APPLIED
PHYSICS
231

ELECTRON
PHYSICS

Outline of A. P. 231 - Electron Physics

No. of Lect.

I. Individual Particle Motion in Electric and Magnetic Fields

1. Non-relativistic motion; fields uniform in space and time — 4

A. Electric field only
B. Magnetic field only
C. Electric and magnetic fields

a) E parallel to H
b) E perpendicular to H; trochoidal trajectories
c) E makes an angle θ relative to H

D. Applications of motion in uniform fields; cathode ray oscilloscope, 180° focussing β ray spectrometer, velocity selector, mass spectrograph

2. Relativistic motion — 2

A. Review of special theory of relativity
B. Covariant form of Newton's equation for electric and magnetic fields
C. The momentum four vector and the relativistic kinetic energy
D. Applications to fission, pair production, cloud chamber momentum measurements
E. Longitudinal and transverse mass

3. Non-relativistic motion; inhomogeneous fields

A. Inhomogeneous electric fields: electron optics — 2

a) Earnshaw's theorem
b) Theorem on identity of trajectories for particles with different e/m values
c) "Snell's Law" for electron optics
d) The lens equation for axially symmetric fields
e) Focal length of a short, symmetrical electrostatic lens

B. Inhomogeneous magnetic fields — 3

a) Field gradient perpendicular to field direction
b) Motion in a curving field
c) Field gradient in the same direction as the field - the magnetic mirror - Fermi cosmic ray acceleration mechanism

4. Applications to the magnetron and klystron — 3

A. The cylindrical 8 cavity magnetron

a) D. C. magnetron - cutoff condition
b) Cavity resonance condition

C. Potential at nucleus due to electrons
D. Lamb diamagnetic shielding of the nucleus
E. Ionization energy of neutral atom

3. Free ions

A. Boundary conditions for calculation of electronic charge distribution

4. The electrostatic field of an impurity atom in a metal "Mott shielding"

V. The Physics of Fully Ionized Gases 5

1. The Debye shielding length
2. Plasma oscillations
3. Equations of motion for charge density, and current density of a plasma in the presence of electric and magnetic fields, pressure gradients and collisions
4. Electromagnetic waves in a plasma

A. Transverse E wave

a) No resistivity - cutoff frequency
b) Finite resistivity

5. Review of principles of controlled thermonuclear fusion

A. Requirements for controlled thermonuclear device
B. The Pinch effect
C. Instability of the pinch

G. B. Benedek

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LECTURE I   2-7-61

References:

Millman	"Electronics"
Myers	"Electron Optics"
Cosslett	"
Spitzer	"Theory of Ionized Gases"
Simon	"Intro. to Thermonuclear Research"
A. Bishop	"Project Sherwood"
Harmon	"Electronic Motion"

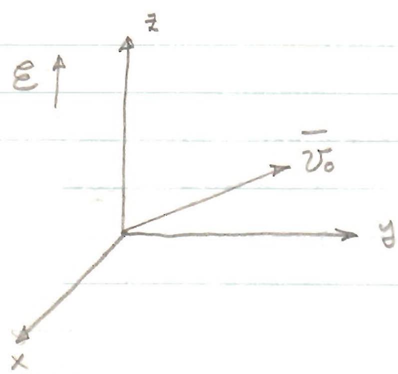
Motion of Charged Particles in Applied Fields:

The equations of motion are:

$$\frac{d(m\vec{v})}{dt} = e\vec{\mathcal{E}} + \frac{e\,\bar{v}\times\mathcal{H}}{c}$$

Units are: esu for $e\mathcal{E}$
emu for H (gauss)
cgs for mechanical quantities

1. Electric field homogeneous in space and constant in time:



v_{0y}, v_{0z} at $t = 0$

$$x = 0$$
$$y = v_{0y}\,t$$
$$z = v_{0z}\,t + \frac{1}{2}\frac{e\,\mathcal{E}_z}{m}t^2$$

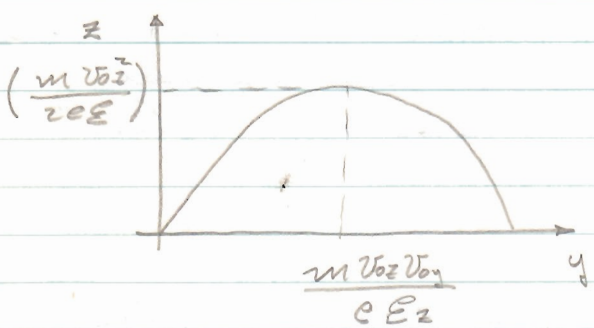
$$m\ddot{x} = 0$$
$$m\ddot{y} = 0$$
$$m\ddot{z} = e\,\mathcal{E}$$

Removing the time:

$$z = v_{oz}\left(\frac{y}{v_{oy}}\right) + \frac{1}{2}\,\frac{e\mathcal{E}_z}{m}\left(\frac{y}{v_{oy}}\right)^2$$

Completing the square:

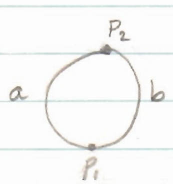
$$\left(z + \frac{m v_{oz}^2}{2e\mathcal{E}_z}\right) = \frac{e\mathcal{E}_z}{2m v_{oy}^2}\left[y + \frac{m}{e\mathcal{E}_z}\,v_{oz}v_{oy}\right]^2$$



2. $\mathcal{E}$ constant in time but inhomogeneous in space $\mathcal{E} = \mathcal{E}(r)$.

now $\nabla \times \mathcal{E} = \frac{1}{c}\frac{\partial B}{\partial t} = 0$

thus: $\int \nabla \times E \cdot dA = \oint E \cdot dr = 0$



$$\therefore \int_{P_1,\,a}^{P_2} E \cdot dr + \int_{P_2,\,b}^{P_1} E \cdot dr = 0$$

Thus the integral is independent of the path and dependent only on scalar work quantities at P_1 and P_2, viz,

$$\int_{P_1}^{P_2} \mathcal{E} \cdot dr = V_1 - V_2$$

or is the work done in transporting a unit charge from P_1 to P_2.

