = \pi/2\, e^{-i\, 3\pi/8}$

Recall we make Taylor exp:

$$f(z) = f(z_0) + \frac{f''(z_0)}{2}(z - z_0)^2$$

what if $f''(z_0)$ is also zero.

Consider:



wavelength λ_e

We assume: $d \gg \lambda_A$, λ_e $\quad \lambda_A \cdot \lambda_e \approx$ same order of magnitude at microwave freq

permeab $= \mu = \mu_0$

$$\frac{|\varepsilon|}{\varepsilon_0} \approx 70 = \text{dielectric const of } H_2O$$

will replace by ∞.

Hydro eq:

$$\vec{v} = -\nabla \varphi$$

$$\nabla^2 \varphi - \frac{1}{C_A^2} \frac{\partial^2}{\partial t^2} \varphi = 0$$

$$\frac{D\rho}{Dt} + \rho \nabla \vec{v} = 0 \quad \rightarrow \quad \frac{\partial \rho}{\partial t} + \rho_0 \nabla \cdot \vec{v} = 0$$

We take for the solution:

$$\varphi = Re\, \Phi\, e^{-i\omega_A t}$$

Then $(\nabla^2 + k_A^2)\Phi = 0 \quad : k_A = \frac{\omega_A}{c_A}$

The BC are: $\rho = \rho_0 \qquad \frac{d\rho}{dt} = 0 \qquad \nabla \cdot \vec{v} = 0$

$$\nabla^2 \varphi = 0$$

$$\nabla^2 \Phi = 0$$

$$\Phi = 0$$

$$\Phi = C \left[\frac{e^{ik_A R}}{R} - \frac{e^{ik_A R'}}{R'} \right]$$

$$R = \sqrt{x^2 + y^2 + (z+d)^2} \quad ; \quad R' = \sqrt{x^2 + y + (z-d)^2}$$

We are interested in $v_z = -\frac{\partial \varphi}{\partial z}$

$$z_0 = z_0(xy) = \frac{2C}{\omega_A} \text{Im} \frac{\partial}{\partial d} \frac{e^{i\left(k_A\sqrt{x^2+y^2-d^2} - i\omega_A t\right)}}{\sqrt{x^2+y^2+d^2}}$$

We say that $\vec{E}_{Trans} = 0$ at $z = z_0(xy)$

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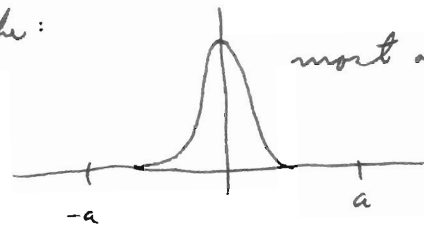
Recall: ① $\int_{-a}^{a} e^{-\lambda(\cosh x - 1)}\,dx \sim \sqrt{\frac{2\pi}{\lambda}}$ $\qquad e^{-\lambda(\cosh a - 1)}$

② $\int_{-a}^{a} e^{i\lambda(\cosh x - 1)}\,dx \sim e^{i\pi/4}\sqrt{\frac{2\pi}{\lambda}}$ $\qquad \frac{1}{\lambda}$

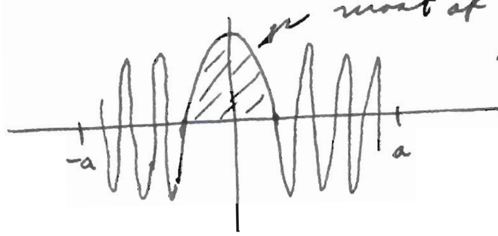
③ $\int_{-a}^{a} \cos\left[\lambda(\cosh x - 1)\right]\,dx \sim \sqrt{\frac{\pi}{\lambda}}$ $\qquad \frac{1}{\lambda}$

For ① integrand like:



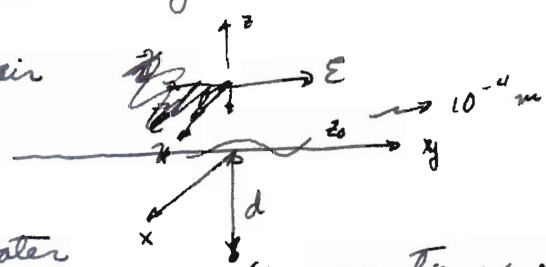
most of area around $x=0$

For ② and ③



most of contribution comes from here. oscillating part cancels out.

Recall scattering problem:



acoustic wave source

$\left|\frac{\epsilon}{\epsilon_0}\right| = 70$

$d \gg \lambda_A,\ \lambda_e,\ |\lambda_A - \lambda_e|$

$f_e \sim 10^{7}$

$\frac{f_A}{f_e} \sim 10^{-5}$

Equations for acoustic wave;

$$\vec{v} = -\nabla\varphi$$

$$\varphi = \mathrm{Re}\, \Phi\, e^{-i\omega_A t}$$

$$(\nabla^2 + k_A^2)\Phi = -C\, \delta(x)\, \delta(y)\, \delta(z+d)$$

BC: $\Phi = 0$ for $z = 0$

$$(\nabla^2 + k_A)\, G_0(x,y\,z) = -\delta(x)\, \delta(y)\, \delta(z)$$

$$G_0 = \frac{e^{ik_A\sqrt{x^2+y^2+z^2}}}{4\pi\sqrt{x^2+y^2+z^2}}$$

$$\Phi = C\left[G_0(x,y,z+d) - G_0(x,y,z-d)\right]$$

$$v_z(0) = -C\, \mathrm{Re}\, e^{-i\omega_A t}\frac{\partial}{\partial z}\left[G_0(x,y,z+d) - G_0(x,y,z-d)\right]_{z=0}$$

$$= -2C\, \mathrm{Re}\, e^{-i\omega_A t}\frac{\partial}{\partial d} G_0(x,y,d)$$

Then $z_0 = \dfrac{2C}{\omega_A}\, \mathrm{Im}\, e^{-i\omega_A t}\dfrac{\partial}{\partial d} G_0(x,y,d)$

For the electromagnetic wave:

$$\left.\begin{aligned} \nabla\times\vec{\mathcal{E}} &= i\omega\mu_0\vec{\mathcal{H}} \\ \nabla\times\vec{\mathcal{H}} &= -i\omega\epsilon_0\vec{\mathcal{E}} \end{aligned}\right\}$$ surface hardly appears to move with respect to EM wave.

BC: $E_x = E_y = 0$ for $z = z_0(x,y,t)$

$$E_y^{inc} = e^{-ik_0 z}$$

$$H_x^{inc} = \frac{1}{\ } e^{-ik_0 z}$$

Define the transforms:

$f(r) \rightarrow F(k)$ (rotationally invariant) no θ dependence

$$F(k) = \frac{1}{2\pi} \iint_{-\infty}^{\infty} dx\, dy\, e^{i(k_x x + k_y y)} f(x) \quad ; \quad k = \sqrt{k_x^2 + k_y^2}$$

$$f(r) = \frac{1}{2\pi} \iint_{-\infty}^{\infty} dk_x\, dk_y\, e^{-i(k_x x + k_y y)} F(k)$$

or: $$F(k) = \frac{1}{2\pi} \int_0^\infty r\,dr \int_{-\pi}^{\pi} d\theta\, e^{ikr\cos\theta} f(r)$$

$$= \int_0^\infty r\,dr\, f(r)\, J_0(kr)$$

$$f(r) = \int_0^\infty k\,dk\, F(k)\, J_0(kr)$$

To check, see Erdelyi Vol II p92 eq (34)

We write the acoustic wave equation as:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} + k_A^2\right) \int_0^\infty k\, dk\, g_0(k,z)\, J_0(kr)$$

$$= -\frac{1}{2\pi} \int_0^\infty k\, dk\, J_0(kr)\, \delta(z)$$

or: $$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) J_0(kr) = -k^2 J_0(kr)$$

$$\left(\frac{\partial^2}{\partial z^2} + k_A^2 - k^2\right) g_0(k,z) = -\frac{1}{2\pi}\,\delta(z)$$

$$g_0(k,z) = \frac{i}{4\pi}\cdot\frac{e^{i\sqrt{k_A^2-k^2}\,|z|}}{\sqrt{k_A^2-k^2}} \quad \begin{cases} \sqrt{k_A^2-k^2} > 0 & k_A > k \\ \sqrt{k_A^2-k^2} < 0 & k_A < k \end{cases}$$

$$G_0(x,y,z) = \frac{i}{4\pi}\int_0^\infty k\,dk\,J_0(kr)\,\frac{e^{i\sqrt{k_A^2-k^2}\,|z|}}{\sqrt{k_A^2-k^2}} = \frac{e^{ik_A\sqrt{r^2-z^2}}}{\sqrt{r^2-z^2}}$$

$$z_0 = \frac{-c}{2\pi\omega_A}\,h_u\,e^{-i\omega_A t}\int_0^\infty k\,dk\,J_0(kr)\,e^{i\sqrt{k_A^2-k^2}\,d}$$

Problem: show that:

$$i\int_0^\infty k\,dk\,J_0(kr)\,\frac{e^{i\sqrt{k_A^2-k^2}\,z}}{\sqrt{k_A^2-k^2}} \sim \frac{e^{ik_A\sqrt{r^2+z^2}}}{\sqrt{r^2+z^2}}$$

For the em field:

$$(\nabla^2 + k_e^2)\,E_y = 0 \quad ; \quad E_y = 0 \text{ for } z = z_0(x,y,t)$$

$$E_y = \underbrace{e^{ik_e z} - e^{-ik_e z}}_{E_y^0} + E_y^{sc}$$

some additional scattering.

$$0 = E_y^0(z_0) + E_y^{sc}(z_0) \simeq E_y^0(z_0) + E_y^{sc}(0)$$

and: $2i \sin k_e z_0 \sim 2i\,k_e z_0$

$E_y^0(z_0) =$

$$(\nabla^2 + k_e^2)\, E_y^{sc} = 0$$

and $E_y^{sc} = -2i\, k_e z_0(x, y, t)$

$$E_y^{sc} = -\frac{i C k_e}{\pi \omega_A} \left\{ \int_0^{k_A} k\,dk\, J_0(kr)\, e^{i\sqrt{k_e^2 - k^2}\, z} \sin\left[\sqrt{k_A^2 - k^2}\, d - \omega_A t\right] \right.$$

$$\left. - \int_{k_A}^{\infty} k\,dk\, J_0(kr)\, e^{i\sqrt{k_e^2 - k^2}\, z}\, e^{-\sqrt{k^2 - k_A^2}\, d} \sin \omega_A t \right\}$$

$2\frac{3}{4}$

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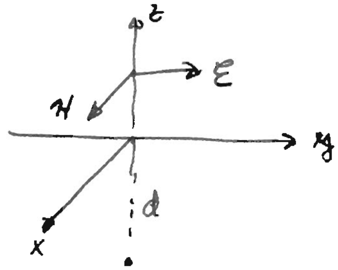


$d \gg \lambda_A, \lambda_e, |\lambda_e - \lambda_A|$

$$\nabla \times \mathcal{E} = \quad \mathcal{H}$$

$$\nabla \times \mathcal{H} = -i\omega\epsilon_0 \mathcal{E}$$

$$E_x = 0$$

$$E_y = E_y^{inc} + E_y^{refl} + E_y^{sc}$$

$$\qquad e^{-ik_e z} \qquad e^{ik_e z}$$

$$E_y^{sc}(r, z) = -\frac{i c k_e}{\pi \omega_A} \int_0^\infty k\,dk\, J_0(kr)\, e^{i\sqrt{k_e^2 - k^2}\, z}\, \mathcal{F}(k)$$

$$r = \sqrt{x^2+y^2}$$

$\sqrt{k_e^2 - k^2}$ for $k_e > k$

$i\sqrt{k^2 - k_e^2}$ $k_e < k$

$$\mathcal{F}(k) = \begin{cases} \sin\left(\sqrt{k_A^2 - k^2}\, d - \omega_A t\right) \\ -\sin \omega_A t\, e^{-\sqrt{k^2 - k_A^2}\, d} \end{cases}$$

$$\frac{e^{ik\sqrt{r^2+z^2}}}{\sqrt{r^2+z^2}} = i\int_0^\infty k\,dk\, J_0(kr)\, \frac{e^{-i\sqrt{k_A^2-k^2}\,|z|}}{\sqrt{k_A^2-k^2}}$$

$$\nabla \cdot \mathcal{E} = 0 \qquad \frac{\partial}{\partial y} E_y^{sc} + \frac{\partial}{\partial z} E_z = 0$$

$$E_z = \frac{c k_e}{\pi \omega_A} \frac{\partial}{\partial z} \int_0^\infty k\,dk\, J_0(kr)\, \frac{e^{i\sqrt{k_e^2-k^2}\, z}}{\sqrt{k_e^2-k^2}}\, \mathcal{F}(k)$$

Case I:
Look at $z << d$

$$E_y^{sc}(z=0) = 2 \lambda k_e \, z_0(x,y)$$

Recall asymptotic $J_0(k\rho)$: Use method of stationary phase.

$$J_0(k\rho) \sim \begin{cases} e^{ik\rho} \\ e^{-ik\rho} \end{cases}$$

Define $e^{i\phi(k)}$; $\phi = \pm k\rho + \sqrt{k_A^2 - k^2}\,|z|$

$$\phi' = \pm \rho - \frac{k}{\sqrt{k_A^2 - k^2}}\,|z| = 0$$

Take $\rho - \frac{k}{\sqrt{k_A^2 - k^2}}|z| = 0$; $k^2 = \frac{k_A^2 \rho^2}{\rho^2 + z^2}$

or: define $k_1 = \frac{k_A \rho}{\sqrt{\rho^2 + z^2}}$ stationary phase point.

$$\phi_+(k_1) = k_A \frac{\rho^2 + z^2}{\sqrt{\rho^2 + z^2}} = k_A \sqrt{\rho^2 + z^2}$$

Under this case we can write:

$$E_y^{sc}(\rho, z) \sim e^{i\sqrt{k_e^2 - k^2}\, z} \Big|_{k=k_1} 2 \lambda k_e \, z_0(x,y)$$

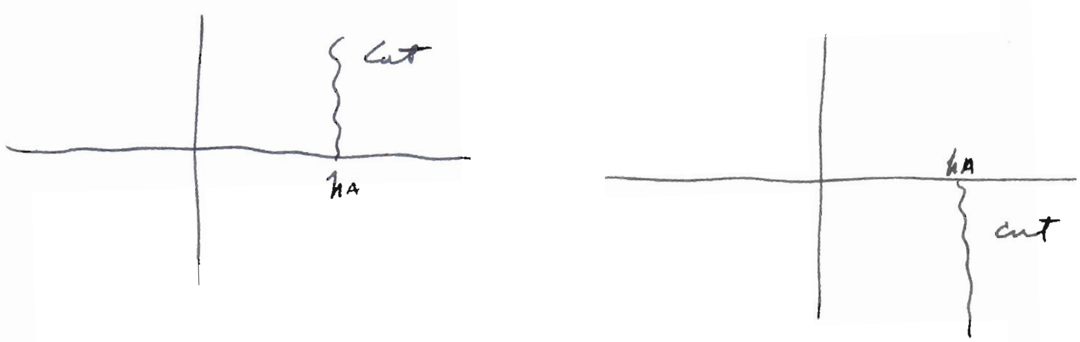
Case II: more general case

$$2i\,\mathcal{F}(k) = \begin{cases} e^{-i\omega_A t}\, e^{i\sqrt{k_A^2-k^2}\,d} - e^{i\omega_A t}\, e^{-i\sqrt{k_A^2-k^2}\,d} & ;\ (k_A > k) \\ e^{-i\omega_A t}\, e^{-\sqrt{k^2-k_A^2}\,d} - e^{i\omega_A t}\, e^{?\sqrt{k^2-k_A^2}\,d} & ;\ (k > k_A) \end{cases}$$

How do we combine this?

$$= e^{-i\omega_A t}\, e^{i\sqrt{k_A^2-k^2}\,d} - e^{i\omega_A t}\, e^{-i\sqrt{k_A^2-k^2}^{\,(-)}\,d}$$

Examine in complex plane



Then:

$$E_y^{sc}(r,z) = \frac{-c\,k_e}{2\pi\omega_A}\left[e^{-i\omega_A t}\,\mathcal{E}_1(r,z) - e^{i\omega_A t}\,\mathcal{E}_2(r,z)\right]$$

where:

$$\mathcal{E}_1 = \int_0^\infty k\,dk\, J_0(kr)\, e^{i\sqrt{k_A^2-k^2}\,z}\, e^{i\sqrt{k_A^2-k^2}\,d}$$

$$\mathcal{E}_2 \quad '' \qquad '' \quad e^{-i\sqrt{k_A^2-k^2}^{\,(-)}\,d}$$

Now choose $k_e > k_A$:

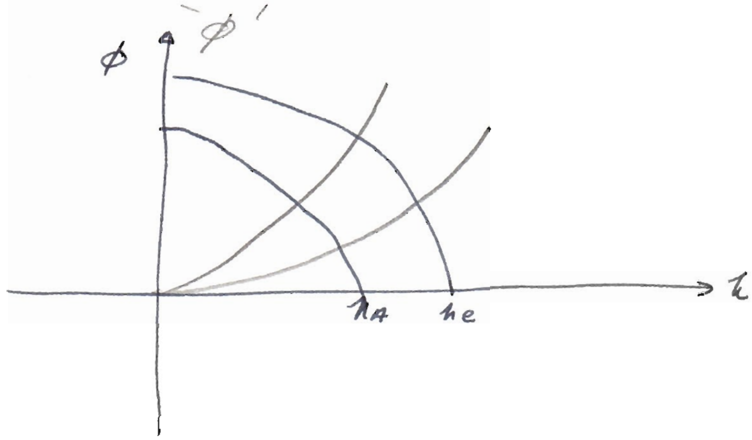
$$\phi_1 = \pm kr + \sqrt{k_e^2-k^2}\,z + \sqrt{k_A^2-k^2}\,d$$

$$\phi_2 = \pm kr + \sqrt{k_e^2-k^2}\,z - \sqrt{k_A^2-k^2}\,d$$

It is possible to solve for point of stat. phase but expression is too complex. Make plot:

$k < k_A$

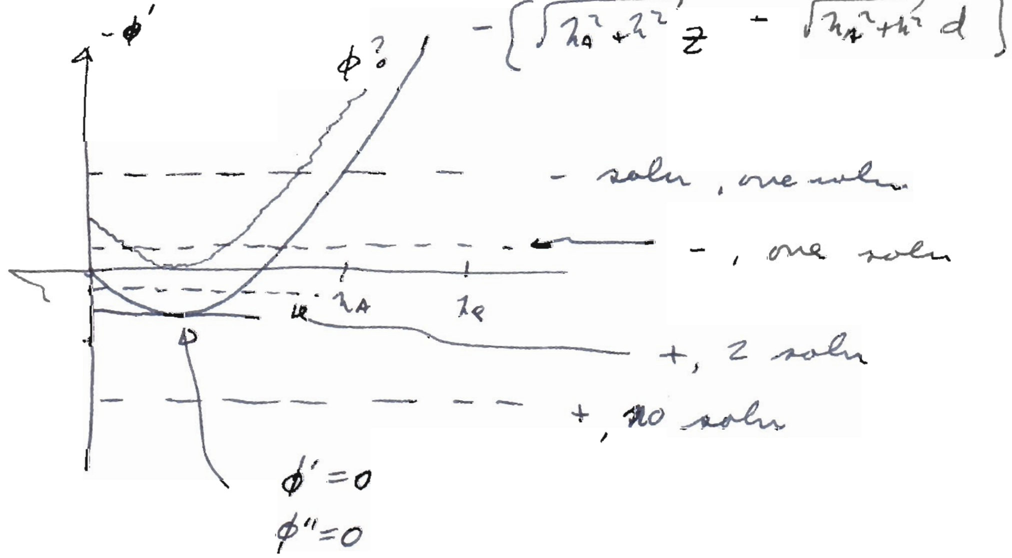
$k < k_e$



In ϕ_1 or $\sqrt{k_e^2+k^2}$, + sign has ~~[illegible]~~ one soln
In ϕ_2 more ~~[illegible]~~ complex: − sign " no "

~~[illegible]~~: for $d > z$: − has one soln
+ " no "

for $d < z$:



Now r enters as a parameter for which ϕ'' vanishes sometimes.

We then study integrals of the form:

$$\int_{-\infty}^{\infty} e^{i\alpha f(x,\alpha)}\,dx$$

$\frac{d}{dx} f(x, \alpha) = 0 \Rightarrow x = x_1(\alpha) \,, \quad x = x_2(\alpha)$

$x_1(0) = x_2(0)$

assuming two roots exist.

$\frac{\partial^2}{\partial x^2} f(x_1, 0) = 0 \qquad \frac{d}{dx} f = (x - x_1)(x - x_2) g$

but $\frac{d^3}{dx^3} f(x_1, 0) \neq 0$

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$$\int_{-a}^{a} \underbrace{e^{i\lambda(\cosh x-1)}}_{\varphi(x)} \, dx \sim \sqrt{\frac{\pi}{\lambda}}\, e^{i\pi/4}$$

$$\varphi(0) + \underbrace{x\,\varphi'(0)}_{0} + \frac{x^2}{2}\varphi''(0)$$

3 term

$$\int_{-a}^{a} e^{i\lambda f(x,\alpha)}\,dx, \quad \alpha \geq 0$$

$f(x,\alpha)$ has 3 cont. derivatives in x

$f'(x,\alpha) = 0$ when $x = x_1(\alpha)$

$x = x_2(\alpha)$

$x_1(0) = x_2(0)$

$x_2(\alpha) > x_1(\alpha)$

$f'''(x,\alpha) \neq 0$

Consider: $e^{i\lambda(\underbrace{ax^3+bx^2+cx+d}_{P(x)})}$

4 term

$P'(x) = 0$ when $x = x_1,\ x = x_2$

$P(x_1) = f(x_1) = f_1$

$P(x_2) = f(x_2) = f_2$

$P'(x) = 3ax^2+2bx+c = 3a(x-x_1)(x-x_2)$

$2b = 3a(x_1+x_2)$

$c = 3a\,x_1 x_2$

$a \to$ fn. of $x_1 - x_2$, $f_1 - f_2$

$$f_1 = a\left(x_1^3 - \tfrac{3}{2}(x_1+x_2)x_1^2 + 3x_1^2x_2\right) + d$$

$$f_2 = a\left(x_2^3 - \tfrac{3}{2}(x_1+x_2)x_2^2 + 3x_1x_2^2\right) + d$$

$$a = \frac{f_1 - f_2}{\left(-\frac{1}{2}x_1^3 + \frac{3}{2}x_1^2x_2\right) - \left(-\frac{1}{2}x_2^3 + \frac{3}{2}x_1x_2^2\right)} = -2\,\frac{f_1-f_2}{(x_1-x_2)^3}$$

$$d = \frac{(x_1^3 - 3x_1^2x_2)f_2 - (x_2^3 - 3x_1x_2^2)f_1}{(x_1-x_2)^2}$$

now let $x_2 \to x_1$: call $x = x_3$ when $f''(x) = 0$

and it is seen that $x_1 < x_3 < x_2$.

$$f''(x_1) = (x_1 - x_3) f'''(x_4) \qquad x_1 < x_4 < x_3$$

$$f''(x_2) = (x_2 - x_3) f'''(x_5) \qquad x_3 < x_5 < x_2$$

Expand f_2 & f_1:

$$f_2 = f_1 + \frac{(x_2 - x_1)^2}{2} f''(x_1) + \frac{(x_2 - x_1)^3}{6} f'''(x_6) \qquad x_1 < x_6 < x_2$$

$$f_1 = f_2 + \frac{(x_1 - x_2)^2}{2} f''(x_2) + \frac{(x_1 - x_2)^3}{6} f'''(x_7) \qquad x_1 < x_7 < x_2$$

$$f_2 - f_1 = \left[\frac{(x_2 - x_1)^2 (x_1 - x_3)}{2} + \frac{(x_2 - x_1)^2}{6} \right] f'''$$

$$f_1 - f_2 = \left[\frac{(x_2 - x_1)^2 (x_1 - x_3)}{2} + \frac{(x - x_2)^2}{6} \right] f'''$$

$$f_1 - f_2 = \frac{1}{2} \left[\frac{(x_1 - x_2)^3}{2} + \frac{(x_2 - x_1)^3}{3} \right] f''' = \frac{(x_1 - x_2)^3}{12} f'''$$

Then: $a = \frac{1}{6} f''$, $d = f(x_1)$

Push $x_2 \to x_1 \to 0$, b and c drop out,

$$d = f(0)$$

$$P = f(0) + \frac{1}{6} x^3 f'''$$

$$\int_{-\infty}^{\infty} e^{-\alpha (ax^3 + bx^2 + cx + d)} \, dx$$

replace $x \to x - \frac{b}{3a}$ to get rid of b.

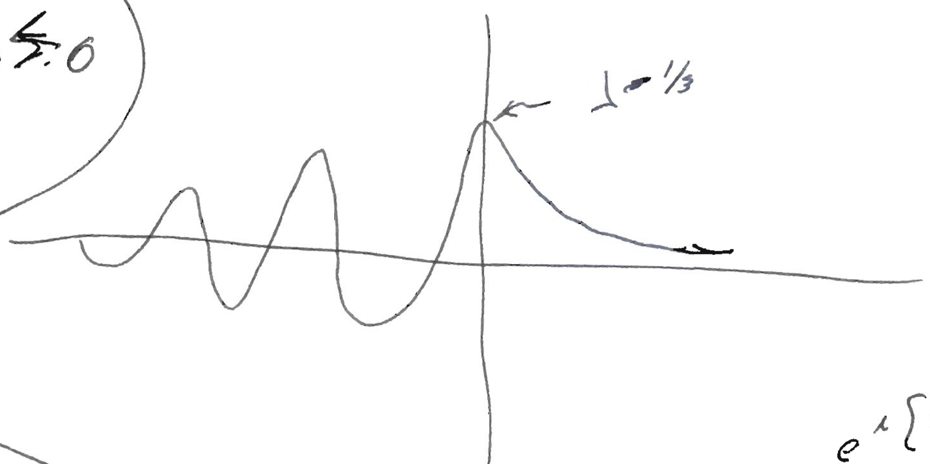
③

$$e^{i\lambda d}\int_{-\infty}^{\infty} e^{i\lambda(ax^3+cx)}\,dx$$

~~[illegible]~~ Look up: $\displaystyle\int_0^{\infty}\cos(t^3+3xt)\,dt = \begin{cases}\sqrt{\frac{x}{3}}\,K_{1/3}(2x^{3/2}) & x\geq 0\\[4pt] -\frac{\pi}{3}\sqrt{-x}\left(J_{1/3}(2(-x)^{3/2}) + J_{-1/3}(2(-x)^{3/2})\right) & x\leq 0\end{cases}$

Take: $\beta = \frac{1}{3}\,c\,a^{-\frac{1}{3}}\lambda^{-2/3}$

$$\begin{cases} 2\cdot 3^{-1/2}\,a^{-1/3}\lambda^{-1/3}\,\beta^{1/2}\,K_{\frac{1}{3}}(2\beta^{3/2}) & \beta\geq 0\\[4pt] -\frac{2\pi}{3}\,a^{-1/3}\lambda^{-1/3}(-\beta)^{1/2}\left[J_{1/3}(2(-\beta)^{\frac{3}{2}}) + J_{-1/3}(2(-\beta)^{3/2})\right] & \beta\leq 0\end{cases}$$



Recall: $\displaystyle -\pi/3\ \cdots \left[\left\{\overbrace{H^{(1)}_{1/3}(2(-\beta)^{3/2}) + H^{(1)}_{-1/3}(2(-\beta)^{3/2})}^{e^{i[2(-\beta)^{3/2}]}}\right\}\right.$

$\displaystyle \left. + \left\{\overbrace{H^{(2)}_{1/3}(2(-\beta)^{3/2}) + H^{(2)}_{-1/3}(2(-\beta)^{3/2})}^{e^{-i[2(-\beta)^{3/2}]}}\right\}\right]$

$H^{(1)}_{\nu}(x) \sim e^{ix}$, $H^{(2)}_{\nu}(x)\sim e^{-ix}$

Final Form: Corrected for $[\varphi''(0)]^{1/2}$

$$-\frac{\pi}{3}\cdots\left[\left\{H^{(1)}_{1/3}+H^{(1)}_{-1/3}\right\}\sqrt{\frac{P'(x_1)}{f''(x_1)}}\right.$$

$$\left.+\left\{H^{(2)}_{1/3}+H^{(2)}_{-1/3}\right\}\sqrt{\frac{P'(x_2)}{f''(x_2)}}\right.$$

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Formal perturbation theory for linear problems:

$$H\psi = E\psi$$

Scattering: $H = H_0 + V$

$$(\nabla^2 + E - V)\psi = 0 \quad ; \quad \hbar = 2m = 1$$

$$\psi = \psi_{incident} + \text{outgoing wave}$$

$$(\nabla^2 + E)\psi(\vec{r}) = V(\vec{r})\psi(\vec{r})$$

$$(\nabla^2 + E)G(\vec{r}) = \delta(\vec{r})$$

$$G(\vec{r}) = -\frac{e^{ikr}}{4\pi r}$$

$$\psi(\vec{r}) = \psi^{inc}(\vec{r}) + \int G(\vec{r}-\vec{r}')V(\vec{r}')\psi^{inc}(\vec{r}')\,d\vec{r}'$$

$$+ \int G(\vec{r}-\vec{r}')V(\vec{r}')G(\vec{r}'-\vec{r}'')V(\vec{r}'')\psi^{inc}(\vec{r}'')\,d\vec{r}'\,d\vec{r}'' + \cdots$$

$$G\psi = \mathcal{G}\psi = \int G(\vec{r}-\vec{r}')\psi(\vec{r}')\,d\vec{r}' \quad ; \quad V\psi = V(\vec{r})\psi(\vec{r})$$

?

$$\psi = \psi^{inc} + GV\psi^{inc} + GVGV\psi^{inc} + GVGVGV\psi^{inc} + \cdots$$

$$= (1 + GV + GVGV + \cdots)\psi^{inc} = (1 - GV)^{-1}\psi^{inc}$$

$$\psi = \psi^{inc} + GV\psi \quad ; \quad \psi^{inc} = (1 - GV)\psi$$

Eigenvalue: $-\frac{d^2}{dx^2}\psi = E\psi \quad ; \quad \psi(x) = C \sin \frac{n}{2}(x - \pi)$

$$E = \left(\frac{n}{2}\right)^2$$

Pert. Theory:

$$H_0\psi_n = E_n\psi_n$$

$$H\psi = E\psi$$

$$\psi = \psi^{(0)} + \psi^{(1)} + \psi^{(2)} + \cdots$$

$$E = E^{(0)} + E^{(1)} + E^{(2)} + \cdots$$

$$H_0\psi^{(0)} = E^{(0)}\psi^{(0)} \quad ; \quad \int \psi_m^* \psi_n \,dx = \delta_{mn}$$

$$(H_0 + V)(\psi^{(0)} + \psi^{(1)} + \cdots) = (E^{(0)} + E^{(1)} + \cdots)(\psi^{(0)} + \psi^{(1)} + \cdots)$$

1st order: $H_0 \psi^{(1)} + V\psi^{(0)} = E^{(0)}\psi^{(1)} + E^{(1)}\psi^{(0)}$

2nd order: $H_0 \psi^{(2)} + V\psi^{(1)} = E^{(0)}\psi^{(2)} + E^{(1)}\psi^{(1)} + E^{(2)}\psi^{(0)}$

Look at 1st order:

$$(H_0 - E^{(0)})\psi^{(1)} = -V\psi^{(0)} + E^{(1)}\psi^{(0)}$$

Assume no degeneracy and let $\int \psi^{(0)*}\psi^{(n)}\,d\vec{r} = 0,\ n > 0$

$$\underbrace{\int \psi^{(0)*}(H_0 - E^{(0)})\psi^{(1)}\,d\vec{r}}_{=0} = \int \psi^{(0)*}(-V\psi^{(0)} + E^{(1)}\psi^{(0)})\,d\vec{r}$$

$$\therefore\quad E^{(1)} = \int \psi^{(0)*}\, V\, \psi^{(0)}\,d\vec{r}$$

Take: $\psi^{(1)} = \sum_m a_m \psi_m$

$$\sum_m a_m (E_m - E_n)\psi_m = -\sum_m \left(\int \psi_m^* V \psi_n\,d\vec{r}\right)\psi_m + E^{(1)}\psi_n$$

$$= -\sum_{m\neq n}\left(\int \psi_m^* V \psi_n\,d\vec{r}\right)\psi_m$$

$$a_m = \frac{-\int \psi_m^* V \psi_n\,d\vec{r}}{E_n - E_m} \qquad n \neq m$$

2nd order: $(H_0 - E^{(0)})\psi^{(2)} = -V\psi^{(1)} + E^{(1)}\psi^{(1)} + E^{(2)}\psi^{(0)}$

$$0 = \int \psi^{(0)*}(-V\psi^{(1)} + E^{(1)}\psi^{(1)} + E^{(2)}\psi^{(0)})\,d\vec{r} = E^{(2)} - \int \psi^{(0)*} V\psi^{(1)}$$

or: $$E^{(2)} = -\sum_{m\neq n} \frac{\left(\int \psi_m^* V \psi_n\,d\vec{r}\right)\left(\int \psi_n^* V \psi_m\,d\vec{r}\right)}{E_m - E_n}$$

Recall: $(H_0 - E^{(0)})G(\vec{r}) = -\delta(\vec{r})$

Define: $(H_0 - E^{(0)})g = -1$

Define: $\psi^{(n)} = |n\rangle$

$$\left(H_0 + V - E^{(0)} - E^{(1)} - E^{(2)} - \cdots\right)\left(|0\rangle + |1\rangle + \cdots\right) = 0$$

$$\left(H_0 - E^{(0)}\right)|0\rangle = 0$$

$$\left(H_0 - E^{(0)}\right)|1\rangle = \left(E^{(1)} - V\right)|0\rangle$$

$$\left(H_0 - E^{(0)}\right)|2\rangle = \left(E^{(1)} - V\right)|1\rangle + E^{(2)}|0\rangle$$

$$\underbrace{\langle 0|\, H_0 - E^{(0)}\, |2\rangle}_{0} = \langle 0|\, E^{(1)} - V\, |1\rangle + \langle 0|\, E^{(2)}\, |0\rangle$$

$$\therefore \quad E^{(2)} = \langle 0|V|1\rangle$$

In general: $\left(H_0 - E^{(0)}\right)|n\rangle = \left(E^{(1)} - V\right)|n-1\rangle + E^{(2)}|n-2\rangle$

$$+ E^{(3)}|n-3\rangle + \cdots + E^{(n)}|0\rangle$$

$$0 = \langle 0|\, H_0 - E^{(0)}\, |n\rangle = -\langle 0|V|n-1\rangle + \langle 0|\, E^{(n)}\, |0\rangle$$

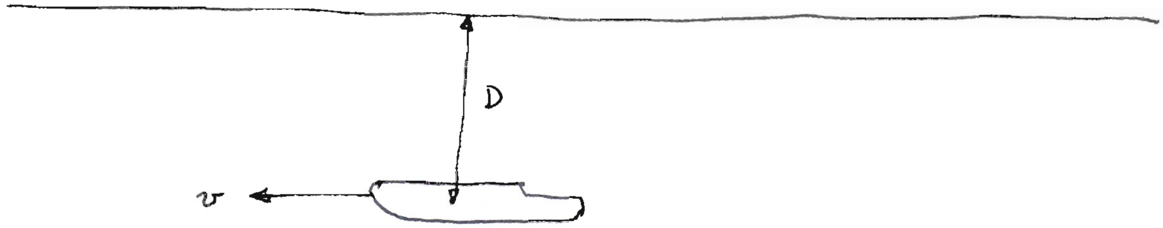
Hence: $E^{(n)} = \langle 0|V|n-1\rangle$

(27) ①

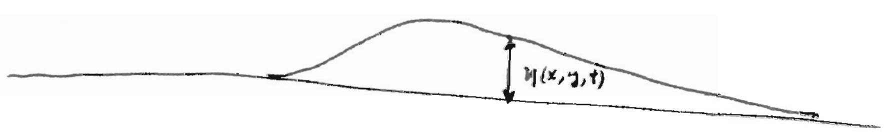
AM 203
11-24-61

Kelvin ship-wave problem:

Body of water, infinitely deep, smooth surface, except for disturbance due to moving submerged object.



We replace, on hydrodyn. principle, the object with a fluid source. Position: $x - ut$
For inviscid fluid $\nabla^2 \varphi = 0$, irrotational, φ exists: $\vec{v} = \operatorname{grad} \varphi$.
For BC:



For small η; $\eta_t = w$

$$\rho \varphi_t + (p + \rho g z) = 0$$

evaluate at η:

$$\rho \varphi_t(\eta) + \underset{0}{\underset{\shortparallel}{p}} + \rho g \eta = 0 \quad ; \quad \eta \simeq -\frac{1}{g}\varphi_t \text{ as BC.}$$

$$\varphi(\eta) = \varphi(0) + \eta \varphi_z(0) + \dots$$ good approximation for this problem.

so: $\eta = -\frac{1}{g}\, \eta\, \varphi_{tz}(0)$

Now, since we replace the moving object with sources, we have:

Hence we write:

$$\nabla^2 \varphi = \delta(x - ut)\,\delta(y - 0)\,\delta(z + D)$$

If no surface:

$$\varphi \sim \frac{1}{\sqrt{(x-ut)^2 + y^2 + (z+D)^2}}$$

However, here we have: BC:

$\varphi_z(0) = -\frac{1}{g}\varphi_{tt}(0)$ η has vanished from problem. } only interested in φ

Define transforms:

$$\bar{\varphi} = \iint e^{-i(\xi x + \eta y)}\, dx\, dy\, \varphi(x,y)$$

$$\bar{\varphi}_{zz} - (\xi^2 + \eta^2)\bar{\varphi} = \delta(z+D)\, e^{-i\xi u t}$$

Two solns: $\bar{\varphi} = A e^{-\sqrt{\xi^2+\eta^2}\,(z+D)} + B e^{+\sqrt{}\,(z+D)}$ $z > -D$

$\bar{\varphi} = C e^{\sqrt{\xi^2+\eta^2}\,(z+D)}$ $z < -D$

~~et~~ We have enough with the BC to get A, B, C.

Now change to $x - ut = x'$ as transform variable so as to eliminate time, taking as scale of length $L = \frac{u^2}{g}$

Then: $\overline{\varphi}_z = -\xi \overline{\varphi}$

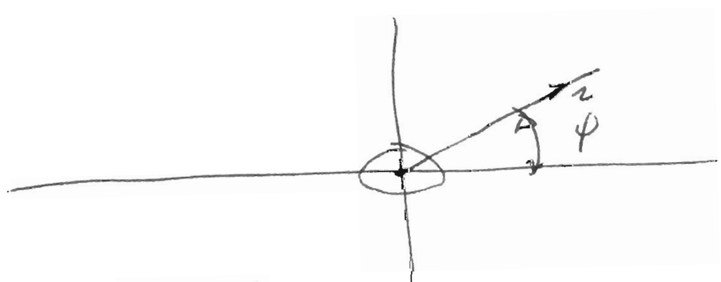
describable BC at free surface.

Now solve for A, B, C and get:

$$\varphi(x', y', z') = \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} \frac{1}{\xi^2 - \sqrt{\xi^2+\eta^2}} \, e^{-\sqrt{}\, D' + i\xi x' + i\eta y'} \, d\xi \, d\eta$$

D' is not large value compared to ~~x', z~~ $x'; y'$ because we are interested in large surface area for detection purposes: $D' \sim 1-20$ for distance down to few hundred feet.

Change to r, ψ system:



$x = r\cos\psi$

$y = r\sin\psi$

$\xi = \rho\cos\theta$

$\eta = \rho\sin\theta$

We take range of values

$\rho: -\infty \to +\infty$; $\theta: -\pi/2 \to \pi/2$

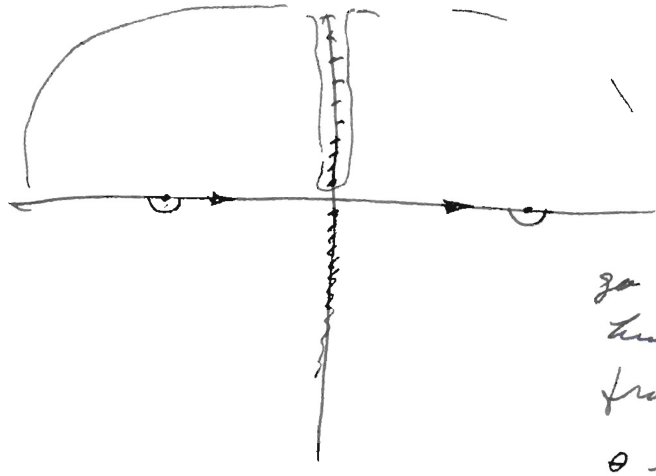
to use Cauchy Int Theor.

Must then take ~~ρ~~ $|\rho|$

④

$$\varphi(x',y',\phi') = \frac{1}{4\pi^2}\iint_{-\infty}^{\infty} \frac{|\rho|\,d\rho\,d\theta}{\rho^2\cos^2\theta - |\rho|}\, e^{-|\rho|D' + i\rho r\cos(\theta-\psi)}$$

In complex ρ plane:



go under because we know waves out in front are small because θ is less than ψ. This gives residues and weaker wave front and strong behind.

We through away branch line contribution because it is small due to $|\rho|$, subject to verification.

Taking residues:

$$\varphi = 2\,\mathrm{Re}\,\frac{i e^{-D/\cos^2\theta_0}}{4\pi\cos^2\theta_0}\int_{-\pi/2}^{\pi/2} e^{i r \frac{\cos(\theta-\psi)}{\cos^2\theta}}\, d\theta$$

evaluating small parameter ψ at saddle point.

$$\tan\psi = \frac{-\sin 2\theta_0}{2(1+\sin^2\theta_0)}$$

$\psi < 19.5^\circ$: 2 saddle pts.

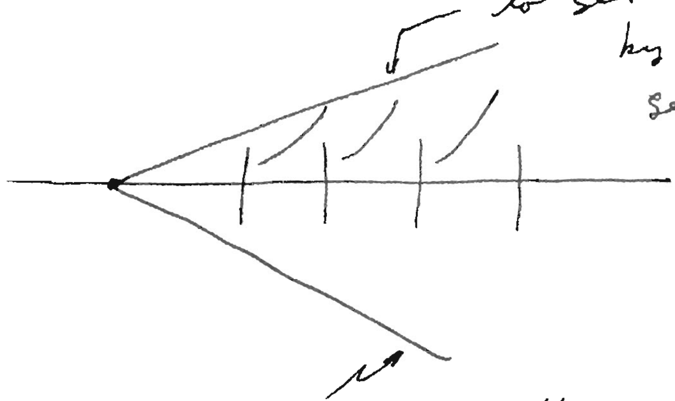
$= 19.5^\circ$: 1 saddle pt, $\theta_0 = 35.25^\circ$

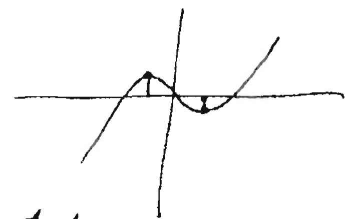
$> 19.5^\circ$: 0 saddle pts

Kelvin established stat. phase to evaluate $\varphi(r, \theta)$.

What he found:

to get behavior here, approx. by cubic in θ as s.p. get closer and closer together





goes as $r^{-1/3}$ so that waves grow as they go out.

AM203
11-27-61

Formal intro. to Eigenvalue pert. Theo.

$H\psi = E\psi$ $\qquad H_0 \psi_n = E_n \psi_n$

$H = H_0 + V$ $\qquad \langle \psi_n | \psi_m \rangle = \delta_{nm}$

hermitian

$|0\rangle = \psi_n \qquad E^0 = E_n$

$\psi = |0\rangle + |1\rangle + |2\rangle + \cdots \;;\; E = E^{(0)} + E^{(1)} + E^{(2)} + \cdots$

If $(H_0 - E^0)\psi = 0$, then $\psi = \text{const}\,|0\rangle$

$(H_0 - E^{(0)})\, g_0 = -1 \qquad \langle 0|n\rangle = \delta_{n0}$

$(H_0 - E^0)\, g_0 |0\rangle = -|0\rangle$

$\langle 0| (H_0 - E^0) g |0\rangle = -\langle 0|0\rangle$

$(H_0 - E^0) \sum c_m \psi_m = \sum (E_m - E_n) c_m \psi_m$

Can define inverse

$(H_0 - E^0)^{-1}\psi$

$Q\psi = \psi - \left(\int \psi_n^* \psi\, dx\right)\psi_n$

$= \psi - |0\rangle\langle 0|\psi\rangle \;;\; Q = 1 - |0\rangle\langle 0|$

or $Q^2 = Q$ and we have projection operator.

Then: $\left\{ \begin{array}{l} (H_0 - E^0) g_0 = -Q \\ \langle 0| g_0 = 0 \end{array} \right\} \quad \begin{array}{l} \psi = \sum c_m \psi_m \\ g_0 \psi = \sum c_m' \psi_m \end{array}$

$\langle 0 | g_1 | \psi \rangle = 0$

$C_n' = 0$

$(H_0 - E^0) g_1 | \psi \rangle = -Q | \psi \rangle$; also: $g_1 (H_0 - E^0) = -Q$

$$\sum (E_m - E_n) C_m' \psi_m = -\sum_{n \neq m} c_m \psi_m$$

If $m \neq n$, then $C_m' = \dfrac{C_m}{E_n - E_m}$

or: $$g_1 \psi = \sum_{m \neq n} \frac{C_m}{E_n - E_m} \psi_m$$

Retrieving:

$$(H_0 + H_1 - E^0 - E^{(1)} - \cdots)(|0\rangle + |1\rangle + |2\rangle + \cdots) = 0$$

$$(H_0 - E^0)|n\rangle = -H_1 |n-1\rangle + E^{(1)} |n-1\rangle + E^{(2)} |n-2\rangle + \cdots + E^{(n)} |0\rangle$$

Operate with $-Q$:

$$-Q|n\rangle = -g_1 H_1 |n-1\rangle + E^{(1)} g_1 |n-1\rangle + \cdots$$

For $n \geq 1$: $Q|n\rangle = (1 - |0\rangle\langle 0|)|n\rangle = |n\rangle$

Then:

$$|n\rangle = g_1 H_1 |n-1\rangle - E^{(1)} g_1 |n-1\rangle - E^{(2)} g_1 |n-2\rangle + \cdots$$

$$E^{(n)} = \langle 0 | H_1 | n-1 \rangle$$

$E^0 = E_m \quad , \quad |0\rangle = \psi_n$

$E^{(1)} = \langle 0 | H_1 | 0 \rangle$

$|1\rangle = -E^{(1)} g_1 |0\rangle + g_1 H_1 |0\rangle$

$= \cancel{g_1 |0\rangle} \quad g_1 H_1 |0\rangle$

$E^{(2)} = \langle 0 | H_1 g_1 H_1 | 0 \rangle$

$|2\rangle = -E^{(1)} g_1 |1\rangle + g_1 H_1 |0\rangle$

$= - g_1 g_1 H_1 |0\rangle \langle 0 | H_1 | 0 \rangle + g_1 H_1 g_1 H_1 |0\rangle$

$E^{(3)} = - \langle 0 | H_1 g_1 g_1 H_1 | 0 \rangle \langle 0 | H_1 | 0 \rangle$

$+ \langle 0 | H_1 g_1 H_1 g_1 H_1 | 0 \rangle$

$|3\rangle = -E^{(2)} g_1 |1\rangle - E^{(1)} g_1 |2\rangle + g_1 H_1 |2\rangle$

$= g_1 g_1 H_1 |0\rangle \langle 0 | H_1 g_1 H_1 | 0 \rangle + g_1 g_1 g_1 H_1 |0\rangle \langle 0 | H_1 | 0 \rangle \langle 0 | H_1 | 0 \rangle$

$- g_1 g_1 H_1 g_1 H_1 |0\rangle \langle 0 | H_1 | 0 \rangle$

$- g_1 H_1 g_1 g_1 H_1 |0\rangle \langle 0 | H_1 | 0 \rangle + g_1 H_1 g_1 H_1 g_1 H_1 |0\rangle$

$E^4 = \langle 0 | H_1 | 3 \rangle$

$$|4\rangle = -E^{(3)} g_1 |1\rangle - E^{(2)} g_1 |2\rangle - E^{(1)} g_1 |3\rangle + g_1 H_1 |3\rangle$$

$$= g_1 g_1 H_1 |0\rangle\langle 0| H_1 g_1 g_1 H_1 |0\rangle\langle 0| H_1 |0\rangle \quad 13$$

$$- g_1 g_1 H_1 |0\rangle\langle 0| H_1 g_1 H_1 g_1 H_1 |0\rangle \quad 10$$

$$+ g_1 g_1 g_1 H_1 |0\rangle\langle 0| H_1 |0\rangle\langle 0| H_1 g_1 H_1 |0\rangle \quad 12$$

$$- g_1 g_1 H_1 g_1 H_1 |0\rangle\langle 0| H_1 g_1 H_1 |0\rangle \quad 9$$

$$+ g_1 g_1 g_1 H_1 |0\rangle\langle 0| H_1 g_1 H_1 |0\rangle\langle 0| H_1 |0\rangle \quad 11$$

$$- g_1 g_1 g_1 g_1 H_1 |0\rangle\langle 0| H_1 |0\rangle\langle 0| H_1 |0\rangle\langle 0| H_1 |0\rangle \quad 8$$

$$+ g_1 g_1 g_1 H_1 g_1 H_1 |0\rangle\langle 0| H_1 |0\rangle\langle 0| H_1 |0\rangle \quad 7$$

$$+ g_1 g_1 H_1 g_1 g_1 H_1 |0\rangle\langle 0| H_1 |0\rangle\langle 0| H_1 |0\rangle \quad 6$$

$$- g_1 g_1 H_1 g_1 g_1 H_1 |0\rangle\langle 0| H_1 |0\rangle \quad 5$$

$$- g_1 H_1 g_1 g_1 H_1 |0\rangle\langle 0| H_1 g_1 H_1 |0\rangle \quad 4$$

$$+ g_1 H_1 g_1 g_1 g_1 H_1 |0\rangle\langle 0| H_1 |0\rangle\langle 0| H_1 |0\rangle \quad 3$$

$$- g_1 H_1 g_1 g_1 H_1 g_1 H_1 |0\rangle\langle 0| H_1 |0\rangle \quad 2$$

$$- g_1 H_1 g_1 H_1 g_1 g_1 H_1 |0\rangle\langle 0| H_1 |0\rangle \quad 1$$

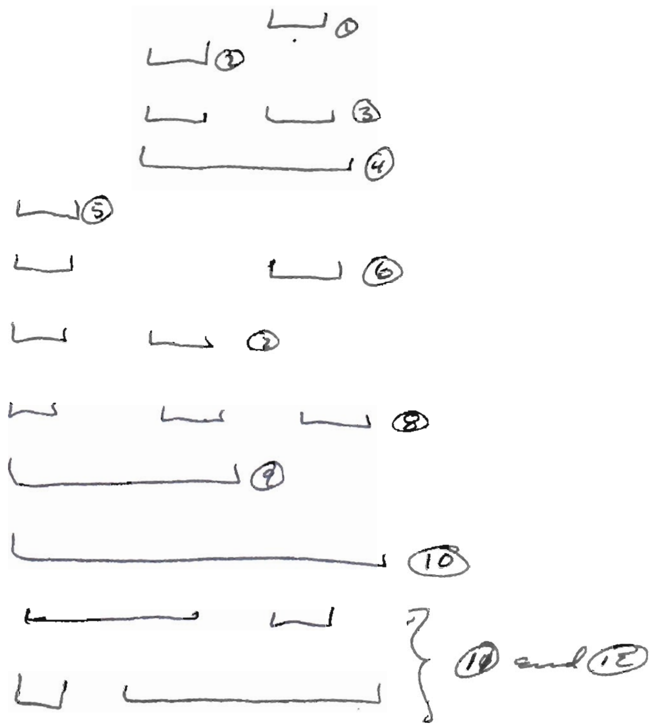
$$+ g_1 H_1 g_1 H_1 g_1 H_1 g_1 H_1 |0\rangle \quad 0$$

Recall $g_1|0\rangle = 0$

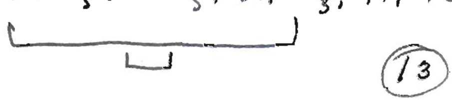
Rules: a. 0 has $g_1 H_1$ repeated n times

b. 1 take out H_1, add bracketed H_1

$g_1 H_1 g_1 H_1 g_1 H_1 g_1 H_1 |0\rangle$



$g_1 H_1 g_1 H_1 g_1 H_1 g_1 H_1 |0\rangle$



(29)

AM 203
11-29-61

①

Recall: assumptions:

1. $H_0 = H_0^+$

2. H, H_0, H_1 are linear

{ Note: we have not assumed $H_1 = H_1^+$

3. No degeneracy, that is, if $(H_0 - E^0)\psi = 0$ then $\psi = \text{const}\,|0\rangle$

→ E not necessarily real since H is not Hermitean

$$(H - E)\psi = 0$$

$$H_0 \psi_n = E_n \psi_n$$

$$\psi = |0\rangle + |1\rangle + \cdots$$

$$E = E^0 + E^1$$

$$Q = 1 - |0\rangle\langle 0|$$

$$Q = Q^+$$

$$(H_0 - E^{(0)})g_1 = -Q$$

$$g_1(H_0 - E^0) = -Q$$

$$\langle 0 | g_1 = 0$$

$$g_1 |0\rangle = 0$$

However, we now assume $H_1 = H_1^+$ and examine the special consequences.

$$E^{(6)} = \langle 0 | H_1 g_1 H_1 g_1 H_1 g_1 H_1 g_1 H_1 | 0\rangle + \cdots$$

Is $E^{(5)} = E^{(5)*}$?

Recall: $\langle a | O | b\rangle^* = \langle b | O^+ | a\rangle$

since: $\left(\int \psi_a^+ O \psi_b\right)^* = \left(\int (O^+\psi_a)^* \psi_b\right)^* = \int \psi_b^* O^+ \psi_a$

What is g_1^+ ?

$$\left.\begin{aligned} g_1^+(H_0 - E^0) &= -Q \\ (H_0 - E^0)g_1^+ &= -Q \\ g_1^+|0\rangle &= 0 \\ \langle 0 | g_1^+ &= 0 \end{aligned}\right\} \text{ hence } g_1 = g_1^+$$

Then:

$$E^{(5)*} = \langle 0 | \left(H_1 \, g_1 \, H_1 \, g_1 \, H_1 \, g_1 \, H_1 \, g_1 \, H_1 \right)^{\dagger} | 0 \rangle + \cdots$$

$$= \langle 0 | H_1 \, g_1 \, H_1 \, g_1 \, H_1 \, g_1 \, H_1 \, g_1 \, H_1 \, | 0 \rangle + \cdots$$

The next term would be:

$$\left[- \langle 0 | H_1 \, g_1 \, H_1 \, g_1 \, g_1 \, H_1 | 0 \rangle \langle 0 | H_1 \, g_1 \, H_1 \, | 0 \rangle \right]^*$$

$$= - \langle 0 | H_1 \, g_1 \, g_1 \, H_1 \, g_1 \, H_1 \, | 0 \rangle \langle 0 | H_1 \, g_1 \, H_1 \, | 0 \rangle$$

but another term will arise later on to give original above, hence, if $H_1 = H_1^{\dagger}$:

$$E^{(5)*} = E^{(5)}$$

Recall $E^{(2)}$: $\quad E^{(2)} = \langle 0 | H_1 \, g_1 \, H_1 \, | 0 \rangle = \sum_{m \neq n} \dfrac{\langle 0 | H_1 | \Psi_m \rangle \langle \Psi_m | H_1 | 0 \rangle}{E^{(0)} - E_m}$

$$= \sum_{m \neq n} \frac{|\langle \Psi | H_1 | 0 \rangle|^2}{E^{(0)} - E_m}$$

If $H_1 = H_1^*$, then $\langle 0 | H_1 | \Psi_m \rangle^* = \langle \Psi_m | H_1^{\dagger} | 0 \rangle = \langle \Psi_m | H_1 | 0 \rangle$

~~what~~ Why not assume $H_1 = H_1^{\dagger}$ to begin with?

1. not needed for formalism
2. There are problems in which $H_1 \neq H_1^{\dagger}$

Examples of $H_1 \neq H_1^{\dagger}$:

Pseudopotential:

$$H = H_0 + V = -\nabla^2 + V$$

$$H_1 = V = \begin{cases} \infty & r < a \\ 0 & r > a \end{cases}$$

$(-\nabla^2 + V)\,\psi = E\psi$

$\psi = 0,\ r < a$

The BV problem is $-\nabla^2\psi = E\psi$, $\psi = 0$, $r = a$

Far from a, we should expect a perturbation expansion. We can't apply blindly:

$$E^{(1)} = \langle 0 | H_1 | 0 \rangle = \int |\psi_m(\vec r)|^2 V\, d\vec r$$

$$= \infty \underbrace{\int_{r<a} |\psi_m(\vec r)|^2 d\vec r}_{\neq 0} \quad \text{, get nonsense}$$

Hence, we must try to replace V with a pseudopotential that behaves the V:

Write for coordinates r, θ, φ: neglecting θ, φ, $E = k^2$

$$\left(-\frac{d^2}{dr^2} - \frac{2}{r}\frac{d}{dr}\right)\psi = k^2\psi \quad , r > a \quad ; \ \psi(a) = 0$$

$$\psi = \begin{cases} \text{const } \dfrac{\sin k(r-a)}{r} & , r > a \\ 0 & , r < a \end{cases}$$

Replace by:

$$\psi' = \text{const } \frac{\sin k(r-a)}{r} \qquad \text{all } r$$

We now have question of what happen at origin.

$$(-\nabla^2 - k^2)\,\psi' \longrightarrow \delta(\vec r) \ \ldots ?$$

$$\int_{r<a} (-\nabla^2 - k^2)\,\psi' d\tau = \int_{r=\epsilon} -\frac{\partial \psi'}{\partial n}\, d\sigma = -4\pi\epsilon^2 \frac{d}{dr}\frac{c\sin k(r-a)}{r}\Big|_{r=\epsilon}$$

$\epsilon\,\psi'(\epsilon) = 0$ as $\epsilon \to 0$ $\qquad = -4\pi\, C \sin ka$

We want to find c:

$$\frac{d}{dr}(r\psi')\Big|_{r=0} = c\,k\cos k(r-a)\Big|_{r=0} = c\,k\cos ka$$

Then: $$c = \frac{1}{k\cos ka}\frac{d}{dr}(r\psi')\Big|_{r=0}$$

Then: $$(-\nabla^2 - k^2)\psi' = 4\pi\frac{\tan ka}{k}\,\delta(r)\frac{d}{dr}(r\psi')$$

$$\left(-\nabla^2 + 4\pi\frac{\tan ka}{k}\,\delta(r)\frac{d}{dr}r\right)\psi' = E\psi'$$

We see that $H_1' = 4\pi\frac{\tan ka}{k}\,\delta(r)\frac{d}{dr}r$

$$\approx 4\pi a\,\delta(\vec{r})\frac{d}{dr}r \qquad \neq H_1'^{\dagger}$$

~~These~~ now ψ and ψ' coincide in a large region so E is still real even tho $H_1' \neq H_1'^{\dagger}$.

Suppose $\delta(r)\frac{d}{dr}r$ operates on regular fn at $r=0$

$$\delta(r)\left(\frac{d}{dr}r\right)f = \delta(r)\left[f(r) + rf'\right] = \delta(r)f(0)$$

But if $f = \frac{1}{r}$, $\delta(r)\left(\frac{d}{dr}r\right)\frac{1}{r} = 0$

References: K. Huang + C. N. Yang ; PR 105, 767 (1957)

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①

AM 203
12-1-61

Recall: $(-\nabla^2 + V - E)\psi = 0$ $\qquad V = \begin{cases} \infty, & r < a \\ 0 & r > a \end{cases}$

$(-\nabla^2 + H_1' - E)\psi' = 0$

$\hookrightarrow 4\pi a\, \delta(r) \frac{\partial}{\partial r} r$

$$\left\{ -\nabla_1^2 - \nabla_2^2 + V(r_{1-2}) - E \right\} \psi = 0$$

~~Go~~ Go to center of mass coordinates and show like above:

$$\left(-\nabla^2 + \underbrace{8\pi a}_{4?}\, \delta(\vec{r}_{12}) \frac{\partial}{\partial r} r - E \right) \psi' = 0$$

Now consider 3-body problem:

$$\left[-\nabla_1^2 - \nabla_2^2 - \nabla_3^2 + V(r_{12}) + V(r_{23}) + V(r_{31}) - E \right] \psi = 0$$

$r_{12} = r_1 - r_2$, etc.

Set,
$$\left\{ -\nabla_1^2 - \nabla_2^2 - \nabla_3^2 + 8\pi a\, \delta(r_{12}) \frac{\partial}{\partial r_{12}} r_{12} + 8\pi a\, \delta(r_{23}) \frac{\partial}{\partial r_{23}} r_{23} \right.$$
$$\left. + 8\pi a\, \delta(r_{31}) \frac{\partial}{\partial r_{31}} r_{31} - E \right\} \psi' = 0$$

Can continue this problem with the perturbation theory outlined before. Find some strange things in the 4th order.

We now go to apply Pert. Theo. to ordinary differential equation. Ord. d.e. ~~of~~ are of form:

$$D_0 y + \epsilon D_1 y = 0 \qquad ; \quad y = f(x)$$

order of $D_1 \leq$ order of D_0

$$y = y_0 + \epsilon y_1 + \cdots$$

$$D_0 y_0 = 0 \qquad ; \qquad D_0 y_1 + \epsilon D_1' y_1 = 0 \quad , \text{ etc.}$$

~~[illegible]~~

Example: $u = f(x)$

$$(x + \epsilon u)\frac{du}{dx} + u = 0$$

$$x\frac{du}{dx} + u + \epsilon u \frac{du}{dx}$$

If we form inverted eq. $(x + \epsilon u) + u\frac{dx}{du} = 0$

can be solved : $\frac{d}{du}(xu) + \epsilon u = 0$

$$xu + \frac{\epsilon u^2}{2} + c = 0$$

BC. $\left.\begin{array}{l} u = 1 \\ x = 1 \end{array}\right\}$ interval of x : $(0, 1)$

$$xu + \frac{\epsilon u^2}{2} - \left(1 + \frac{\epsilon}{2}\right) = 0 \quad ; \quad x = -\frac{\epsilon u}{2} + \frac{1}{u}\left(1 + \frac{\epsilon}{2}\right)$$

$$u = \frac{-x + \sqrt{x^2 + 2\epsilon + \epsilon^2}}{\epsilon} = +\frac{x}{\epsilon}\left(-1 + \sqrt{1 + \frac{2\epsilon + \epsilon^2}{x^2}}\right)$$

Expand in p.s. about ϵ

$$u = \frac{1}{x} + \epsilon\left(\frac{1}{2x} - \frac{1}{2x^2}\right) + \epsilon^2\left(-\frac{1}{2x^2} + \frac{1}{2x^3}\right) + \cdots$$

no good for $x = 0$

Do pert. expansion; try to fix for $x = 0$.

$$u = u_0(x) + \epsilon u_1(x) + \epsilon^2 u_2(x) + \cdots$$

$$= \frac{u^{(0)}(x)}{x} + \epsilon\frac{u^{(1)}(x)}{x^3} + \epsilon^2\frac{u^{(2)}(x)}{x^5} + \cdots$$

taking out singularities gives derivatives of higher order. $u^{(1)}(x)$ has no sing. at $x = 0$

$$x\frac{d}{dx}\left(\frac{u^0}{x} + \epsilon\frac{u^1}{x^3} + \cdots\right) + \frac{u^0}{x} + \epsilon\frac{u^1}{x^3} + \cdots$$

$$+ \epsilon\left(\frac{u^0}{x} + \epsilon\frac{u^1}{x^3} + \cdots\right)\frac{d}{dx}\left(\frac{u^0}{x} + \epsilon\frac{u^1}{x^3} + \cdots\right) = 0$$

The term in:

$\underline{\epsilon^n}$:

$$-x\frac{u^{(n)}}{x^{2n+2}}(2n+1)+\frac{u^{(n)}}{x^{2n+1}}-\frac{u^{(0)}}{x}\frac{u^{(n-1)}}{x^{2n}}(2n-1)$$

$$-\frac{u^{(1)}}{x^3}\frac{u^{(n-2)}}{x^{2n-2}}(2n-3)-\cdots-\frac{u^{(n-1)}}{x^{2n}}\frac{u^{(0)}}{x}=0$$

$$u^{(n)}=-\frac{1}{2}\left(u^{(n-1)}u^{(0)}+u^{(n-2)}u^{(1)}+\cdots+u^{(0)}u^{(n-1)}\right)$$

Do by generating function:

$$U=\sum_{n=0}^{\infty}u^{(n)}z^n$$

$$U-1=-\frac{1}{2}zU^2\quad;\quad U(z=0)=1$$

$$u=\frac{1}{x}U\left(\frac{\epsilon}{x^2}\right),\quad U(z)=\frac{-1+\sqrt{1+2z}}{z}$$

$$u=\frac{1}{x}\,\frac{-1+\sqrt{1-\frac{2\epsilon}{x^2}}}{\epsilon/x^2}=\frac{-1+\sqrt{x^2+2\epsilon}}{\epsilon}$$

This is a very general method, and is the only one applicable to many recent problems like superconductivity.

We now consider the <u>Poincaré - Lighthill Method</u>

$$u=u_0(\xi)+\epsilon u_1(\xi)+\cdots$$

$$x=x_0(\xi)+\epsilon x_1(\xi)+\cdots$$

We change to the new independent variable ξ

Apply this to the above d.e.

$x \ll 1$
$u \ll 1$
$\xi \ll 1$

$$(\xi + \epsilon x_1 + \epsilon u_0)(u_0' + \epsilon u_1') + (x_0 + \epsilon u_1)(1 + \epsilon x_1') = 0$$

$$u_0 = \frac{1}{\xi}$$

First order:

$$\xi u_1' + u_1 - \left(x_1 + \frac{1}{\xi}\right)\frac{1}{\xi^2} + \frac{1}{\xi} x_1' = 0$$

Want to choose x_1 so that $\frac{1}{\xi^3}$ term doesn't appear. This means:

$$\frac{-\xi x_1' - x_1}{\xi^2} - \frac{1}{\xi^3} = 0 \quad ; \quad \left(\frac{x_1}{\xi}\right)' - \frac{1}{\xi^3} = 0$$

$$x_1 = \xi\left(-\frac{1}{2\xi^2} + \frac{1}{2}\right) = \frac{1}{2}\left(\xi - \frac{1}{\xi}\right)$$

This makes:

$$u_1 = 0.$$

Then: $\quad u = \frac{1}{\xi} \quad ; \quad x = \xi + \frac{\epsilon}{2}\left(\xi - \frac{1}{\xi}\right) = \frac{1}{u} - \frac{\epsilon}{2}\left(u - \frac{1}{u}\right)$

$$u x - 1 + \frac{\epsilon}{2}(u^2 - 1) = 0$$

which identifies with above.

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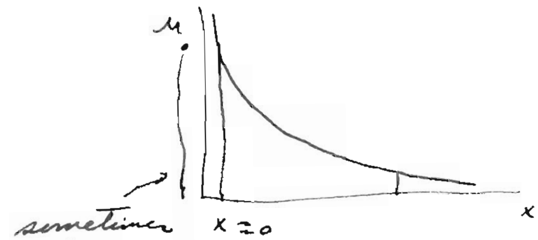
Indication of Failure of PL method:



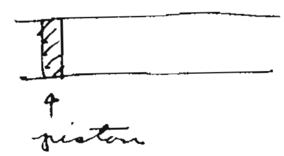
$u = u_0 + \epsilon u_1 + \cdots$

$x = z + \epsilon z_1 + \epsilon^2 x_2 + \cdots$

PL in Partial Diff. Eq.

Success only in hyperbolic cases.

1-D motion of gas. eq. of mot.



$$f_j(x_i, \epsilon) = g_i(z_i, \epsilon)$$

$$x_i = x_i(z_i, \epsilon)$$

ϵ is some input parameter, or amplitude, which is small.

Decide a priori that z_i will be characteristic variables.

End up with 4 d.e. We expect to deal with:

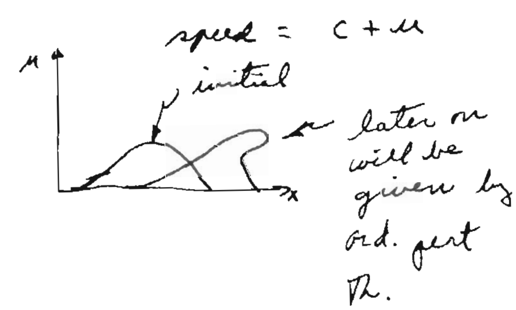
vel. $u(\alpha, \beta, \epsilon) = \ldots$

ρ " $= \ldots$

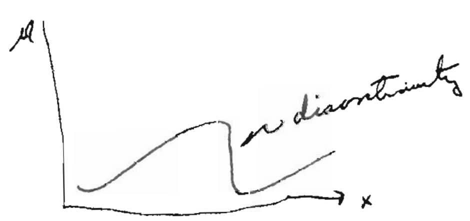
x " $= \ldots$

t " $= \ldots$

Ordinary pert. th. fails because



Real solution should look like:



Even if ord. pert. expansion in ϵ converged, it would still need many, many terms to describe discontinuity. So very inconvenient, may not even give spacial periodicity.

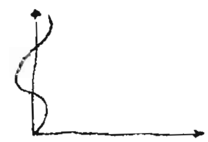
These difficulties motivate us to try pl method. Eq. of mot.

$$\rho_t + (\rho u)_x = 0 \qquad \text{(mass)}$$

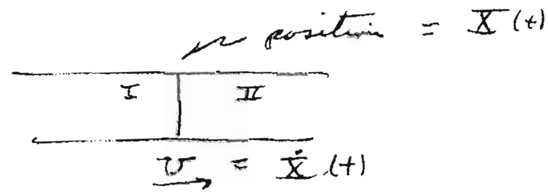
$$\rho u_t + \rho u u_x + p_x = 0 \quad \text{(momentum)}$$

$$P/p_0 = (\rho/\rho_0)^\gamma \quad \text{(eqation of state) (cons. of E)}$$

BC. $u(a\,\cancel{\sin}\,\omega t, t) = \omega a \cos \omega t$



These would be all eq. needed for continuous solu's. However, not continuous, so we must choose type of discontinuity:



How do we conserve mass thru discontinuity?

$$\rho(X(t)_-, t)\left[u(X(t)_-, t) - v(t)\right] = \rho(X(t)_+, t)\left[u(X(t)_+, t) - v(t)\right]$$

or: $\rho_1(u_1 - v) = \rho_2(u_2 - v)$

There will be many such discontinuities.

Cons. of mom. gives:

$$\rho_1(u_1 - v)u_1 - \rho_2(u_2 - v)u_2 = p_2 - p_1$$

Cons. of E gives:

$$\underbrace{\rho_1(u_1 - v)\frac{u_1^2}{2}}_{\text{convected kinetic}} + \underbrace{c_v\,\rho_1(u_1 - v)T_1}_{\text{convected internal}} + \underbrace{p_1 u_1}_{\text{work done by pressure}} = \rho_2(u_2 - v)\frac{u_2^2}{2} + \rho_2(u_2 - v)c_v T_2 + p_2 u_2$$

It is known that:

$$(u_1 - v)(u_2 - v) = c^{*2}$$

c^* = speed

The entropy change is of 3rd order in the shock intensity:

$$\Delta S \approx (u_1 - u_2)^3$$
$$\sim (p_1 - p_2)^3$$
$$\sim (\rho_1 - \rho_2)^3$$

Characteristics:

~~Propagation~~ Propagation des Ondes

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If we cons. small pert. then, multiple valued solution don't occur. This implies pert. series cannot converge since known solution is multiply-valued.

Recall:

$$(\rho u)_x + \rho_t = 0$$

$$\rho u u_x + \rho u_t + p_x = 0$$

$$\frac{p}{p_0} = (\rho/\rho_0)^\gamma$$

PL used in ord. d.e. to remove sing. from domain of interest. ~~The~~ Reason PL method used in p.d.e. is not clear, probably desperation.

$u = \epsilon \sin \omega t$

or problem gives hyperbolic eq.

Denote the characteristic equation as:

$\alpha =$ const.

$\beta =$ const.

The d.e. for the characteristics:

$$\left.\begin{aligned} x_\alpha &= (u+c)\,t_\alpha \\ x_\beta &= (u-c)\,t_\beta \end{aligned}\right\} \quad \gamma \frac{p}{\rho} = \frac{dp}{d\rho} = c^2$$

In terms of the characteristic variables:

$$\frac{u_\alpha}{2} + \frac{c_\alpha}{\gamma - 1} = 0$$

$$\frac{u_\beta}{2} - \frac{c_\beta}{\gamma - 1} = 0$$

Ref. Phyllis Fox : J. Math & Phys. Oct 1955

We can get 4 eq.

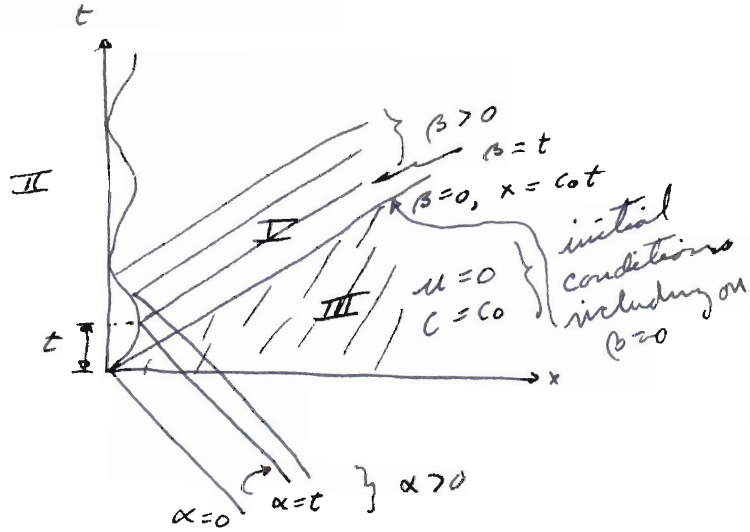
$$\begin{matrix} x \\ t \\ u \\ c \end{matrix} \left(\alpha, \beta, \omega, \epsilon \right)$$

We will do initial value problem since this gives good def. of outgoing waves for non-linear systems.

Expand:

$$X = X_0(\alpha\beta) + \epsilon X^{(1)}(\alpha,\beta) + \epsilon^2 X^{(2)}(\alpha,\beta) + \dots$$

and same for t, u, c



Can we say $\beta = t$, $\alpha = t$?

Suppose it isn't: can still write:

$t = f(\alpha)$

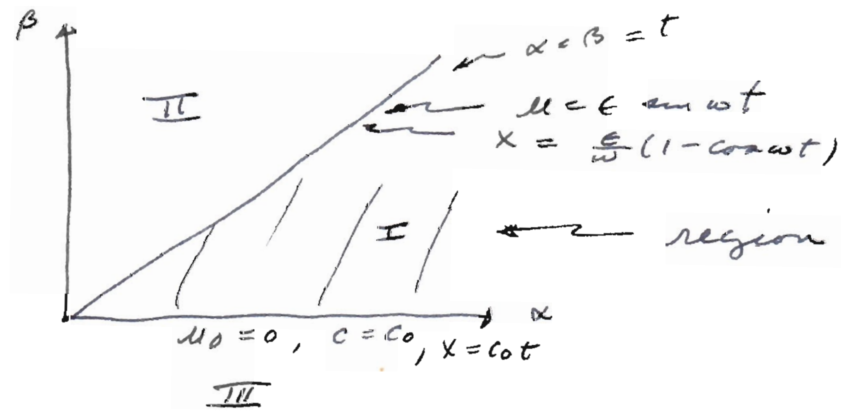
$t = g(\beta)$

and then say f, g are new independent ~~values~~ variables without loss of generality, seeing that above 4 d.e.'s are unchanged.

Then we say:

$t = \alpha = \beta$ on the piston path.

So:



We now try to solve: Set:

$$\left.\begin{aligned} \frac{u_\alpha^{(0)}}{2} + \frac{c_\alpha^{(0)}}{\gamma - 1} &= 0 \\ \frac{u_\beta^{(0)}}{2} - \frac{c_\beta^{(0)}}{\gamma - 1} &= 0 \end{aligned}\right\}$$

on $\alpha = \beta$; $u^{(0)} = 0$, $X^{(0)} = 0$, $t^{(0)} = \alpha = \beta$

on $\beta = 0$; $u^{(0)} = 0$, $c^{(0)} = 0$

$\boxed{u^{(0)} = 0 \quad c^{(0)} = c_0}$

③

$$x_\alpha^{(0)} = c_0 t_\alpha^{(0)} \qquad x^{(0)} - c_0 t^{(0)} = f(\beta)$$

$$x_\beta^{(0)} = -c_0 t_\beta^{(0)} \qquad x^{(0)} + c_0 t^{(0)} = g(\alpha)$$

$$x^{(0)} = \frac{f(\beta) + g(\alpha)}{2} \; ; \qquad c_0 t^{(0)} = \frac{g(\alpha) - f(\beta)}{2}$$

On boundary $\alpha = \beta$; $x^{(0)} = 0$; $f(\beta) = -g(\alpha)$

From $\alpha = \beta = t^{(0)}$

$$c_0 t^{(0)} = -\frac{[f(\alpha) + f(\beta)]}{2} \; ; \qquad -f(\alpha) = c_0 \alpha, \quad g(\beta) = c_0 \beta$$

We then get:

$$\boxed{\begin{aligned} x^{(0)} &= -c_0\beta + c_0\alpha \\ c_0 t^{(0)} &= \frac{c_0(\alpha+\beta)}{2} \end{aligned}}$$

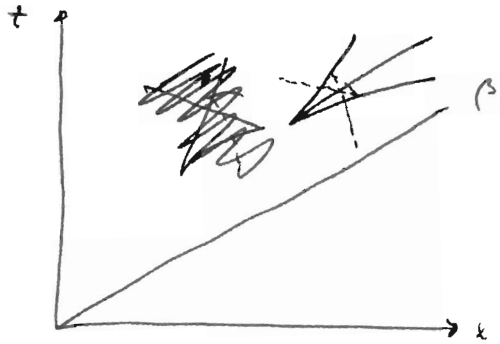
We have all zero:

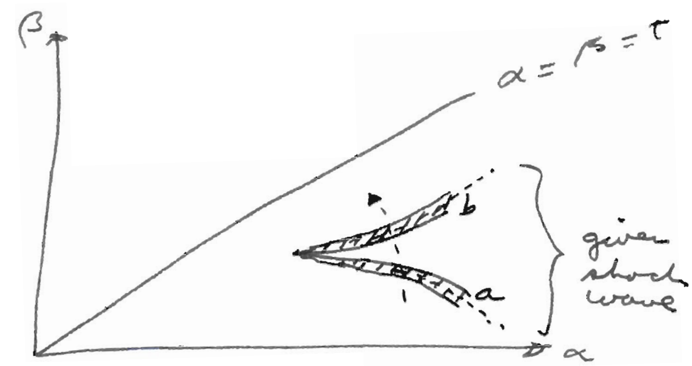
We ~~the~~ do 1st order: get same d.e. as before: subject to new BC. Set:

$$u^{(1)} = \sin \omega\beta \; ; \qquad c^{(1)} = \frac{\gamma-1}{2} \sin \omega\beta$$

$$x^{(1)} = \frac{1}{\omega} + \frac{\gamma+1}{8}\left[(\alpha-\beta)\sin\omega\beta - \frac{1}{\omega}\cos\omega\alpha + \frac{\gamma-7}{\gamma+1}\frac{\cos\omega\beta}{\omega}\right]$$

$$t^{(1)} = \frac{\gamma+1}{8}\left[-(\alpha-\beta)\sin\omega\beta - \frac{1}{\omega}\cos\omega\alpha + \frac{1}{\omega}\cos\omega\beta\right]$$


t
$\beta=0$
x


β
$\alpha=\beta=\tau$
b
a
given shock wave
α

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Calculus of Variations

1. Classical Mechanics

Lagrangian $L = L(q_i, \dot{q}_i, t)$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \quad ; \; i = 1 \cdots n \qquad (1)$$

Hamiltonian:

$$H = H(p_i, q_i, t) = \sum p_i \dot{q}_i - L$$

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

Vary the Hamiltonian:

$$\delta H = \sum \dot{q}_i \delta p_i + \cancel{\sum p_i \delta \dot{q}_i} \underbrace{- \sum \frac{\partial L}{\partial q_i}\delta q_i - \cancel{\sum \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i}}_{-\delta L}$$

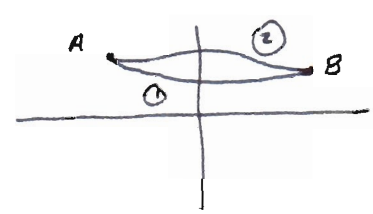
$$= \sum \dot{q}_i \delta p_i - \sum \frac{\partial L}{\partial q_i}\delta q_i$$

We see that:

$$\left.\frac{\partial H}{\partial q_i}\right|_{p_i,t} = -\left.\frac{\partial L}{\partial q_i}\right|_{\dot{q},t} = -\dot{p}_i \quad ; \quad \left.\frac{\partial H}{\partial p_i}\right|_{q_i,t} = \dot{q}_i$$

Hamilton's Principle: $\delta \int L \, dt = 0 \longleftrightarrow (1)$

Consider the path in q space:



differential difference in path.

$$\delta \int L\,dt = \int_{②} L\,dt - \int_{①} L\,dt$$

q_i ① $\dot{q}_i$ ①

$q_i + \delta q_i$ ② $\dot{q}_i + \frac{d}{dt}\delta q_i$ ②

Then:

$$\delta \int L\,dt = \int_{①} \delta L\,dt = \sum_i \int_{①} \left(\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\frac{d}{dt}\delta q_i \right) dt$$

int. by parts:

$$= \underbrace{\frac{\partial L}{\partial \dot{q}_i}\delta q_i \Big|_A^B}_{0 \text{ since } \delta q_i = 0 \text{ on boundaries.}} + \sum_i \int_{①} \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i\,dt = 0$$

which gives Hamilton's principle.

Now if $L = L(q_i, \dot{q}_i)$, then energy is conserved.

$H = H(q_i, p_i)$

Compute: $$\frac{dH}{dt} = \sum_i \left(\frac{\partial H}{\partial q_i}\dot{q}_i + \frac{\partial H}{\partial p_i}\dot{p}_i \right) = \sum_i \left(-\dot{p}_i \dot{q}_i + \dot{p}_i \dot{q}_i \right) = 0$$

Then $H = E$ (energy conserved) constant of motion.

and: $\sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L = E$ (holds on ①)

If this also holds on ②, then Hamilton's principle becomes:

$$\delta \int \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i}\,dt = 0$$

and becomes the <u>Action Principle</u>.

2. Classical Fields: ∞ degrees of freedom

Here we must work with Lagrangian and Hamiltonian densities

$$\mathcal{L} = \mathcal{L}\,(q_i, \dot{q}_i, \nabla q_i)$$

Lagrangian density:

Hamilton's principle is: $\delta \int dt \int_V d\vec{x}\, \mathcal{L} = 0$

$$\delta \int dt \int_V d\vec{x}\, \mathcal{L} = \int dt \int d\vec{x} \sum_i \left(\frac{\partial \mathcal{L}}{\partial q_i} \delta q_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \frac{d}{dt} \delta q_i + \frac{\partial \mathcal{L}}{\partial \nabla q_i} \cdot \nabla \delta q_i \right)$$

$$= \int dt \int_V d\vec{x}\, \underbrace{\left(\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \nabla \cdot \frac{\partial \mathcal{L}}{\partial \nabla q_i} \right)}_{0} \delta q_i$$

since $\displaystyle \frac{\partial \mathcal{L}}{\partial \nabla q_i} = \left(\frac{\partial \mathcal{L}}{\partial \frac{\partial q_i}{\partial x_1}}, \frac{\partial \mathcal{L}}{\partial \frac{\partial q_i}{\partial x_2}}, \frac{\partial \mathcal{L}}{\partial \frac{\partial q_i}{\partial x_3}} \right)$

$$\mathcal{L} = -\frac{1}{2} \sum_\mu \left(\frac{\partial q}{\partial x^\mu} \right)^2 - \frac{1}{2} m^2 q^2 = -\frac{1}{2} |\nabla q|^2 + \frac{1}{2} \dot{q}^2 - \frac{1}{2} m^2 q^2$$

Eq. of mot.

$$m^2 q - \frac{d}{dt} \dot{q} + \underbrace{\nabla \cdot \nabla}_{\nabla^2} q = 0 \quad ; \quad \Box^2 = \nabla^2 - \frac{d^2}{dt^2}$$

$$\Box^2 q - m^2 q = 0$$

3. EM field: $\mu_0 = \epsilon_0 = c = 1$

M's equations:

$$\begin{cases} \nabla \cdot \vec{B} = 0 \\ \nabla \times \vec{E} = -\dot{\vec{B}} \end{cases} \qquad \boxed{\begin{cases} \nabla \cdot \vec{D} = 0 \\ \nabla \times \vec{H} = \dot{\vec{D}} \end{cases}}$$

Will use these pair since $H = B$, $E = D$ in free space.

Potentials:

$$\vec{B} = \nabla \times \vec{A}$$

$$\left.\begin{array}{l} \vec{B} = \nabla \times \vec{A} \\ \nabla \times (\vec{\mathcal{E}} + \dot{\vec{A}}) = 0 \end{array}\right\} \quad \vec{E} = -\dot{\vec{A}} - \nabla\varphi$$

The dynamic variables are $\vec{A}, \varphi$:

$$\underset{\substack{\uparrow \\ \text{density}}}{\mathcal{L}} = \mathcal{L}(\underbrace{\vec{A}, \dot{\vec{A}}, \nabla\vec{A}, \varphi, \dot{\varphi}, \nabla\varphi}_{20 \text{ variables}})$$

However, field is really described by $\mathcal{E}, H, B, D$.

$$= \mathcal{L}(\vec{B}, \vec{\mathcal{E}})$$

$$= \mathcal{L}(\underbrace{\nabla \times \vec{A}, -\dot{\vec{A}} - \nabla\phi}_{6 \text{ variables}})$$

The missing of $\dot{\varphi}$ in last term ~~are~~ causes trouble.

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3. Electromagnetic Field: $\nabla\cdot\vec{B}=0$ $\qquad \vec{B}=\nabla\times\vec{A}$
$\nabla\times\vec{E}=-\dot{\vec{B}}$ $\qquad \vec{E}=-\dot{\vec{A}}-\nabla\varphi$ } dynamic variables $\vec{A},\varphi$

The Lagrangian density should be of form:

$$L=L(\vec{E},\vec{B})=L(-\dot{\vec{A}}-\nabla\varphi,\ \nabla\times\vec{A})$$

From Example 2:

$$\frac{\partial L}{\partial q_\alpha}-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\alpha}-\nabla\cdot\frac{\partial L}{\partial \nabla q_\alpha}=0 \quad ; \text{ from } \delta\int L\,dt\,d\vec{x}=0$$

where here: $q_\alpha\to A$: $\therefore$ $\frac{\partial L}{\partial \vec{A}}=0$; $\frac{\partial L}{\partial \dot{\vec{A}}}=-\frac{\partial L}{\partial \vec{E}}$

$$\nabla\cdot\frac{\partial L}{\partial A_1}=\frac{\partial}{\partial x_2}\frac{\partial L}{\partial\frac{\partial A_1}{\partial x_2}}+\frac{\partial}{\partial x_3}\frac{\partial L}{\partial\frac{\partial A_1}{\partial x_3}}=-\frac{\partial}{\partial x_2}\frac{\partial L}{\partial B_3}+\frac{\partial}{\partial x_3}\frac{\partial L}{\partial B_2}$$

$$=-\left(\nabla\times\frac{\partial L}{\partial\vec{B}}\right)_1 \quad , \therefore \quad \nabla\cdot\frac{\partial L}{\partial \vec{A}}=-\nabla\times\frac{\partial L}{\partial\vec{B}}$$

Then:

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial\vec{E}}+\nabla\times\frac{\partial L}{\partial\vec{B}}=0} \qquad q_\alpha=\vec{A}$$

$$\frac{\partial L}{\partial\varphi}=\frac{\partial L}{\partial\dot{\varphi}}=0 \quad ; \quad \frac{\partial L}{\partial\nabla\varphi}=-\frac{\partial L}{\partial\vec{E}}$$

$$\therefore \quad \boxed{\nabla\cdot\frac{\partial L}{\partial\vec{E}}=0} \qquad q_\alpha=\varphi$$

We infer: $\qquad$ we have: $\qquad$ and:

$$\left.\begin{array}{l}\boxed{\nabla\times\vec{H}=\dot{\vec{D}}}\\ \boxed{\nabla\cdot\vec{D}=0}\end{array}\right\} \quad \left.\begin{array}{l}\boxed{\vec{D}=\frac{\partial L}{\partial\vec{E}}}\\ \boxed{\vec{H}=-\frac{\partial L}{\partial\vec{B}}}\end{array}\right\} \quad L=\frac{1}{2}(E^2-B^2)$$

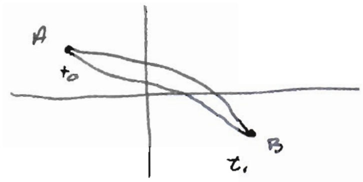
$$dL=\frac{\partial L}{\partial\vec{B}}d\vec{B}+\frac{\partial L}{\partial\vec{E}}d\vec{E}=-\vec{H}\,d\vec{B}+\vec{D}\,d\vec{E}$$

These procedures demonstrate the relevancy of the variational principle to phys. We return to our consideration of VP in detail.

1. Classical Part. mech.

Two VP's

① $\delta \int L\, dt = 0$



② $\delta \int \sum q_i \frac{\partial L}{\partial \dot q_i}\, dt = 0$; subject to.. $\left(\sum \dot q_i \frac{\partial L}{\partial \dot q_i} - L = E \right)$

Natural system: $L = T - V$

quad in $\dot q$ (for T); ind. of $\dot q$ (for V)

Then $H = T + V$

$$\sum q_i \frac{\partial L}{\partial \dot q_i} = H + L = 2T$$

The ② method is often more easy to apply. We ask what is connection between ① and ②?

$L = L(q_i, \dot q_i)$; $i = 1 \cdots n$; $j = 2 \cdots n$

We change to new set of variables:

$$q_i' = \frac{dq_i}{dq_1} = \frac{\dot q_i}{\dot q_1} \; , \quad q_1' = 1$$

Write: $L = \Omega(q_i, \dot q_1, q_j')$

$$\frac{\partial \Omega}{\partial \dot q_1} = \frac{\partial L}{\partial \dot q_1} + \sum_j q_j' \frac{\partial L}{\partial \dot q_j} = \frac{1}{\dot q_1} \sum_i \dot q_i \frac{\partial L}{\partial \dot q_i}$$ note similarity

$$\dot q_1 = \dot q_1 (q_i, q'_j ; E) \qquad \sum \dot q_i \frac{\partial L}{\partial \dot q_i} - L = E$$

Take: $L' = \left.\frac{\partial \Omega}{\partial \dot q_1}\right|_{\dot q_1 = \dot q_1 (q_i, q'_j ; E)} = L'(q_i, q'_j)$

$$\dot q_1 \overbrace{\frac{\partial \Omega}{\partial \dot q_1}}^{L'} - \Omega = E$$; Take der. w.r.t. q'_j

$$\underbrace{\frac{\partial \dot q_1}{\partial q'_j} \frac{\partial \Omega}{\partial \dot q_1} + \dot q_1 \frac{\partial^2 \Omega}{\partial \dot q_1 \partial q'_j} + \dot q_1 \frac{\partial^2 \Omega}{\partial \dot q_1^2} \frac{\partial \dot q_1}{\partial q'_j} - \frac{\partial \Omega}{\partial \dot q_1} \frac{\partial \dot q_1}{\partial q'_j}}_{0}$$

$$- \frac{\partial \Omega}{\partial \dot q'_j} = 0$$

$$\dot q_1 \frac{\partial^2 \Omega}{\partial \dot q_1 \partial q_i} + \dot q_1 \frac{\partial^2 \Omega}{\partial \dot q_1^2} \frac{\partial \dot q_1}{\partial q_i} - \frac{\partial \Omega}{\partial q_i} = 0$$

$$\frac{\partial L'}{\partial q'_j} = \frac{\partial^2 \Omega}{\partial \dot q_1 \partial \dot q'_j} + \frac{\partial^2 \Omega}{\partial \dot q_1^2} \frac{\partial \dot q_1}{\partial q'_j} = \frac{1}{\dot q_1} \frac{\partial \Omega}{\partial \dot q'_j} = \frac{\partial L}{\partial \dot q_j}$$

$$\frac{\partial L'}{\partial q_i} = \frac{1}{\dot q_1} \frac{\partial \Omega}{\partial q_i} = \frac{1}{\dot q_1} \frac{\partial L}{\partial q_i}$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot q_j} = \frac{\partial L}{\partial q_j} \quad ; \quad \boxed{\frac{d}{dq_1} \frac{\partial L'}{\partial q'_j} = \frac{\partial L'}{\partial q_j}}$$

reduced Lagrange eq.

We have eliminated time and found a new Lagrangian that satisfies Lagrange eq.

so: new Ham. Princ. is $\delta \int L' \, dq_1 = 0$

plugging in for L' we get:

$$\delta \int \frac{1}{\dot{q}_1} \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} \, dq_1 = 0$$

$$\downarrow$$

$$\delta \underbrace{\int \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} \, dt}_{\text{action}} = 0$$

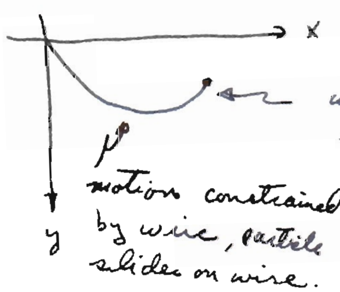
(Action Principle)
Lagrange: 1761

sometimes called principle of least action.

$\boxed{\text{Homework: Read about "kinetic focus" Whittaker "Analytical Dynamics"}}$

We now take up a simple example, not related to mech. however.

Path of quickest descent



or what is path so particle reaches here quickest?

Take $2g = 1$ for gravity

$$v = \sqrt{y}$$

$$\frac{ds}{dt}$$

$$dt = \frac{ds}{\sqrt{y}} = \sqrt{\frac{1+y'^2}{y}}\, dx \qquad ds^2 = dx^2 + dy^2, \quad y' = \frac{dy}{dx}$$

$$I = \int_0^{x_0} \underbrace{\sqrt{\frac{1+y'^2}{y}}}_{L} \, \underbrace{dx}_{t} \quad (1)$$

Can reduce by one variable: and get for constraint:

$$y' \frac{\partial L}{\partial y'} - L = E \quad (2)$$

Problem: derive (2) from (1) using usual procedure.

Continuing:

$$\frac{\dfrac{y'^2}{y}}{\sqrt{\dfrac{1+y'^2}{y}}} - \sqrt{\frac{1+y'^2}{y}} = E$$

$$dy = \frac{1}{E}\sqrt{\frac{1}{y} - E^2}\; dx$$

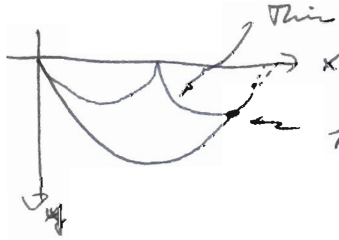
Define: $\quad d\theta = \dfrac{dx}{y} \quad ; \quad d\theta = \dfrac{dy}{\sqrt{\frac{y}{E^2} - y^2}}$

$$\theta = \cos^{-1}(1 - 2E^2 y) \quad ; \quad y = \frac{1}{2E^2}(1 - \cos\theta)$$

and:
$$x = \frac{1}{2E^2}(\theta - \sin\theta)$$

parametric equations of cycloid

This not physically allowed.



path thru point fixes E

(35)

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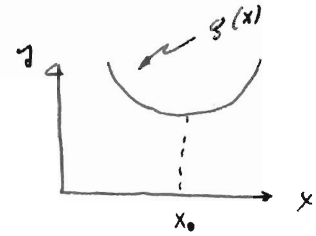
(1)



x_0 is minimum, around $(x_0-\epsilon, x_0+\epsilon)$

$g(x) \geq g(x_0)$

$g'(x) = 0$ at x_0

$g''(x) \geq 0$ at x_0 ← necessary

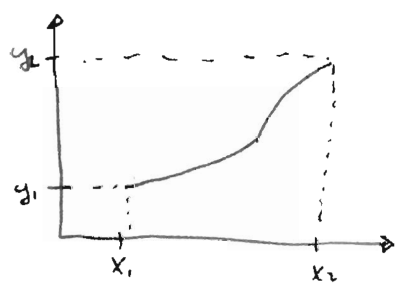
$g''(x) > 0$ at x_0 ← sufficient

We will consider:

$$I = \int_{x_1}^{x_2} f(x,y,y')\,dx$$ with fixed end points $\left.\begin{matrix}(x_1, y_1)\\(x_2, y_2)\end{matrix}\right\} x_2 > x_1$

Soln: $y = y(x)$ is cont. but y' is not nec. cont.

Ex. $f = y^2 (y'-1)^2$ (Erdmann)



Now vary I: $y \to y + \delta y$; $y' \to y' + \frac{d}{dx}\delta y = y' + \delta y'$

$$I(y+\delta y) - I(y) = \int_{x_1}^{x_2} \{ f(x, y+\delta y, y'+\delta y') - f(x,y,y')] \, dx$$

$$= \underbrace{\int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\delta y' \right] dx}_{\delta I} {}^{=0} + \underbrace{\frac{1}{2} \int_{x_1}^{x_2} \left\{ \frac{\partial^2 f}{\partial y^2}(\delta y)^2 + 2\frac{\partial^2 f}{\partial y \partial y'}(\delta y)(\delta y') + \frac{\partial^2 f}{\partial y'^2}(\delta y')^2 \right] dx}_{\delta^2 I} \geq 0$$

$g'(x) = 0$ corr. to $\delta I = 0$: extremum

$g''(x) \geq 0$ " " $\delta^2 I \geq 0$: minimum

Euler - Lagrange Eq.

$$\int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} \delta y + \frac{\partial f}{\partial y'} \delta y' \right] dx = 0 \quad ; \quad \left. \begin{array}{l} \delta y = 0 \text{ at } x = x_1 \\ \phantom{\delta y = 0 \text{ at }} x = x_2 \end{array} \right\} \text{fixed end pts.}$$

$$\frac{dh}{dx} = \frac{\partial f}{\partial y} \quad ; \quad h = \int_{x_0}^{x} \frac{\partial f}{\partial y} dx$$

Then: int by parts

$$\underbrace{h \, \delta y \Big|_{x_1}^{x_2}}_{0} - \int_{x_1}^{x_2} \left(h - \frac{\partial f}{\partial y'} \right) \delta y' \, dx = 0$$

Since $\int_{x_1}^{x_2} \delta y' \, dx = \delta y \Big|_{x_1}^{x_2} = 0$, we choose:

$$\delta y' = \alpha \left[h + \frac{\partial f}{\partial y'} + c \right] \quad ; \quad c = \frac{- \int_{x_1}^{x_2} dx \left(h - \frac{\partial f}{\partial y} \right)}{x_2 - x_1}$$

α: small constant

We can add a constant c to with no change and form square:

$$\int_{x_1}^{x_2} \left(h - \frac{\partial f}{\partial y'} + c \right)^2 dx = 0$$

then integrand $= 0$

$$\boxed{\int_{x_0}^{x} \frac{\partial f}{\partial y} dx - \frac{\partial f}{\partial y'} + c = 0} \quad \text{take } \frac{d}{dx} \text{ & set} \quad \boxed{\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} = 0}$$

Euler - Lagrange eq.

See that $\frac{\partial f}{\partial y'}$ is cont. since $\int_{x_0}^{x} \frac{\partial f}{\partial y}\,dx$, C are cont.

<u>Weierstrass ~~Cond~~ Corner Cond. I</u>:

→ $\frac{\partial f}{\partial y'}$ cont at corner

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0 \longrightarrow \frac{\partial f}{\partial y} - \frac{\partial^2 f}{\partial x\,\partial y'} - y'\frac{\partial^2 f}{\partial y\,\partial y'} - y''\frac{\partial^2 f}{\partial y'^2} = 0$$

For the special case: $\frac{\partial^2 f}{\partial y'^2} \equiv 0$, then $\underbrace{f = M(x,y) + N(x,y)\,y'}_{\text{linear in } y'}$

Identity Case: subs. f in above:

$$\frac{\partial M}{\partial y} + \frac{\partial N}{\partial y}y' - \frac{\partial N}{\partial x} - y'\frac{\partial N}{\partial y} = 0 \quad ; \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

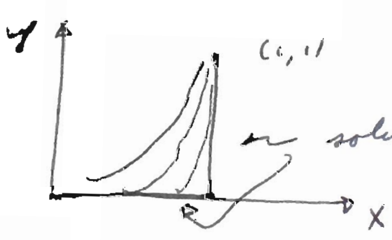
V: $M = \frac{\partial V}{\partial x}$, $N = \frac{\partial V}{\partial y}$; exact

Subst. $f = \frac{\partial V}{\partial x} + \frac{\partial V}{\partial y}y'$ in $I = \int_{x_1}^{x_2} f\,dx$

Then:
$$I = \int_{x_1}^{x_2}\left(\frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy\right) = V(x_2\,y_2) - V(x_1\,y_1)$$

No ~~Identity case~~: ~~$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$~~ If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ is not an identity Then can determine $y = y_1(x)$

Ex. $I = \int_0^1 y^2\,dx$, $(0,0)$ to $(1,1)$



Here $M = y^2$; $\frac{\partial M}{\partial y} = 0 = 2y$

$y = 0$ is solution

solution for $I = 0$, but not admissible because not single valued.

From here on assume $\frac{\partial^2 f}{\partial y'^2} \neq 0$

but doesn't pass thru end pts, hence not solution.

Recall path of quickest descent: $f = \sqrt{\dfrac{1+y'^2}{y}}$

$$\frac{\partial f}{\partial y'} = \frac{1}{\sqrt{y}} \frac{y'}{\sqrt{1+y'^2}}$$ must be continuous

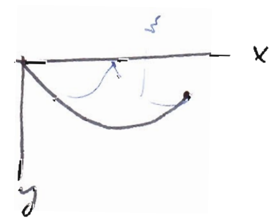
Hence

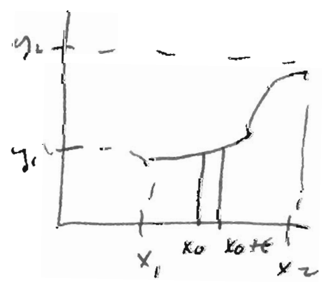
$$\frac{\partial f}{\partial y'}(x_0, y_0, y_1') = \frac{\partial f}{\partial y'}(x_0, y_0, y_2')$$

then:

$$\frac{y_1'}{\sqrt{1+y_1'^2}} = \frac{y_2'}{\sqrt{1+y_2'^2}} \Rightarrow y_1' = y_2'$$

No corner if $f = g(x\,y)\sqrt{1+y'^2}$





$\delta y = 0$ except $(x_0, x_0+\epsilon)$

Then: $$\delta^2 I \rightarrow \int_{x_0}^{x_0+\epsilon} \left(\frac{\partial^2 f}{\partial y'^2} (\delta y')^2 \right) dx \geq 0$$

$$\delta y = \int \delta y' dx \sim (\delta y')\epsilon$$

and we get Legendre's condition of necessity: $\dfrac{\partial^2 f}{\partial y'^2} \geq 0$
for the minimum.

Exchange of Variables

Recall $$I = \int_{x_1}^{x_2} f(x, y, y')dx = \int_{y_1}^{y_2} \bar{f}(\bar{x}, \bar{y}, \bar{y}')d\bar{x}$$

$x_2 > x_1$
$y_2 > y_1$

Obviously: $\bar{x} = y$, $\bar{y} = x$ $\quad \bar{y}' = \dfrac{d\bar{y}}{d\bar{x}} = \dfrac{dx}{dy} = \dfrac{1}{y'}$

$\bar{f}\, dy = f\, dx$, $\bar{f} = \dfrac{f}{y'}$: Better have $y' \geq 0$

The Euler-Lagrange equation is:

$$\frac{\partial \bar{f}}{\partial \bar{y}} - \frac{d}{d\bar{x}} \frac{\partial \bar{f}}{\partial \bar{y}'} = 0 \quad ; \quad \frac{\partial \bar{f}}{\partial \bar{y}} = \frac{\partial}{\partial x} \frac{f}{y'}$$

$$\frac{1}{y'} \frac{\partial f}{\partial x} - \frac{d}{dy} \frac{\partial \frac{f}{y'}}{\partial \frac{1}{y'}} = 0 \qquad = \frac{\partial}{\partial \bar{y}} \bar{f} = \frac{1}{y'} \frac{\partial f}{\partial x}$$

$$\frac{\partial f}{\partial x} - \frac{d}{dx}\left(f - y' \frac{\partial f}{\partial y'}\right) = 0$$

$$\frac{\partial f}{\partial x} - \left\{ \frac{\partial f}{\partial x} + y' \frac{\partial f}{\partial y} + y'' \frac{\partial f}{\partial y'} - y'' \frac{\partial f}{\partial y'} - y \frac{d}{dx} \frac{\partial f}{\partial y'} \right\} = 0$$

or $\dfrac{\partial f}{\partial y} - \dfrac{d}{dx} \dfrac{\partial f}{\partial y'} = 0$ so same E-L equation back again.

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Recall: determine condition for minimum of

$$I = \int_{x_1}^{x_2} f(x, y, y')\, dx \qquad x_2 > x_1$$

1. $\dfrac{\partial f}{\partial y} = \dfrac{d}{dx}\dfrac{\partial f}{\partial y'}$ — Euler Eq.

2. $\dfrac{\partial^2 f}{\partial y'^2} \geq 0$ Leg. (for minimum)

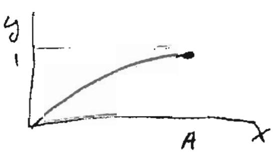
nec. under each.

3. $\dfrac{\partial f}{\partial y'}$ ~~const.~~ cont. at corner Weierstrass I

$y' \dfrac{\partial f}{\partial y'} - f$ ~~const.~~ cont. at corner " II (under each.)

Example: $f = y'^2 (y'-1)^2$; $(0,0)$ to $(A, 1)$

suppose only know 1 & 2, nothing about corners.



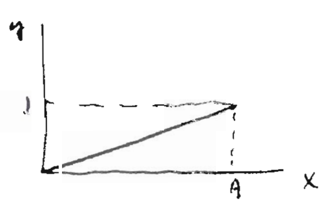
1. $\dfrac{\partial f}{\partial y'} = \text{const} \quad \therefore \quad y' = a$

$y = ax + b \, , \quad y = \dfrac{x}{A}$

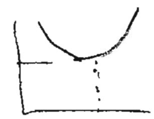
2. $\dfrac{\partial f}{\partial y'} = 4y'^3 - 6y'^2 + 2y'$

$\dfrac{\partial^2 f}{\partial y'^2} = 12y'^2 - 12y' + 2 \geq 0$ not if $\dfrac{3-\sqrt{3}}{6} \overset{?}{\leq} y' \overset{?}{\leq} \dfrac{3+\sqrt{3}}{6}$

This is satisfied for $A >$ certain A_0 , for ex, $A_0 = 5$ is OK, because $y' = \dfrac{1}{A}$



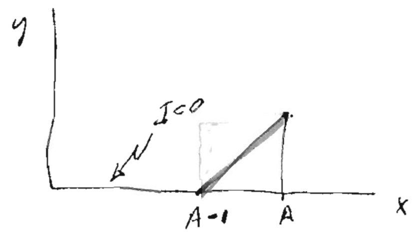
$\delta I = 0$, $I = 0$



Examine int.

$$I = \int_0^A y'^2 (y'-1)^2 \, dx$$

$I \geq 0$: For $I = 0$, $f = 0$, if $y' = 0, 1$



Look at 3.

should have:

$$\left.\begin{aligned} 4y_r'^3 - 6y_r'^2 + 2y_r' &= 4y_l'^3 - 6y_l'^2 + 2y_l' \\ 3y_r'^4 - 4y_r'^3 + y_r'^2 &= 3y_l'^4 - 4y_l'^3 + y_l'^2 \end{aligned}\right.$$

$y_r' \neq y_l'$ because of different roots.

(exch of variable)

$y_r' \neq y_l'$ → under this constraint only two solns:

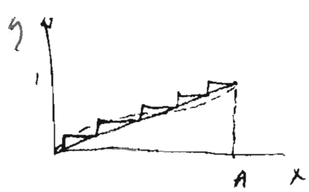
(must have corner)

$$\begin{cases} y_r' = 0 \\ y_l' = 1 \end{cases} \qquad \begin{cases} y_r' = 1 \\ y_l' = 0 \end{cases}$$

types of corners

admissible

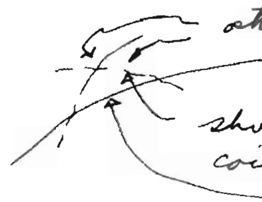
The straight can't be soln (give $I = 0$), but one neighboring could be:



straight line gives weak minimum. We have found strong minimum, δy infinitesimal, $\frac{d}{dx}\delta y$ not ~~necessarily~~ infinitesimal (at corner, e.g.)
Hence need further condition for strong minimum.

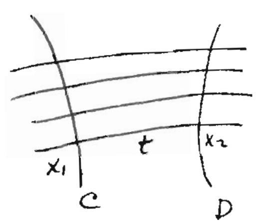
4. Weierstrass condition

Suppose have extremal curve:


other extremal curves

should have minimum when coincides with

More generally:



introduce parameter t:

$$X_1 = x_1(t)$$
$$X_2 = x_2(t)$$

$$f(x, y, y'; t)$$

$$I = \int_{x_1(t)}^{x_2(t)} f\, dt$$

$$\frac{df}{dt} = \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial x}\frac{dx}{dt}$$

Want $\displaystyle \frac{dI}{dt} = \frac{dx}{dt} f \Big|_1^2 + \int_{x_1(t)}^{x_2(t)} \frac{df}{dt}\, dx$

$$\frac{\partial f}{\partial y} = \left.\frac{d}{dx}\frac{\partial f}{\partial y'}\right\} + \frac{\partial f}{\partial y'}\frac{dy'}{dt}$$

$$= \left[f\frac{dx}{dt} + \frac{\partial f}{\partial y'}\left(\frac{dy}{dt} - y'\frac{dx}{dt}\right)\right]_1^2 \quad \geq 0 ?$$

using E-L condition.

(2):

variation with t: $\frac{dy}{dt}$, $\frac{dx}{dt}$

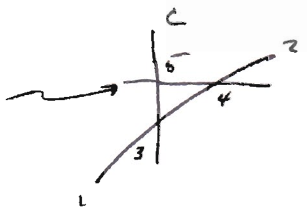
$y' = \frac{dy}{dx}$ along extremals

$$\int_{x_1(t)}^{x_2(t)} \frac{d}{dx}\frac{\partial f}{\partial y'}\frac{dy}{dt}\, dx$$

int. by parts.

④

not straight lines



Path 3−5−4 : J(5)

$$J(5) = \int_{x_3}^{x_5} f(x, Y, Y')\,dx + I(54)$$

C: $Y = Y(x)$

The Weierstrass condition says: $\left.\dfrac{\partial J(5)}{\partial X_5}\right|_{x_5 = x_3} \geq 0$
(Here x is parameter +)

Since:

$$f(x_3, y_3, Y_3') - f(x_3, y_3, y_3') - \frac{\partial f}{\partial y'}(x_3\, y_3\, y_3')\,(Y_3' - y_3') \geq 0$$

since X_3 is an arbitrary point.

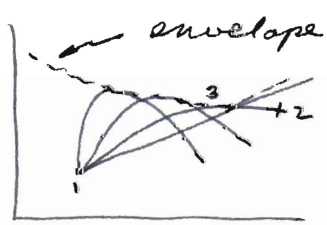
$$f(x, y, Y') - f(x, y, y') - \frac{\partial f}{\partial y'}(Y' - y') \geq 0 \quad \text{for all } Y' \neq y'$$

This is cond. ④

5. Jacobi condition:

suppose family of extremal curves:

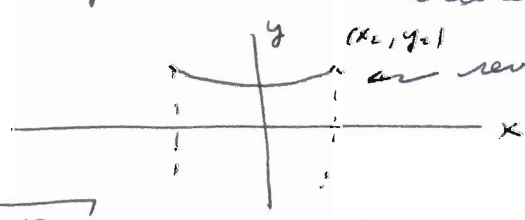
If 3 is conjugate to 1, then $x_3 \geq x_2$
Called kinetic focus in mechanics



Jac. cond.
in finding min from 1 to 2
there is no 3 tangent to ---- curve.

Problem

soap bubble between two rings.



as revolve about x-axis
find minimum for curve.

$f = y\sqrt{1 + y'^2}$; critical pt: $y_2 \sim 1.5\, x_2$

Ref. E.T. Whittaker Anal. dyn.

O. Bolza Lectures in calc. of var. Reprint

G. A. Bliss " "

In other notation I. → 1, 3 II. → 4

III. → 2 IV → 5

II_N : $E(\bar{x}, \bar{y}_\cdot, \bar{y}_\cdot' Y') \geq 0$

for arbitrary $Y' \neq \bar{y}'$ and

$(\bar{x} - x) < \epsilon$

$|\bar{y} - y| < \epsilon$

$(\bar{y}' - y') < \epsilon$

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misc. Remarks:

1. $I = \int f(x, y, z, y', z')\,dx$ $\quad (x_1, y_1, z_1)$ $(x_2\, y_2\, z_2)$

I. $\frac{\partial f}{\partial y} = \frac{d}{dx}\frac{\partial f}{\partial y'}$; $\frac{\partial f}{\partial z} = \frac{d}{dx}\frac{\partial f}{\partial z'}$

$\frac{\partial f}{\partial y'}$, $\frac{\partial f}{\partial z'}$, $y'\frac{\partial f}{\partial y'} + z'\frac{\partial f}{\partial z'} - f$ cont. at corner

Weierstrauss E fn.

II. $E(x, y, z, y', z', Y', Z') = f(x, y, z, Y', Z')$

$\frac{\partial^2 f}{\partial y'^2} \geq 0$

$- f(x, y, z, y', z') - (Y' - y')\frac{\partial f}{\partial y'}$

$- (Z' - z')\frac{\partial f}{\partial z'} \geq 0$ for $(Y' Z') \neq (y', z')$

III. $\eta^2 \frac{\partial^2 f}{\partial y'^2} + 2\eta\zeta \frac{\partial f}{\partial y' \partial z'} + \zeta^2 \frac{\partial^2 f}{\partial z'^2} \geq 0$ for $\eta^2 + \zeta^2 = 1$

IV. All other cond. inv. under change of variables.

2. Lagrange Multipliers

Min. $f(x, y)$ with $g(x, y) = 0$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$$

$$0 = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy$$

$$\frac{\frac{\partial f}{\partial x}}{\frac{\partial g}{\partial x}} = \frac{\frac{\partial f}{\partial y}}{\frac{\partial g}{\partial y}} \equiv \lambda \quad ; \quad \frac{\partial}{\partial x}(f - \lambda g) = \frac{\partial}{\partial y}(f - \lambda g) = 0$$

②

$h = f - \lambda g$ ext h with $g = 0$

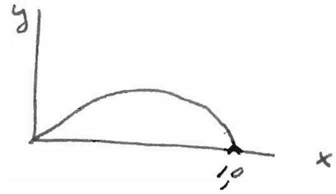
$\uparrow$ LM

min. $I = \int f(x, y\, y')\, dx$ with $J = \int g(x, y\, y')\, dx = J_0$

Isoparametric problem.

Example:

$\delta I - \lambda \delta J = 0$



$f = -y$

$g = \sqrt{1 + y'^2}$

$$0 = \int dx\, \delta y' \left(\frac{\partial f}{\partial y'} - \int_{x_0}^{x} \frac{\partial f}{\partial y}\, dx' \right)$$

$$0 = \int dx\, \delta y' \left(\frac{\partial g}{\partial y'} - \int_{x_0}^{x} \frac{\partial g}{\partial y}\, dx' \right)$$

$$0 = \int dx\, \delta y' \quad ?$$

$$\frac{\partial f}{\partial y'} - \int_{x_0}^{x} \frac{\partial f}{\partial y}\, dx' - \lambda \left(\frac{\partial g}{\partial y'} - \int_{x_0}^{x} \frac{\partial g}{\partial y}\, dx \right) = C$$

$h = f - \lambda g$

I: $\frac{\partial h}{\partial y} = \frac{d}{dx} \frac{\partial h}{\partial y'}$

$\frac{\partial h}{\partial y'}$, $h + y' \frac{\partial h}{\partial y'}$ cont.

3. Constraints

Want to min. $\int f(x, y, z, y', z')\, dx$
subject to

$$g(x, y, z, y', z') = 0$$

previously λ was number, here λ will be fn.

Replace g by system of constraints:

say:

$$J(x_0) = g(x_0, y, z, y', z') = \int dx\ \delta(x - x_0)\, g(x, y, z\, y' z')$$

get continuum no. of constraints
which is in same form as before.

$$h = f - \lambda g$$

$$h(x y z\ y' z') = f(x y z\ y' z') - \int dx_0\ \lambda(x_0)\ \delta(x - x_0)\, g(x y z\, y' z')$$

$$= f(x, y, z\ y' z') - \lambda(x)\, g(x y z\ y' z')$$

same as before except λ is fn. of x.

Example: $g(x y z) = 0$.

$$\frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} = -\lambda(x)\frac{\partial g}{\partial y}$$

$$\frac{d}{dx}\frac{\partial f}{\partial z'} - \frac{\partial f}{\partial z} = -\lambda(x)\frac{\partial g}{\partial z}$$

Example: find extremum: with differential constraint

$$\min.\int f(x, y z\, y' z')\, dx \quad ; \quad \alpha\, dy + \beta\, dz = \gamma\, dx$$

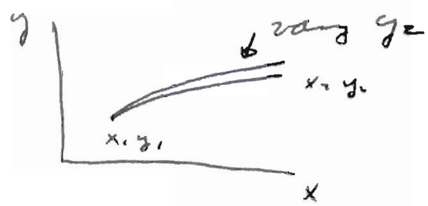
$$\frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} = -\lambda(x)\,\alpha$$

$$\frac{d}{dx}\frac{\partial f}{\partial z'} - \frac{\partial f}{\partial z} = -\lambda(x)\,\beta$$

4. Natural Boundary Conditions

Suppose we are looking for ~~one~~ ~~ext~~ a min of int

$$I = \int_{x_1}^{x_2} f(x, y\ z\ y'\ z')\, dx \qquad \begin{matrix} (x_1\ y_1) \\ (x_2\ y_2) \\ \uparrow \\ \text{vary } y_2 \end{matrix}$$



Let $y \to y + \delta y$

$$\delta I = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\, \delta y + \frac{\partial f}{\partial y'}\, \delta y' \right) dx$$

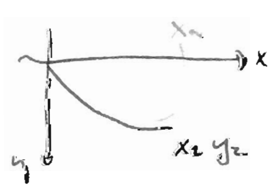
$$= \int_{x_1}^{x} \frac{\partial f}{\partial y}\, dx'\, \delta y \Big|_{x_1}^{x_2} + \int_{x_1}^{x_2} \underbrace{\left(\frac{\partial f}{\partial y'} - \int \frac{\partial f}{\partial y}\, dx' \right)}_{C} \delta y'\, dx = 0$$

$$= \int_{x_1}^{x_2} \frac{\partial f}{\partial y}\, dx'\, \delta y_2 + C\, \delta y_2 = 0$$

given

$$\frac{\partial f}{\partial y'} = 0 \quad \text{at RH end (natural BC)}$$

Example:



Recall soln: $x = \frac{1}{2c^2}(\theta - \sin\theta)$

$y = \frac{1}{2c^2}(1 - \cos\theta)$

Will set:

$y' = 0$ at x_2.

$$f = \sqrt{\frac{1 + y'^2}{y}}$$

Do this for homework.

two different ways.

1. natural BC
2. Find time it takes to go to fixed pt.

Problem

2. Then minimize t with respect to x_2 to get min. time.

Show that $t = \underbrace{c'}_{\text{some const.}} \sqrt{x_2}$

Recall linear int. eq at beginning of term.

$$f(x) - \lambda \int K(x,y) f(y)\, dy = 0 \quad ; \quad K(x,y) = K^*(y,x)$$

Want to min $\underbrace{-(\varphi, K\varphi)}_{-\frac{1}{\lambda} \text{ min and positive}} = -\int \varphi^*(x)\, K(x,y)\, \varphi(y)\, dx\, dy$

subject to:

$$\int |\varphi(x)|^2 dx - 1 = 0$$

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Boundary value problems:

$$\epsilon u'' + (1-x^2)u + u^2 = 1$$

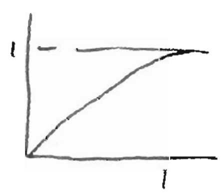
$$u(-1) = u(1) = 0$$

$$u \overset{?}{=} u_0(x) + \epsilon u_1(x) + \cdots$$

$$u^2 + (1-x^2)u - 1 = 0$$

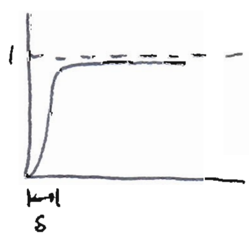
recognize that somewhere u must be steep so that u'' becomes large. We hope that $\epsilon u''$ is important only near the boundary.

u'' becomes much larger that $u = 1$, so that $\epsilon u''$ is a large number.
how is this?



u'' not larger than u

However:

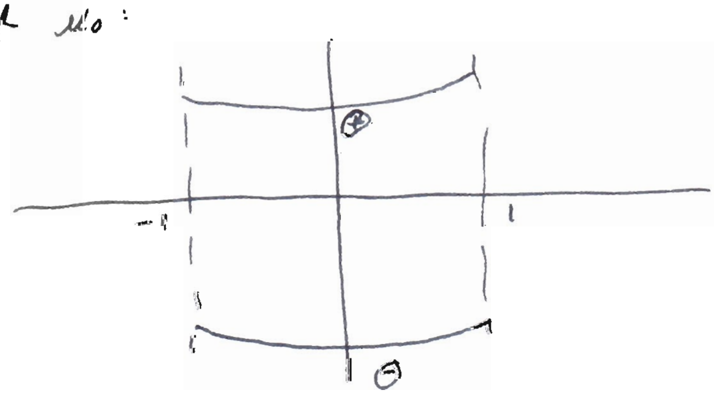


first derivative $\sim \frac{1}{\delta}$
second " $\sim \frac{1}{\delta^2}$

We hope that steepest u will be confined to the immediate neighbourhood of the boundary.
Hence:
gives:
$$u_0 = \frac{-(1-x^2) \pm \sqrt{(1-x^2)^2 + 4}}{2}$$

ln x -1x -1x

Plot of u_0:



Now want to consider near boundary. Try to scale problem to make $\epsilon u''$ appear large. Want to do this with respect to ϵ as this determines steepness of u: Choose:

$$\eta = (1+x)\,\epsilon^{\beta} \quad \text{as new ind. var.}$$

substitute.

$$\epsilon^{1+2\beta} u_{\eta\eta} + \epsilon^{-\beta}\eta\left[2-\epsilon^{-\beta}\eta\right]u + u^2 = 1$$

choose $\beta = -1/2$

$$u_{\eta\eta} + \epsilon^{1/2}\eta\left\{2-\epsilon^{1/2}\eta\right]u + u^2 = 1$$

This would give sol'n of form:

$$u = \begin{cases} u_0(x) & x > 7\epsilon^{+1/2} \\ u_1(x) & x < 20\epsilon^{1/2} \end{cases}$$

now try for sol'n of form $u = u_0 + W(\eta)$:

$$\epsilon u_0'' + \epsilon^{(1+2\beta)} w'' + \epsilon^{-\beta}\eta\left[2-\epsilon^{-\beta}\eta\right]\left[u_0 + w\right] + u_0^2 + 2u_0 W + W^2 = 1$$

Choose $\beta = -1/2$

$$w'' + 2u_0 W + W^2 + \mathcal{O}(\epsilon^{1/2}) + O(\epsilon) = 0$$

now we neglect terms in ϵ so last two terms we can drop. We will have to write u_0 as fn of η:

$$(1+x)\,\epsilon^{-1/2} = \eta \quad ; \quad x = \eta\,\epsilon^{1/2} - 1$$

$$u_0 = \frac{\pm\sqrt{4+\epsilon\eta^2\left[2-\sqrt{\epsilon}\,\eta\right]^2} - \epsilon^{1/2}\eta\left\{2-\sqrt{\epsilon}\,\eta\right\}}{2}$$

In neighborhood of $\eta < \sim 1000$, we have:

$$u_0 \simeq \pm 1$$

Do $+1$ first:

$$W'' + 2W + W^2 = 0$$

BC are $W(0) = -1$; $W \to 0,\ \eta \to \infty$

$$\frac{(W')^2}{2} + W^2 + \frac{W^3}{3} = 0$$

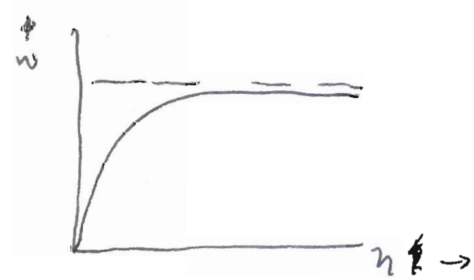
not + for W' $\quad 1 \quad -1/3$

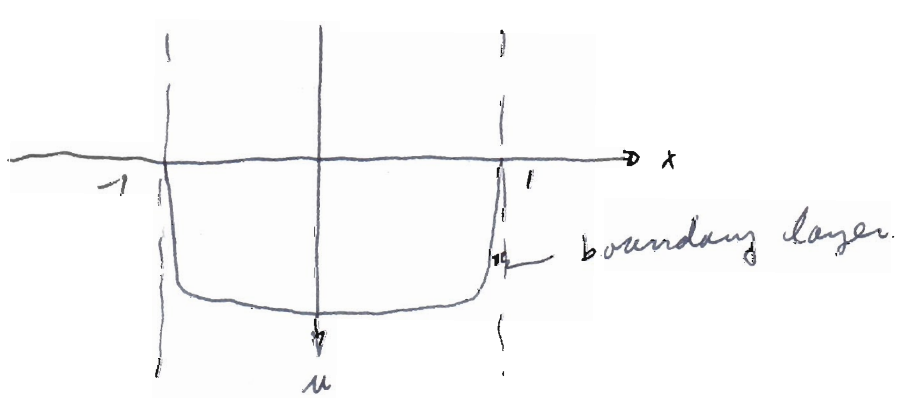
This soln does not satisfy BC, hence we say $+1$ is no good. For -1:

$$\frac{(W')^2}{2} - W^2 + \frac{W^3}{3} = 0$$

OK: If carrying out int.:

$$W = 3 - 3\tanh^2\left\{\frac{\eta}{\sqrt{2}} + \operatorname{arctanh}\sqrt{\frac{2}{3}}\right\}$$





Vander Pol Oscillator :

$$u'' - \mu u'(1-u^2) + u = 0$$

last year we did for μ small. Now we do for large μ and hope to make Boundary layer problem using periodic BC.

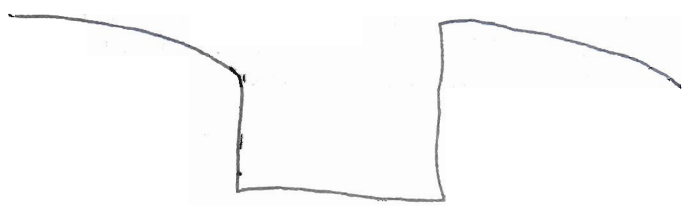
Let : $\eta = t\mu^{\alpha}$ and let $\alpha = -1$

$$\epsilon W'' - W'(1-W^2) + W = 0$$

$u(t) \rightarrow W(\eta)$

$u(x) \rightarrow u(\eta)$

Soln is something like ~~the~~

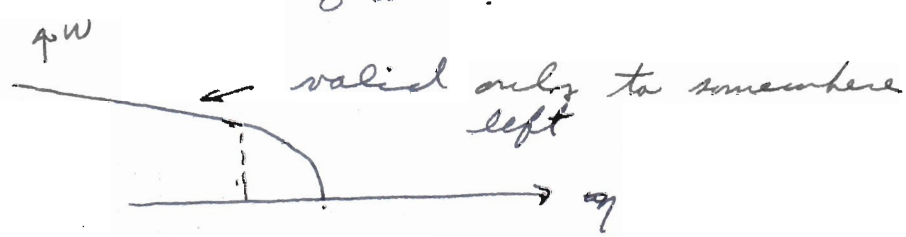


$$dw\left(\frac{1}{w} - w\right) = d\eta \quad \text{omitting } \epsilon \text{ term}$$

$$\ln w - \frac{w^2}{2} = \eta - \frac{1}{2}$$

↑ arbitrary const.

looks like



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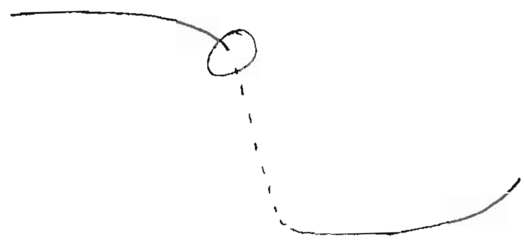
Recall: $\epsilon\, w''(\eta) - w'(\eta)\left(1-w^2(\eta)\right) + w(\eta) = 0$

$\ln w - \frac{w^2-1}{2} = \eta$ ~~first~~ zero order sol'n



Rescale w: $w = 1 + \epsilon^{\nu} h(\xi)$; $\eta - \eta_0 = \epsilon^{\mu}\xi$ — small, $\ll 1$

to get corner ○ without match zero order sol'n

$\nu > 0$ is necessary
$\mu > 0$ is conjectured

Substitution gives:

$$\epsilon^{1+\nu-2\mu} h'' - \epsilon^{\nu-\mu} h'\left(1 - [1+\epsilon^{\nu} h]^2\right) + 1 + \epsilon^{\nu} h = 0$$

$$-\epsilon^{\nu-\mu} h'\left(-2\epsilon h - \epsilon^{2\nu} h^2\right)$$

Want: $1+\nu-2\mu = 0$; $2\nu - \mu = 0$; $\nu = 1/3$, $\mu = 2/3$

So: $\underbrace{h'' + 2hh' + 1}_{\text{integrate}} = -\underbrace{\epsilon^{1/3}\left(h^2 h' + h\right)}_{\text{ignore}}$

$h' + h^2 + \xi = C$ or can always choose η_0 such that $C=0$. We take $C=0$ and choose η_0 later.

$h' + h^2 + \xi = 0$ (Riccati's equation)

Set $h = \frac{v'}{v}$; $\frac{v''}{v} - \left(\frac{v'}{v}\right)^2 + \xi = 0$

$v'' + \xi v = 0$ gives Airy fns.

Z mean any linear combination of Hankel fns.

$$v = \xi^{1/2} Z_{1/3}\left(\tfrac{2}{3}\xi^{3/2}\right) = \xi^{1/2}\Big[A\,K_{1/3}\left(\tfrac{2}{3}i\,\xi^{3/2}\right) + B\,\underbrace{K_{1/3}\left(-\tfrac{2}{3}i\,\xi^{3/2}\right)}_{I_{1/3}\left(\frac{2}{3}i\,\xi^{3/2}\right)}\Big]$$

K's are most convenient choice.

Note that v is an entire fn. $K_{1/3}$ ~~grows~~ decays exponentially and $I_{1/3}$ grows exponentially for negative value of ξ

We now try to patch $\frac{v'}{v}$ to w in overlap region.

We have:

$$\ln w \to \ln\left(1+\epsilon^{1/3}h\right) = \epsilon^{1/3}h - \frac{\epsilon^{2/3}h^2}{2} + \dots$$

$$\frac{w^2-1}{2} \to \frac{2\epsilon^{1/3}h + \epsilon^{2/3}h^2/2}{2}, \quad \text{use } \eta = \epsilon^{u}(\xi-\xi_0)$$

get: $h^2 = -(\xi-\xi_0)$

To fix branch, pick $\xi^{1/2} \sim i$ for $\xi<0$

We want $|\xi| \gg 0$ to use asymptotic, but not large enough so that $|\eta| \ll 1$ is no longer true. Then is all right.

Change $A \to iA$, $B \to -iB$, then $K_{1/3}$ decays exp. and $I_{1/3}$ grows.

$$iA\,\xi^{1/2}K_{1/3}\left(\tfrac{2}{3}i\,\xi^{3/2}\right) \sim AC\,e^{-2/3(-\xi)^{3/2}}(-\xi)^{-1/4}$$

$$-B\,\xi^{1/2}I_{1/3}(\quad) \sim BC'\,e^{2/3(-\xi)^{3/2}}(-\xi)^{-1/4}$$

$$\frac{\partial V}{\partial(-\xi)} \sim -AC(-\xi)^{1/4}e^{-\frac{2}{3}(-\xi)^{3/2}} + BC'(-\xi)^{1/4}e^{\frac{2}{3}(-\xi)^{3/2}}$$

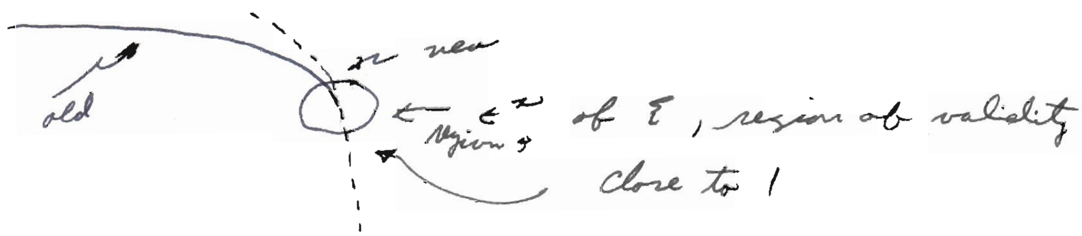
$$\eta = \frac{V'}{V} \sim -(-\xi)^{1/2} \text{ for large } -\xi \text{ if } B \neq 0$$

$$\sim +(-\xi)^{1/2} \text{ if } B=0$$

$B=0$ fits our situation because h must be positive

We have soln for slow-moving part and for corner.

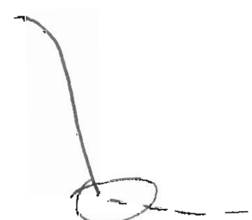
$K_{1/3}$ is real along real axis and has ~~poles~~ zeros on + real axis.



We now go to find next region: integrate W equation
steep part

$$\epsilon W' - W + \frac{W^3}{3} = C$$

$$\frac{dW}{P_3(W)} = d\eta$$



then find this region near -2.

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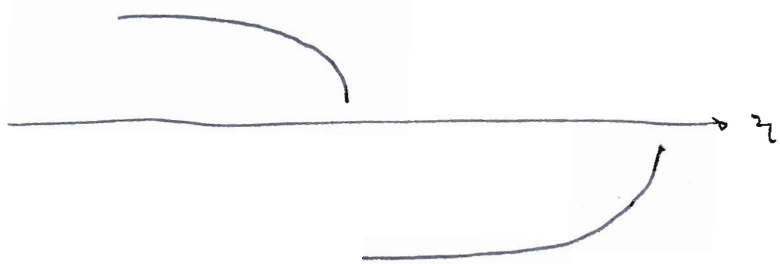
$$\epsilon w'' - w'(1-w^2) + w = 0$$

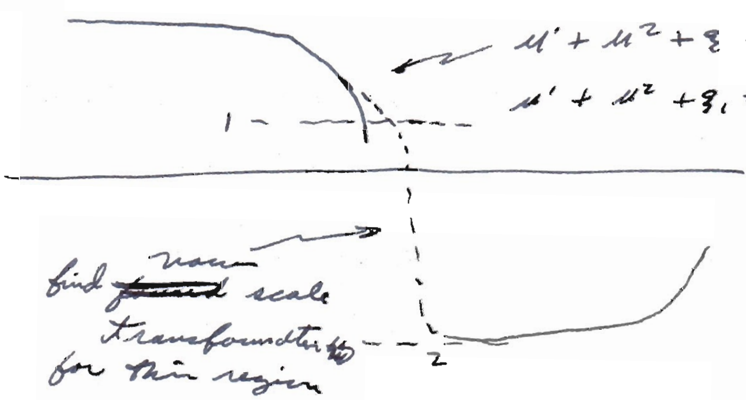
2 solns: $w(\eta)$; $-w(\eta-\eta_0)$

periodic solution:



$w(\eta) = -w(\eta-\eta_0)$ since η_0 for half-period.

Have found $\qquad \ln w - \frac{w^2-1}{2} = \eta$



$\mu' + \mu^2 + \xi = 0$

$\mu' + \mu^2 + \xi_1 = 0$ near $\xi = \xi_1$ on the left.

ξ_1 is first (smallest) root of $J_{1/3}\left(\frac{2}{3}\xi^{3/2}\right) + I_{1/3}\left(\frac{2}{3}\xi^{3/2}\right)$

Use: $w = g(\tau)$; $\tau = (\eta-\eta_2)\epsilon^{\gamma}$, $\gamma < 0$

$$\epsilon^{1+2\gamma} g'' - \epsilon^{\gamma} g'(1-g^2) + g = 0$$

Choose $\gamma = -1$, set:

$$g'' - g'(1-g^2) = 0 \quad , \; g \to q$$

$$q' - q + q^3/3 = c$$

Now $w = 1 + \epsilon^{1/3}\mu(\xi) = g(\tau)$; $\tau = (\eta-\eta_2)\epsilon^{-1} = (\xi\epsilon^{2/3} - \eta_2)\epsilon^{-1}$

$$\epsilon^{1/3}\mu'\epsilon^{1/3} - (1+\epsilon^{1/3}\mu) + \frac{(1+\epsilon^{1/3}\mu)^3}{3} = c$$

$$\epsilon^{2/3}\mu' - 1 + 1/3 + \epsilon^{2/3}\mu^2 + \frac{\epsilon\mu^3}{3} = c$$

$\epsilon^{1/3}$ higher order than other term to be consistent with previous approximation

$$\epsilon^{2/3}(\mu' + \mu^2) = c + \frac{2}{3}$$

$$\mu' + \mu^2 = \left(c + \frac{2}{3}\right)\epsilon^{-2/3}$$

From above in overlapping region of validity, we must have:

$$\xi_1 = -(c + 2/3)\epsilon^{-2/3}$$

$$c = -\frac{2}{3} - \xi_1\epsilon^{2/3}$$

Substitute back in equation:

$$3\,dq = -\left[q^3 - 3q + 2 + 3\xi_1\epsilon^{2/3}\right]d\tau$$

$$= -\left[(q+2)(q-1)^2 + 3\xi_1\epsilon^{2/3}\right]$$

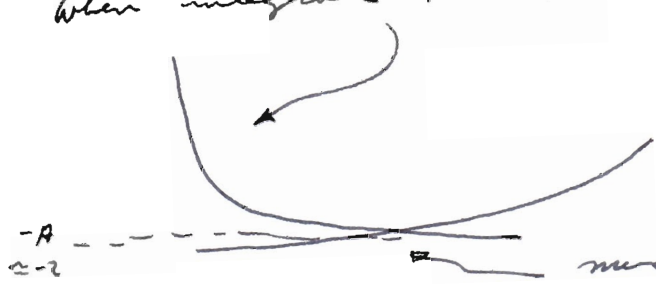
$$q = -(2+\beta)$$

$$-\beta(3+\beta)^2 = -3\xi_1\epsilon^{2/3}, \quad \beta \simeq \frac{\xi_1}{3}\epsilon^{2/3}$$

$$\frac{3\,dq}{(q-q_1)(q-q_2)(q-q_3)} = d\tau$$

, goes to log, and then exp.

When integrate this set:



must blend in here

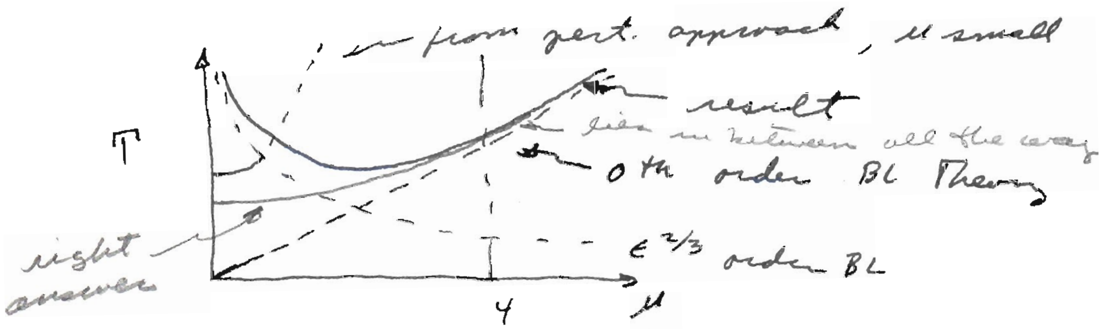
Could do by perturbation ~~of~~ or by making another boundary layer problem.

Scale: $w = -A + \epsilon^{s} \phi(z)$; $z = (y - y_3)\epsilon^{S}$

use $s = 1$, $S = -1$ (will not modify the period at all)

What does entire result look like:

$$y'' - \mu y'(1-y^2) + y = 0 \quad ; \quad \epsilon = \frac{1}{\mu^2}$$



Higher order terms in the asymptotic expansion helps for large μ but leads to large error for little ϵ

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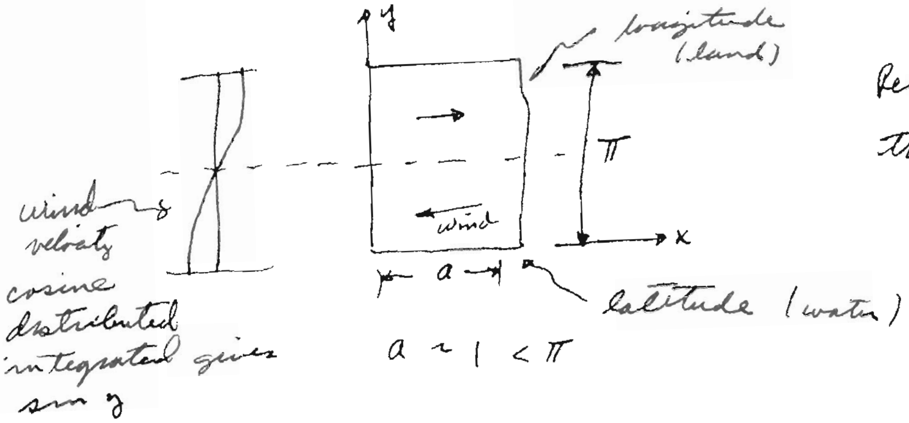
Ocean wave problem: Will not do physics to get equations.



$a \sim 1 < \pi$

Rectangle to describe the Atlantic basin.

$$\sum_{\hat{e}'} (\hat{e}\cdot\hat{e}')^2 = 1 - (\hat{e}\cdot\hat{n}')^2$$

Equation: $\sigma \Delta\Delta\psi + \psi_y(\epsilon\Delta\psi - f)_x - \psi_x(\epsilon\Delta\psi - f)_y = \sin y$

2-D equations having integrated over depth

f contains coriolis effect: $f = f_0 + \beta y$

varies with y because earth is sphere and spin velocity vector varies with y

$\sigma\Delta\Delta\psi$ describes laminar flow ($\Delta\psi$ is vorticity) describes diffusion but is very simplified. We use σ to help make turbulence, eddy viscosity, very small $\sim 10^{-8}$, $\epsilon \sim 10^{-5}$, or: $\sigma \ll \epsilon \ll 1$.

If σ controls the BL and not ϵ, have $\frac{\sigma}{\ell^4}$ scaling in BL
if ϵ comes in, it is as $\frac{\epsilon}{\ell^3}$

suppose we throw out σ and keep ϵ and do BL find that when do integration LHS vanishes. Hence ϵ determines BL of certain size (rather wide) and σ determines another BL of smaller size nearer boundary.

We start:

$$\psi = \psi^{(0)}(x,y) + \psi^{(1)}(\xi, y)$$

Dominant σ, ϵ.

Set: $\beta \psi_x^{(0)} = \sin y$; $\psi^{(0)} = \frac{1}{\beta} \sin y \left(x - b(y)\right)$

satisfies BC on bottom and top where the theory says no fluid crosses so no BL

Could choose $b(y)$ to make no BL on $x = a$.

However, Take BL on $x = 0$ only for now.

Use: $\eta = \epsilon^{\nu} x$

Then:

$$\left(\psi_y^0 + \psi_y^{(1)}\right)\left(\epsilon^{1+2\nu} \psi_{\eta\eta}^{(1)} + \epsilon\psi_{yy}^{(1)} + \epsilon\, \psi^{(0)}\right)_\eta \epsilon^{\nu}$$

$$-\left(\psi_x^0 + \epsilon^{\nu} \psi_\eta^{(1)}\right)\left(\psi_{\eta\eta}^{(1)} \epsilon^{1+2\nu} - f\right)_y = \sin y$$

$$\left(\psi_y^0(0,y) + \psi_y^{(1)}\right)\left(\epsilon^{1+3\nu} \psi_{\eta\eta\eta}^{(1)}\right) - \psi_\eta^{(1)} \psi_{\eta\eta y}^{(1)} \epsilon^{1+3\nu} + \epsilon^{\nu} \beta \psi_\eta^{(1)} = 0$$

Take $\nu = -1/2$ to make ϵ cancel.

To solve, must perform transformation:

$$\psi_{\eta\eta}^{(1)} = F(\psi^{(1)}, y)$$

$$\left(\psi_y^0(0,y) + \cancel{\psi_y^{(1)}}\right) F_{\psi^{(1)}} \psi_\eta^{(1)} - \psi_\eta^{(1)}\left(F_y + \cancel{\psi_y^{(1)} F_{\psi^{(1)}}}\right) + \psi_\eta^{(1)} \beta = 0$$

$$\psi_y^0(0,y)\, F_{\psi^{(1)}} - F_y + \beta = 0$$

$$\boxed{\cancel{\text{now: } F_y = F_{\psi^0} \psi_y^0 \qquad \psi_y^0 F_{\psi^{(1)}} - F_{\psi^{(0)}} \psi_y^0 + \beta = 0}}$$

Look for solns: $F = \beta y \mp F(\psi^0 + \psi^{(1)})$

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$$\psi_y(\epsilon\Delta\psi - f)_x - \psi_x(\epsilon\Delta\psi - f)_y = \sin y$$

$$\psi = \psi^{(0)}(x,y) + \psi^{(1)}(\xi y) \quad ; \quad \xi = \epsilon^{-1/2}x$$

$$[\psi^{0}_y + \psi^{(1)}_y][\epsilon^{-1/2}\psi^{(1)}_{y\xi\xi} + \epsilon^{1/2}\psi^{(1)}_{yyy} + \epsilon\Delta\psi^{(0)}_x] - \{\psi^{(0)}_x + \epsilon^{-1/2}\psi_\xi^{(1)}\}[\psi_{\xi\xi y} + \epsilon\psi^{(1)}_{yyy} + \epsilon\Delta\psi^{(0)}_y - f_y] = \sin y$$

$$(\psi^{(0)}_y + \psi^{(1)}_y)(\psi^{(1)}_{y\xi\xi}) - \psi^{(1)}_\xi\psi^{(1)}_{\xi\xi y} + \psi^{(1)}_\xi f_y = 0$$

$$-\psi^{(0)}_x(\psi^{(1)}_{\xi\xi y} - f_y) = \sin y + f(\psi^{(1)})$$

$$\xi >> 1 \; ; \quad \psi^{(0)} = \frac{\sin y}{\beta}(x-a)$$

a is a positive constant

Substitute:

$$\psi^{(1)}_{\xi\xi} = F(\psi^{(1)}, y) \quad ; \quad F(\psi^{(1)}\xi) = f(y) + G(\psi^{(1)} + \psi^{(0)}) - f_0$$

$$\beta y + G(\psi^{(0)}) = 0 \quad ; \quad -\frac{a}{\beta}\sin y = \mu$$

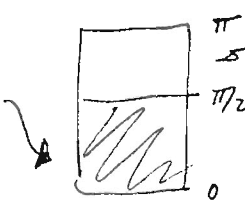
$$y = -\sin^{-1}\frac{\beta}{a}\mu$$

$$\beta \sin^{-1}\frac{\beta}{a}\mu = G(\mu)$$

$$\therefore F(\psi^{(1)}, y) = \beta y + \beta\sin^{-1}\left[\frac{\beta}{a}[\psi^{0} + \psi^{(1)}]\right]$$

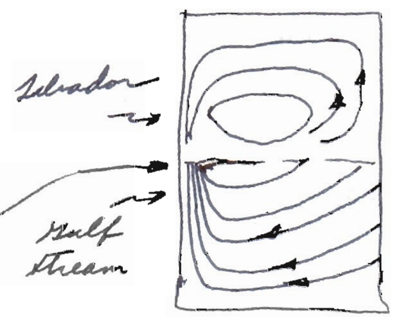
$$= \psi^{(1)}_{\xi\xi}$$

Gives a solution for



can get this region by inverting coordinates and changing $a \to -a$.

Picture of Soln: Flow Picture



get disjointed flow here
not physically possible. Must consider
this as another BL like in
Van der Pol oscillator.

Have no diffusion of vorticity past this point. So must
consider BL in this corner.

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(43) ①

WKB method

$$u''(x) - \left(\lambda^2 Q(x)\right) u(x) = 0$$

$$Q(0) = 0$$

$$Q'(0) = 1$$

$$\lambda \gg 1$$

Take: $u = e^{\lambda \int_{x_0}^{x} g(t,\lambda)\,dt}$

given: $\lambda^2 g^2 + \lambda g' = \lambda^2 Q$

look for: $g = g_0 + \frac{1}{\lambda} g_1 + \dots$

$$g_0^2 = Q$$

$$2 g_0 g_1 + g_0' = 0$$

$$g_1 = -\frac{g_0'}{2g_0} = -\frac{1}{2}(\log g_0)'$$

$$u = g_0^{-1/2} e^{\lambda \int_{x_0}^{x} g_0\,dt} + \underbrace{O\left(\frac{1}{\lambda}\right)}_{\text{drop when } \lambda \text{ is large}}$$

or: $u = Q^{-1/4} e^{\pm \lambda \int \sqrt{Q}\,dx}$

<u>if Q has no zeroes</u>

What if Q has zeroes get sing as $Q^{-1/4}$ as approach origin.

$Q^{1/4} e^{-\lambda \int \sqrt{Q}\,dx}$

Hence this approach doesn't work. Must use something like boundary layers.

Pick: $\lambda^n x = \xi$

$$\lambda^{2n} u_{\xi\xi} - \lambda^2 \underbrace{(x + a_2 x^2 + \cdots)}_{\lambda^{-n}\xi + a_2 \lambda^{-2n}\xi^2 + \cdots)\, u} u = 0$$

Choose $n = 2/3$

$$\underbrace{u_{\xi\xi} - \xi u}_{\text{zeroth order}} - a_2 \lambda^{-2/3}\xi^2 u + \cdots = 0$$

$$u = \xi^{1/2} Z\left(\tfrac{2}{3}\lambda \xi^{3/2}\right)$$

$$\downarrow$$

$$K_{1/3}\left(\tfrac{2}{3}\xi^{3/2}\right)$$
$$I_{1/3}\left(\tfrac{2}{3}\xi^{3/2}\right)$$

so that in  we have these two soln

There is also an overlapping region of validity:
Here we have to $O(\lambda^{-2/3}\xi^2)$ while previously had $O(\frac{1}{\lambda^2} g_2)$
ie $\frac{1}{\lambda^2} g_2 \ll \frac{1}{\lambda} g_1$; $|x| \gg \lambda^{-\alpha}$ for some power

$|\xi| \lambda^{-2/3} \ll 1$; $|\xi| \ll \lambda^{2/3}$, $|x| \ll 1$

Can connect overlapping regions of validity by using asymptotic expansion of Hankel fns or $K_{1/3}$ which is like $Q^{1/4} e^{-\lambda \frac{3}{2}\int x^{3/2} dx}$. This solution is good except for negative real axis which is found by using Hankel fn identities and new asymptotic expansion. Also identities in terms of $e^{i n \pi}$

Form: $K_{1/3}(\lambda\varphi)$; $\varphi = \frac{2}{3}\int_0^x \sqrt{Q}\,dt$

must have $x^{1/2}$
behave properly at origin
This is accomplished by $\frac{\varphi^{1/2}}{Q^{1/4}}$

This gives $u = Q^{-1/4} e^{\lambda \int \cdots}$ asymptotically

For the other region we have $I_{1/3}$

Of what d.e. are these $K_{1/3}$ and $I_{1/3}$ solution of:

$$u_{xx} - \lambda^2 Q u = \frac{5}{16}\left[\frac{(Q')^2 - \frac{4}{9}\varphi Q^3}{Q^2} - \frac{4}{5}\frac{Q''}{Q}\right]u$$

Another Problem from Hydrodynamics:

$$u - (\lambda^2 i Q)u = 0$$

If do this problem as above, get

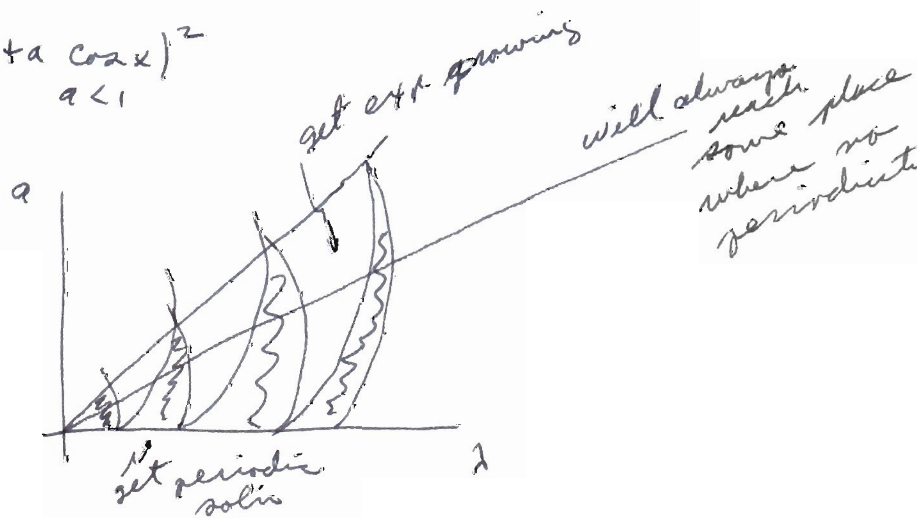
Cannot analytically continue from one region into another.

Suppose we consider original equation, but now with:

$$Q(x) = (1 + a\cos x)^2$$
$a < 1$

Gives Hill's equation.

Now WKB method will always give periodic soln regardless of degree of approximation.



Direct method of Variation:

$$\delta \int_a^b F(y, y', x)\, dx = 0$$

$$y = \sum_{n=0}^{31} a_n \varphi_n(x)$$

Find $F(y, y', x)$

Change question to

Find $F(y, y'; x)$ in form of φ_n's to minimize δI.

AM 203 Course Outline

Integral Equations

1. setting up and iteration
2. Wiener Hopf Method
3. Wide and Narrow Kernels
4. Theory

Method of Steepest Descent

method of Stationary Phase

Poincaré - Lighthill Method

Formal Perturbation Theory for Linear Problems

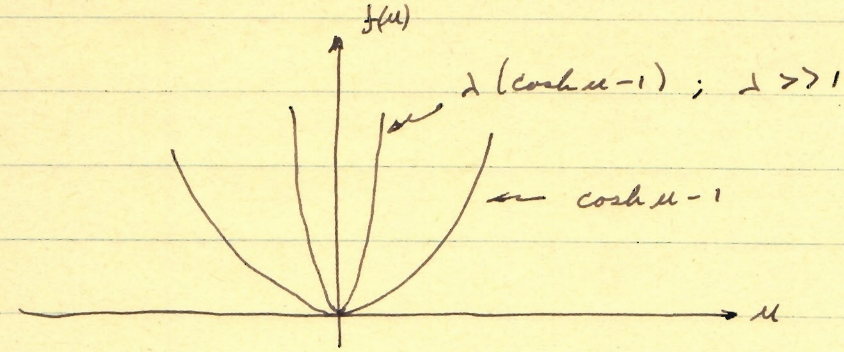
Calculus of Variations

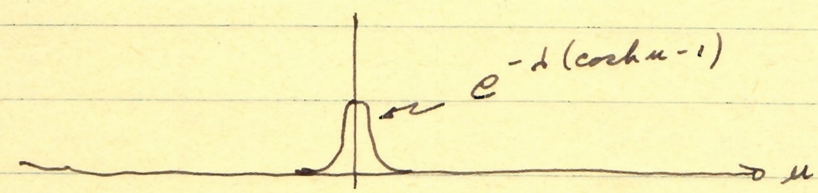
Boundary Value Problems (Boundary Layer)

WKB Method

Method of Steepest Descent

Consider: $$\int_{-1}^{1} e^{-\lambda \cosh u}\,du = e^{-\lambda}\int_{-1}^{1} e^{-\lambda(\cosh u - 1)}\,du$$





Integrand acts almost like δ fn:

$$e^{-\lambda}\int_{-1}^{+1} e^{-\lambda(\cosh u-1)}\,du \approx e^{-\lambda}\int_{-\infty}^{\infty} e^{-\lambda \frac{u^2}{2}}\,du = \sqrt{\frac{2}{\lambda}}\,e^{-\lambda}\int_{-\infty}^{\infty} e^{-x^2}\,dx$$

$$= \sqrt{\frac{2\pi}{\lambda}}\,e^{-\lambda}$$

More General Method

Consider:

$$\int_a^b g(z)\, e^{-\lambda\varphi(z)}\,dz$$

$g(z)$, $\varphi(z)$ are analytic, and have saddle points.

z^2 has s.p. at origin

Find the path s.t. Im z is fixed and Re z goes thru s.p.
Call path s and s.p. s_0.

Write: $\varphi(s) = \phi(s) + i\psi(s)$

$$\int_a^b g(z)\, e^{-\lambda\varphi(z)}\, dz \rightarrow g(z_0)\, e^{-i\lambda\psi(s_0) - \lambda\phi(s_0)} \int_{-\infty}^{\infty} e^{-\lambda(\phi(s) - \phi(s_0))}\, ds$$

Expand $\phi(s) - \phi(s_0)$ about s.p., knowing $\phi'(s_0) = 0$

$$\approx \frac{\phi''(s_0)}{2}(s - s_0)^2$$

Then: $$\int_a^b g(z)\, e^{-\lambda\varphi(z)}\, dz \rightarrow g(z_0)\, e^{-i\lambda\psi(s_0) - \lambda\phi(s_0)} \int_{-\infty}^{\infty} e^{-\frac{\lambda\phi''}{2}x^2}\, dx$$

$$= g(z_0)\sqrt{\frac{2\pi}{\lambda\phi''}}\; e^{i\lambda\psi(s_0) - \lambda\phi(s_0)}$$

<u>Method of Stationary Phase</u>

Do:
$$\int_c e^{-\lambda(\cosh z - 1)}\, dz$$

where c is some appropriate contour along at the ends of which the integral vanishes.

$$\cosh z - 1 = \cosh(x + iy) - 1$$

$$\cosh(x + iy) = \frac{e^x e^{iy} + e^{-x} e^{-iy}}{2}$$

$$= \frac{1}{2}\left\{ e^x \cos y + i e^x \sin y + e^{-x}\cos y - i e^{-x}\sin y \right\}$$

$$= \cosh x \cos y + i \sinh x \sin y$$

$\therefore \quad \operatorname{Im} \varphi(z) = \sinh x \sin y = \text{constant}$

$\operatorname{Re} \varphi(z) = \cosh x \cos y - 1$

The saddle point is $z_0 = 0$, so the path of s.d. is:

$$\sinh x \sin y = 0$$

$$\text{or} \quad x = 0,\ y = 0$$

$$\varphi''(z) = \cosh z \quad , \quad \varphi''(z_0) = 1$$

$$\therefore \quad \int_C e^{-\lambda(\cosh z - 1)}\, dz \to \int_{-\infty}^{\infty} e^{-\frac{\lambda x^2}{2}}\, dx = \sqrt{\frac{2\pi}{\lambda}}$$

Now do:

$$I = \int_{-\infty}^{\infty} e^{i\lambda(\cosh x - 1)}\, dx \quad ;$$

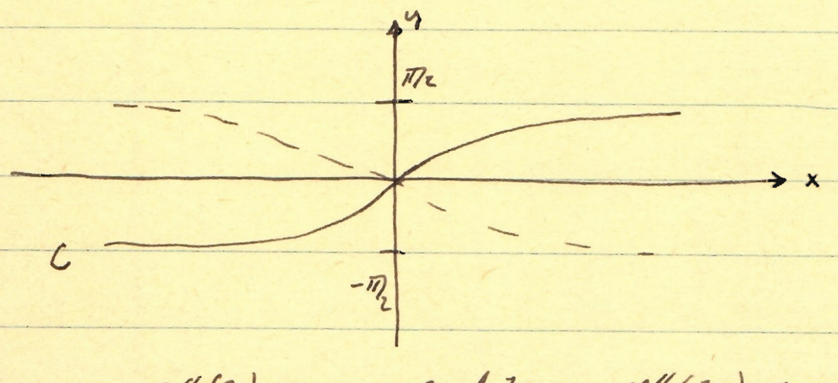
Look at $\displaystyle\int_C e^{i\lambda(\cosh z - 1)}\, dz \quad ; \ \varphi(z) = -i(\cosh z - 1)$

$\operatorname{Im} \varphi(z) = \cosh x \cos y - 1 = \text{const.}$

$\operatorname{Re} \varphi(z) = \sinh x \sin y$

$z_0 = 0$

hence path of sd is: $\cosh x \cos y - 1 = 0$



$$\varphi''(z) = -i \cosh z \quad , \quad \varphi''(z_0) = -i$$

In the neighborhood of the s.p. : $x^2 - y^2 \approx 0$, $x = y$

$$I = \int_C e^{-\lambda(-i)\frac{z^2}{2}} dz \qquad , \quad \text{take } C = x = y$$

$$I = \int_{-\infty}^{\infty} e^{i\lambda x^2 (1+i)^2} dx\,(1+i) \quad ?$$

Poincaré - Lighthill Method :

Consider :

$$(x + \epsilon u)\frac{du}{dx} + u = 0 \quad ; \quad u(1) = 1$$

Try ordinary pert. Theory : $u = u_0 + \epsilon u_1(x) + \cdots$

$u_0(1) = 1$

$u_1(1) = 0$, etc

$$(x + \epsilon u_0 + \epsilon^2 u_1 + \cdots)\left(\frac{du_0}{dx} + \epsilon\frac{du_1}{dx} + \cdots\right) + u_0 + \epsilon u_1 + \cdots = 0$$

$$x\frac{du_0}{dx} + u_0 = 0 \qquad ; \qquad \frac{du_0}{dx} + \frac{u_0}{x} = 0$$

$$x\frac{du_1}{dx} + \cancel{\epsilon}\, u_0\frac{du_0}{dx} + \cancel{\epsilon}\, u_1 = 0$$

$$x\frac{du_1}{dx} + u_1 = -u_0\frac{du_0}{dx}$$

$$\frac{du_0}{u_0} = -\frac{dx}{x} \quad ; \quad u_0 = c\,e^{-\int\frac{dx}{x}} = c\,e^{-\ln x}$$

$$= \frac{c}{x} = \frac{1}{x}$$

$$x\frac{du_1}{dx} + u_1 = \frac{1}{x^3} \quad ; \quad \frac{du_1}{dx} + \frac{u_1}{x} = \frac{1}{x^4}$$

$$du_1 + \frac{u_1}{x}dx = \frac{1}{x^4}dx$$

$$x\,du_1 + u_1\,dx = \frac{dx}{x^3} \quad ; \quad u_1 x = \frac{-1}{2x^2} + c$$

$$= \frac{-1}{2x^2} + \frac{1}{2} \quad ; \quad u_1 = \frac{-1}{2x^3} + \frac{1}{2x}$$

$$u = \frac{1}{x} + \epsilon\left(\frac{1}{2x} - \frac{1}{2x^3}\right) + \cdots$$

Keep getting reciprocal powers in x for higher ϵ

PL method consists of:

$$u = u_0(\xi) + \epsilon u_1(\xi) + \cdots$$

$$x = \xi + \epsilon x_1(\xi) + \cdots$$

Then:

$$(\xi + \epsilon x_1 + \epsilon_2 x_2 \cdots + \epsilon u_0 + \epsilon^2 u_1 \cdots)\left(\frac{du_0}{d\xi} + \epsilon\frac{du_1}{d\xi} + \cdots\right)$$

$$+ u_0 + \epsilon u_1 + \cdots = 0$$

$$\xi\frac{du_0}{d\xi} + u_0 = 0 \quad : \quad u_0 = \frac{1}{\xi}$$

$$\xi\frac{du_1}{d\xi} + x_1\frac{du_0}{d\xi} + u_0\frac{du_0}{d\xi} + u_0$$

$$\frac{du}{dx} = \frac{du}{d\xi}\,\frac{d\xi}{dx} = \frac{\dfrac{du_0}{d\xi} + \epsilon\dfrac{du_1}{d\xi} + \epsilon^2\dfrac{du_2}{d\xi} + \cdots}{1 + \epsilon\dfrac{dx_1}{d\xi} + \epsilon^2\dfrac{dx_2}{d\xi} + \cdots}$$

$$\frac{du_0}{d\xi}$$

$$1 + \epsilon\frac{dx_1}{d\xi} + \epsilon^2\frac{dx_2}{d\xi} \;\Big)\; \frac{du_0}{d\xi} + \epsilon\frac{du_1}{d\xi} + \epsilon^2\frac{du_2}{d\xi} + \cdots$$

$$\frac{du_0}{d\xi} + \epsilon\frac{du_0}{d\xi}\frac{dx_1}{d\xi} + \epsilon^2\frac{du_0}{d\xi}\frac{dx_2}{d\xi}$$

$$\epsilon\left(\frac{du_1}{d\xi} - \frac{du_0}{d\xi}\frac{dx_1}{d\xi}\right) + \epsilon^2\left(\cdots\right.$$

Hence: $$\frac{du}{dx} = \frac{du_0}{d\xi} + \epsilon\left(\frac{du_1}{d\xi} - \frac{du_0}{d\xi}\frac{dx_1}{d\xi}\right)$$

$$u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots$$

$$x = \xi + \epsilon x_1 + \cdots$$

$$(x + \epsilon u)\frac{du}{dx} + u = 0$$

$$\left(\xi + \epsilon x_1 + \epsilon u_0\right)\left[\frac{du_0}{d\xi} + \epsilon\left(\frac{du_1}{d\xi} - \frac{du_0}{d\xi}\frac{dx_1}{d\xi}\right)\right] + u_0 + \epsilon u_1 = 0$$

$$\xi\frac{du_0}{d\xi} + u_0 = 0 \quad ; \quad u_0 = \frac{1}{\xi}$$

$$x_1\frac{du_0}{d\xi} + \xi\frac{du_1}{d\xi} - \xi\frac{du_0}{d\xi}\frac{dx_1}{d\xi} + u_0\frac{du_0}{d\xi} + u_1 = 0$$

$$\xi\frac{du_1}{d\xi} + u_1 = -(x_1 + u_0)\frac{du_0}{d\xi} + \xi\frac{du_0}{d\xi}\frac{dx_1}{d\xi}$$

$$\xi \frac{du_1}{d\xi} + u_1 = \left(x_1 + \frac{1}{\xi}\right)\frac{1}{\xi^2} - \frac{1}{\xi}\frac{dx_1}{d\xi}$$

Choose x_1 to cancel $\frac{1}{\xi^3}$:

$$-\frac{1}{\xi}\frac{dx_1}{d\xi} + \frac{x_1}{\xi^2} = -\frac{1}{\xi^3} \quad ; \quad \frac{dx_1}{d\xi} - \frac{x_1}{\xi} = \frac{1}{\xi^2}$$

$$dx_1 - \frac{x_1}{\xi} d\xi = \frac{1}{\xi^2} d\xi$$

$$IF = \frac{1}{\xi}$$

$$\frac{dx_1}{\xi} - \frac{x_1}{\xi^2} d\xi = \frac{d\xi}{\xi^3} = d\left(\frac{x_1}{\xi}\right)$$

$$\frac{x_1}{\xi} = -\frac{1}{\cancel{2}\xi^2} + c = -\frac{1}{2\xi^2} + \frac{1}{2} = \frac{\xi^2 - 1}{2\xi^2}$$

$$x_1 = \frac{\xi^2 - 1}{2\xi} \quad ,$$

$$(x^2 + \epsilon w)\frac{dw}{dx} + w - (2x^3 + x^2) = 0 \quad ; \quad w(1) = A \neq 1$$

over $(0, 1)$

$w = w_0 + \epsilon w_1$ $\qquad$ $w_0(1) = A \neq 1$, $w_1(1) = 0$

$x = \xi + \epsilon x_1$ $\qquad$ $\xi = 1$; $x = 1$; $x = 1$, $x_1 = 0$

$$\frac{dw}{dx} = \frac{dw_0}{d\xi} + \epsilon\left(\frac{dw_1}{d\xi} - \frac{dw_0}{d\xi}\frac{dx_1}{d\xi}\right)$$

$$x^2 = \xi^2 + 2\xi x_1 \epsilon$$

$$x^3 = \xi^3 + 3\xi^2 x_1 \epsilon$$

$$\left[\xi^2 + 2\xi x_1\epsilon + \epsilon w_0\right]\left[\frac{dw_0}{d\xi} + \epsilon\left(\frac{dw_1}{d\xi} - \frac{dw_0}{d\xi}\frac{dx_1}{d\xi}\right)\right] + w_0 + \epsilon w_1$$

$$- 2\xi^3 - 6\xi^2 x_1\epsilon - \xi^2 - \cancel{4}\,2\xi x_1\epsilon = 0$$

$$\xi^2\frac{dw_0}{d\xi} + w_0 = 2\left(\xi^3 + \tfrac{1}{2}\xi^2\right)$$

$$\xi^2\frac{dw_1}{d\xi} + w_1 = \xi^2\frac{dw_0}{d\xi}\frac{dx_1}{d\xi} - w_0\frac{dw_0}{d\xi} - 2\xi x_1\frac{dw_0}{d\xi}$$
$$+ 6\xi^2 x_1 + \cancel{4}\,2\xi x_1$$

$$\frac{dw_0}{d\xi} + \frac{w_0}{\xi^2} = 2\left(\xi + \tfrac{1}{2}\right) \quad ; \quad dw_0 + \frac{w_0}{\xi^2}d\xi = 2\left(\xi + \tfrac{1}{2}\right)d\xi$$

$$IF = e^{\int \frac{d\xi}{\xi^2}} = e^{-\frac{1}{\xi}} \quad ; \quad d\left(w_0 e^{-\frac{1}{\xi}}\right) = 2e^{-1/\xi}\left(\xi + \tfrac{1}{2}\right)d\xi$$

$$w_0 e^{-\frac{1}{\xi}} = 2\int(\xi+1)e^{-1/\xi}\,d\xi$$

$$= \int d\left(\xi^2 e^{-1/\xi}\right) = \xi^2 e^{-1/\xi} + K$$

$$w_0 = \xi^2 + Ke^{+1/\xi} \quad : \quad A = 1 + Ke^{+1} \quad ; \quad Ke = A-1 \quad ; \quad K = \frac{A-1}{e}$$

Recap: $\xi^2\dfrac{dw_0}{d\xi} + w_0 = 2\xi^3 + \xi^2$; $w_0 = \xi^2 + Ke^{1/\xi}$

$$\xi^2\frac{dw_1}{d\xi} + w_1 = \left(2\xi^3 + \xi^2 - w_0\right)\frac{dx_1}{d\xi} - w_0\frac{dw_0}{d\xi} - 2\xi x_1\frac{dw_0}{d\xi} + 6\xi^2 x_1 + 2\xi x_1$$

Ordinary Result:

$$\xi^2 \frac{dw_1}{d\xi} + w_1 = -w_0 \frac{dw_0}{d\xi}$$

$$\frac{dw_0}{d\xi} = 2\xi + K e^{1/\xi} \cdot \frac{-1}{\xi^2} = 2\xi - \frac{K}{\xi^2} e^{1/\xi}$$

$$\xi^2 \frac{dw_1}{d\xi} + w_1 = 2\xi^3 - K e^{1/\xi} + 2K\xi e^{1/\xi} - \frac{K^2}{\xi^2} e^{2/\xi}$$

The worst singularity here is $e^{2/\xi}$ and we want to choose x_1 to cancel this.

Choose $\overline{w_0} = K e^{1/\xi}$ so that $w_0 = \xi^2 + \overline{w_0}$

$$\frac{dw_0}{d\xi} = 2\xi + \frac{d\overline{w_0}}{d\xi}$$

Then:

$$\xi^2 \frac{dw_1}{d\xi} + w_1 = (2\xi^3 - \overline{w_0}) \frac{dx_1}{d\xi} - \overline{w_0} \frac{d\overline{w_0}}{d\xi} - \xi^2 \frac{d\overline{w_0}}{d\xi} - 2\xi \overline{w_0} - 2\xi^3$$

$$-4\xi^2 x_1 - 2\xi x_1 \frac{d\overline{w_0}}{d\xi} + 6\xi^2 x_1 + 2\xi x_1$$

We suspect $x_1 \sim e^{1/\xi}$. We only want worst ($e^{2/\xi}$) to cancel, so choose:

$$-\overline{w_0} \frac{dx_1}{d\xi} - (2\xi x_1 + \overline{w_0}) \frac{d\overline{w_0}}{d\xi} = 0$$

$$K e^{1/\xi} \frac{dx_1}{d\xi} + 2\xi x_1 \cdot K e^{+1/\xi} \cdot \frac{-1}{\xi^2} + \frac{-K^2 e^{+2/\xi}}{\xi^2} = 0$$

$$\frac{dx_1}{d\xi} - \frac{2}{\xi}x_1 = \frac{\kappa e^{1/\xi}}{\xi^2}$$

$$dx_1 - \frac{2x_1}{\xi}d\xi = \frac{\kappa e^{1/\xi}}{\xi^2}d\xi$$

$$IF = e^{-2\ln\xi} = \frac{1}{\xi^2}$$

$$d\left(\frac{x_1}{\xi^2}\right) = \int \frac{\kappa e^{1/\xi}}{\xi^4}d\xi \qquad \text{etc}$$

Back to $(x + \epsilon u)\frac{du}{dx} + u = 0$

$u_0 = \frac{1}{\xi}$, $u_1 = u_2 = \cdots = 0$

$x_1 = \frac{\xi^2 - 1}{2\xi}$, $x_2 = x_3 \cdots = 0$

$u = \frac{1}{\xi}$; $x = \xi + \epsilon\left(\frac{\xi^2-1}{2\xi}\right)$

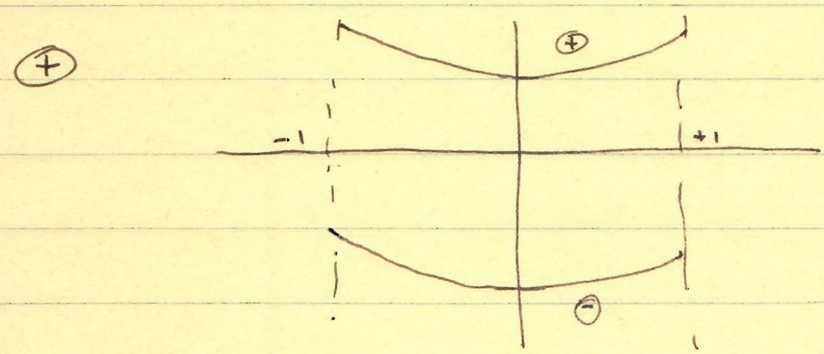
Boundary Layer Problems

Consider:

$$\epsilon u'' + (1-x^2)u + u^2 = 1 \quad , \quad u(-1) = u(1) = 0$$

$$(1-x^2)u_0 + u_0^2 = 1 \quad ; \quad u_0^2 + (1-x^2)u_0 - 1 = 0$$

$$u_0 = \frac{-(1-x^2)}{2} \pm \frac{1}{2}\left[(1-x^2)^2 + 4\right]^{1/2}$$



Scale to -1 with $(1+x)\epsilon^{\beta} = \eta \quad : \quad x = \eta\epsilon^{-\beta} - 1$

$$\epsilon^{1+2\beta} u_{\eta\eta} + (1 - \eta^2\epsilon^{-2\beta} + 2\eta\epsilon^{-\beta} - 1)u + u^2 = 1$$

$$\epsilon^{1+2\beta} u_{\eta\eta} + \eta\epsilon^{-\beta}(2 - \eta\epsilon^{-\beta})u + u^2 = 1$$

Try for solution of $u = u_0 + w(\eta)$

$$u_0 = -\tfrac{1}{2}\eta\epsilon^{-\beta}(2 - \eta\epsilon^{-\beta}) \pm \tfrac{1}{2}\left[\eta^2\epsilon^{-2\beta}(2-\eta\epsilon^{-\beta})^2 + 4\right]^{1/2}$$

$$\epsilon u_0'' + \epsilon^{1+2\beta} u_{\eta\eta} + \eta\epsilon^{-\beta}(2 - \eta\epsilon^{-\beta})(u_0 + w) + u_0^2 + 2wu_0 + w^2 = 0$$

Choose $\beta = -\frac{1}{2}$:

$$\cancel{\epsilon} \; W'' + W^2 + 2 W u_0 + O(\epsilon^{1/2}) + O(\epsilon) = 0$$

or: $W'' + 2 W u_0 + W^2 = 0$

For $\eta < \sim 1000$, or very close to the boundary $x = -1$ we have: $u_0 \doteq \pm 1$ so:

$$W'' \pm 2W + W^2 = 0 \qquad \text{which sign?}$$

Try $(+1)$:

$$W'' + 2W + W^2 = 0$$

$$\frac{(W')^2}{2} + W^2 + \frac{W^3}{3} = 0$$

can't be $\quad 1 \quad -\frac{1}{3}$

The new BC are: since $u = u_0 + w(x) = +1 + w(0) = 0$

$$W(0) = -1$$

$$\frac{(W')^2}{2} - W^2 + \frac{W^3}{3} = 0$$

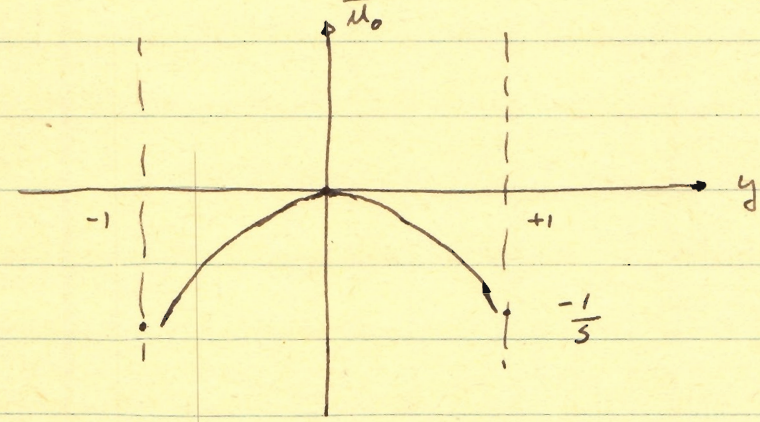
Consider: $\epsilon u_{yy} - (2-y^2) u_x = y^2$, $u(0,y) = u(x,-1) = u(x,1) = 0$

$0 < \epsilon \ll 1$

$$\bar{u}(s,y) = \int_0^\infty u(x,y) e^{-sx} dx$$

$$\epsilon \bar{u}_{yy} - (2-y^2) s \bar{u} = y^2 \quad ; \quad \bar{u}(s,-1) = \bar{u}(s,1) = 0$$

$\bar{u}_0$: $\quad -(2-y^2) s \bar{u}_0 = y^2 \quad ; \quad \bar{u}_0 = \dfrac{y^2}{(y^2-2)s}$



Change variables $\quad \eta = (1+y)\epsilon^{\beta} \quad ; \quad y = \eta e^{-\beta} - 1$

$$\epsilon^{1+2\beta} \bar{u}_{\eta\eta} - (2 - \eta^2 \epsilon^{-2\beta} + 2\eta\epsilon^{-\beta} - 1) s \bar{u} = y^2$$

$$y^2 = \eta^2 \epsilon^{-2\beta} - 2\eta\epsilon^{-\beta} + 1$$

$$\bar{u}_0(\eta) = \frac{1}{s}\left[\frac{\eta^2\epsilon^{-2\beta} - 2\eta\epsilon^{-\beta} + 1}{\eta^2\epsilon^{-2\beta} - 2\eta\epsilon^{-\beta} - 1}\right]$$

Take $\beta = -1/2$

$$\bar{u}_{\eta\eta} - (1 - \eta^2 \epsilon + 2\eta\epsilon^{1/2}) s \bar{u} = \eta^2\epsilon - 2\eta\epsilon^{1/2} + 1$$

$$\bar{u}_0(\eta) = \frac{1}{s}\left[\frac{\eta^2\epsilon - 2\eta\epsilon^{1/2} + 1}{\eta^2\epsilon - 2\eta\epsilon^{1/2} - 1}\right]$$

$$\begin{array}{r|l} & \quad 1 \\ \hline \eta^2\epsilon - 2\eta\epsilon^{1/2} - 1 & \eta^2\epsilon - 2\eta\epsilon^{1/2} + 1 \\ & \underline{\eta^2\epsilon - 2\eta\epsilon^{1/2} - 1} \\ & \qquad\qquad 2 \end{array}$$

$$\bar{u} = \bar{u}_0 + \bar{w} \quad ; \quad \bar{u}(s,-1) = \bar{u}_0(s,-1) + \bar{w}(s,0) = 0$$

$$\bar{w}(s,0) = \frac{1}{s}$$

$$\bar{w}_{\eta\eta} + (\eta^2\epsilon - 2\eta\epsilon^{1/2} - 1)s\left(\bar{u}_0 + \bar{w}\right) = \eta^2\epsilon - 2\eta\epsilon^{1/2} + 1$$

$$\bar{w}_{\eta\eta} + (\eta^2\epsilon - 2\eta\epsilon^{1/2} + 1) + s(\eta^2\epsilon - 2\eta\epsilon^{1/2} - 1)\bar{w} = \eta^2\epsilon - 2\eta\epsilon^{1/2} + 1$$

$$\bar{w}_{\eta\eta} - s\bar{w} = 0 \quad ; \quad \bar{w} = A\cosh\sqrt{s}\,\eta$$

WKB method:

$$u''(x) - \lambda^2 Q(x)\, u(x) = 0 \quad ; \quad Q(0) = 0, \quad Q'(0) = 1$$

Take $u = e^{\lambda\int_0^x g(t,\lambda)\,dt}$

$$u' = \lambda g(x,\lambda)\, u$$

$$u'' = \lambda g'(x,\lambda)\, u + \lambda^2 g^2(x,\lambda)\, u$$

$$\lambda^2 g^2(x,\lambda) + \lambda g'(x,\lambda) = \lambda^2 Q(x)$$

$$\boxed{g^2 + \frac{1}{\lambda} g' = Q}$$

or: $g^2 + \epsilon g' = Q$; $\epsilon = \frac{1}{\lambda}$, $\lambda >> 1$

$g = g_0 + \epsilon g_1 + \cdots$

$g_0^2 = Q$

$(g_0 + \epsilon g_1)^2 + \epsilon g_0' = Q$; $2 g_0 g_1 + g_0' = 0$

$g_1 = -\frac{g_0'}{2 g_0} = -\frac{1}{2}(\log g_0)'$

$\int g\, dx = \int \sqrt{Q}\, dx + \frac{1}{\lambda}\left(-\frac{1}{4}\log Q\right)$

$u = e^{\lambda \int g\, dx} = Q^{-1/4} e^{\lambda \int \sqrt{Q}\, dx}$

for Q with no zeroes.

Suppose, however, Q has a zero at $x=0$. Then try something like BL: Put $\lambda^n x = \xi$ in:

$u''(x) - \lambda^2 Q(x)\, u(x) = 0$; $Q(0) = 0$, $Q'(0) = 1$

Then: $Q(x) = x + a_2 x^2 + \cdots$

$$\lambda^{2n} u_{\xi\xi} - \lambda^2 \underbrace{(x + a_2 x^2 + \cdots)}_{\xi \lambda^{-n} + a_2 \xi^2 \lambda^{-2n} + \cdots} u = 0$$

$$\lambda^{2n} u_{\xi\xi} - \xi u \lambda^{2-n} - a_2 \xi^2 \lambda^{2-2n} u - \cdots = 0$$

$$u_{\xi\xi} - \xi u \lambda^{2-3n} - a_2 \xi^2 u \lambda^{2-4n} - \cdots = 0$$

choose $n = 2/3$: $u_{\xi\xi} - \xi u - a_2 \xi^2 \lambda^{-2/3} u = 0$

$$u_0 = \xi^{1/2} \underbrace{Z_{1/3}\left(\tfrac{2}{3} i \xi^{3/2}\right)}$$

$K_{1/3}\left(\tfrac{2}{3} \xi^{3/2}\right)$

$I_{1/3}\left(\tfrac{2}{3} \xi^{3/2}\right)$

Some Homework Problems

(1) $(xu')' + \lambda x u = 0$; $u'(0) = u(1) = 0$

$(xG')' = \delta(x-\xi)$

$xG'' + G' = \delta(x-\xi)$

$xG' = A$; $G = A \ln x + B$

$G' = \frac{A}{x}$

$u(1) \Rightarrow G(1) = 0 = B$: $G = A \ln x$; $x > \xi$

$G' = 0$; $x < \xi$

$G = A'$

$$\begin{vmatrix} 1 & -\ln\xi \\ 0 & \frac{1}{\xi} \end{vmatrix} \begin{vmatrix} A' \\ A \end{vmatrix} = \begin{vmatrix} 0 \\ 1 \end{vmatrix} \quad ; \quad \Delta = \frac{1}{\xi}$$

$\Delta_{A'} = +\ln\xi$

$\Delta_A = 1$

$$\begin{vmatrix} \ln\xi & -1 \\ \frac{1}{\xi} & 0 \end{vmatrix} \begin{vmatrix} A \\ A' \end{vmatrix} = \begin{vmatrix} 0 \\ \frac{1}{\xi} \end{vmatrix}$$

$\Delta = \frac{1}{\xi}$; $\Delta_A = \frac{1}{\xi}$, $\Delta_{A'} = \frac{\ln\xi}{\xi}$

$A = 1$

$A' = \ln\xi$

$$u'' + u + \lambda x^4 u = 0 \;; \qquad u(1) = u(-1) = 0$$

$$G'' + G = \delta(x-\xi)$$

$$G = A \sin x + B \cos x$$

$$u(1) = 0 \quad : \quad G = A \sin 1 + B \cos 1 = 0$$

$$u(-1) = 0 \quad : \quad G = -A \sin 1 + B \cos 1 = 0$$

Try : $\quad G = A \sin(x-1) \quad ; \quad x > \xi$

$\qquad\quad G = A' \sin(x+1) \quad ; \quad x < \xi$

$$\begin{Vmatrix} \sin(\xi-1) & -\sin(\xi+1) \\ \cos(\xi-1) & -\cos(\xi+1) \end{Vmatrix} \begin{Vmatrix} A \\ A' \end{Vmatrix} = \begin{Vmatrix} 0 \\ 1 \end{Vmatrix}$$

$$\Delta = \sin(\xi+1)\cos(\xi-1) - \sin(\xi \pm 1)\cos(\xi+1) = \sin 2$$

$$\Delta_A = \sin(\xi+1) \;; \qquad \Delta_{A'} = \sin(\xi-1)$$

$$G(\xi|x) = \frac{1}{\sin 2} \begin{cases} \sin(\xi+1)\sin(x-1) & x > \xi \\ \sin(\xi-1)\sin(x+1) & x < \xi \end{cases}$$

$$u = -\lambda \int_{-1}^{1} \xi^4 \, G(\xi|x) \, u(\xi) \, d\xi$$

Note that the Kernel may be symmeterized by writing

$$u(x) = x^{-2} v(x)$$

$$v = -\lambda \int_{-1}^{1} x^2 \xi^2 G(\xi|x)\, u(\xi)\, d\xi \;=\; \lambda \int_{-1}^{1} K(\xi|x)\, v(\xi)\, d\xi$$

where $K(\xi|x) = -x^2 \xi^2 G(\xi|x)$

Let $K\varphi_n = \int_{-1}^{1} K(\xi|x)\, \varphi_n(\xi)\, d\xi = \varphi_{n+1}(x)$

1

$$u = -\lambda \int_0^x (\ln x)\, \xi\, u(\xi)\, d\xi \;-\; \lambda \int_x^1 \ln\xi\; \xi\, u(\xi)\, d\xi$$

$$u' = -\tfrac{1}{x}\lambda \int_0^x \xi\, u(\xi)\, d\xi - \lambda \ln x \;\; x\, u(x)$$

$$+ \lambda \ln x \;\; x\, u(x) \;=\; -\tfrac{1}{x}\lambda \int_0^x \xi\, u(\xi)\, d\xi$$

$$u'' = \frac{1}{x^2}\lambda \int_0^x \xi\, u(\xi)\, d\xi \;-\; \lambda\, u(x)$$

$$x u'' + u' + \lambda x u = 0$$

(3) $$u(x) = \cos \alpha x + \lambda \int_{-\infty}^{\infty} K_0(a|x-t|)\, u(t)\, dt$$

Define the Fourier Transforms:

$$\bar{u}(\xi) = \int_{-\infty}^{\infty} u(x) e^{-i\xi x} dx$$

$$u(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{u}(\xi) e^{i\xi x} d\xi$$

$$\bar{u}(\xi) = \bar{C}(\xi) + \lambda \bar{K}(\xi) \bar{u}(\xi)$$

$$\bar{C}(\xi) = \lim_{\beta \to 0} \int_{-\infty}^{\infty} e^{-\beta|x|} e^{-i\xi x} \cos \alpha x \, dx$$

$$\bar{K}(\xi) = \frac{\pi}{(\xi^2 + a^2)^{1/2}}$$

$$\bar{u}(\xi) = \bar{C}(\xi) + \frac{\lambda \pi}{(\xi^2 + a^2)^{1/2}} \bar{u}(\xi)$$

$$\bar{u}(\xi) = \bar{C}(\xi) \left[\frac{1}{1 - \frac{\lambda\pi}{(\xi^2+a^2)^{1/2}}} \right] = \bar{C}(\xi) \left[\frac{(\xi^2+a^2)^{1/2}}{(\xi^2+a^2)^{1/2} - \lambda\pi} \right]$$

$$\bar{C}(\xi) \left[1 + \frac{\lambda\pi}{(\xi^2+a^2)^{1/2} - \lambda\pi} \right]$$

$$\bar{u}(x) = \cos \alpha x + \frac{\cancel{\lambda\pi}}{2\pi} \int_{-\infty}^{\infty} e^{i\xi x} d\xi \lim_{\beta \to 0} \int_{-\infty}^{\infty} e^{-\beta|x'|} e^{-i\xi x'} \cos \alpha x' dx'$$

$$\cdot \frac{\lambda\pi}{(\xi^2+a^2)^{1/2} - \lambda\pi}$$

$$= \cos \alpha x + \lambda\pi \int_{-\infty}^{\infty} \cos \alpha x' \, M(x-x')\, dx'$$

where: $M(x-x') = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi\, e^{i\xi(x-x')} \cdot \frac{1}{(\xi^2+a^2)^{1/2} - \lambda\pi}$

Now consider:

$$\mu(x) = \cos\alpha x + \lambda \int_0^\infty K_0(a|x-t|)\, \mu(t)\, dt$$

Consider the general equation:

$$\mu(x) = g(x) + \lambda \int_0^\infty K(x-t)\, \mu(t)\, dt$$

Define
$$\mu(x) = 0 \; ; \; x<0$$
$$g(x) = 0 \; ; \; x<0$$
$$H(x) = 0 \; ; \; x>0$$

$$\mu(x) = g(x) + H(x) + \lambda \int_{-\infty}^{\infty} K(x-t)\, \mu(t)\, dt$$

$$\bar{\mu}_\ominus(\xi) = \bar{g}_\ominus(\xi) + \bar{H}_\oplus(\xi) + \lambda \bar{K}(\xi)\, \bar{\mu}_\ominus(\xi)$$

$$\bar{\mu}_\ominus(\xi) = \frac{\bar{g}_\ominus(\xi) + \bar{H}_\oplus(\xi)}{1 - \lambda \bar{K}(\xi)}$$

Assume: $1 - \lambda\bar{K}(\xi) = \bar{L}_\oplus(\xi)\, \bar{L}_\ominus(\xi)$

$$\bar{L}_\ominus(\xi)\, \bar{\mu}_\ominus(\xi) = \frac{\bar{H}_\oplus(\xi)}{\bar{L}_\oplus(\xi)} + \frac{\bar{g}_\ominus(\xi)}{\bar{L}_\oplus(\xi)}$$

Consider: $e^{-ax} = \int_0^\infty K(|x-t|)\, u(t)\, dt \quad ; \; 0 < x < \infty$

$$g(x) = \lambda \int_0^\infty K(x-t)\, u(t)\, dt$$

$$g(x) = 0 \quad ; \; x < 0$$

$$u(x) = 0 \quad ; \; x < 0$$

$$H(x) = 0 \quad ; \; x > 0$$

$$g(x) = H(x) + \lambda \int_{-\infty}^\infty K(x-t)\, u(t)\, dt$$

$$\bar{g}_{\ominus}(\xi) = \bar{H}_{\oplus}(\xi) + \lambda \bar{K}(\xi)\, \bar{u}_{\ominus}(\xi) = \bar{H}_{\oplus}(\xi) + \lambda \bar{G}_{\ominus}(\xi)\, \bar{G}_{\oplus}\, \bar{u}_{\ominus}$$

$$\lambda \bar{G}_{\ominus} \bar{u}_{\ominus} = -\frac{\bar{H}_{\oplus}}{\bar{G}_{\oplus}} + \frac{\bar{g}_{\ominus}}{\bar{G}_{\oplus}}$$

$$\bar{K}(\xi) = \frac{1}{(\xi^2+1)^{1/3}} = \underbrace{\left(\frac{1}{(\xi-i)^{1/3}}\right)_{\ominus}}_{\bar{G}_{\ominus}} \underbrace{\left(\frac{1}{(\xi+i)^{1/3}}\right)_{\oplus}}_{\bar{G}_{\oplus}}$$

$$g_{\ominus}(\xi) = \int_0^\infty e^{-ax} e^{-i\xi x}\, d\cancel{x} = \int_0^\infty e^{-(a+i\xi)}\, dx$$

$$= \frac{1}{a+i\xi}$$

$$\frac{\bar{g}_{\ominus}}{\bar{G}_{\oplus}} = \frac{(\xi+i)^{1/3}}{a+i\xi} = \frac{(\xi+i)^{1/3}}{i(\xi-ia)}$$

$$= \left[\frac{(\xi+i)^{1/3} - (ia+i)^{1/3}}{i(\xi-ia)}\right]_{\oplus} + \left[\frac{(ia+i)^{1/3}}{i(\xi-ia)}\right]_{\ominus}$$

$$\therefore \quad \frac{\bar{u}_\ominus}{(\xi - i)^{1/3}} - \frac{(ia+i)^{1/3}}{i(\xi - ia)} = E(\xi)$$

For $u_\ominus(x)$ to be integrable at origin requires:

$$\lim_{|\xi| \to \infty} \bar{u}_\ominus \to 0$$

$$\bar{u}_\ominus = E(\xi)\,(\xi - i)^{1/3} + \frac{(\xi - i)^{1/3}(ia+i)^{1/3}}{i(\xi - ia)}$$

requires $E(\xi) = 0$. Hence:

$$\bar{u}_\ominus = \frac{(\xi - i)^{1/3}(ia+i)^{1/3}}{i(\xi - ia)}$$

$$u(x) = \frac{(ia+i)^{1/3}}{2\pi i} \int_{-\infty}^{\infty} e^{i\xi x} \frac{(\xi - i)^{1/3}}{(\xi - ia)}\, d\xi$$

Consider: $u(x) = \lambda \int_0^\infty K(x-t)\, u(t)\, dt$

$$K(x-t) = \left\{1 - 3(x-t)^2\right\} e^{-a|x-t|}$$

$u(x) = 0 \; ; \; x < 0$

$v(x) = 0 \; ; \; x > 0$

$$\bar{u}_\ominus + \bar{v}_\oplus = \lambda \bar{K}\, \bar{u}_\ominus \; ; \qquad \bar{u}_\ominus (1 - \lambda \bar{K}) = -\bar{v}_\oplus$$

Let $1 - \lambda \bar{K} = \bar{G}_\oplus \bar{G}_\ominus \; ; \qquad \bar{G}_\ominus \bar{u}_\ominus = \dfrac{-\bar{v}_\oplus}{\bar{G}_\oplus} = E(\xi)$

$$\bar{K} = \int_{-\infty}^{\infty} e^{-i\xi x} (1-3x^2)\, e^{-a|x|}\, dx$$

$$= \int_0^{\infty} e^{-(a+i\xi)x} (1-3x^2)\, dx + \int_{-\infty}^{0} e^{(a-i\xi)x} (1-3x^2)\, dx$$

$$= \underbrace{\frac{1}{a+i\xi} + \frac{1}{a-i\xi}}_{\frac{2a}{\xi^2+a^2}} \pm \underbrace{3\int_0^{\infty} x^2 e^{-(a+i\xi)x}\, dx}_{\frac{6}{(a+i\xi)^3}} \pm \underbrace{3\int_{-\infty}^{0} x^2 e^{(a-i\xi)x}\, dx}_{\underbrace{3\int_0^{\infty} x^2 e^{-(a-i\xi)x}\, dx}_{\frac{6}{(a \cancel{+} i\xi)^3}}}$$

So then:

$$K = \frac{2a}{\xi^2+a^2} - 6\left\{\frac{(a-i\xi)^3 + (a+i\xi)^3}{(\xi^2+a^2)^3}\right\}$$

$$\begin{array}{r} a^2 - 2i\xi - \xi^2 \\ a - i\xi \\ \hline a^3 - 2ia\xi - a\xi^2 \\ -i\xi a^2 - 2\xi^2 + i\xi^3 \end{array} \qquad \begin{array}{r} a^2 + 2i\xi - \xi^2 \\ a + i\xi \\ \hline a^3 + 2ia\xi - a\xi^2 \\ i\xi a^2 - 2\xi^2 - i\xi^3 \end{array} \qquad \begin{array}{r} (a^2+\xi^2)^2 \\ = a^4 + 2a^2\xi^2 + \xi^4 \\ a^2 + \xi^2 \\ \hline a^6 + 2a^4\xi^2 + a^2\xi^4 \\ a^4\xi^2 \quad 2a^2\xi^4 + \xi^6 \\ \hline a^6 + 3a^4\xi^2 + 3a^2\xi^4 + \xi^6 \end{array}$$

$$K = \frac{2a}{\xi^2+a^2} - \cancel{6}\,12\left\{\frac{a^3 - a\xi^2 - 2\xi^2}{(a^2+\xi^2)^3}\right\}$$

$$= \frac{2a^5 + 4a^3\xi^2 + 2a\xi^4 \pm 12a^3 + 12a\xi^2 + 24\xi^2}{(a^2+\xi^2)^3}$$

$$1 - \lambda K = \frac{a^6 + 3a^4\xi^2 + 3a^2\xi^4 + \xi^6 - 2a^5\lambda - 4a^3\lambda\xi^2 - 2\xi^4\lambda + 12a^3\lambda - 12a\xi^2\lambda - 24\xi^2\lambda}{(\xi^2+a^2)^3}$$

$$= \frac{\xi^6 + (3a^2 - 2a\lambda)\xi^4 + (3a^4 - 4a^2\lambda - 12a\lambda - 24\lambda)\xi^2 + (a^6 - 2a^5\lambda + 12a^3\lambda)}{(\xi^2+a^2)^3}$$

Consider: $$u(y) = \lambda\beta \int_{-1}^{+1} K_0(\beta(y-t))\, u(t)\, dt$$

<u>Infinite Domain:</u>

$$w(y) = \lambda\beta \int_{-\infty}^{\infty} K_0[\beta(y-t)]\, w(t)\, dt$$

$$\bar{w}(\xi) = \lambda\beta \frac{\pi}{(\xi^2+\beta^2)^{1/2}} \bar{w}(\xi)$$

$$\frac{\lambda\beta\pi}{(\xi^2+\beta^2)^{1/2}} = 1 \quad ; \quad w(y) = \cos \xi y$$

$$\xi^2 = (\lambda\beta\pi)^2 - \beta^2 = (\lambda^2\pi^2 - 1)\beta^2$$

<u>Semi-infinite Domain:</u>

$$v(y) = \lambda\beta \int_0^{\infty} K_0[\beta(y-t)]\, w(t)\, dt$$

$$\bar{V}_\ominus + \bar{h}_\oplus = \frac{\lambda\beta\pi}{(\eta^2+\beta^2)^{1/2}} \bar{V}_\ominus$$

$$\left\{1 - \frac{\lambda\beta\pi}{(\eta^2+\beta^2)^{1/2}}\right\} \bar{V}_\ominus = -\bar{h}_\oplus$$

$$\frac{(\eta^2+\beta^2)^{1/2} - \lambda\beta\pi}{(\eta^2+\beta^2)^{1/2}} = \frac{\eta^2+\beta^2-\lambda^2\beta^2\pi^2}{(\eta^2+\beta^2)^{1/2}\left[(\eta^2+\beta^2)^{1/2}+\lambda\beta\pi\right]}$$

$$= \frac{\eta^2-\xi^2}{(\eta^2+\beta^2)^{1/2}\left[(\eta^2+\beta^2)^{1/2}+\lambda\beta\pi\right]}$$

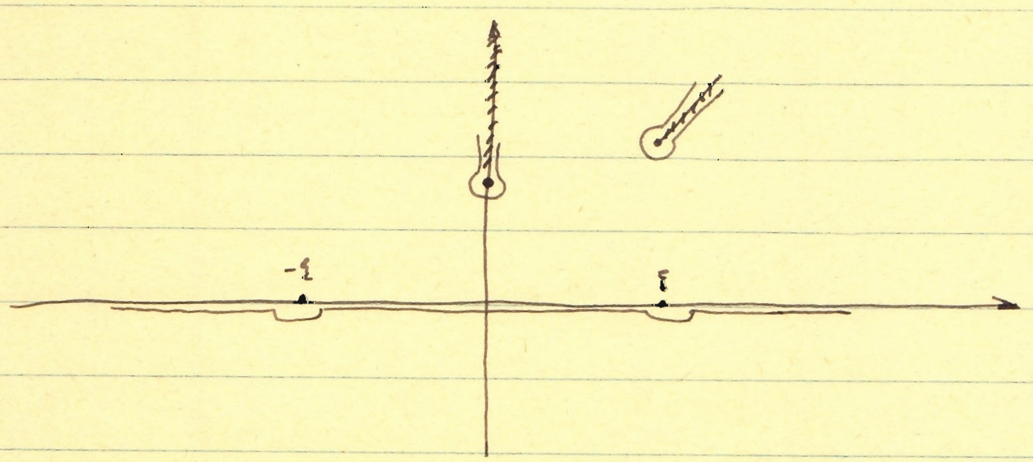
$$1 - \lambda \bar{K} = \frac{\eta^2 - \xi^2}{(\eta + \lambda\beta)^{1/2}_{\oplus} \, (\eta - \lambda\beta)^{1/2}_{\ominus} \, L_{\oplus} \, L_{\ominus}}$$

$$\frac{(\eta^2 - \xi^2)\, \bar{V}_{\ominus}}{(\eta - \lambda\beta)^{1/2}_{\ominus} \, L_{\ominus}} = E(\eta)$$

$$\bar{V}_{\ominus} = \frac{(\eta - \lambda\beta)^{1/2} \, E(\eta) \, L_{\ominus}}{\eta^2 - \xi^2} \qquad E(\eta) = 1$$

$$\bar{V}_{\ominus} = \frac{(\eta - \lambda\beta)^{1/2} \, L_{\ominus}}{\eta^2 - \xi^2}$$

$$V(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{(\eta - \lambda\beta)^{1/2} \, L_{\ominus}(\eta) \, e^{\lambda \eta y} \, d\eta}{(\eta - \xi)(\eta + \xi)}$$



$$V(y) = A \cos\left[\xi(y + \epsilon)\right] + \frac{1}{2\pi} \int_c \frac{(\eta - \lambda\beta) L_{\ominus}(\eta) \, e^{\lambda \eta y} \, d\eta}{\eta^2 - \xi^2}$$

$$A \cos\left[\xi(y + \epsilon)\right] + \chi(y)$$

In the finite domain:

$$u(y) = W(y) - V($$

$$u(y) = A \cos \xi y + \chi(y+1) + \chi(1-y)$$

Consider: $1 = -\frac{1}{2\pi} \int_0^\infty K_0[\beta|x-t|]\, f(t)\, dt \quad , x > 0$

$$g(x) = \int_0^\infty K(x-t) f(t)\, dt \quad : \quad \left.\begin{array}{l} g(x) = 0, \; x<0 \\ f(x) = 0, \; x<0 \end{array}\right\} H(x) = 0, x>0$$

$$\bar{g}_\ominus + \bar{H}_\oplus = \bar{K}\, \bar{f}_\ominus(\xi) = \bar{G}_\ominus \bar{G}_\oplus \bar{f}_\ominus$$

$$\bar{f}_\ominus \bar{G}_\ominus = \frac{\bar{H}_\oplus}{\bar{G}_\oplus} + \frac{\bar{g}_\ominus}{\bar{G}_\oplus}$$

$$\bar{K}(\xi) = -\frac{1}{2\pi} \frac{\pi}{(\xi^2+\beta^2)^{1/2}} = -\frac{1}{2}\left(\frac{1}{(\xi+i\beta)^{1/2}}\right)_\oplus \left(\frac{1}{(\xi-i\beta)^{1/2}}\right)_\ominus$$

$$\bar{g}_\ominus = \int_0^\infty e^{-i\xi x}\, dx = \left(\frac{1}{i\xi}\right)_\ominus$$

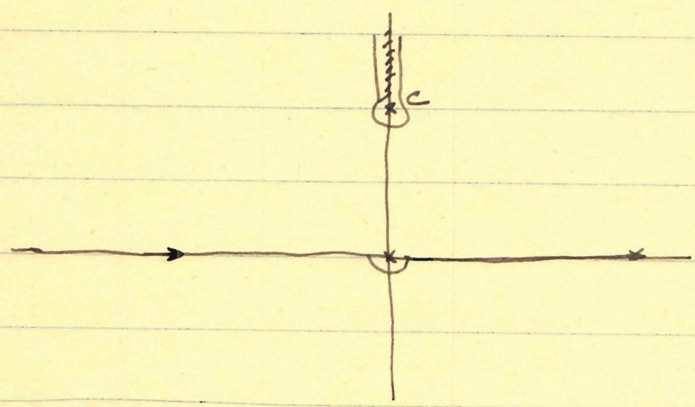
$$\frac{\bar{g}_\ominus}{\bar{G}_\oplus} = \frac{(\xi+i\beta)^{1/2}}{i\xi} = \left[\frac{(\xi+i\beta)^{1/2} - (i\beta)^{1/2}}{i\xi}\right]_\oplus + \left[\frac{(i\beta)^{1/2}}{i\xi}\right]_\ominus$$

$$\frac{-\frac{1}{2}\bar{f}_\ominus}{(\xi-i\beta)^{1/2}} - \frac{(i\beta)^{1/2}}{i\xi} = E(\xi)$$

$$-\frac{1}{2}\bar{f}_\ominus = (\xi-i\beta)^{1/2} E(\xi) + \frac{(\xi-i\beta)^{1/2}(i\beta)^{1/2}}{i\xi} \quad ; \quad E(\xi) = 0$$

$$\bar{f}_\ominus = \frac{-2(\xi - i\beta)^{1/2}(i\beta)^{1/2}}{i\xi}$$

$$f(x) = \frac{-2(i\beta)^{1/2}}{2\pi i}\int_{-\infty}^{\infty} \frac{(\xi - i\beta)^{1/2}}{\xi} e^{i\xi x}\, d\xi$$



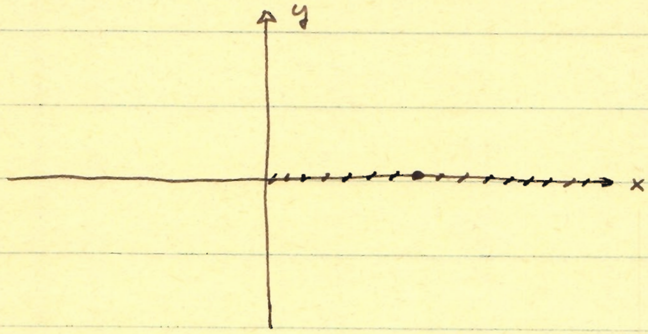
$$\int_{-\infty}^{\infty} \frac{(\xi - i\beta)^{1/2}}{\xi} e^{i\xi x}\, d\xi = \underbrace{2\pi i \sum R}_{2\pi i\, (-i\beta)^{1/2}} - \int_c \frac{(\xi - i\beta)^{1/2}}{\xi} e^{i\xi x}\, d\xi$$

$$f(x) = -2\beta + \frac{2(i\beta)^{1/2}}{2\pi i}\int_c \frac{(\xi - i\beta)}{\xi} e^{i\xi x}\, d\xi$$

$u_{xx} + u_{yy} - k^2 u = 0$; $u(x,0) = 1$, $x > 0$

$u \to 0$ as $\Im\,(x+iy)^{1/2} \to \infty$

u regular except on $y = 0$, $x > 0$



Define transforms:

$$u^*(x,\eta) = \int_{-\infty}^{0^-} u(x,y)\, e^{-i\eta y}\, dy + \int_{0^+}^{\infty} u(x,y)\, e^{-i\eta y}\, dy$$

$$\bar{u}(\xi,\eta) = \int_{-\infty}^{\infty} u^*(x,\eta)\, e^{-i\xi x}\, dx$$

$$u^+(\xi,y) = \int_{-\infty}^{\infty} u(x,y)\, e^{-i\xi x}\, dx$$

$$\int_{-\infty}^{0^-} u_{yy}\, e^{-i\eta y}\, dy = u_y\, e^{-i\eta y}\Big]_{-\infty}^{0^-} + i\eta \int_{-\infty}^{0^-} u_y\, e^{-i\eta y}\, dy$$

$$\left.\begin{array}{ll} u = e^{-i\eta y} & dv = u_{yy}\, dy \\ du = -i\eta e^{-i\eta y}\, dy & v = u_y \end{array}\right\} \quad f(x) = u_y(x,0^+) - u_y(x,0^-)$$

* transform the above equation:

$$u^+_{xx} - \eta^2 u^+ - k^2 u^* = f(x)$$

\- Transform:

$$-\xi^2 \bar{u} - \eta^2 \bar{u} - k^2 \bar{u} = \bar{f}(\xi)$$

$$\bar{u}(\xi,\eta) = \frac{-\bar{f}(\xi)}{\xi^2+\eta^2+k^2}$$

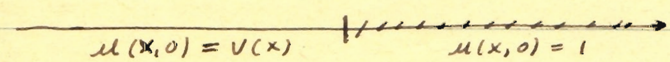
$$u^{+}(\xi,y) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\bar{f}(\xi)\, e^{i\eta y}\, d\eta}{\eta^2+\xi^2+k^2}$$

$$= -\frac{1}{2\pi}\bar{f}(\xi)\int_{-\infty}^{\infty} \frac{e^{i\eta y}\, d\eta}{\eta^2+\xi^2+k^2} = -\frac{1}{2\pi}\bar{f}(\xi)\int_{-\infty}^{\infty} \frac{e^{i\eta y}\, d\eta}{\left(\eta+i\sqrt{\xi^2+k^2}\right)\left(\eta-i\sqrt{\xi^2+k^2}\right)}$$

$$= \frac{-\bar{f}(\xi)\, e^{-|y|\sqrt{\xi^2+k^2}}}{2\sqrt{\xi^2+k^2}}$$

Once we have $f(x)$, we can do the problem.

Along $y=0$:



$$u^{+}(\xi,0) = \int_{-\infty}^{0} v(x)\, e^{-i\xi x}\, dx + \int_{0}^{\infty} e^{-i\xi x}\, dx$$

$$= \bar{v}_{\oplus} + \left(\frac{1}{i\xi}\right)_{\ominus} = \frac{-\bar{f}_{\ominus}}{2\sqrt{\xi^2+k^2}} = \frac{-\bar{f}_{\ominus}}{2\,(\xi+ik)^{1/2}_{\oplus}\,(\xi-ik)^{1/2}_{\ominus}}$$

$$\bar{f}(\xi) = \frac{-2\sqrt{ik}\,(\xi-ik)^{1/2}}{i\xi}$$

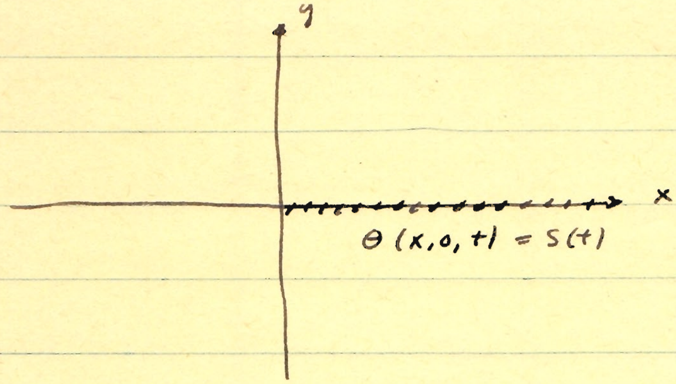
$$u(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\sqrt{ik}\,(\xi-ik)^{1/2}\, e^{-|y|\sqrt{\xi^2+k^2}}\, d\xi}{i\xi\,(\xi-ik)^{1/2}(\xi+ik)^{1/2}}\, e^{i\xi x}$$

$$u(x,y) = \frac{\sqrt{ik}}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{-|y|\sqrt{\xi^2+k^2}}\, d\xi}{\xi\,(\xi+ik)^{1/2}}\, e^{i\xi x}$$

$$\theta_{xx} + \theta_{yy} - \theta_x - \theta_t = 0 \quad ; \quad \theta = \theta(x,y,t)$$

$$\theta(x,0,t) = s(t) \quad \text{for } x > 0$$

$$s(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$



$$\tilde{\theta}(x,y,s) = \int_0^\infty e^{-st}\,\theta(x,y,t)\,dt$$

$$\theta^*(x,\eta,s) = \int_{-\infty}^{0^-} e^{-i\eta y}\,\tilde{\theta}(x,y,s)\,dy + \int_{0^+}^{\infty} e^{-i\eta y}\,\tilde{\theta}(x,y,s)\,dy$$

$$\bar{\theta}(\xi,\eta,s) = \int_{-\infty}^{\infty} e^{-i\xi x}\,\theta^*(x,\eta,s)\,dx$$

$$\theta^+(\xi,y,s) = \int_{-\infty}^{\infty} e^{-i\xi x}\,\tilde{\theta}(x,y,s)\,dx$$

$$\tilde{\theta}_y(x,0^+,s) - \tilde{\theta}_y(x,0^-,s) = f(x,s)$$

$\sim$: $\quad \tilde{\theta}_{xx} + \tilde{\theta}_{yy} - \tilde{\theta}_x - s\tilde{\theta} = 0$

$*$: $\quad \theta^*_{xx} - \eta^2\theta^* - \theta^*_x - s\theta^* = f(x,s)$

$-$: $\quad -\xi^2\bar{\theta} - \eta^2\bar{\theta} - i\eta\bar{\theta} - s\bar{\theta} = \bar{f}(\xi,s)$

$$\bar{\theta}(\xi,\eta,s) = \frac{-\bar{f}(\xi,s)}{\xi^2+\eta^2+i\eta+s}$$

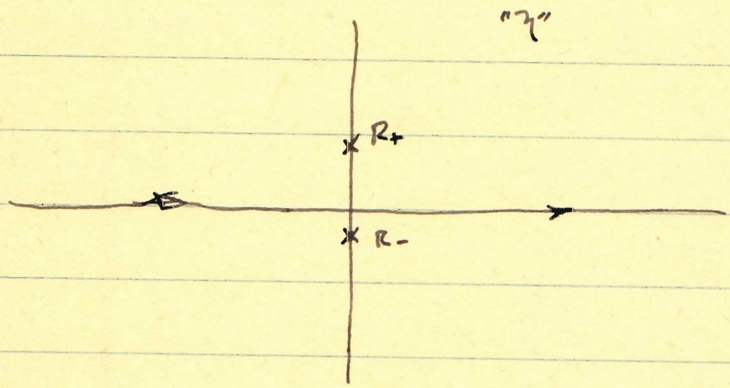
$$\tilde{\theta}(x,0,s) = \int_0^\infty e^{-st}\,dt = \frac{1}{s}$$

$$\boxed{\cancel{\theta^+(\xi,0,s) = \frac{1}{s}\int_{-\infty}^{\infty} e^{-i\xi x}\,dx = \frac{1}{i\xi s}}}$$

$$\eta^2+i\eta+\xi^2+s \quad ; \quad \eta = \frac{-i}{2} \pm \frac{1}{2}\left[-1-4\xi^2-4s\right]^{1/2}$$

$$= \frac{i}{2}\left\{1 \pm \left[4\xi^2+4s+1\right]^{1/2}\right\} = R_1, R_2 = R_+, R_-$$

$$\cancel{\theta^+ =} \quad \theta^+(\xi,y,s) = \frac{-f(\xi,s)}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i\eta y}\,d\eta}{(\eta-R_+)(\eta-R_-)}$$



$$\theta^+(\xi,y,s) = \frac{-f(\xi,s)}{2\pi}\cdot 2\pi i\,\frac{e^{i|y|R_+}}{R_+-R_-}$$

$$R_+ - R_- = +i\left[4\xi^2+4s+1\right]^{1/2}$$

$$\Theta^+(\xi, y, s) = \frac{-\bar{f}(\xi,s)\, e^{-|y|\left\{\frac{1}{2} + \frac{1}{2}\left[4\xi^2 + 4s + 1\right]^{1/2}\right\}}}{\left[4\xi^2 + 4s + 1\right]^{1/2}}$$

0

$\theta(x,0,s) = V(x)$ $\qquad$ $\theta(x,0,s) = \frac{1}{s}$

$$\Theta^+(\xi,0,s) = \bar{V}_{\oplus}(\xi) + \left(\frac{1}{i\xi s}\right)_{\ominus} = \frac{-\bar{f}(\xi,s)}{\left[4\xi^2 + 4s + 1\right]^{1/2}}$$

$$\left[4\xi^2 + 4s + 1\right]^{1/2} = 2\left\{\xi^2 + s + \tfrac{1}{4}\right\}^{1/2} = 2\left\{\left(\xi + i\underbrace{\sqrt{s + 1/4}}_{p}\right)^{1/2}_{\oplus}\left(\xi - i\sqrt{s+1/4}\right)^{1/2}_{\ominus}\right\}$$

$$\frac{\bar{f}(\xi,s)}{(\xi - ip)^{1/2}} + \frac{(ip)^{1/2}}{i\xi s} = E(\xi)$$

Choose $E(\xi) = 0$ for now: $\quad \bar{f}(\xi,s) = \dfrac{(ip)^{1/2}\,(\xi - ip)^{1/2}}{i\xi s}$

$$\therefore \boxed{\Theta(x,y,t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{st}\,dt \cdot \frac{(ip)^{1/2}}{2\pi i s}\int_{-\infty}^{\infty} \frac{e^{i\xi x - |y|\left\{\frac{1}{2} + (\xi^2 + p^2)^{1/2}\right\}}}{\xi\,(\xi + ip)^{1/2}}\,d\xi}$$

① Consider: $e^{-ax} = \int_0^\infty K(x-t)\,u(t)\,dt \;;\; \bar{K}(\xi) = \dfrac{1}{(\xi^2+1)^{1/3}}$

$$g(x) = \int_0^\infty K(x-t)\,u(t)\,dt \quad : \quad \begin{array}{l} u(x) = 0 \;;\; x<0 \\ g(x) = 0 \;;\; x<0 \\ H(x) = 0 \;;\; x>0 \end{array}$$

$$\bar{g}_\ominus + \bar{H}_\oplus = \bar{K}(\xi)\,\bar{u}_\ominus = \bar{G}_\oplus \bar{G}_\ominus \bar{u}_\ominus$$

$$\bar{K}(\xi) = \frac{1}{(\xi - i)^{1/3}_\ominus}\;\frac{1}{(\xi + i)^{1/3}_\oplus}$$

$$\bar{g}_\ominus = \int_0^\infty e^{-(a+i\xi)x}\,dx = \frac{1}{a+i\xi} = \frac{1}{i(\xi - ia)}$$

$$\frac{\bar{g}_\ominus}{\bar{G}_\oplus} = \left[\frac{(\xi+i)^{1/3} - (ia+i)^{1/3}}{i(\xi - ia)}\right]_+ + \left[\frac{(ia+i)^{1/3}}{i(\xi-ia)}\right]_\ominus$$

$$\frac{\bar{u}_\ominus}{(\xi-i)^{1/3}} = \frac{(ia+i)^{1/3}}{i(\xi-ia)} \;;\quad \bar{u}_\ominus = \frac{(\xi-i)^{1/3}(ia+i)^{1/3}}{i(\xi-ia)}$$

$$\boxed{u(x) = \frac{(ia+i)^{1/3}}{2\pi i}\int_{-\infty}^{\infty}\frac{(\xi-i)^{1/3}\,e^{i\xi x}\,d\xi}{\xi - ia}}$$

$$u(x) = \frac{(ia+1)^{1/3}}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{i\xi x + \frac{1}{3}\log(\xi - i)}}{\xi - ia}\,d\xi$$

$$= \frac{(ia+1)^{1/3}}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{x\left[i\xi + \frac{1}{3x}\log(\xi-i)\right]}}{\xi - ia}\,d\xi$$

$$\int_{-\infty}^{\infty} f(z)\, e^{-\lambda g(z)}\, dz \quad : \quad z_0 = \text{s.p.}$$

z_1 = pole of $f(z)$

$$u(x) = \frac{(ia+i)^{1/3}}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{-x\left[-i\xi - \frac{1}{3x}\log(\xi - i)\right]}}{\xi - ia}\, d\xi$$

$$g(\xi, x) = -i\xi - \frac{1}{3x}\log(\xi - i)$$

$$g'(\xi) = -i - \frac{1}{3x(\xi - i)} = 0 \quad ; \quad 3x(\xi - i) = i$$

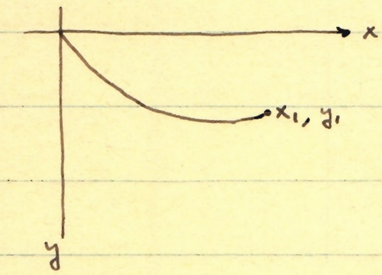
$$; \quad \xi_0 = i\left(\frac{1}{3x} + 1\right)$$

$$g''(\xi) = \frac{1}{3x(\xi - i)^2} \to \frac{1}{3x\left(\frac{-1}{(3x)^2}\right)} = -3x$$

$\operatorname{Im} g(\xi) = \text{const.}$

$$u(x) \approx \frac{(ia+i)^{1/3}}{2\pi i} \int_C e$$

Calculus of Variations



$$\tfrac{1}{2} m v^2 = m g y$$

$$v = \sqrt{2gy} = \frac{ds}{dt}$$

$$ds = \sqrt{dx^2 + dy^2}$$

$$= dx\sqrt{1 + (y_x)^2} \quad ; \quad y_x = \frac{dy}{dx}$$

$$\therefore \quad \frac{dx}{dt} = \sqrt{2g}\sqrt{\frac{y}{1+(y_x)^2}}$$

$$\sqrt{2g}\, t = \int_0^{x_1} \sqrt{\frac{1+(y_x)^2}{y}}\, dx = \int_0^{x_1} f(x, y, y_x)\, dx$$

$$\delta \int_0^{x_1} f(x, y, y_x)\, dx = \int_0^{x_1} \left[\frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y_x}\delta y_x \right] dx$$

$$= \int_0^{x_1} \left\{ \frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y_x} \right\} \delta y\, dx = 0$$

Hence: $$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y_x} = 0$$

now: $$\frac{df}{dx} = \frac{\partial f}{\partial x} + y_x \frac{\partial f}{\partial y} + y_{xx}\frac{\partial f}{\partial y_x}$$

$$-\left(y_x \frac{\partial f}{\partial y} + y_{xx}\frac{\partial f}{\partial y_x} \right) = -\frac{df}{dx} + \frac{\partial f}{\partial x}$$

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{d}{dx}\left(y\frac{\partial f}{\partial y} + y_x \frac{\partial f}{\partial y_x} \right) - y\frac{d}{dx}\frac{\partial f}{\partial y} - y_x\frac{d}{dx}\frac{\partial f}{\partial y_x}$$

$$0 = \frac{\partial f}{\partial x} - \frac{d}{dx}\left(f - y_x\frac{\partial f}{\partial y_x} \right) + \frac{d}{dx}\left(y\frac{\partial f}{\partial y} \right) - \frac{d}{dx}\frac{\partial f}{\partial y_x}$$

$$0 = \frac{\partial f}{\partial x} - \frac{d}{dx}\left(f - y_x \frac{\partial f}{\partial y_x}\right) + y_x \frac{\partial f}{\partial y} - y_x \frac{d}{dx}\frac{\partial f}{\partial y_x}$$

or: $$\frac{\partial f}{\partial x} - \frac{d}{dx}\left(f - y_x \frac{\partial f}{\partial y_x}\right) = 0$$

Here we have: $f - y_x \dfrac{\partial f}{\partial y_x} = c$

$$\frac{\partial f}{\partial y_x} = \frac{2\, y_x / y}{2\sqrt{\dfrac{1+(y_x)^2}{y}}}$$

$$\sqrt{\frac{1+(y_x)^2}{y}} - \frac{(y_x)^2/y}{\sqrt{\dfrac{1+(y_x)^2}{y}}} = c$$

$$= \frac{\frac{1}{y}}{\sqrt{\dfrac{1+y_x^2}{y}}} = \frac{1}{\sqrt{y(1+y_x^2)}} = c$$

$$1 = c^2\left\{y + y\left(\frac{dy}{dx}\right)^2\right\} \; ;$$

<u>Weierstrauss Corner Cond. # I</u>

$\dfrac{\partial f}{\partial y'}$ must be continuous at a corner:

$$I = \int_{x_1}^{x_2} f(x, y, y')\, dx$$

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$$

or: $$\frac{\partial f}{\partial y} - \frac{\partial^2 f}{\partial x\, \partial y'} - y' \frac{\partial^2 f}{\partial y\, \partial y'} - y'' \frac{\partial^2 f}{\partial y'^2} = 0$$

Consider: $\dfrac{\partial^2 f}{\partial y'^2} = 0$: $\dfrac{\partial f}{\partial y} - \dfrac{\partial^2 f}{\partial x \partial y'} - y' \dfrac{\partial^2 f}{\partial y \partial y'} = 0$

$$f = M(x,y) + N(x,y)\,y'$$

$$\frac{\partial M}{\partial y} + y' \frac{\partial N}{\partial y} - \frac{\partial N}{\partial x} - y' \frac{\partial N}{\partial y} \quad ; \quad \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 0$$

$\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$: For some V : $\dfrac{\partial V}{\partial x} = M$; $\dfrac{\partial V}{\partial y} = N$

Then we have:

$$I = \int \left[\frac{\partial V}{\partial x} + \frac{\partial V}{\partial y} y' \right] dx = \int \left(\frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy \right)$$

$$= \int dV = V(x_2 y_2) - V(x_1 y_1)$$

Wierstrass Corner Condition: $\dfrac{\partial f}{\partial y'}$ continuous at a corner.

Legendre's Condition

$$\delta I = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} \delta y + \frac{\partial f}{\partial y'} \delta y' \right] dx$$

$$+ \frac{1}{2} \int_{x_1}^{x_2} \left\{ \frac{\partial^2 f}{\partial y^2} (\delta y)^2 + 2 \frac{\partial^2 f}{\partial y \partial y'} \delta y \, \delta y' + \frac{\partial^2 f}{\partial y'^2} (\delta y')^2 \right\} dx$$

must have $\dfrac{\partial^2 f}{\partial y'^2} \geq 0$ for minimum.

Exchange of variables:

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0 \quad : \qquad \frac{\partial \bar f}{\partial \bar y} - \frac{d}{d\bar x}\frac{\partial \bar f}{\partial \bar y'} = 0$$

$$\bar f\, dy = f\, dx \quad ; \quad \bar f = \frac{f}{y'} \qquad \bar y' = \frac{1}{y'} \ , \quad \bar x = y \ , \quad \bar y = x$$

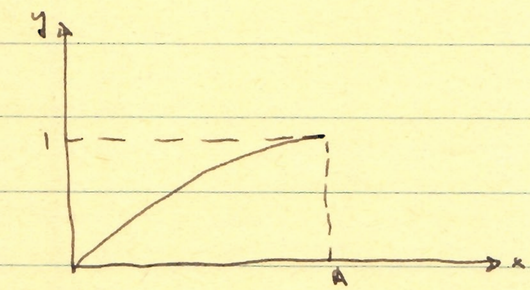
$$\frac{\partial}{\partial x}\frac{f}{y'} - \frac{d}{dy}\frac{\partial}{\partial \frac{1}{y'}}\frac{f}{y'} = 0$$

$$\frac{1}{y'}\frac{\partial f}{\partial x} - \frac{d}{dy}\left[f + \frac{1}{y'}\frac{\partial f}{\partial \frac{1}{y'}}\right] = 0$$

$$\partial \frac{1}{y'} = -\frac{1}{y'^2}\,\partial y'$$

$$\frac{\partial f}{\partial x} - \frac{d}{dx}\left[f - y'\frac{\partial f}{\partial y'}\right] = 0$$

Consider: $f = y'^2(y'-1)^2$; $(0,0)$ to $(A,1)$



$$f - y'\frac{\partial f}{\partial y'} = c$$

$$y'^2(y'-1)^2 - y'\left[2y'(y'-1)^2 + y'^2\cdot 2(y'-1)\right]$$

$$= y'^2(y'-1)^2 + 2y'^3 - 2y'^2 = c$$

$$y'^4 - 2y'^3 + y' + 2y'^3 - 2y'^2 = \cancel{\quad} = C$$

Hence y' is some constant: $y' = a$

$y = ax + b$; $y = \frac{x}{A}$

$$\frac{\partial f}{\partial y'} = 4y'^3 - 6y'^2$$

$$\frac{\partial^2 f}{\partial y'^2} = 12y'^2 - 12y' \geq 0 \quad \text{or} \quad y'(y'-1) \geq 0$$

$$y' \geq 1$$

That is: $A \leq 1$ for minimum to exist

$$2y'^3 - 4y'^2 + 2y' + 2y'^3 - 2y'^2 \quad \text{OK}$$

$$4y'^3 - 6y'^2 + 2y'$$

~~The rate~~.

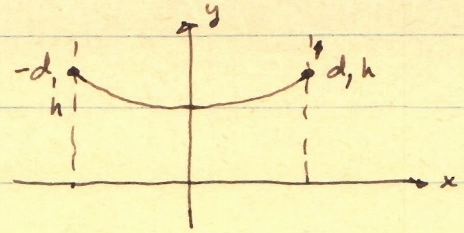
$$f = \varphi(x,y)\sqrt{1+y'^2}$$

$$\frac{\partial f}{\partial y'} = \varphi(x,y)\frac{y'}{\sqrt{1+y'^2}} \to y'_r = y'_e$$

$$f - y'\frac{\partial f}{\partial y'} = \varphi(x,y)\left\{\sqrt{1+y'^2} - \frac{y'^2}{\sqrt{1+y'^2}}\right\} = \frac{\varphi(x,y)}{\sqrt{1+y'^2}} \quad ?$$

$$\to y'_r = y'_e$$

9. Consider.



$$A = 2\pi \int_{-d}^{d} y\,(1+y'^2)^{1/2}\,dx \quad ; \quad f = y\,(1+y'^2)^{1/2}$$

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0 \quad ; \quad \frac{\partial f}{\partial x} - \frac{d}{dx}\left(f - y'\frac{\partial f}{\partial y'}\right) = 0$$

$$\therefore \; f - y'\frac{\partial f}{\partial y'} = c \quad ; \quad \frac{\partial f}{\partial y'} = \frac{yy'}{\sqrt{1+y'^2}}$$

$$y\,(1+y'^2)^{1/2} - \frac{yy'^2}{\sqrt{1+y'^2}} = \frac{y}{\sqrt{1+y'^2}} = c$$

$$y^2 = c^2 + c^2\left(\frac{dy}{dx}\right)^2 \quad ; \quad \frac{dy}{dx} = \frac{1}{c}\sqrt{y^2-c^2}$$

$$dx = \frac{c\,dy}{\sqrt{y^2-c^2}} = \frac{dy}{\sqrt{\left(\frac{y}{c}\right)^2-1}} = c\,\frac{dz}{\sqrt{z^2-1}}$$

$\cancel{\sinh^2\theta - \cosh^2\theta = 1}$ $\quad \cosh^2\theta - \sinh^2\theta = 1$

$$\sinh^2\theta = \cosh^2\theta - 1$$

Let $z = \cosh\theta$; $dz = \sinh\theta\, d\theta$

$$x = c\,\cosh^{-1} \underset{\frac{y}{c}}{\underset{\parallel}{z}} + b \qquad y = c\cosh\frac{x-b}{c}$$

$\cancel{y = c\cosh}$ $\quad \cancel{y = c\cosh}$ $\quad \cancel{y = c\cosh\frac{x}{c} + b}$

$$\left.\begin{aligned} h &= c\cosh\frac{d-b}{c} \\ h &= c\cosh\frac{d+b}{c} \end{aligned}\right\} \quad b = 0$$

$h = c \cosh \frac{d}{c}$; gives equation for c, $\therefore$ $y = c \cosh \frac{x}{c}$

Legendre Test: $\quad \dfrac{\partial^2 f}{\partial y'^2} = \dfrac{y}{\sqrt{1+y'^2}} - \dfrac{y'y}{(1+y'^2)^{3/2}}$

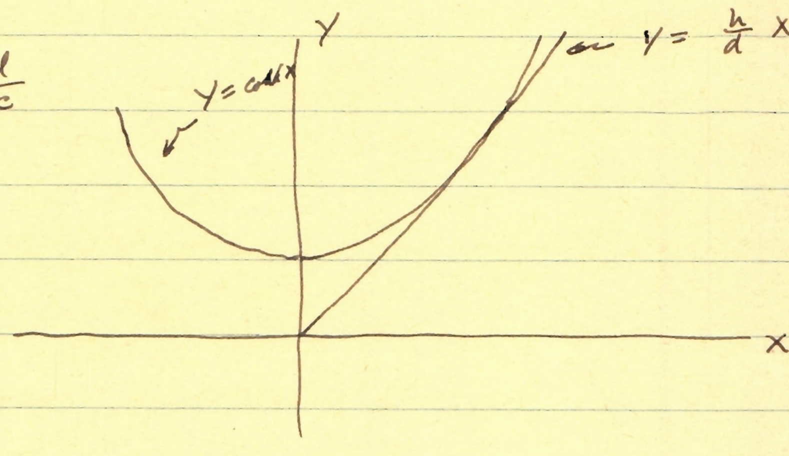
$$= \frac{y}{(1+y'^2)^{3/2}}$$

$y' = \sinh \frac{x}{c}$, $\quad \cosh^2 x - \sinh^2 x = 1$

$$\frac{\partial^2 f}{\partial y'^2} = \frac{c \cosh \frac{x}{c}}{\cosh^3 \frac{x}{c}} = \frac{c}{\cosh^2 \frac{x}{c}} \geq 0$$

$\frac{h}{c} = \cosh \frac{d}{c}$

$Y = \frac{h}{c}$, $X = \frac{d}{c}$



Weierstrass Condition:

$$E(x, y, y'; Y') = f(x, y, Y') - f(x, y, y') - \frac{\partial f}{\partial y'}(Y' - y') \geq 0$$

for all $Y' \neq y'$.

Jacobi Test : Find envelope for family of curves

③ Boundary Layer:

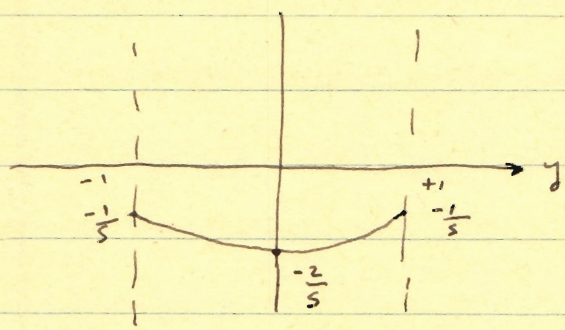
$$\epsilon u_{yy} - (2-y^2) u_x = y^2$$

$$u(0,y) = u(x,-1) = u(x,1) = 0$$

$$\bar{u}(s,y) = \int_0^\infty e^{-sx} u(x,y)\, dx$$

$$\epsilon \bar{u}_{yy} - (2-y^2) s \bar{u} = y^2$$

$$-(2-y^2) s \bar{u}_0 = y^2 \quad ; \quad \bar{u}_0 = \frac{y^2}{s(y^2-2)}$$



Introduce $\eta = (y+1)e^{\beta}$; $y = \eta e^{-\beta} - 1$; $\eta = (y+1)\epsilon^{-1/2}$

$$\epsilon^{1-2\beta} \bar{u}_{\eta\eta} - (2 - \eta^2 e^{-2\beta} + 2\eta e^{-\beta} - 1) s \bar{u} = \eta^2 e^{-2\beta} - 2\eta e^{-\beta} + 1$$

$$u_0 = \frac{\eta^2 \epsilon^{-2\beta} - 2\eta\epsilon^{-\beta} + 1}{s(\eta^2 \epsilon^{-2\beta} - 2\eta\epsilon^{-\beta} - 1)}$$

Let $\bar{u} = \bar{u}_0 + \bar{w}$

$$\epsilon \bar{\mathcal{U}}_0'' + \epsilon^{1-2\beta} \bar{\mathcal{U}}_{\eta\eta} + (\eta^2 \epsilon^{-2\beta} - 2\eta \epsilon^{-\beta} - 1)_s \left\{ \frac{\eta^2 \epsilon^{-2\beta} - 2\eta \epsilon^{-\beta} + 1}{s(\eta^2 \epsilon^{-2\beta} - 2\eta \epsilon^{-\beta} - 1)} \right.$$

$$\left. + \bar{\omega} \right\} = \eta^2 \epsilon^{-2\beta} - 2\eta \epsilon^{-\beta} + 1$$

$$\epsilon \mathcal{U}_0'' + \epsilon^{1+2\beta} \bar{\mathcal{U}}_{\eta\eta} + (\eta^2 \epsilon^{-2\beta} - 2\eta \epsilon^{-\beta} - 1)\, s\, \bar{\omega} = 0$$

Choose $\beta = -\frac{1}{2}$; $\bar{\omega}_{\eta\eta} + (\eta^2 \epsilon - 2\eta \epsilon^{\frac{1}{2}} - 1)\, s \bar{\omega} + \epsilon \mathcal{U}_0'' = 0$

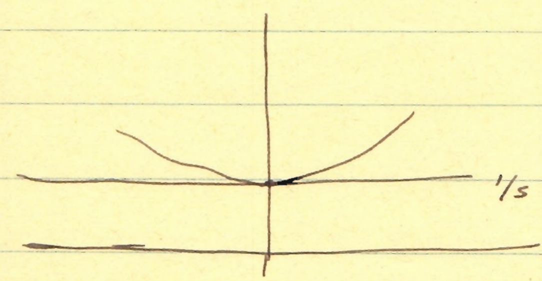
or: $\bar{\omega}_{\eta\eta} - s\bar{\omega} = 0$; $\bar{\omega} = A \cosh\sqrt{s}\,\eta + B \sinh\sqrt{s}\,\eta$

must satisfy BC at $\eta = 0$, $y = -1$, $\bar{\mathcal{U}} = 0$, $\bar{\mathcal{U}}_0 = -\frac{1}{s}$

$\therefore$ $\bar{\omega} = \frac{1}{s}$ at $\eta = 0$; $\frac{1}{s} = A$

hence:

$$\bar{\omega} = \frac{1}{s} \cosh\sqrt{s}\,\eta \quad ; \quad \bar{\omega} = \frac{1}{s} \cosh\sqrt{s}\,(y+1)\epsilon^{-1/2}$$



$$\bar{u} = \frac{y^2}{s(y^2-2)} + \frac{1}{s} \cosh\sqrt{s}\,(y+1)\,\epsilon^{-1/2}$$

$$\mathcal{U}(x,y) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{s*} ds \left\{ \frac{y^2}{s(y^2-2)} + \frac{1}{s} \cosh\sqrt{s}\,(y+1)\,\epsilon^{-1/2} \right\}$$

Van der Pol Oscillator:

$$u'' - \mu u'(1-u^2) + u = 0 \quad , \text{ periodic BC}$$

$\mu >> 1$: set $t = \eta\mu =$

$$\epsilon w_{\eta\eta} - w'(1-w^2) + w = 0 \quad ; \quad \epsilon = \frac{1}{\mu^2}$$

$$w_0'(1-w_0^2) + w_0 = 0 \quad ; \quad \frac{dw_0}{d\eta} = \frac{1}{\frac{1}{w_0} - w_0}$$

$$dw_0\left(\frac{1}{w_0} - w_0\right) = d\eta \quad ; \quad \ln w_0 - \frac{w_0^2}{2} = \eta$$

WKB Method:

$$u'' - \lambda^2 Q(x)\, u = 0 \quad ; \quad Q(0) = 0 \; ; \; Q'(0) = 1 \; ; \; \lambda >> 1$$

Try $u = e^{\lambda \int_0^x g(t,\lambda)\,dt}$

$$u' = \lambda g\, e^{\lambda \int g\, dx} \quad ; \quad u'' = \lambda g' e^{\cdots} + \lambda^2 g^2 e^{\cdots}$$

$$\lambda^2 g^2 + \lambda g' = \lambda^2 Q \quad ; \quad g^2 + \frac{g'}{\lambda} = Q$$

$$= g^2 + \epsilon g' = Q \quad ; \quad \epsilon = \frac{1}{\lambda}$$

$$g = g_0 + \epsilon g_1 + \cdots$$

$$g_0^2 + 2\epsilon g_1 g_0 + \epsilon g_0' = Q \quad ; \quad g_0 = \sqrt{Q} \quad ; \quad g_0' + 2 g_1 g_0 = 0$$

$$g_1 = -\frac{g_0'}{2g_0} = -\frac{1}{2}(\log g_0)'$$

$$u = e^{\lambda \int \left(\sqrt{Q} - \frac{1}{2\lambda}(\log g_0)'\right) dx} = g_0^{-1/2}\, e^{\lambda \int \sqrt{Q}\, dx}$$

$$u = \frac{1}{Q^{1/4}}\, e^{\lambda \int \sqrt{Q}\, dx}$$

Consider $u''(x) - \lambda(\operatorname{erf} x)\, u(x) = 0$

$$u = \frac{1}{(\operatorname{erf} x)^{1/4}}\, e^{\lambda^{1/2} \int \sqrt{\operatorname{erf} x}\, dx}$$

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-r^2}\, dr$$

$$\int_0^x e^{-r^2}\, dr = \int_0^x \left[1 - r^2 + \frac{r^4}{2!} - \frac{r^6}{3!} + \cdots\right] dr$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \cdots$$

so $Q(x)$ has a zero at $x = 0$.

$$u'' - \lambda Q u = 0$$

Choose $\lambda^n x = \xi$; $\lambda^{2n} u_{\xi\xi} - \lambda(x + a_2 x^2 + \cdots)\, u = 0$

$$u_{\xi\xi} - \left(\lambda^{1-2n-n}\,\xi + a_2 \lambda^{1-2n-2n}\,\xi^2 + \cdots\right) u = 0$$

Choose: $3n = 1$; $n = \frac{1}{3}$; $u_{\xi\xi} - \xi u + a_2 \lambda^{-1/3} \xi^2 + \cdots = 0$

$$u_{\xi\xi} - \xi u = 0$$

$$\boxed{u = Q^{-1/4}\, e^{\pm \lambda \int \sqrt{Q}\, dx}}$$

$$u''(x) + \lambda^2 Q(x) u = 0$$

Try: $u = e^{\lambda \int g(x)\, dx}$

$$u' = \lambda g e^{\cdots} \quad ; \quad u'' = \lambda^2 g^2 e^{\cdots} + \lambda g' e^{\cdots}$$

oscillates for $x < 0$,

$$\lambda^2 g^2 + \lambda g' + \lambda^2 Q(x) = 0$$

$$g^2 + \frac{g'}{\lambda} + Q(x) = 0$$

$$g = g_0 + \epsilon g_1$$

$$g_0^2 + 2\epsilon g_1 g_0 + \epsilon g_0' + Q(x) = 0 \quad ; \quad g_0 = \pm i \sqrt{Q}$$

$$g_1 = -\frac{g_0'}{2 g_0} = -\frac{1}{2} (\log g_0)'$$

$$\log g_0 = \log \pm i \sqrt{Q} = \log \sqrt{Q} \pm i \frac{\pi}{2}$$

$$=$$

$$u = e^{\lambda \int dx \left[\pm i \sqrt{Q} - \frac{1}{2\lambda} (\log g_0)' \right]}$$

$$= e^{\lambda \int dx \pm i \sqrt{Q}}\, e^{-\frac{1}{2} \log g_0} = e^{\pm i \lambda \int \sqrt{Q}\, dx}\, e^{-\frac{1}{2} \log \sqrt{Q}}\, e^{\mp i \pi/4}$$

$$u = Q^{-1/4}\, e^{\pm i \left[\lambda \int \sqrt{Q}\, dx - \pi/4 \right]}$$

For $x \leq 0$; $u_{\xi\xi} + \xi u = 0$; $u \sim \xi^{1/2} J_{1/3}\left(\frac{2}{3}\xi^{3/2}\right)$

PL Method

$$(x + \epsilon u)\frac{du}{dx} + u = 0 \quad ; \quad u(1) = 1$$

Take $u = u_0 + \epsilon u_1 + \cdots$

$x = \xi + \epsilon x_1 + \cdots$

$$\frac{du}{dx} = \frac{\frac{du}{d\xi}}{\frac{dx}{d\xi}} = \frac{\frac{du_0}{d\xi} + \epsilon \frac{du_1}{d\xi} + \cdots}{1 + \epsilon \frac{dx_1}{d\xi}}$$

$$= \frac{du_0}{d\xi} - \epsilon \frac{du_0}{d\xi}\frac{dx_1}{d\xi}$$

$$(\xi + \epsilon x_1 + \epsilon u_0)\left(\frac{du_0}{d\xi} - \epsilon \frac{du_0}{d\xi}\frac{dx_1}{d\xi} + \epsilon \frac{du_1}{d\xi}\right) + u_0 + \epsilon u_1 = 0$$

$$\xi \frac{du_0}{d\xi} + u_0 = 0$$

$$\xi \frac{du_1}{d\xi} + u_1 = \xi \frac{du_0}{d\xi}\frac{dx_1}{d\xi} - x_1 \frac{du_0}{d\xi} - u_0 \frac{du_0}{d\xi}$$

$$\frac{du_0}{d\xi} + \frac{u_0}{\xi} = 0 \quad ; \quad e^{\int \frac{d\xi}{\xi}} = \xi$$

$$du_0 + \frac{u_0}{\xi} d\xi = 0 \quad ; \quad \xi\, du_0 + u_0\, d\xi = 0$$

$$d(\xi u_0) = 0 \quad ; \quad u_0 = \frac{1}{\xi}$$

$$\xi \frac{du_1}{d\xi} + u_1 = -\frac{1}{\xi}\frac{dx_1}{d\xi} + \frac{x_1}{\xi^2} + \frac{1}{\xi^3}$$

Want $\quad \frac{1}{\xi}\frac{dx_1}{d\xi} - \frac{x_1}{\xi^2} = \frac{1}{\xi^3}$

$$dx_1 - \frac{x_1}{\xi}\, d\xi = \frac{d\xi}{\xi^2}$$

$$e^{\int \frac{-dx}{\xi}} = \frac{1}{\xi} \quad ; \quad d\left(\frac{x_1}{\xi}\right) = \frac{d\xi}{\xi^3}$$

$$\frac{x_1}{\xi} = -\frac{1}{2\xi^2} + c \quad ; \quad c = \frac{1}{2} \quad ; \quad \frac{x_1}{\xi} = -\frac{1}{2\xi^2} + \frac{1}{2}$$

Calculus of Variations:

Shortest distance between two points.

$$ds = \sqrt{dx^2 + dy^2} = \left(1 + \left(\frac{dy}{dx}\right)^2\right)^{1/2} dx = (1 + y'^2)\, dx$$

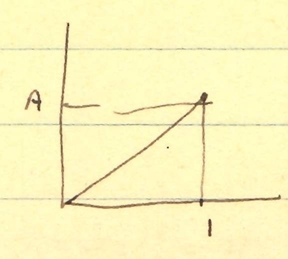
$$S = \int_{x_1}^{x_2} (1 + y'^2)^{1/2} dx = \int_{x_1}^{x_2} f(x, y, y')\, dx$$

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0 \quad ; \quad \frac{\partial f}{\partial x} - \frac{d}{dx}\left\{ f - y' \frac{\partial f}{\partial y'} \right\} = 0$$

$$\frac{\partial f}{\partial y'} = \frac{y'}{(1 + y'^2)^{1/2}} = c \quad ; \quad \therefore \; y' = a$$

$y = ax + b \quad ; \quad$ Let $(x_1, y_1) = 0,0 \quad ; \quad x_2, y_2 = 1, A$

$\therefore \; y = Ax$



Legendre Test : $\dfrac{\partial^2 f}{\partial y'^2} = \dfrac{1}{(1+y'^2)^{1/2}} - \dfrac{y'^2}{(1+y'^2)^{3/2}} = \dfrac{1}{(1+y'^2)^{3/2}}$

or: $\dfrac{1}{(1+A^2)^{3/2}} \geqslant 0$;

Corner Test : OK because of $(1+y'^2)^{1/2}$

E - Test :

$$f(x, y, \underset{\substack{\downarrow \\ p}}{Y'}) - f(x, y, y') - \frac{\partial f}{\partial y'}(\underset{\substack{\downarrow \\ p}}{Y'} - y') \geqslant 0 \qquad , \; p \neq y'$$

$$(1+p'^2)^{1/2} - (1+y'^2)^{1/2} - (p-y')\frac{y'}{(1+y'^2)^{1/2}}$$

$$= \left[(1+p'^2)(1+y'^2)\right]^{1/2} - 1 - y'^2 - py' + y'^2 \geqslant 0$$

$$\left[(1+p'^2)(1+y'^2)\right]^{1/2} - (1+py') \geqslant 0$$

$$\left[1 + p'^2 + y'^2 + p'^2 y'^2\right]^{1/2} \geqslant (1+py')$$

$$\cancel{1} + p'^2 + y'^2 + \cancel{p'^2 y'^2} \geqslant \cancel{1} + 2py' + \cancel{p^2 y'^2}$$

$$p^2 + y'^2 \geqslant 2py'$$

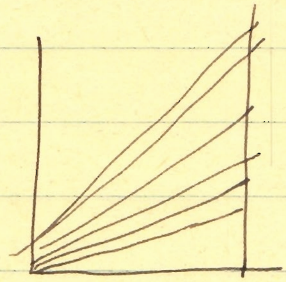
$$p^2 - 2py' + y'^2 \geqslant 0$$

$$(p - y') \geqslant 0 \quad ; \quad p \geqslant y'$$

Jacobi Test

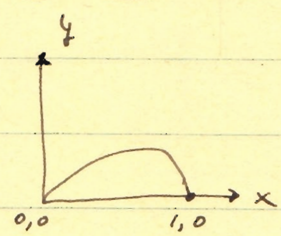
$$g(x,y) = y - Ax = 0$$

$$\frac{\partial}{\partial A} g(x,y) = x = 0$$



Isoparametric Problem:

Maximum area for fixed length



$$A = \int_0^1 \underset{= f}{y}\, dx$$

$$L = \int_0^1 \underset{= g}{\sqrt{1+y'^2}}\, dx$$

$$\delta \int_0^1 h\, dx = \delta \int_0^1 (f + \lambda g)\, dx = 0$$

$$h = y + \lambda \sqrt{1+y'^2} \quad ; \quad \frac{d}{dx}\left(h - y' \frac{\partial h}{\partial y'}\right) = 0 \text{ etc.}$$

Formal Perturbation Theory:

$$H\psi = E\psi \; ; \qquad H = H_0 + V \quad ; \qquad H_0 \psi_n = E_n \psi_n$$

Expand: $\psi^{(n)} = |n\rangle \quad ; \quad \langle m|n\rangle = \delta_{mn}$

$$(H_0 + V - E^{(0)} - E^{(1)} \cdots)(|0\rangle + |1\rangle + |2\rangle \cdots) = 0$$

$$(H_0 - E^{(0)})|0\rangle = 0$$

$$(H_0 - E^{(0)})|1\rangle = -V|0\rangle + E^{(1)}|0\rangle$$

$$(H_0 - E^{(0)})|2\rangle = -V|1\rangle + E^{(2)}|0\rangle + E^{(1)}|1\rangle$$

$$\vdots$$

$$(H_0 - E^{(0)})|n\rangle = -V|n-1\rangle + E^{(n)}|0\rangle + \cdots + E^{(1)}|n-1\rangle$$

Hence:

$$E^{(n)} = \langle 0|V|n-1\rangle$$

Now we have:

$$(H_0 - E^{(0)})G(\vec{x}) = -\delta(\vec{x})$$

Define: $(H_0 - E^{(0)})g_0 = -1$

Then $(H_0 - E^{(0)})g_0|0\rangle = -|0\rangle$

Define the operator: $(H_0 - E^{(0)})g_1 = -1 + |0\rangle\langle 0| = -Q$

Then: $\langle 0|g_1 = 0$

$$(H_0 - E^{(0)})g_1|\psi\rangle = -|\psi\rangle + |0\rangle\langle 0|\psi\rangle$$

Now: $\psi = \sum_m C_m \psi_m$; $g_1 \psi = \sum_m C_m' \psi_m$

$$(H_0 - E^{(0)})\, g_1 |\psi\rangle = -|\psi\rangle + |0\rangle\langle 0|\psi\rangle$$

or $\sum_m (E_m - E_n)\, C_m' \psi_m = -\sum_{m \neq n} C_m \psi_m$

or: $C_m' = \dfrac{C_m}{E_n - E_m}$

Then $g_1 \psi = \sum_{m \neq n} \dfrac{C_m}{E_n - E_m} \psi_m$

Also: $g_1 (H_0 - E^0) = -Q$

$$(H_0 - E^{(0)})\, |n\rangle = -V|n-1\rangle + E^{(n)}|0\rangle + \cdots + E^{(1)}|n-1\rangle$$

Operate with g_1

$$|n\rangle = +g_1 V|n-1\rangle + E^{(1)} g_1 |n-1\rangle + \cdots$$

$\cancel{E^{(n)} = \langle 0 \ldots}$ $E^{(n)} = \langle 0|V|n-1\rangle$

$E^{(1)} = \langle 0|V|0\rangle$; $E^{(2)} = \langle 0|V|1\rangle$

$|1\rangle = -g_1 V|0\rangle + \langle 0|V|0\rangle g_1 |0\rangle$

$$E^{(2)} = \underbrace{\langle 0|V|0\rangle\langle 0|g_1|0\rangle}_{0} + \langle 0|V g_1 V|0\rangle$$

① $$e^{-ax} = \int_0^\infty K(x-t)\, u(t)\, dt \quad ; \quad \bar{K}(\xi) = \frac{1}{(\xi^2+1)^{1/3}}$$

$$\int_0^\infty e^{-(a+i\xi)x}\, dx = \frac{1}{a+i\xi}$$

$$\frac{1}{i(\xi - ia)}$$

$$\bar{H}_\oplus(\xi) + \left(\frac{1}{a+i\xi}\right)_\ominus = \left\{\frac{1}{(\xi^2+1)^{1/3}}\right\} \bar{u}_\ominus$$

$$= \left(\frac{1}{(\xi+i)^{1/3}}\right)_\oplus \left(\frac{1}{(\xi-i)^{1/3}}\right)_\ominus \bar{u}_\ominus$$

$$\frac{\bar{u}_\ominus}{(\xi-i)^{1/3}} - \frac{(ia+i)^{1/3}}{i(\xi - ia)} = E(\xi)$$

For u to be integrable at $x=0$, $u \sim \frac{1}{x^{1-\epsilon}}$; $0 \leq \epsilon \leq 1$

$$\bar{u} \sim c\, \xi^{1-\epsilon-1} \sim c\, \xi^{-\epsilon}$$

$$\lim \frac{\xi^{-\epsilon}}{\xi^{1/3}} = \frac{1}{\xi^{\epsilon+1/3}} \quad ; \quad \epsilon = \frac{2}{3} \quad ; \quad u \sim \frac{1}{x^{1/3}}$$

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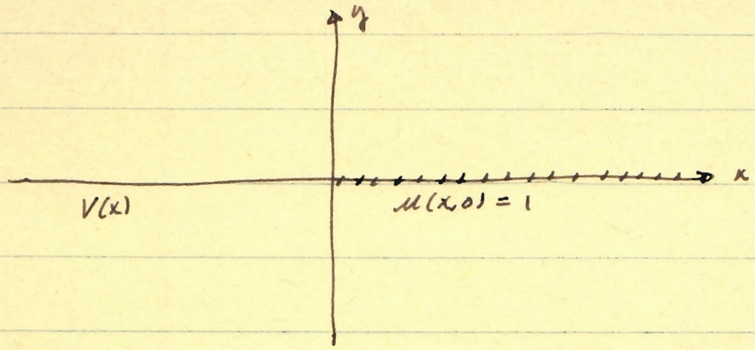
② $u'' - \lambda \operatorname{erf} x \; u = 0$

$u_{xx} + u_{yy} - k^2 u = 0 \quad ; \quad u(x,0) = 1 \quad \text{for } x > 0$



Define:

$$u^*(x,\eta) = \int_{-\infty}^{0^-} + \int_{0^+}^{\infty} u(x,y)\, e^{-i\eta y}\, dy$$

$$\bar{u}(\xi,\eta) = \int_{-\infty}^{\infty} u^*(x,\eta)\, e^{-i\xi x}\, dx$$

$$u^+(\xi,y) = \int_{-\infty} u(x,y)\, e^{-i\xi x}\, dx$$

$u_y(x,0^+) - u_y(x,0^-) = f(x)$

$\therefore \quad u^*_{xx} - \eta^2 \bar{u}^* - k^2 u^* = f(x)$

$-\xi^2 \bar{u} - \eta^2 \bar{u} - k^2 \bar{u} = \bar{f}(\xi)$

or $\quad \bar{u}(\xi,\eta) = \dfrac{-\bar{f}(\xi)}{\xi^2 + \eta^2 + k^2} = \dfrac{-\bar{f}(\xi)}{\left[\eta - i\sqrt{\xi^2+k^2}\right]\left[\eta + i\sqrt{\xi^2+k^2}\right]}$

$$u^+(\xi,y) = \frac{-\bar{f}(\xi)}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\eta y}\, d\eta}{[\quad][\quad]} = \frac{-\bar{f}(\xi)\, e^{-|y|\sqrt{\xi^2+k^2}}}{2\sqrt{\xi^2+k^2}}$$

$$u^+(\xi,0) = \frac{-\bar{f}(\xi)}{2\sqrt{\xi^2+k^2}}$$

$$= \underbrace{\int_{-\infty}^{0} v(x)\, e^{-i\xi x}\, dx}_{\bar{V}_{\oplus}(\xi)} + \underbrace{\int_0^{\infty} e^{-i\xi x}\, dx}_{\left(\frac{1}{i\xi}\right)_{\ominus}}$$

$$\bar{V}_{\oplus} + \left(\frac{1}{i\xi}\right)_{\ominus} = \frac{-\frac{1}{2}\bar{f}_{\ominus}}{(\xi+ik)^{1/2}_{\oplus}\,(\xi-ik)_{\ominus}}$$

$$\frac{\frac{1}{2}\bar{f}_{\ominus}}{(\xi-ik)^{1/2}} + \frac{(ik)^{1/2}}{i\xi} = E(\xi)$$

$$\bar{f}(\xi) = 2(ik)^{1/2}\,\frac{(\xi-ik)^{1/2}}{i\xi}$$

$$u^{+}(x,y) = -\frac{(ik)^{1/2}}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{i\xi x - |y|\sqrt{\xi^2+k^2}}}{\xi\,(\xi+ik)^{1/2}}\, d\xi$$

APPLIED MATHEMATICS 203

FINAL EXAMINATION

January 24, 1962

1. Let $F(x) = G(x) + \int_0^\infty K(x-t)\, F(t)\, dt$ where $G(x) = 0$ in $x < 0$ and where the Fourier transforms of F, G, K exist. Develop the solution of this equation via the Wiener-Hopf technique, giving a clear but concise statement of each important argument. Under what circumstances (and why, and how) can this method still give valuable information when the upper limit of the integral is replaced by L ?

2. Let

$$F(a,b,\lambda) = \int_{-\infty}^{\infty} \frac{e^{-\lambda\left[\sqrt{x^2 + a^2} - 3ix\right]}}{(x^2 + b^2)^{1/2}}\, dx$$

The value of the integrand is to be taken as $b^{-1} e^{-\lambda a}$ at $x = 0$, and the cuts adopted are to be such that the integrand becomes continuous on the real axis.

Evaluate F for real positive a, b, λ, with $\lambda \gg 1$, by the method of steepest descent. In particular specify the saddle point used, sketch the path of steepest descent, and specify those values of a, b for which you believe the result to be valid.

(OVER)

APPLIED MATHEMATICS 203

Final Examination (Cont'd)

3. Let

$$\varepsilon \Delta\Delta u - u_x = \sin y$$

in $0 < x < 1$ and $0 < y < \pi$ with $u = 0$ ~~and (grad u) · $\vec{n}$ =~~ 0 on the boundary, and $u_x(0,y) = u_x(1,y) = u_{yy}(x,0) = u_{yy}(x,\pi) = 0$ Δ is the Laplace operator. ε is positive and $\ll 1$.

Find an efficient description of $u(x,y)$.

$x \sin y \left(e^{\frac{1-x}{\varepsilon^{1/3}}} - 1\right)$

$\varepsilon u_{xxxx} - u_x = \sin y$

AM 203 FINAL EXAMINATION

January, 1961

(1) Let: $e^{-ax} = \int_0^\infty K(|x-t|)\, u(t)\, dt$ in $0 < x < \infty$

where $\bar{K}(\xi) = \dfrac{1}{(\xi^2+1)^{1/3}}$

Find $u(x)$; an integral representation will suffice. Identify the singularity in $u(x)$ at $x=0$. Determine the behaviour of $u(x)$ for $x \gg 1$. Let the foregoing problem be replaced by that for which

$$e^{-ax} = \int_0^{.1} K(|x-t|)\, w(t)\, dt$$

Describe clearly, but concisely, how you would attempt to find $w(x)$ in $0 < x < .1$.

(2) Let $u''(x) = \lambda\, (\mathrm{erf}\, x)\, u(x)$. Find for large λ, a suitable approximation for $u(x)$

(a) in the region $x > 0$

(b) in $x < 0$

(c) about $x = 0$

Deduce a representation, valid uniformly for all x when $\lambda \gg 1$, for that function $u(x)$ for which $u(x) \to 0$ as $x \to \infty$.

(3) Let $\epsilon u_{yy} - (2 - y^2)\, u_x = y^2$ with $u(0,y) = u(x,-1) = u(x,1) = 0$ and with $0 < \epsilon \ll 1$. Use the boundary layer technique to find $u(x,y)$.

(4) Let $u_{xx} + u_{yy} + u_{zz} + \lambda x^2 u = 0$ (1)
with $u = 0$ on the surface of the cube bounded by the planes $x = \pm 1$, $y = \pm 1$, $z = \pm 1$. Construct the variational principle whose Euler equation is (1) and use the "direct methods" of the variational calculus to approximate the eigenvalue λ.

Fall 1961

APPLIED MATHEMATICS 203

Problem Set No. 1

1. Consider the 0-th order Bessel equation

$$(xu')' + \lambda xu = 0$$

with the boundary conditions

$$u'(0) = u(1) = 0$$

Transform into an integral equation. Find (approximately) the lowest eigenvalue by the iteration method, starting with a function

$$u^{(0)}(x) = \begin{cases} 1 & (0 \leq x \leq 1) \\ 0 & \text{otherwise} \end{cases}$$

Compare the result with the tabulated value of the first zero of J_0.

2. Repeat Number 1 for the problem

$$u'' + u + \lambda x^4 = 0$$

$$u(-1) = u(1) = 0 .$$

3. Solve the equation

$$u(x) = \cos \alpha x + \lambda \int K_0(a|x-t|)\, u(t)dt$$

for the infinite domain and for the semi-infinite domain. Compare the asymptotic behavior of the solutions as $x \to \infty$.

If the complete solutions are too hard to get, find only the asymptotic forms.

4. Find at least one non-trivial solution of the homogeneous Wiener-Hopf equation

$$u(x) = \lambda \int_0^\infty K(x-t)u(t)dt$$

$$K(x,t) = [1-3(x-t)^2]\, e^{-a|x-t|} .$$

5. Write the solution to the equation

$$u(y) = \lambda\beta \int_{-1}^{1} K[\beta(y-t)]u(t)dt$$

in the form

$$u(y) = \cos(\quad) + \chi_1 + \chi_2$$

where cos ( ) is the solution for an infinite interval, and χ_1 and χ_2 are corrections corresponding to semi-infinite domains. Evaluate the accuracy of the result by substituting it in the integral equation.

6. Use the Wiener-Hopf method to solve

$$u_{xx} + u_{yy} - k^2u = 0$$
$$u(x,0) = 1 \quad \text{for} \quad x > 0$$
$$u \rightarrow 0 \quad \text{as} \quad \mathrm{Im}\,[(x+iy)^{1/2}] \longrightarrow \infty$$
$$u \text{ regular except on } x > 0,\ y = 0$$

Do this directly without use of an integral equation and then do it using an integral equation equivalent.

7. Read carefully the section on "The Milne Problem" in Sneddon's The Fourier Transform. With the full results for the K_0 kernel supposedly in your notes, how might you best approximate the solutions of the Milne problem?

8. Let $\theta = \theta(x,y,t)$

and $\Delta\theta - \theta_x - \theta_t = 0$ in $-\infty < t < \infty$, $-\infty < x < \infty$, $-\infty < y < \infty$ except on $y = 0$, $x > 0$.

$$\theta(x,0,t) = S(t) \text{ for } x > 0$$

$$S(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

$$\theta \longrightarrow 0 \quad \text{as} \quad \mathrm{Im}\,(x+iy)^{1/2} \longrightarrow \infty .$$

Find an integral representation of $\theta(x,y,t)$ and evaluate it for large t using the method of steepest descent.

Problem 1

Consider the 0th order Bessel equation:

$$(xu')' + \lambda x u = 0$$

with the boundary condition

$$u'(0) = u(1) = 0$$

Transform into an integral equation. Find approximately the lowest eigenvalue by the iteration method, starting with a function:

$$u^{(0)}(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Compare the result with the tabulated value of the first zero of J_0:

Let us transform the differential equation into an integral equation and so, let us look at the Green's function,

$$(xG')' = \delta(x-\xi) \qquad (1)$$

From which we deduce:

$$G(x|\xi) = \begin{cases} \ln \xi & x < \xi \\ \ln x & x > \xi \end{cases} \qquad (2)$$

and so the integral equation is:

$$u(x) = -\lambda \int_0^x (\ln x)\, \xi\, u(\xi)\, d\xi - \lambda \int_x^1 (\ln \xi)\, \xi\, u(\xi)\, d\xi$$

Let us now define the operator $\hat{G}$, such that:

$$\hat{G}\, \varphi_n = \int_0^1 \xi\, G(\xi|x)\, \varphi_n(\xi)\, d\xi = \varphi_{n+1}$$

We first take: $\varphi_1 = 1$; operating on φ_1 with $\hat{G}$, we get:

$$\varphi_2 = \frac{1}{4}(x^2 - 1) \qquad \text{[which does satisfy the BC]}$$

Operating on φ_2 once more with $\hat{G}$ gives:

$$\varphi_3 = \frac{1}{4^3}(x^4 - 4x^2 + 3)$$

Now, if we define the "scalar product" of two function $u(x)$ and $v(x)$ as:

$$\langle u | v \rangle = \int_0^1 u(x)\, v(x)\, dx$$

Then we have:

$$\lambda_1 \approx \frac{\langle \varphi_3, \varphi_2 \rangle}{\langle \varphi_3, \varphi_3 \rangle} = \frac{\frac{1}{4^4}\int_0^1 (x^2 - 5x^4 + 7x^2 \cancel{4} - 3)\, dx}{\frac{1}{4^6}\int_0^1 (x^8 + 16x^4 + 9 - 8x^6 + 6x^4 - 24x^2)\, dx}$$

or: $\lambda_1 \approx 5.7$

We now solve the differential equation: we get:

$$u = J_0(\sqrt{\lambda}\, x)$$

when $x = 1$, we have: $J_0(\sqrt{\lambda}) = 0$

$$\therefore \sqrt{\lambda_1} = 2.40 \quad \text{or} \quad \lambda_1 = 5.76$$

pretty good.

Problem 2

Find the Green's function of the equation:

$$u'' + u + \lambda x^4 u = 0 \qquad (1)$$
$$u(1) = u(-1) = 0$$

$$G'' + G = -\delta(x-\xi) \qquad (2)$$

Two independent sol'ns are:

$$G(x|\xi) = \begin{cases} A \sin(1+x) & ; \ -1 < x < \xi < +1 \\ B \sin(1-x) & ; \ -1 < \xi < x < +1 \end{cases}$$

But $G(x|\xi)$ is symmetric in x and ξ, so we can write:

$$G(x|\xi) = C \begin{cases} \sin(1+x)\sin(1-\xi) & x < \xi \\ \sin(1-x)\sin(1+\xi) & \xi < x \end{cases}$$

To determine C, use the fact that:

$$\left[G'\right]_{x=\xi} = -1$$

That is, there is a jump in G' at $x=\xi$, is equal to -1

Then:

$$C\left[\cos(1-\xi)\sin(1+\xi) + \cos(1+\xi)\sin(1-\xi)\right] = 1$$

$$C \sin 2 = 1 \ , \quad C = \frac{1}{\sin 2} \qquad (3)$$

Hence:

$$G(x|\xi) = \frac{1}{\sin 2}\begin{cases} \sin(1+x)\sin(1-\xi) \ ; & x < \xi \\ \sin(1-x)\sin(1+\xi) \ ; & x > \xi \end{cases}$$

$$\therefore \quad u(x) = \lambda \int_{-1}^{+1} \xi^4 \, G(\xi|x)\, u(\xi)\, d\xi \qquad (4)$$

Note that kernel is not symmetric but nevertheless we are going to use the iteration procedure (which was proved only for Hermitean kernels) This is justified by the fact that this kernel can be "symmetrized."

Let $u(x) = x^{-2} v(x)$

Then:

$$x^{-2} v(x) = \lambda \int_{-1}^{+1} \xi^{4} G(\xi|x) \xi^{-2} v(\xi)\, d\xi \qquad (5)$$

$$v(x) = \lambda \int_{-1}^{+1} \underbrace{x^{2} \xi^{2} G(\xi|x)}_{K(\xi/x)} v(\xi)\, d\xi$$

Let $\hat{K} \varphi_n = \int_{-1}^{+1} K(\xi|x)\, \varphi_n(\xi)\, d\xi = \varphi_{n+1}(x)$

Let us take $\varphi_1 = 1$:

$$\varphi_2 = \int_{-1}^{x} \frac{x^2 \xi^2}{\sin 2} \sin(1-x) \sin(1+\xi)\, d\xi + \int_{x}^{1} \frac{x^2 \xi^2}{\sin 2} \sin(1-\xi) \sin(1+x)\, d\xi$$

$$\varphi_2 = \frac{x^2 \sin(1-x)}{\sin 2} \left\{ (2-x^2) \cos(1+x) - 1 + 2x \sin(1+x) \right\}$$

$$+ \frac{x^2 \sin(1+x)}{\sin 2} \left\{ (2-x^2) \cos(1-x) - 1 - 2x \sin(1-x) \right\}$$

$$\varphi_2 = \frac{(2-x^2) x^2}{\sin 2} \left\{ \sin(1-x) \cos(1+x) + \sin(1+x) \cos(1-x) \right\}$$

$$- \frac{x^2}{\sin 2} \left\{ \sin(1-x) + \sin(1+x) \right\}$$

$$\varphi_2 = x^2 (2-x^2) - \frac{2x^2}{\sin 2} \sin 1 \cos x \; ; \quad \varphi_2 = x^2(2-x^2) - x^2 \frac{\cos x}{\cos 1}$$

$\langle \varphi_2, \varphi_2 \rangle \simeq 30{,}800 \quad ; \quad \langle \varphi_2, \varphi_1 \rangle = 4.8 \cdot 10^{-2}$

$\lambda \simeq 1.6 \cdot 10^{-6}$

Problem 3

Solve the equation:

$$u(x) = \cos \alpha x + \lambda \int K_0(a|x-t|)\, u(t)\, dt$$

for the ∞ and semi-∞ domain. Compare the asymptotic behaviour of the solutions as $x \to \infty$:

Infinite Domain

Fourier transform:

$$\bar{u}(\xi) = \bar{C}(\xi) + \frac{\lambda \pi}{(\xi^2+a^2)^{1/2}}\, \bar{u}(\xi) \qquad (1)$$

where:
$$\bar{C}(\xi) = \lim_{\beta \to 0} \int_{-\infty}^{\infty} e^{-\beta|x|} e^{-i\xi x} \cos \alpha x\, dx \qquad (2)$$

Then:
$$\bar{u}(\xi) = \frac{(\xi^2+a^2)^{1/2}}{(\xi^2+a^2)^{1/2} - \lambda\pi}\, \bar{C}(\xi), \quad \text{i.e.,}$$

$$\bar{u}(\xi) = \frac{(\xi^2+a^2)^{1/2}\left[(\xi^2+a^2)^{1/2} + \lambda\pi\right]}{\xi^2 + (a^2 - \lambda^2\pi^2)}\, \bar{C}(\xi)$$

$$= \bar{C}(\xi) + \lambda\pi\, \frac{\lambda\pi + \sqrt{\xi^2+a^2}}{\xi^2 + (a^2-\lambda^2\pi^2)}\, \bar{C}(\xi) \qquad (3)$$

Let:
$$M(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\xi x}\left[\lambda\pi + \sqrt{\xi^2+a^2}\right]}{\xi^2 + (a^2 - \lambda^2\pi^2)}\, d\xi \qquad (4)$$

Then:
$$u(x) = \cos \alpha x + \lambda\pi \int_{-\infty}^{\infty} \cos \alpha y\; M(x-y)\, dy$$

or:
$$u(x) = \cos \alpha x + \lambda\pi \int_{-\infty}^{\infty} \cos[\alpha(x-y)]\, M(y)\, dy$$

$$u(x) = \cos \alpha x + \lambda\pi \cos \alpha x \int_{-\infty}^{\infty} \cos \alpha y\; M(y)\, dy$$

$$+ \lambda\pi \sin \alpha x \int_{-\infty}^{\infty} \sin \alpha y\; M(y)\, dy$$

But now notice:

$$\int_{-\infty}^{\infty} M(y)\, e^{-i\alpha y}\, dy = \bar{M}(\alpha) = \frac{d\pi + \sqrt{\alpha^2+a^2}}{\alpha^2+a^2-d^2\pi^2}$$

$$\therefore \quad \mu(x) = \cos\alpha x + d\pi \cos\alpha x \; \frac{d\pi + \sqrt{\alpha^2+a^2}}{\alpha^2+a^2-d^2\pi^2}$$

$$\boxed{\mu(x) = \frac{\sqrt{\alpha^2+a^2}\left[\sqrt{\alpha^2+a^2} + d\pi\right]}{\alpha^2+a^2-d^2\pi^2} \cos\alpha x}$$

<u>Semi-infinite Domain</u>

$$\mu(x) = \cos\alpha x + d\int_0^{\infty} K_0(a|x-t|)\, \mu(t)\, dt$$

We extend the region of validity of the equation.

$$V(x) + \mu(x) = c(x) + h(x) + d\int_{-\infty}^{\infty} K_0(a|x-t|)\, \mu(t)\, dt$$

where $V(x) = \begin{cases} 0 & x>0 \\ ? & x<0 \end{cases}$ $\qquad c(x) = \begin{cases} \cos\alpha x & x>0 \\ 0 & x<0 \end{cases}$

$\mu(x) = \begin{cases} \mu(x) & x>0 \\ 0 & x<0 \end{cases}$ $\qquad h(x) = \begin{cases} 0 & x>0 \\ ? & x<0 \end{cases}$

now Fourier transform:

$$\bar{\mu}_{(-)} + \bar{V}_{(+)} = \bar{h}_{(+)} + \frac{1}{2}\left[\frac{1}{i(\xi-\alpha)} + \frac{1}{i(\xi+\alpha)}\right]_{(-)} + \frac{d\pi}{(\xi^2+a^2)^{1/2}}\,\bar{\mu}_{(-)}$$

i.e.

$$\bar{V}_{(+)} - \bar{h}_{(+)} = \frac{1}{2i}\left[\frac{1}{\xi-\alpha} + \frac{1}{\xi+\alpha}\right]_{(-)} - \left\{1 - \frac{d\pi}{(\xi^2+a^2)^{1/2}}\right\}\bar{\mu}_{(-)}$$

But: $$1 - \frac{d\pi}{(\xi^2+a^2)^{1/2}} = \frac{(\xi^2+a^2)^{1/2} - d\pi}{(\xi^2+a^2)^{1/2}} = \frac{\xi^2+a^2-d^2\pi^2}{(\xi^2+a^2)^{1/2}\left[\sqrt{\xi^2+a^2} + d\pi\right]}$$

Let: $\bar{L}_{(+)}(\xi)\,\bar{L}_{(-)}(\xi) = (\xi^2+a^2)^{1/2} + d\pi$

First Consider: $\quad d^2\pi^2 - a^2 = \gamma^2 > 0$

$$\left\{ \bar{V}_{(+)} - \bar{h}_{(+)} \right\} L_{(+)} \sqrt{\xi + ia}_{(+)} = \underbrace{\frac{L_{(+)} \sqrt{\xi + ia}_{(+)}}{2i(\xi - \alpha)}}_{A} + \underbrace{\frac{L_{(+)} \sqrt{\xi + ia}_{(+)}}{2i(\xi + a)}}_{B} - \frac{(\xi^2 - \gamma^2)\,\bar{\mu}_{(-)}}{L_{(-)} \sqrt{\xi - ia}}$$

$$A = \left[\frac{L_{(+)}(\xi) \sqrt{\xi + ia} - L_{(+)}(\alpha) \sqrt{\alpha + ia}}{2i(\xi - \alpha)} \right]_{(+)} + \left[\frac{L_{(+)}(\alpha) \sqrt{\alpha + ia}}{2i(\xi - \alpha)} \right]_{(-)}$$

$$B = \left\{ \frac{L_{+}(\xi) \sqrt{\xi + ia} - L_{(+)}(-\alpha) \sqrt{-\alpha + ia}}{2i(\xi + \alpha)} \right]_{(+)} + \left[\frac{L_{(+)}(-\alpha) \sqrt{-\alpha + ia}}{2i(\xi + \alpha)} \right]_{(-)}$$

Therefore:

$$\left\{ \bar{V}_{(+)} - \bar{h}_{(+)} \right\} L_{(+)}(\xi) \sqrt{\xi + ia}_{(+)} - \left[\frac{L_{(+)}(\xi) \sqrt{\xi + ia} - L_{(+)}(\alpha) \sqrt{\alpha + ia}}{2i(\xi - \alpha)} \right]_{(+)}$$

$$- \left\{ \frac{L_{(+)}(\xi) \sqrt{\xi + ia} - L_{(+)}(-\alpha) \sqrt{-\alpha + ia}}{2i(\xi + \alpha)} \right]_{(+)} = E(\xi)$$

$$= \left[\frac{L_{(+)}(\alpha) \sqrt{\alpha + ia}}{2i(\xi - \alpha)} \right]_{(-)} + \left[\frac{L_{(+)}(-\alpha) \sqrt{-\alpha + ia}}{2i(\xi + \alpha)} \right]_{(-)} - \frac{(\xi^2 - \gamma^2)}{L_{(-)}(\xi) \sqrt{\xi - ia}}\, \bar{\mu}_{(-)}$$

At this stage, we do not have enough information to determine $E(\xi)$. Furthermore it is quite clear that at ∞ the behavior will be like a $\cos \alpha x$ (due to the first 2 terms on the RHS: pole at $\pm \alpha$, which will give $e^{i\alpha x}$, $e^{-i\alpha x}$). Let us require μ to be integrable, then $E = 0$:

$$\bar{\mu} = \frac{L_{(-)}(\xi) \sqrt{\xi - ia}}{\xi^2 - \gamma^2} \left\{ \frac{L_{(+)}(\alpha) \sqrt{\alpha + ia}}{2i(\xi - \alpha)} + \frac{L_{(+)}(-\alpha) \sqrt{-\alpha + ia}}{2i(\xi + \alpha)} \right]$$

$$\mu(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{\mu}(\xi)\, e^{i\xi x}\, d\xi$$

$$\mu(x) = \Bigg[\frac{L_{(-)}(\gamma)\sqrt{\gamma - ia}}{2\gamma} \left\{ \frac{L_{(+)}(\alpha)\sqrt{\alpha + ia}}{2(\gamma-\alpha)} + \frac{L_{(+)}(-\alpha)\sqrt{-\alpha+ia}}{2(\gamma+\alpha)} \right\} e^{i\gamma x}$$

$$+ \frac{L_{(-)}(-\gamma)\sqrt{-\gamma - ia}}{2\gamma} \left\{ \frac{L_{(+)}(\alpha)\sqrt{\alpha + ia}}{2(\gamma+\alpha)} + \frac{L_{(+)}(-\alpha)\sqrt{-\alpha+ia}}{2(\gamma-\alpha)} \right\} e^{-i\gamma x}$$

$$+ \frac{(a^2+\alpha^2)^{1/2}\left(\sqrt{\alpha^2+a^2} + d\pi\right)}{(\alpha^2+a^2-d^2\pi^2)} \cos \alpha x$$

$$+ \frac{\sqrt{a^2-\alpha^2}}{4\alpha(\alpha^2-\gamma^2)} \left\{ L_{(-)}(\alpha)\, L_{(+)}(-\alpha)\, e^{i\alpha x} - L_{(-)}(-\alpha)\, L_{(+)}(\alpha)\, e^{-i\alpha x} \right\} \Bigg]$$

AM 203 Homework

Problem 4

Let us extend the region of validity of the int. eq., for $x < 0$, and Fourier Transform and get:

$$\bar{u}_{(-)}(\xi) + \bar{V}_{(+)}(\xi) = \lambda \bar{K}(\xi)\, \bar{u}_{(-)}(\xi) \qquad (1)$$

now:

$$\bar{K}(\xi) = \frac{2a}{\xi^2+a^2} + \overset{3?}{\vee}\; \frac{4a(3\xi^2-a^2)}{(\xi^2+a^2)^3} \qquad (2)$$

$$= \frac{2a\left[\xi^4 + (2a^2+6)\xi^2 + a - 2a^2\right]}{(\xi^2+a^2)^3}$$

and so:

$$-\bar{V}_{(+)}(\xi) = \left[1 - \lambda \bar{K}(\xi)\right] \bar{u}_{(-)}(\xi) - \bar{V}_{(+)}(\xi) = \frac{Q(\xi)\qquad \bar{u}_{(-)}(\xi)}{(\xi-ia)^3_{(-)}\,(\xi+ia)^3_{(+)}} \qquad (3)$$

where:

$$Q(\xi) = \xi^6 + (3a^2 - 2a\lambda)\xi^4 - (\overset{36?}{16}a\lambda + 4\lambda a^3 - 3a^4)\xi^2$$

$$- (2\lambda a^5 - \overset{12?}{4}\lambda a^2 - a^6)$$

note that if: $Q(\xi_1) = 0$, where ξ_1 is real, then:

$$Q(-\xi_1) = 0$$

Also if: $Q(\xi_2) = 0$

$$Q(\bar{\xi}_2) = 0$$

$$Q(-\bar{\xi}_2) = 0$$

and $Q(-\xi_2) = 0$

Therefore there might only be two possibilities that could arise:

1. $Q(\xi) = (\xi^2-\xi_1^2)(\xi^2-\xi_2^2)(\xi^2-\xi_3^2)$ $\quad \xi_1, \xi_2, \xi_3$ real and +

2. $Q(\xi) = (\xi^2-\xi_1^2)(\xi^2-\xi_2^2)(\xi^2-\bar{\xi}_2^2)$ $\quad \begin{cases} \xi_1 \text{ real and } + \\ \xi_2 \text{ complex} \end{cases}$

at this point, refer to Titchmarsh, Th. of F. Int, p 339 for theory.

<u>1st Case</u> $\quad Q(\xi) = (\xi^2 - \xi_1^2)(\xi^2 - \xi_2^2)(\xi^2 - \xi_3^2)$

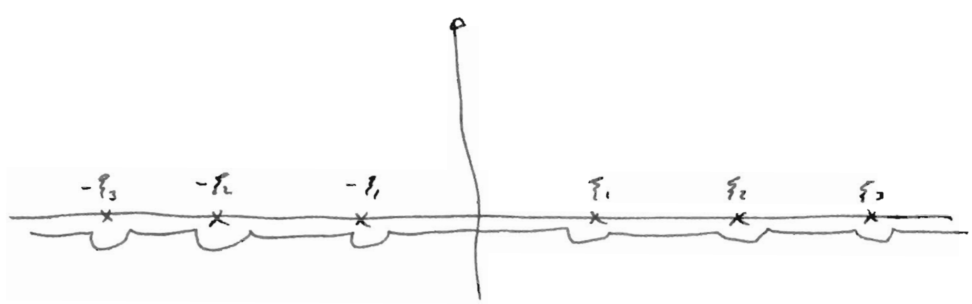
Then (3) becomes:

$$-(\xi + ia)^3_{(+)} \; \bar{V}_{(+)} = \frac{(\xi^2-\xi_1^2)(\xi^2-\xi_2^2)(\xi^2-\xi_3^2)}{(\xi - ia)^3} \; \bar{u}_{(-)} = \underline{P}(\xi) \qquad (4)$$

where $\underline{P}(\xi)$ is not any entire function, but a polynomial the degree of which does not exceed 3.

$$\therefore \quad u(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\underline{P}(\xi)\,(\xi - ia)^3}{(\xi^2-\xi_1^2)(\xi^2-\xi_2^2)(\xi^2-\xi_3^2)} \, e^{i\xi x} \, d\xi \qquad (5)$$

The path of integration goes below the singularities *
(since for $x<0$, we should get $u=0$)



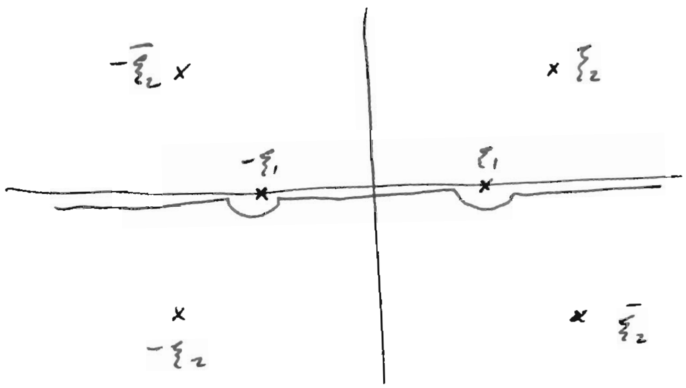
$$\therefore \quad u(x) = i \left[\frac{(\xi_1 - ia)^3 \underline{P}(\xi_1)\, e^{i\xi_1 x} + (\xi_1 + ia)^3 \underline{P}(-\xi_1)\, e^{-i\xi_1 x}}{2\xi_1 (\xi_1^2 - \xi_2^2)(\xi_1^2 - \xi_3^2)} \right.$$

$$+ \frac{(\xi_2 - ia)^3 \underline{P}(\xi_2)\, e^{i\xi_2 x} + (\xi_2 + ia)^3 \underline{P}(-\xi_2)\, e^{-i\xi_2 x}}{2\xi_2 (\xi_2^2 - \xi_1^2)(\xi_2^2 - \xi_3^2)}$$

$$\left. + \frac{(\xi_3 - ia)^3 \underline{P}(\xi_3)\, e^{i\xi_3 x} + (\xi_3 + ia)^3 \underline{P}(-\xi_3)\, e^{-i\xi_3 x}}{2\xi_3 (\xi_3^2 - \xi_1^2)(\xi_3^2 - \xi_2^2)} \right] \qquad (6)$$

2nd Case

$$Q(\xi) = (\xi^2 - \xi_1^2)(\xi^2 - \xi_2^2)(\xi^2 - \bar{\xi}_2^2) \quad \begin{cases} \xi_1 \text{ real } \& + \\ \xi_2 \text{ complex} \end{cases}$$



Write (3) as:

$$-(\xi + ia)^3_{(+)} \, (\xi + \xi_2)^{-1}_{(+)} \, (\xi - \bar{\xi}_2)^{-1}_{(+)} \, \bar{V}_{(+)} = \frac{(\xi^2 - \xi_1^2)(\xi - \xi_2)(\xi + \bar{\xi}_2)}{(\xi - ia)^3} \bar{\mu}_{(-)}$$

$$= \Pi(\xi)$$

here $\Pi(\xi)$ is a polynomial, which degree does not exceed 1.

$$u(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Pi(\xi)(\xi - ia)^3}{(\xi^2 - \xi_1^2)(\xi - \xi_2)(\xi - \bar{\xi}_2)} e^{i\xi x} \, d\xi \qquad (8)$$

$$= i \left\{ \frac{\Pi(\xi_1)(\xi_1 - ia)^3 e^{i\xi_1 x} + \Pi(-\xi_1)(\xi_1 + ia)^3 e^{-i\xi_1 x}}{2\xi_1 (\xi_1 - \xi_2)(\xi_1 + \bar{\xi}_2)} \right.$$

$$\left. + \frac{\Pi(\xi_2)(\xi_2 - ia)^3 e^{i\xi_2 x}}{(\xi_2^2 - \xi_1^2)(\xi_2 + \bar{\xi}_2)} + \frac{\Pi(-\bar{\xi}_2)(\bar{\xi}_2 + ia)^3 e^{-i\bar{\xi}_2 x}}{(\bar{\xi}_2^2 - \xi_1^2)(\xi_2 + \bar{\xi}_2)} \right] \qquad (9)$$

Depending on the relative magnitude of a and d we either have case I or case II. Here we are not in a position to specify any further $P(\xi)$ or $\Pi(\xi)$.

Problem 5

Write the solution to the equation:

$$\mu(y) = \lambda\beta \int_{-1}^{+1} K_0[\beta(y-t)]\,\mu(t)\,dt$$

in the form:

$$\mu(y) = \cos(\ ) + \chi_1 + \chi_2$$

where $\cos(\ )$ is the solution for an infinite interval and χ_1 & χ_2 are correction corresponding to semi-infinite domains. Evaluate the accuracy of the result by substituting it in the integral equation.

① Infinite Domain

$$w(y) = \lambda\beta \int_{-\infty}^{\infty} K_0\{\beta(y-t)\}\,w(t)\,dt \qquad (1)$$

change variables: $y - t = \tau$ (2)

$$w(y) = -\lambda\beta \int_{+\infty}^{-\infty} K_0(\beta\tau)\,w(y-\tau)\,d\tau \qquad (3)$$

now let's try as a possible solution, $w(y) = e^{iky}$ where k is not known yet:

$$e^{iky} = \lambda\beta\, e^{iky} \int_{-\infty}^{\infty} e^{-ik\tau} K_0[\beta\tau]\,d\tau$$

$$e^{iky} = \frac{\lambda\beta\pi}{(\beta^2+k^2)^{1/2}}\, e^{iky}$$

so e^{iky} is a solution and e^{-iky} is also a solution

$$\cos ky = \frac{\lambda\beta\pi}{(\beta^2+k^2)^{1/2}} \cos ky$$

provided that:

$$d\beta\pi = (\beta^2+k^2)^{1/2}$$

$$k^2 = (d^2\pi^2-1)\beta^2 \qquad (4)$$

so if we take $|d\pi| \gtreqless 1$, then:

$$\boxed{w(y) = \cos ky\,,\ \text{where } k^2 = (d^2\pi^2-1)\,\beta^2} \qquad (5)$$

2 Semi-infinite Domain $[\,|d\pi|>1\,]$

Consider:

$$v(y) = d\beta\int_0^\infty K_0\,[\beta(y-t)]\,v(t)\,dt \qquad (6)$$

Let me follow the usual procedure, and extend the range of validity of the integral (6), then Fourier transform to get the Wiener-Hopf equation.

$$\bar{V}_{(-)}(\eta) + \bar{h}_{(+)}(\eta) = \frac{d\beta\pi}{(\beta^2+\eta^2)^{1/2}}\,\bar{V}_{(-)}(\eta) \qquad (7)$$

$$\bar{h}_{(+)} = \frac{d\beta\pi - (\beta^2+\eta^2)^{1/2}}{(\beta^2+\eta^2)^{1/2}}\,\bar{V}_{(-)}$$

or $$\bar{h}_{(+)} = \frac{k^2-\eta^2}{(\beta^2+\eta^2)^{1/2}\left[d\beta\pi + (\beta^2+\eta^2)^{1/2}\right]}\,\bar{V}_{(-)} \qquad (8)$$

Let $$L = \frac{1}{d\beta\pi + (\beta^2+\eta^2)^{1/2}} = \left[L_{(+)}(\eta)\,L_{(-)}(\eta)\right]^{-1?}$$

Then: $$(\eta+i\beta)^{1/2}_{(+)}\,L_{(+)}(\eta)\,\bar{h}_{(+)}(\eta) = E(\eta)$$

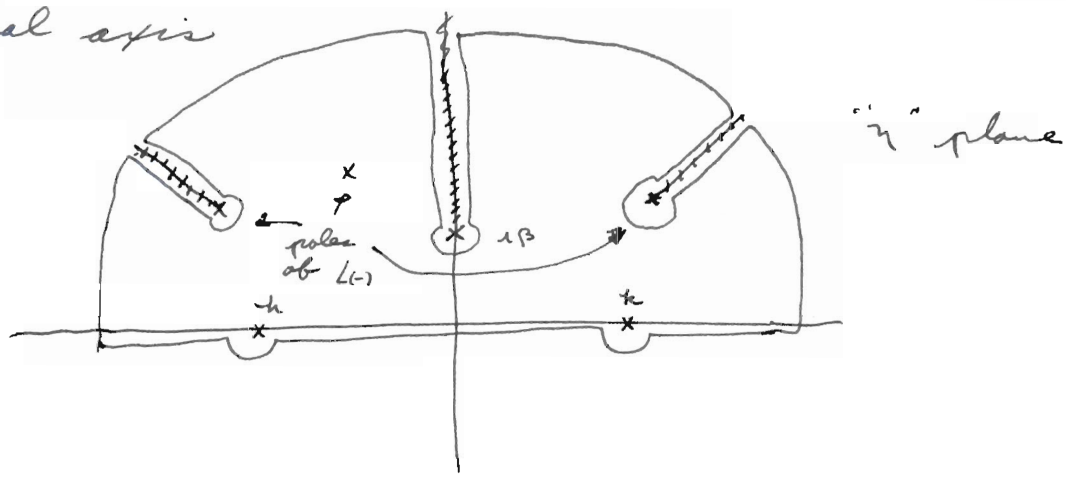
$$= \frac{k^2-\eta^2}{(\eta-i\beta)^{1/2}_{(-)}\,L_{(-)}(\eta)}\,\bar{V}_{(-)}(\eta) \qquad (9)$$

where $E(\eta)$ is a polynomial of degree smaller than 1, that is, is a constant, and since the problem is homogeneous, we can take this constant equal to -1

$$\therefore \quad \bar{V}_{(-)}(\eta) = \frac{(\eta - i\beta)^{1/2} L_{(-)}(\eta)}{\eta^2 - h^2} \qquad (10)$$

$$\therefore \quad \boxed{V(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{(\eta - i\beta)^{1/2} L_{(-)}(\eta)}{\eta^2 - h^2} e^{i\eta y}\, d\eta}$$

The path of integration goes below the poles on the real axis



$$\therefore \quad V(y) = \frac{1}{2\pi} \times 2\pi i \times \frac{(h - i\beta)^{1/2} L_{-}(h)}{2h} e^{ihy}$$

$$+ \frac{1}{2\pi} \times 2\pi i \times \frac{(-h - i\beta)^{1/2} L_{-}(-h)}{-2h} e^{-ihy}$$

$$+ \frac{1}{2\pi}\int_C \frac{(\eta - i\beta)^{1/2} L_{-}(\eta)\, e^{i\eta y}}{\eta^2 - h^2}\, d\eta$$

$$V(y) = A\cos[h(y+\epsilon)] + \frac{1}{2\pi}\int_C \frac{(\eta - i\beta)^{1/2} L_{(-)}(\eta)\, e^{i\eta y}}{\eta^2 - h^2}\, d\eta$$

$$= A\cos[h(y+\epsilon)] + X(y)$$

A and ϵ are known, but are extremely messy to evaluate.

3 Consider the Finite Domain

we write:

$$u(y) = A \cos ky + \chi(y+1) + \chi(1-y)$$

Furthermore, we want $\cos[k(1+\epsilon)] = 0$

This is a transcendental relation for λ.

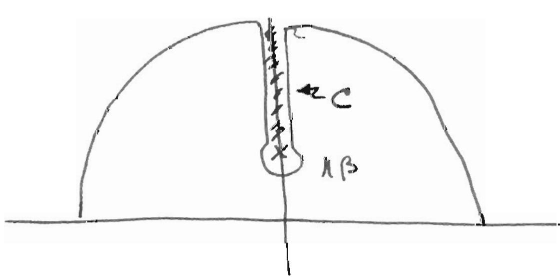
at this point, we use expression for $L_{(-)}$ [supposedly in notes]

$$L_{(-)}(\eta) = \frac{1}{\lambda\pi\beta} \exp \int_0^{\eta/\beta} P_{(-)}(\alpha) \frac{k}{\beta} \, d\alpha$$

where:

$$P_{(-)}(\alpha) = -\frac{\pi\lambda}{2} \left\{ \frac{\dfrac{\frac{\pi}{2} - i \ln(\alpha + \sqrt{1+\alpha^2})}{\pi\sqrt{1+\alpha^2}} - \dfrac{\frac{\pi}{2} - i \ln(\frac{k}{\beta} + \lambda\pi)}{\lambda\pi^2}}{\alpha - k/\beta} + \dots \right\}.$$

The only singularity of $L_{(-)}$ is a branch at $\eta = i\beta$



$$\chi(y) = \frac{1}{2\pi} \int_0^\infty i \frac{R^{1/2} e^{\frac{3\pi i}{4}} L_{(-)}(i\beta + Re^{3\pi i/2}) e^{-\beta y}}{-\{k^2 + (\beta+R)^2\}} e^{-Ry} \, dR$$

$$+ \frac{1}{2\pi} \int_\infty^0 i \frac{R^{1/2} e^{\pi i/4} L_{(-)}(i\beta + Re^{\pi i/2}) e^{-\beta y}}{-\{k^2 + (\beta+R)^2\}} e^{-Ry} \, dR$$

$\therefore \chi(y) \to 0$ as $e^{-\beta y}$ as $y \to 0$

The "left over" when we substitute the expression that we "guessed" for u in the integral equation, are of the form:

$$\int_1^{\infty} \chi(y+1)\, K_0[\beta(y-t)]\, dy \qquad -1 < t < +1$$

and
$$\int_{-1}^{-\infty} \chi(1-y)\, K_0\{\beta(y-t)\}\, dy \qquad -1 < t < +1$$

$$\int_1^{\infty} \chi(y+1)\, K_0\{\beta(y-t)\}\, dy < M \int_1^{\infty} e^{-\beta(y+1)} K_0\{\beta(y-1)\}\, dy$$

$$< M e^{-2\beta} \int_0^{\infty} e^{-\beta x} \underbrace{K_0(\beta x)\, dx}_{1/\beta}$$

$$\therefore \int_1^{\infty} \chi(y+1)\, K_0\{\beta(y-t)\}\, dy \sim O\left(\frac{e^{-2\beta}}{\beta}\right)$$

Problem 6

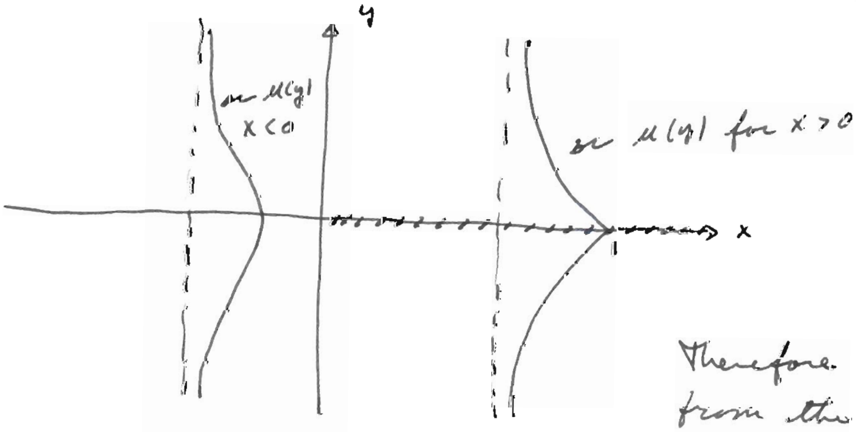
$$u_{xx} + u_{yy} - k^2 u = 0$$

$$u(x,0) = 1 \quad , \; x > 0$$

$$u \to 0 \quad \text{as} \quad \text{for } (x+iy)^{1/2} \to \infty$$

The solutions of this partial differential equation are either even or odd in y. But since, for $x>0$:

$$u(x,0+) = u(x,0^-) = 1$$, u is an even fn. of y.



Therefore we can ~~expect~~ right from the beginning that u_y might have a jump at $y=0$.

Let us define:

$$u_y(x,0^+) - u_y(x,0^-) = f(x) \qquad (1)$$

note that $f(x) = 0$ for $x<0$

Define the following Fourier transforms:

$$u^*(x,\eta) = \int_{-\infty}^{0^-} u(x,y)\, e^{-i\eta y}\, dy + \int_{0^+}^{\infty} u(x,y)\, e^{-i\eta y}\, dy \qquad (2)$$

$$\bar{u}(\xi,\eta) = \int_{-\infty}^{\infty} u^*(x,\eta)\, e^{-i\xi x}\, dx \qquad (3)$$

$$u^+(\xi,y) = \int_{-\infty}^{\infty} u(x,y)\, e^{-i\xi x}\, dx \qquad (4)$$

"star transform" the d.e.

$$u^*_{xx} - \eta^2 u^* - k^2 u^* = f(x) \qquad (5)$$

Note that we could now go back to our d.e. and write:

$$\boxed{u_{xx} + u_{yy} - k^2 u = f(x)\,\delta(y)} \qquad (6)$$

This eq. would then hold everywhere, including the semi-∞ line $y=0$, $x>0$. We now "bar transform" eq. (6):

$$-(k^2+\eta^2+\xi^2)\,\bar{u}(\xi,\eta) = \bar{f}(\xi) \qquad (7)$$

or:
$$\bar{u}(\xi,\eta) = \frac{-\bar{f}(\xi)}{\eta^2+\xi^2+k^2} \qquad (7)'$$

Note that:

$$u^+(\xi,y) = \int_{-\infty}^{\infty} u(x,y)\,e^{-i\xi x}\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{u}(\xi,\eta)\,e^{i\eta y}\,d\eta$$

$$\therefore \quad u^+(\xi,y) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\bar{f}(\xi)\,e^{i\eta y}\,d\eta}{\eta^2+\xi^2+k^2} \qquad (8)$$

Using Jordan's Lemma, after closing the path of integration in the upper & lower half plane depending on the sign of y, we get:

$$u^+(\xi,y) = -\frac{\bar{f}(\xi)\,e^{-|y|\sqrt{\xi^2+k^2}}}{2\sqrt{\xi^2+k^2}} \qquad (9)$$

Note that once $f(x)$ is known, or $\bar{f}(\xi)$, the problem is completely solved. So far we have not used the relation that $u(x,0)=1$ for $x>0$

$$\therefore \quad u^+(\xi,0) = \int_{-\infty}^{0} v(x)\,e^{-i\xi x}\,dx + \int_{0}^{\infty} 1\,e^{-i\xi x}\,dx \qquad (10)$$

where: $\quad u(x,0) = v(x) + w(x)$, such that:

$$v(x) = \begin{cases} 0 & x>0 \\ ? & x<0 \end{cases} \; ; \qquad w(x) = \begin{cases} 1 & x>0 \\ 0 & x<0 \end{cases}$$

$\therefore$ from (10) and (9) we get:

$$\bar{V}_{(+)}(\xi) + \left(\frac{1}{i\xi}\right)_{(-)} = \frac{-\bar{f}_{(-)}(\xi)}{2\sqrt{\xi^2+k^2}} \qquad (11)$$

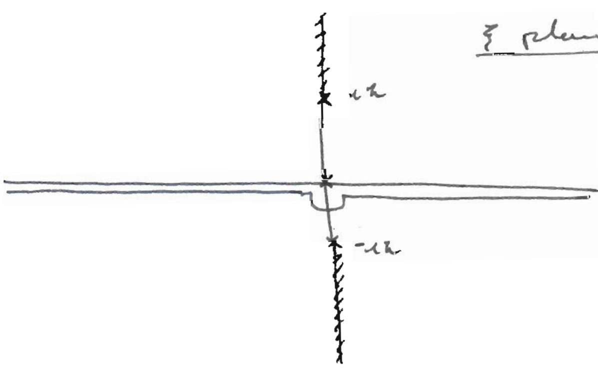
This is the Wiener-Hopf equation for the problem

$$\bar{V}_{(+)}\sqrt{\xi+ik}_{(+)} + \left[\frac{\sqrt{\xi+ik}-\sqrt{ik}}{i\xi}\right]_{(+)} = \frac{-\bar{f}_{(-)}}{2\sqrt{\xi-ik}} - \left[\frac{\sqrt{ik}}{i\xi}\right]_{(-)} = E \qquad (12)$$

Let $E=0$, see discussion. Then:

$$\boxed{\bar{f}(\xi) = -\frac{2\sqrt{ik}\sqrt{\xi-ik}}{i\xi}} \quad (13). \quad \text{Therefore:}$$

$$u(x,y) = \frac{\sqrt{ik}}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{i\xi x-|y|\sqrt{\xi^2+k^2}}}{\xi\sqrt{\xi+ik}}\,d\xi \qquad (14)$$

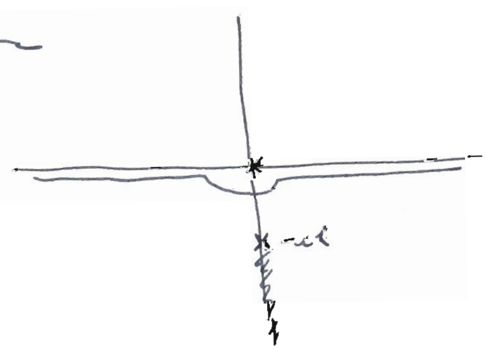


Consider path of integration: in which way should we go around the pole at origin? We want $u(x,0)=1$ for $x>0$, so if we put $y=0$ in (14) we get:

$$u(x,0) = \frac{\sqrt{ik}}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{i\xi x}}{\xi\sqrt{\xi+ik}}\,d\xi$$

If we go below origin, then for $x>0$, using Jordan's Lemma, we get:

$$u(x,0) = 2\pi i \times \frac{\sqrt{ik}}{2\pi i}\times\frac{1}{\sqrt{ik}} = 1$$



Singularity at $x=0$: We want $u_y(x,0)$, or $f(x)$, to be integrable so $f(x) \sim \frac{1}{x^{1-\epsilon}}$, $1 \geq \epsilon > 0$, $(x\to 0)$

Using the Tauberian Theorem, $\bar{f}(\xi) \sim C\,\xi^{-\epsilon}$, $\xi \to \infty$
and so:

$$\frac{-\bar{f}_{(-)}}{2\sqrt{\xi - ik}} - \frac{\sqrt{ik}}{i\xi} \sim \frac{-C\,\xi^{-\epsilon}}{2\,\xi^{1/2}} - \frac{\sqrt{ik}}{i\xi}$$

$\therefore$ the only possibility is $\epsilon = 1/2$, and $E \equiv 0$

Integral Equation Approach

Define the Green Function $G(x, y \mid x_0, y_0)$ such that:

$$G_{xx} + G_{yy} - k^2 G = -\delta(x - x_0)\,\delta(y - y_0) \qquad (1)$$

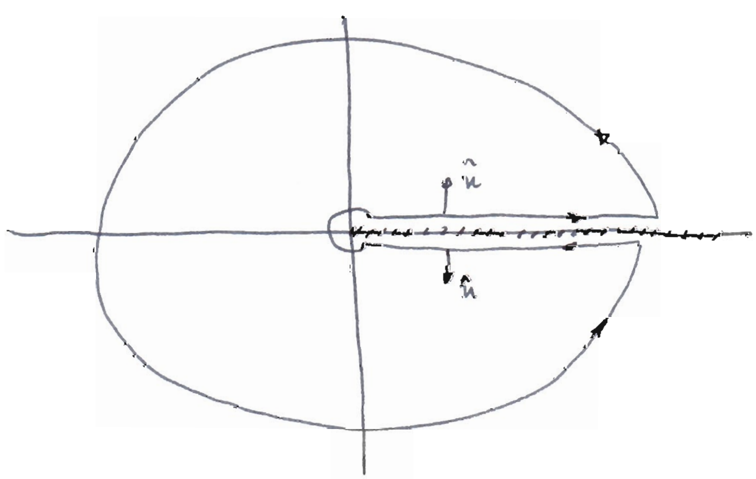
and $G \to 0$ as $x^2 + y^2 \to \infty$

Then: $$\iint_A \left\{ G\left[u_{xx} + u_{yy} - k^2 u\right] - u\left[G_{xx} + G_{yy} - k^2 G\right] \right\} dx\,dy$$

$$= \iint_A u\,\delta(x - x_0)\,\delta(y - y_0)\,dx\,dy = u(x_0, y_0)$$

But $\quad G\nabla^2 u - u\nabla^2 G = \nabla \cdot \left\{ G\nabla u - u\nabla G \right]$

$$\therefore \oint_C (G\nabla u - u\nabla G) \cdot \hat{n}\,ds = u(x_0, y_0)$$



$$\therefore \; u(x_0,y_0) = \int_\infty^0 \left\{ -G(x,0^-)\, u_y(0^-) + u(0^-)\, G_y(0^-) \right\} (-dx)$$

$$+ \int_0^\infty \left\{ G(x,0^+)\, u_y(0^+) - u(x,0^+)\, G_y(x,0^+) \right\} (+dx)$$

But since $u(x,0^+) = u(x,0^-)$ and that both G and G_y are continuous, we gets:

$$\boxed{u(x_0,y_0) = \int_0^\infty dx \; G(x,0 \mid x_0,y_0) \left\{ u_y(x,0^+) - u_y(x,0^-) \right\}}$$

Let us define $f(x) = u_y(x,0^+) - u_y(x,0^-)$

Then: $u(x_0,y_0) = \int_0^\infty G(x,0 \mid x_0 y_0)\, f(x)\, dx$

For: $x_0 > 0$, $y_0 > 0$, ~~at~~ we ~~to~~ have:

$$\boxed{1 = \int_0^\infty G(x,0 \mid x_0,0)\, f(x)\, dx} \qquad x_0 > 0$$

This is the equation for $f(x)$.

Here: $G(x,y \mid x_0,y_0) = -\frac{1}{2\pi} K_0\left[k \sqrt{(x-x_0)^2 + (y-y_0)^2} \right]$

$$\therefore \quad \boxed{1 = -\frac{1}{2\pi} \int_0^\infty K_0\left[k \mid x - x_0 \mid \right] f(x)\, dx \; ; \quad x_0 > 0}$$

Problem 7

The Milne Problem (see Sneddon's Fourier Transform)

The integral equation is:

$$\psi_0(x) = \int_0^\infty \psi_0(t)\, \frac{E(|x-t|)}{2}\, dt \qquad (1)$$

where:

$$E(|x|) = \int_{|x|}^\infty \frac{e^{-u}}{u}\, du \qquad (2)$$

The purpose of the problem is to express $\psi_0(x)$ in terms of $M_\lambda(x)$, where $M_\lambda(x)$ is the solution of:

$$\lambda\, M_\lambda(x) = \int_0^\infty \frac{1}{\pi} K_0\left[x-t\right] M_\lambda(t)\, dt \qquad (3)$$

Wiener-Hopf Equation for the Milne Problem:

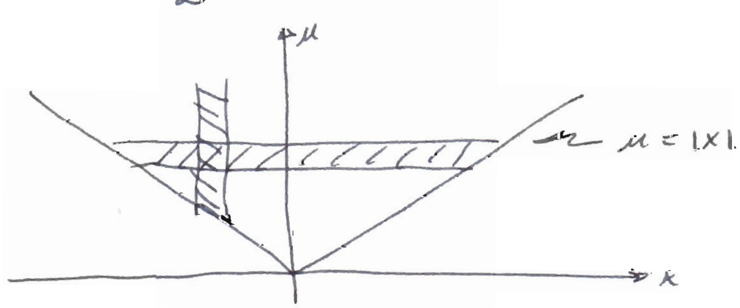
Following the usual procedure, we extend the range of validity of the equation (1) by introducing a function $h(x)$ such that:

$$h(x) = \begin{cases} 0 & x > 0 \\ ? & x < 0 \end{cases}$$

and we Fourier Transform:

$$\bar{h}_{(+)}(\xi) + \bar{\psi}_0(\xi) = \frac{1}{2} \bar{E}(\xi)\, \bar{\psi}_0(\xi) \qquad (4)$$

where $$\bar{E}(\xi) = \int_{-\infty}^{\infty} dx\, e^{-i\xi x} \int_{|x|}^\infty \frac{e^{-u}}{u}\, du \qquad (5)$$



~~It~~ Interchanging the two integrals, we get:

$$\bar{E}(\xi) = \int_0^\infty \frac{e^{-u}}{u}\,du \int_{-u}^{+u} e^{-i\xi x}\,dx$$

$$= \int_0^\infty \frac{e^{-u}}{u}\left\{\frac{e^{-i\xi u}}{-i\xi} - \frac{e^{i\xi u}}{-i\xi}\right\}du = 2\int_0^\infty \frac{e^{-u}\sin\xi u}{u\xi}\,du$$

$$\therefore \frac{d}{d\xi}\left[\xi\bar{E}(\xi)\right] = 2\int_0^\infty e^{-u}\cos\xi u\,du = \frac{2}{1+\xi^2}$$

$$\therefore \xi\bar{E}(\xi) = 2\tan^{-1}\xi + \text{constant}$$

or $$\bar{E}(\xi) = \frac{2\tan^{-1}\xi}{\xi} + \frac{\text{constant}}{\xi}$$

When $\xi = 0$, $\bar{E}(0)$ = Area under $E(x)$, which is finite, so the constant is equal to 0.

$$\therefore \bar{E}(\xi) = \frac{2\tan^{-1}\xi}{\xi}$$

so that the transform of the kernel of the integral equation is:

$$\bar{K}(\xi) = \tfrac{1}{2}\bar{E}(\xi) = \frac{\tan^{-1}\xi}{\xi}$$

<u>Approximate Solution: Substitute Kernel</u>: Let us evaluate the area under the kernel, the first two moments, and the singularity at ∞:

$\bar{K}(0)$ = area = 1

$\bar{K}'(0)$ = first moment = 0, $\quad \bar{K}(\xi) \sim \dfrac{\pi}{2|\xi|}$ as $\xi \to \infty$

$\bar{K}''(0)$ = second moment = $-2/3$

Let us now consider the kernel $K^* = \dfrac{\beta}{2\pi} K_0\left[\beta x\right]$

Then:

$$\bar{K}^*(\xi) = \frac{\beta}{\lambda(\xi^2+\beta^2)^{1/2}}$$

$$\bar{K}^*(0) = \lambda^{-1}$$

$$\bar{K}^{*\prime}(0) = 0, \qquad \bar{K}^*(\xi) \sim \frac{\beta}{\lambda|\xi|} \text{ as } \xi \to \infty$$

$$\bar{K}^{*\prime\prime}(0) = -\frac{1}{\lambda\beta^2}$$

Let us therefore take $\lambda = 1$, and $\beta = \frac{\pi}{2}$; in this way the area under both kernels is the same, the 1st moment is the same, and the behavior at ∞ is identical.

$$\bar{K}^{*\prime\prime}(0) \simeq -.4$$

$$\bar{K}^{\prime\prime}(0) \simeq -.7$$

If we go back to (3) we have:

$$u_1\left(\frac{\pi x}{2}\right) = \int_0^\infty \frac{1}{2} K_0\left[\frac{\pi}{2}(x-t)\right] u_1\left(\frac{\pi t}{2}\right) dt$$

But now we can approximate K by K^*, that is:

$$\psi_0(x) \simeq \int_0^\infty \frac{1}{2} K_0\left[\frac{\pi}{2}(x-t)\right] \psi_0(t)\, dt$$

$$\therefore \quad \psi_0(x) \simeq u_1\left(\frac{\pi x}{2}\right)$$

(Supposedly, we know everything about $u_1\left(\frac{\pi x}{2}\right)$)

Problem 8

$$\nabla^2\theta - \theta_x - \theta_t = 0 \quad \text{in:} \quad 0 < t < \infty$$

$$-\infty < x < \infty$$

$$\theta(x,0,t) = \delta(t) \text{ for } x>0 \qquad -\infty < y < \infty \text{ , except } y=0,\ x>0$$

$$\theta \to 0 \quad \text{as} \quad \operatorname{Im}(x+iy)^{1/2} \to \infty$$

Find an integral representation of $\theta(x,y,t)$ and evaluate it for large t using method of steepest decent.

θ is an even fn. of y, since the equation is unchanged when we replace y by $-y$, and the BC is even in y.

The only possible jump is

$$\theta_y(x,0^+,t) - \theta_y(x,0^-,t) = f(x,t) \qquad (1)$$

Note that $f(x,t) \equiv 0$ for $x<0$

Define the following transform:

$$\theta^*(x,\eta,t) = \int_{-\infty}^{0^-} + \int_{0^+}^{\infty} e^{-i\eta y}\,\theta(x,y,t)\,dy \qquad (2)$$

$$\bar\theta(\xi,\eta,s) = \int_0^\infty e^{-st}dt \int_{-\infty}^{\infty} e^{-i\xi x}\,\theta^*(x,\eta,t)\,dx \qquad (3)$$

First "star transform" the differential equation:

$$-\eta^2\theta^* + \theta^*_{xx} - \theta^*_x - \theta^*_t = f(x,t) \qquad (4)$$

Now "bar transform" eq. (4)

$$-(\eta^2 + \xi^2 + i\xi + s)\,\bar\theta(\xi,\eta,s) = \bar f(\xi,s) \qquad (5)$$

$$\bar\theta(\xi,\eta,s) = \frac{-\bar f(\xi,s)}{\eta^2 + \xi^2 + i\xi + s}$$

Let us invert over η:

$$\theta^+(\xi, y, s) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\bar{f}(\xi, s)}{\eta^2 + \xi^2 + i\xi + s} e^{i\eta y} \, d\eta$$

and get:

$$\theta^+(\xi, y, s) = -\frac{1}{2} \frac{\bar{f}(\xi, s) \, e^{-|y|\sqrt{\xi^2 + i\xi + s}}}{(\xi^2 + i\xi + s)^{1/2}} \qquad (6)$$

In particular:

$$\theta^+(\xi, 0, s) = -\frac{1}{2} \frac{\bar{f}(\xi, s)}{(\xi^2 + i\xi + s)^{1/2}} \qquad (7)$$

But: $\theta^+(\xi, 0, s) = \int_0^{\infty} e^{-st} dt \left[\int_{-\infty}^{+0} e^{-i\xi x} \tau(x,t) + \int_0^{\infty} e^{-i\xi x} S'(t) \, dx \right]$

$$= \tau^+(\xi, s)_{(+)} + \left(\frac{1}{is\xi}\right)_{(-)} \qquad (8)$$

Therefore, the Wiener-Hopf equation of the problem is:

$$[\tau^+(\xi, s)]_{(+)} + \left[\frac{1}{is\xi}\right]_{(-)} = -\frac{1}{2} \frac{\bar{f}(\xi, s)_{(-)}}{(\xi^2 + i\xi + s)^{1/2}} \qquad (9)$$

s is merely a parameter here.

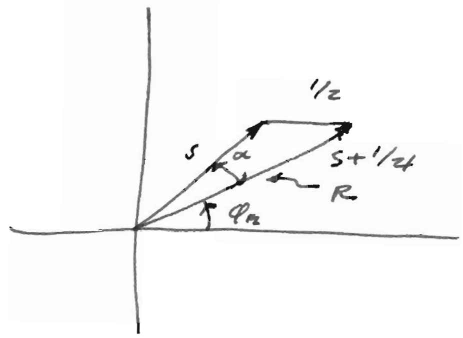
$$[\tau^+(\xi, s)]_{(+)} + \left[\frac{1}{is\xi}\right]_{(-)} = -\frac{1}{2} \frac{\bar{f}(\xi, s)_{(-)}}{(\xi + ia_2)^{1/2}(\xi - ia_1)^{1/2}} \qquad (9)'$$

where:

$$\left.\begin{aligned} a_1 &= (s + \tfrac{1}{4})^{1/2} - \tfrac{1}{2} \\ a_2 &= (s + \tfrac{1}{4})^{1/2} + \tfrac{1}{2} \end{aligned}\right\} \qquad (10)$$

The path of integration for a Laplace transform is always such that: $Re(s) > 0$

s-plane



$$\therefore \quad s = \rho e^{i\alpha} \quad ; \quad -\pi/2 < \alpha < +\pi/2$$

$$s + \tfrac{1}{4} = R e^{i\varphi} \quad ; \quad |\varphi| < |\alpha| \; ; \; R \geq \tfrac{1}{4}$$

$$\therefore \quad (s + \tfrac{1}{4})^{1/2} = R^{1/2} e^{i\varphi/2}$$

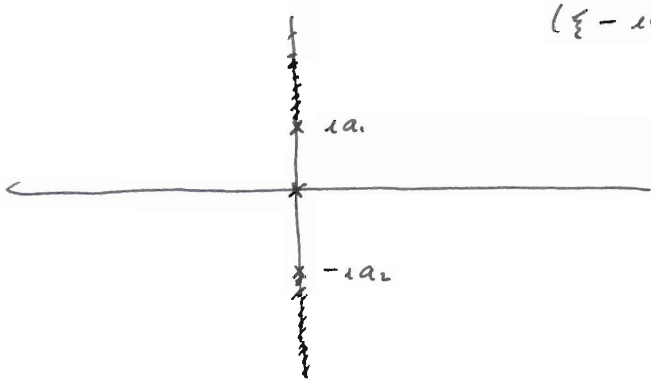
and so: $\quad Re\,(s + 1/4)^{1/2} \geq \frac{1}{2}$

That is:

$$\begin{aligned} &Re\, a_1 > 0 \\ &Re\, a_2 > 0 \qquad (11) \end{aligned}$$

Back to equation (9)', we see that $(\xi + i a_2)^{1/2} = (\xi + i a_2)^{1/2}_{(+)}$

$$(\xi - i a_1)^{1/2} = (\xi - i a_1)^{1/2}_{(-)}$$



Therefore:

$$\gamma^{+}_{(+)} (\xi + i a_2)^{1/2}_{(+)} + (\xi + i a_2)^{1/2}_{(+)} \left\{ \frac{1}{i s \xi} \right\} = -\frac{1}{2} \frac{\bar{f}(\xi, s)_{(-)}}{(\xi - i a_1)^{1/2}_{(-)}}$$

That is:

$$(\xi + i a_2)^{1/2}_{(+)} \gamma^{+}_{(+)} + \left[\frac{(\xi + i a_2)^{1/2} - (i a_2)^{1/2}}{i s \xi} \right]_{(+)} = E$$

$$= -\left[\frac{(i a_2)^{1/2}}{i s \xi} \right]_{(-)} - \frac{1}{2} \frac{\bar{f}(\xi, s)}{(\xi - i a_1)^{1/2}_{(-)}} \qquad (12)$$

where E is an entire fn.

Now the heat flux should be integrable near $x = 0$, i.e., the singularity of $f(x,t)$ is such that:

$$f(x,t) \sim C\, x^{-1+\epsilon} \quad ; \; 0 < \epsilon \leq 1$$

$$\therefore \quad \bar{f}(\xi, s) \sim \xi^{1-\epsilon-1}$$

the only way to get an entire fn. type of behavior at ∞ is to take $\epsilon = 1/2$, and $E = 0$. Hence:

$$\boxed{\bar{f}(\xi, s) = - \frac{2(ia_2)^{1/2}(\xi - ia_1)^{1/2}}{is\xi}} \qquad (13)$$

Putting $\bar{f}(\xi, s)$ into (6), we get:

$$\theta^{+}(\xi, y, s) = -\frac{1}{2} \times \frac{-2(ia_2)^{1/2}(\xi - ia_1)^{1/2}}{is\xi(\xi + ia_2)^{1/2}(\xi - ia_1)^{1/2}} \exp\left\{-|y|\sqrt{\xi^2 + i\xi + s}\right\}$$

That is:

$$\boxed{\theta(x, y, t) = \frac{1}{4\pi^2 i} \int_{-i\infty}^{i\infty} ds\, e^{st} \int_{-\infty}^{\infty} \frac{(ia_2)^{1/2} e^{i\xi x - |y|\sqrt{\xi^2 + i\xi + s}}}{is\xi(\xi + ia_2)^{1/2}} d\xi} \qquad (14)$$

We could now proceed to evaluate $\theta(x, y, t)$ for large t, but it would be a rather hard job due to the many parameters involved. Let us instead look at the heat transfer on the semi-∞ plate, i.e., at $f(x, t)$:

$$f(x, t) = -\frac{1}{2\pi^2 i} \int_{-i\infty}^{i\infty} ds\, e^{st} \int_{-\infty}^{\infty} \frac{(ia_2)^{1/2}(\xi - ia_1)^{1/2} e^{i\xi x}}{is\xi} d\xi \qquad (15)$$

We go below the pole on the real axis, since $f(x, t) \equiv 0$ for $x < 0$. From Foster and Campbell, p. 57, # 549 we get:

$$f(x, t) = -\frac{1}{\pi i} \int_{-i\infty}^{i\infty} i^{1/2} \frac{(ia_2)^{1/2}}{is} \left\{ \frac{e^{-a_1 x}}{(\pi x)^{1/2}} + a_1^{1/2}\, \mathrm{Erf}(a_1^{1/2} x^{1/2}) \right\} e^{st} ds$$

$$f(x, t) = \frac{i}{\pi} \int_{-i\infty}^{i\infty} \left\{ \left(\frac{a_2}{\pi x}\right)^{1/2} e^{-a_1 x} + a_1^{1/2} a_2^{1/2}\, \mathrm{Erf}(a_1^{1/2} x^{1/2}) \right\} \frac{e^{st}}{s} ds$$

$$(a_1 a_2)^{1/2} = s^{1/2}$$

$$\boxed{f(x, t) = \frac{i}{\pi} \int_{-i\infty}^{i\infty} \left[\left(\frac{a_2}{\pi x}\right)^{1/2} e^{-a_1 x} + s^{1/2}\, \mathrm{Erf}(a^{1/2} x^{1/2}) \right] \frac{e^{st}}{s} ds} \qquad (16)$$

Method of Steepest Descent:

$$\operatorname{Erf}(\sqrt{a_1 x}) = 1 - \frac{2}{\sqrt{\pi}} \int_{\sqrt{a_1 x}}^{\infty} e^{-v^2}\, dv = 1 - \frac{a_1^{1/2}}{\pi^{1/2}} \int_x^{\infty} e^{-a_1 \mu} \frac{d\mu}{\mu^{1/2}}$$

Also: $e^{-a_1 x} = a_1 \int_x^{\infty} e^{-a_1 \mu}\, d\mu$, since $x > 0$

Therefore:

$$f(x,t) = \frac{i}{\pi} \int_{-i\infty}^{i\infty} \frac{e^{st}}{s^{1/2}}\, ds + \frac{i}{\pi} \int_x^{\infty} d\mu\, e^{\mu/2} \int_{-i\infty}^{i\infty} \left[\frac{a_1 a_2^{1/2}}{\sqrt{\pi x}} - \frac{a_1^{1/2} s^{1/2}}{\sqrt{\pi \mu}} \right] \frac{1}{s} \cdot e^{st - \mu\sqrt{s+1/4}}\, ds$$

Note that $a_1^{1/2} s^{1/2} = a_1 a_2^{1/2}$, and the singularities of the integrand are:

① A branch pt. at $s = -1/4$
② a pole at $s = 0$

$$f(x,t) = \frac{-2}{\pi^{1/2} t^{1/2}} + \frac{i}{\pi} \int_x^{\infty} d\mu\, e^{\mu/2} \int_{-i\infty}^{i\infty} \frac{a_1 a_2^{1/2}}{\pi^{1/2}} \left\{ \frac{1}{\sqrt{x}} - \frac{1}{\sqrt{\mu}} \right\} \frac{e^{st - \mu\sqrt{s+1/4}}}{s}\, ds \qquad (17)$$

Consider then, the following integral:

$$K = \int_{-i\infty}^{i\infty} \frac{a_1 a_2^{1/2}}{s} e^{st - \mu\sqrt{s+1/4}}\, ds \qquad (18)$$

Define: $g(s) = s - \frac{\mu}{t}\sqrt{s+1/4}$ (19)

$$g'(s) = 1 - \frac{\mu}{2t}(s+1/4)^{-1/2} \; ; \quad \therefore \left(s_0 + \tfrac{1}{4}\right)^{1/2} = \frac{\mu}{2t}$$

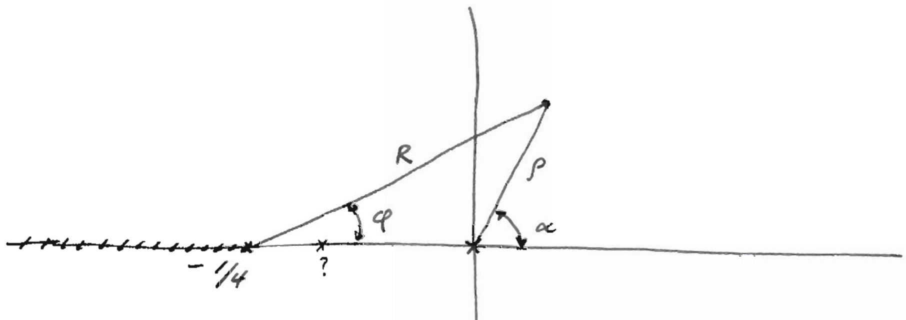
Saddle pt: $s_0 = \frac{\mu^2}{4t^2} - \frac{1}{4}$ (20)

Let us furthermore recall that:

$$\begin{cases} s + 1/4 = R e^{i\varphi} \\ s = \rho e^{i\alpha} \end{cases} \qquad (21)$$

Then: $\operatorname{Im} g(s) = \rho \sin\alpha - \frac{\mu}{t} R^{1/2} \sin\frac{\varphi}{2}$

s-plane



But: $\rho \sin\alpha = R \sin\varphi$

$\therefore \operatorname{Im} g(s) = R^{1/2} \sin\varphi/2 \left\{ 2R^{1/2}\cos\varphi/2 - \frac{\mu}{t} \right\}$

The path of steepest descent is therefore a parabola.

$$R^{1/2}\cos\varphi/2 = \frac{\mu}{2t} \qquad (22)$$

$$g''(s) = \frac{\mu}{4t}\left(s+\frac{1}{4}\right)^{-3/2} \; ; \quad g''(s_0) = \frac{2t^2}{\mu^2}$$

$$\therefore \quad K \cong \frac{a_1(s_0)\left[a_2(s_0)\right]^{1/2}}{s_0} e^{-t\left\{\frac{\mu^2}{4t^2}+\frac{1}{4}\right\}} \int_c e^{\frac{t^2}{\mu^2}(s-s_0)^2} ds$$

$$\cong i\,\frac{\mu}{t}\,\pi^{1/2} e^{-\frac{\mu^2}{4t}-\frac{t}{4}} \frac{\left[\frac{\mu}{2t}-\frac{1}{2}\right]\left[\frac{\mu}{2t}+\frac{1}{2}\right]^{1/2}}{\left[\frac{\mu^2}{4t^2}-\frac{1}{4}\right]}$$

$$K \sim i\,\frac{\mu}{t^{3/2}}\,\pi^{1/2} e^{-\frac{\mu^2}{4t}-\frac{t}{4}} \left[\frac{\mu}{2t}+\frac{1}{2}\right]^{1/2} \Big/ \left[\frac{\mu}{2t}+\frac{1}{2}\right]$$

$$\therefore \quad f(x,t) = \frac{-2}{\pi^{1/2}t^{1/2}} + \frac{i^2}{\pi}\int_x^\infty d\mu \, e^{\mu/2} \frac{\mu}{t^{3/2}} e^{-\frac{\mu^2}{4t}-\frac{t}{4}} \frac{\left\{\frac{1}{\sqrt{x}}-\frac{1}{\sqrt{\mu}}\right\}}{\left[\frac{\mu}{2t}+\frac{1}{2}\right]^{1/2}}$$

$$\sim \frac{-2}{\pi^{1/2}t^{1/2}} + \frac{i^2}{\pi}\int_x^\infty \frac{\mu}{t^{3/2}} \frac{e^{-t\left[\frac{\mu}{2t}-\frac{1}{2}\right]^2}}{\left[\frac{\mu}{2t}+\frac{1}{2}\right]^{1/2}} \left\{\frac{1}{\sqrt{x}}-\frac{1}{\sqrt{\mu}}\right\} d\mu$$

$$\sim \frac{-2}{\pi^{1/2}t^{1/2}} + \frac{i^2}{\pi}\left\{\frac{1}{\sqrt{x}}-\frac{1}{\sqrt{t}}\right\}\int_0^\infty \frac{1}{t^{1/2}} e^{-t\left[\frac{\mu}{2t}-\frac{1}{2}\right]^2} d\mu \sim \frac{-2}{\pi^{1/2}t^{1/2}} + \frac{2i^2}{\pi}\left[\frac{1}{\sqrt{x}}-\frac{1}{\sqrt{t}}\right]\int_{\frac{x-t}{2t^{1/2}}}^\infty e^{-\mu^2} d\mu$$

Finally:

$$f(x,t) \sim -\frac{2}{\pi^{1/2}}\left[\frac{1}{t^{1/2}} + \frac{1}{(\pi x)^{1/2}}\operatorname{Erfc}\left(\frac{x-t}{2t^{1/2}}\right) + \frac{1}{(\pi t)^{1/2}}\operatorname{Erfc}\left(\frac{t-x}{2t^{1/2}}\right)\right]$$

Problem: Surface of Revolution of Minimal Area

We are looking for a curve $y = y(x)$ joining two points $A(-d, h)$ & $B(d, h)$ such that, when rotated about the x-axis, the area of the generated surface is a minimum.

The area of the generated surface is:

$$A = 2\pi \int_{-d}^{d} y(1+y'^2)^{1/2}\,dx \qquad (1)$$

therefore, in this problem we have:

$$f = y(1+y'^2)^{1/2} \qquad (2)$$

Which is of the form $\varphi(x,y)(1+y'^2)^{1/2}$: Therefore the curve cannot have any corners.

1. Euler's Equation - Extremals. Since f is not an explicit fn. of x, we can write down a first integral of Euler's Equation:

$$y(1+y'^2)^{1/2} = y' \frac{yy'}{(1+y'^2)^{1/2}} + c \qquad (3)$$

where c is a constant. Therefore:

$$y = c(1+y'^2)^{1/2}$$

$$\text{or } y = c \cosh \frac{x-b}{c} \qquad (4)$$

The extremals are catenaries. We now determine b and c by the requirement that the curve should pass thru A & B. This raises the question: How many catenaries (4) can one draw thru A & B?

We have:
$$h = c \cosh \frac{-d-b}{c}$$
$$h = c \cosh \frac{d-b}{c}$$

$$\therefore\ b = 0 \quad \text{and} \quad h = c \cosh \frac{d}{c} \qquad (5)$$

This is a transcendental equation for c.

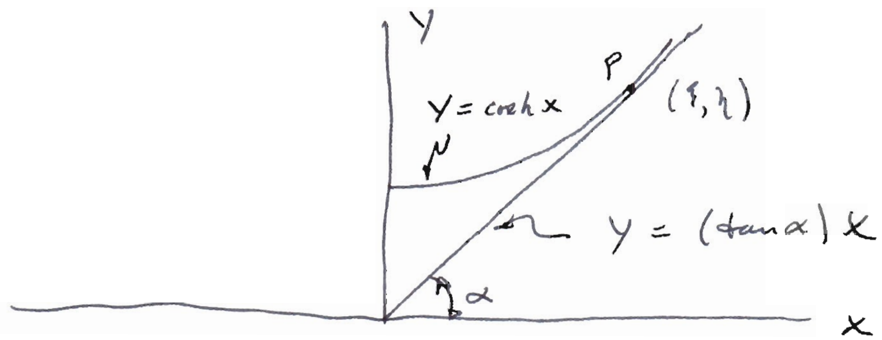
2. Determination of c: Existence & Uniqueness of Solution

Let us write: $\frac{h}{c} = Y$; $\frac{d}{c} = X$ (6)

Then each point of intersection of the two curves

$$\begin{cases} Y = \cosh X \\ Y = \frac{h}{d} X \end{cases} \qquad (7)$$

gives a root of (5). From Fig I, we can see immediately that if $\frac{h}{d} < \tan\alpha$, Eq. (5) has no root and for $\frac{h}{d} > \tan\alpha$, there are two roots: $c_>$ & $c_<$



Let us first determine α. We have:

$$\frac{\eta - 0}{\xi - 0} = \tan\alpha = \text{slope of the tg. at P i.e. } \sinh\xi$$

~~[illegible]~~ but $\eta = \cosh\xi$ and so

$$\tan\alpha = \sinh\xi \quad \text{where} \quad \cosh\xi = \xi \sinh\xi \qquad (8)$$

Let us solve (8) by trial:

$\xi = 1.1$: $\cosh 1.1 - 1.1 \sinh 1.1 = +.198$

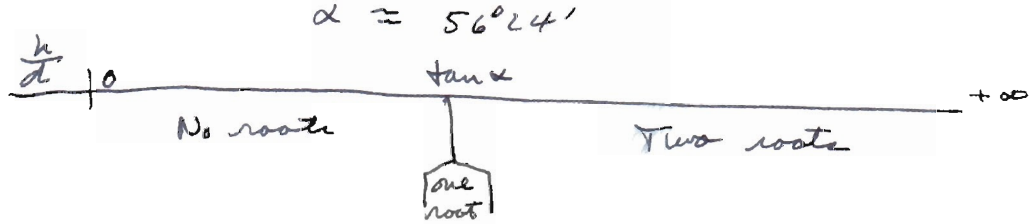
$\xi = 1.2$: $\cosh 1.2 - 1.2 \sinh 1.2 = -.801$

Using a linear interpolation we have about

$$\xi = 1.12 \qquad (9)$$

$$\tan\alpha = 1.5 \qquad (10)$$

$$\alpha \simeq 56°24'$$



We are going to consider $\frac{b}{a} > \tan\alpha$ and try to determine whether both values of c give a strong minimum.

3. Legendre Test We evaluate $f_{y'y'}$:

$$f_{y'y'} = \frac{y}{(1+y'^2)^{3/2}} = \frac{c_i}{\cosh^2 \frac{x}{c_i}}$$

since c_i are positive, it follows that $f_{y'y'} > 0$ (11)

4. Weierstrass Test : We compute the E-fn.

$$E(x, y, y', p) = f(x, y, p) - f(x, y, y') - (p - y') f_{y'}(x, y, y')$$

ie. $E_i = c_i (1+p^2)^{1/2} \cosh \frac{x}{c_i} - c_i \cosh^2 \frac{x}{c_i} - \left\{p - \sinh \frac{x}{c_i}\right\} c_i \sinh \frac{x}{c_i}$

ie: $E_i = c_i \left\{ (1+p^2)^{1/2} \cosh \frac{x}{c_i} - p \sinh \frac{x}{c_i} - 1 \right\}$ (12)

E must be positive for all $p \neq y' =$ ~~ad~~ $\sinh \frac{x}{c_i}$ ie, if we let $p = \sinh \frac{q}{c_i}$, then $q \neq x$

With this substitution, we have :

$$E_i = c_i \left\{ \cosh \left\{ \frac{x-q}{c_i} \right\} - 1 \right\}$$

and for $x \neq q$, $E > 0$ (13)

5: Jacobi Test : Let $g(x, y, c) = y - c \cosh \frac{x}{c} = 0$ (14)

The envelope of the family of catenary (14) is given by:

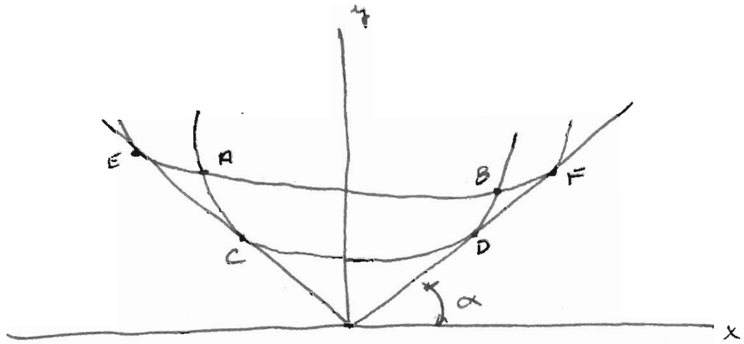
$$\left. \begin{aligned} g(x, y, c) &= 0 \\ \frac{\partial}{\partial c} g(x, y, c) &= 0 \end{aligned} \right\} \quad (15)$$

ie.

$$\left. \begin{aligned} y &= c \cosh \frac{x}{c} \\ \cosh \frac{x}{c} - \frac{x}{c} \sinh \frac{x}{c} &= 0 \end{aligned} \right\} \quad (16)$$

Using (8), we can write $\frac{x}{c} = \pm \xi$ and so:

$$\left.\begin{aligned} y &= \pm x \,\frac{\cosh \xi}{\xi} = \pm x \sinh \xi = \pm (\tan\alpha)\, x \\ y &= \pm (\tan\alpha)\, x \end{aligned}\right\} \quad (17)$$



The curve ACDB does not satisfy the Jacobi condition since C & D are two pts where the curve touches its envelope and they are between A & B. This curve corrsp. to $C_<$.

On the other hand, EABF satisfies the Jacobi condition, and is therefore the unique solution of the problem.

6. <u>Conclusion</u>: For $h > 1.5d$, there exists a catenary $y = c \cosh \frac{x}{c}$ where c is the largest root of the transcendental equation:

$$h = c \cosh \frac{d}{c}$$

which, when rotated around the x-axis generates the surface of revolution of minimal area.

$$u'' + \lambda u = 0 \quad , \quad u(0) = u(\pi) = 0$$

$$\frac{d^2 K(x,t)}{dx^2} = \delta(x-t)$$

$$K \longrightarrow Ax + B$$

$x = 0$: $\quad B = 0 \quad : \quad \therefore \quad K_1 = Ax \qquad x < t$

$x = \pi$: $\quad A\pi + B = 0, \quad \therefore \quad K_2 = A'(x - \pi) \qquad x > t$

$$K_1(t) = K_2(t)$$

$$K_2'(t) - K_1'(t) = 1$$

$$\begin{vmatrix} t & \pi - t \\ 1 & -1 \end{vmatrix} \begin{vmatrix} A \\ A' \end{vmatrix} = \begin{vmatrix} 0 \\ 1 \end{vmatrix}$$

$$\Delta = -\pi \quad , \quad \Delta_A = t - \pi \quad , \quad \Delta_{A'} = t$$

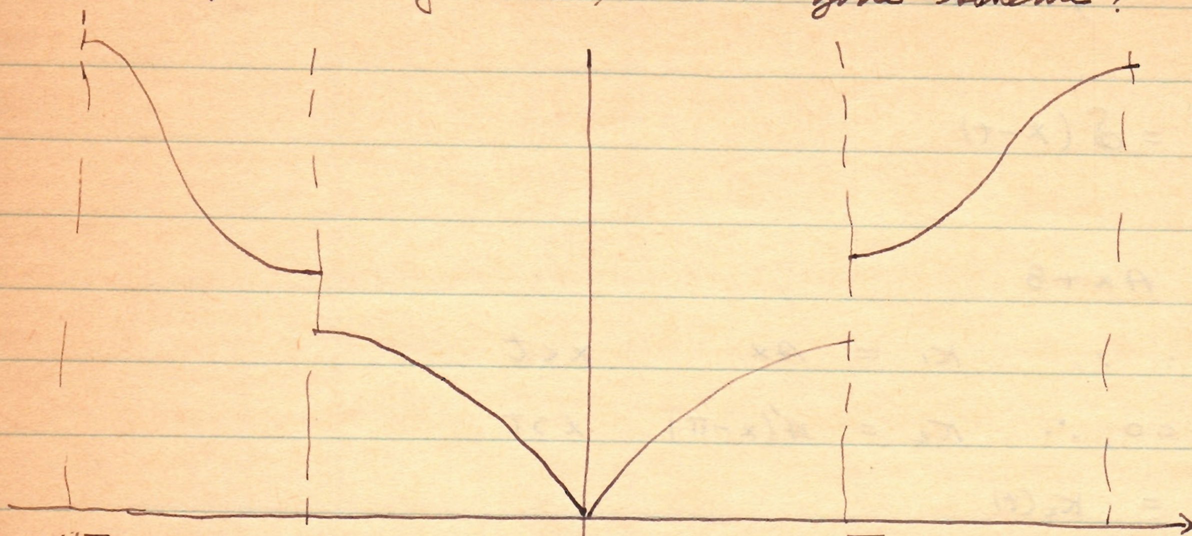
$$\therefore \quad K(x,t) = \frac{1}{\pi} \begin{cases} x(\pi - t) & x < t \\ t(\pi - x) & x > t \end{cases}$$

Assume: $\quad u(x) = \lambda \int_0^\pi K(x,t)\, u(t)\, dt$

$$u'(x) = \lambda \int_0^\pi K'(x,t)\, u(t)\, dt$$

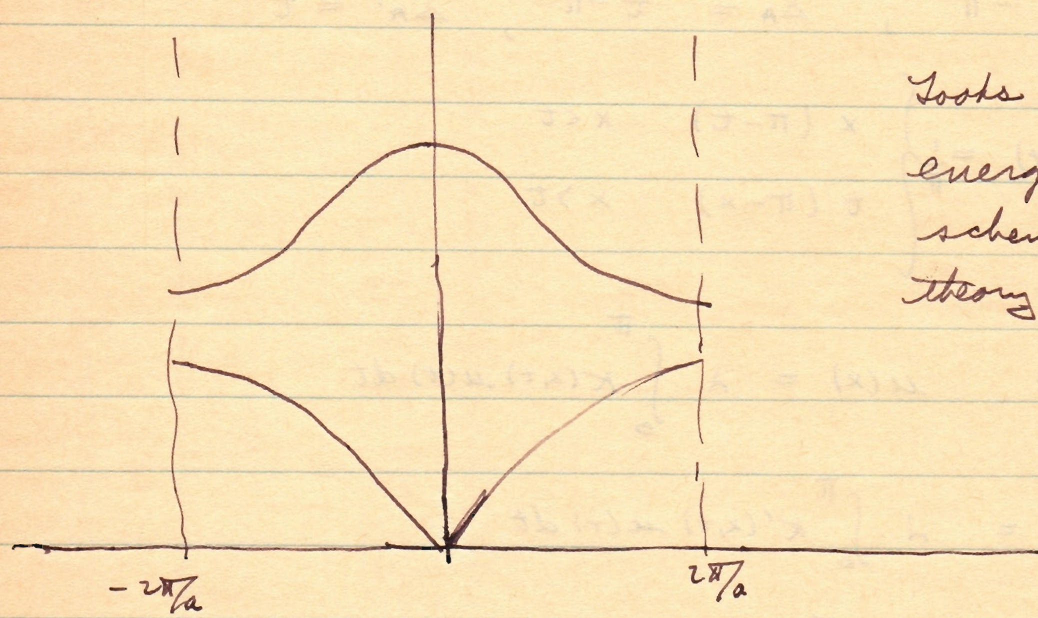
$$u''(x) = \lambda \int_0^\pi K''(x,t)\, u(t)\, dt$$

Another Way: Extended zone scheme:



This gives us the right number of solutions.

Reduced zone scheme:



Looks a lot like energy band scheme of electron theory of solids

Sub.: $\lambda \int_0^\pi \left[K''(x,t) + \lambda K(x,t) \right] u(t)\,dt$

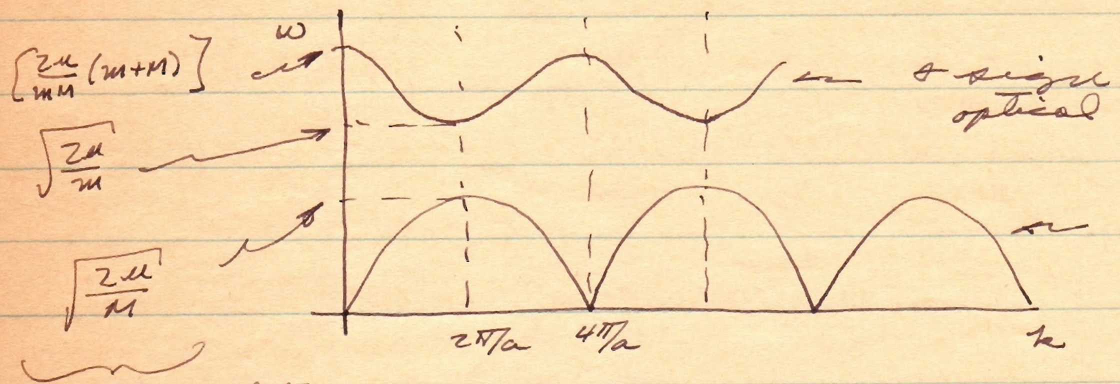
$$= u(x) + \lambda \int_0^\pi K(x,t)\, u(t)\,dt \equiv 0$$

Also: $\frac{A}{B} = \frac{2M\cos(2\pi l/2N)}{\left[M - m \pm \sqrt{\qquad\qquad}\right]}$.

periodicity on ω goes from $0 < l < N$ (eigenval.)

eigenfunctions: $0 < l < 2N$

This means we have twice as many solutions as needed. Plot the dispersion curve.



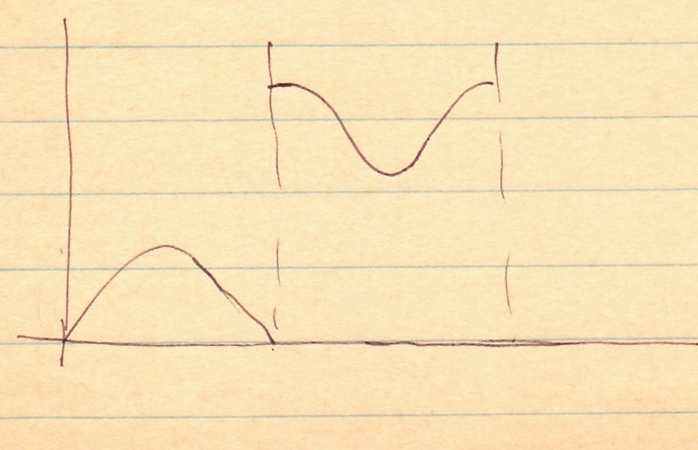
these relations are important because of bandwidth considerations

recall $v = \frac{\partial\omega}{\partial k}$

and as $k \to 0$, $\omega \sim ck$ (acoustic.)

What about multi-valued function above?

One way:



Assume one branch does not exist where other does.

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Course Outline:

Phonons (two weeks)

Electrons (4 weeks)

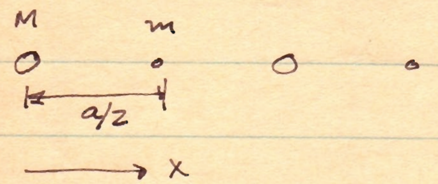
Electron-Phonon Interaction (4 weeks)

Transport Properties (remainder)

Look at Wentzel's book on Q. Th. of Fields

Phonons:

One dimensional - two atoms per cell.



Assume nearest neighbor interaction with ~~constant~~ harmonic field.

eq. of motion:

(m) $$m \frac{d^2}{dt^2} x_{2n+1} = -\mu \left[(x_{2n+1} - x_{2n}) - (x_{2n+2} - x_{2n+1}) \right]$$

(M) $$M \frac{d^2}{dt^2} x_{2n} = -\mu \left[(x_{2n} - x_{2n-1}) - (x_{2n+1} - x_{2n}) \right]$$

Impose BVK condition: 2N masses in all, N of m, and N of M. Thus:

$$x_l = x_{l+2N}$$

not necessary, use pragmatically. We assume solution:

$$x_{2n+1} = A e^{i\left[2\pi l \frac{2n+1}{2N} - \omega t\right]}$$

$$X_{2N} = B e^{i\left[2\pi l \frac{2n}{2N} - \omega t\right]}$$

l is integer, will become reciprocal lattice vector. Usually write wave ~~vectors~~ as $e^{i(k\cdot r - \omega t)}$

Here: $r = \frac{Na}{2}$, $k = \frac{4\pi l}{Na}$

l is arbitrary. Resubstituting determines A and B. Get:

$$-m\omega^2 A e^{i\left[2\pi l \frac{2n+1}{2N} - \omega t\right]} = -\mu\left[A e^{i\left[2\pi l \frac{2n+1}{2N} - \omega t\right]} - B e^{i\left[2\pi l \frac{2n}{2N} - \omega t\right]}\right.$$

$$- \ \ldots \ .$$

Find for secular equation:

$$-m\omega^2 A + 2\mu A = 2\mu B \cos 2\pi l/2N$$

or

$$(m\omega^2 - 2\mu) A + 2\mu \cos(2\pi l/2N) B = 0$$

and: $$2\mu \cos(2\pi l/2N) A + (M\omega^2 - 2\mu) B = 0$$

$$\begin{vmatrix} m\omega^2 - 2\mu & 2\mu\cos(2\pi l/2N) \\ 2\mu\cos(2\pi l/2N) & M\omega^2 - 2\mu \end{vmatrix} = 0$$

Get: $$\omega^2 = \frac{\mu}{mM}\left[M + m \pm \sqrt{M^2 + m^2 + 2Mm\cos(2\pi l/N)}\right]$$

note that the eigenvectors are periodic in $2N$ while ω is periodic in N.