Then: $\mathcal{E}\cdot(r_1 - r_2) = V_1 - V_2$

or $$\boxed{\mathcal{E} = \frac{V_1 - V_2}{r_1 - r_2} = -\nabla V}$$

We desire a first integral of motion:

$$m\frac{dv}{dt} = e\,\mathcal{E}(r), \quad m\left(\frac{dv}{dt}\cdot dr\right) = e\mathcal{E}\cdot dr$$

or $$m\int v\,dv = e\int_{P_1}^{P_2}\mathcal{E}\cdot dr = e(V_1 - V_2)$$

or $\frac{1}{2}m(v_2^2 - v_1^2) = e(V_1 - V_2)$ which is the first integral of the motion or just the conservation of motion. All equations are such that the electronic charge must be used as negative.

3. Magnetic Field Only

Homogeneous: $m\left(\frac{dv}{dt}\right) = \frac{e\bar{v}\times\bar{H}}{c}$

If H is independent of time, it can do no work. The reason is:

$P = F\cdot v$ for the power.

or $$\underbrace{\frac{e\,v\times H}{c}\cdot v}_{=0} = m\left(v\cdot\frac{dv}{dt}\right)$$

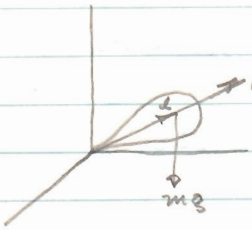
so that v is $\perp$ to $\frac{dv}{dt}$.

Another Way: $\int_{t_1}^{t_2} v\cdot dv = 0$ or $v_1^2 - v_2^2 = 0$

or the square of the velocity does not change.

mechanical analogy (Top is gravitational Field spinning at high velocity)

$$\frac{dL}{dt} = \Omega \times L$$



$$\frac{dL}{dt} = \tau = mg \times l$$

$$L = I\omega \ \bar{1}_\omega$$

$$l = |l| \ 1_\omega$$

$$\frac{d 1_\omega}{dt} = \underbrace{\frac{m\bar{g} l}{I\omega}}_{\Omega} \times \bar{1}_\omega$$

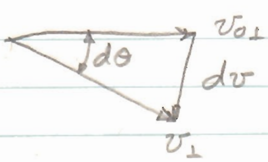
This applies only in case of high ω where the direction of the torque is along the body axis.

Magnetic Analogy: $\frac{d\bar{M}}{dt} = \gamma \bar{M} \times \bar{H}$

For the perpendicular velocity:

$$\frac{dv_\perp}{dt} = \frac{eH}{mc} v_\perp \qquad \text{or} \qquad \frac{dv_\perp}{v_\perp} = \frac{eH}{mc} dt$$

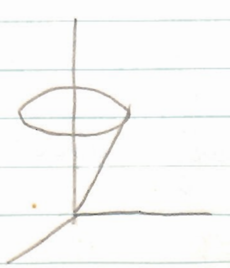
We assume an initial velocity $v_{0\perp}$. Then:



$$d\theta = \frac{dv_\perp}{v_\perp} = \frac{eH}{mc} dt$$

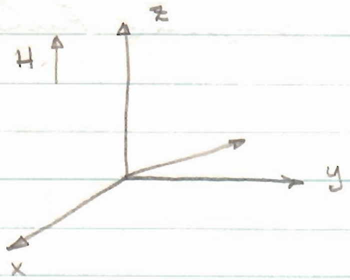
$$\text{or} \quad \frac{d\theta}{dt} = \frac{eH}{mc} = \omega$$

Thus, the particle precesses as a top.



$$\boxed{R = \frac{v_{0\perp}}{\omega} = \frac{v_{0\perp} mc}{eH}}$$

Formal solution:



Given v_{oy}, v_{oz} ; $v_{ox} = 0$

$$\frac{dv_x}{dt} = \frac{eH}{mc} v_y$$

$$\frac{dv_y}{dt} = -\frac{eH}{mc} v_x$$

$$\frac{dv_z}{dt} = 0 \quad , \quad z = v_{oz} t$$

now: $$\frac{d(v_x + i v_y)}{dt} = -i\omega (v_x + i v_y)$$

$$(v_x + i v_y) = (v_x + i v_y)_{t=0}\, e^{-i\omega t}$$

Put in initial conditions:

$$(v_x + i v_y)(0) = v_{oy}\, e^{-i\omega t}$$

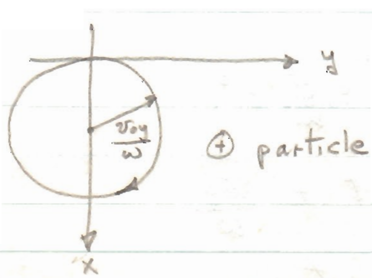
and get: $$\frac{dx}{dt} = v_{oy} \sin \omega t$$

$$\frac{dy}{dt} = v_{oy} \cos \omega t$$

$$\left.\begin{aligned} x &= -\frac{v_{oy}}{\omega} \cos \omega t + \frac{v_{oy}}{\omega} \\ y &= \left(\frac{v_{oy}}{\omega}\right) \sin \omega t \end{aligned}\right\} \quad x = 0 \text{ when } t = 0$$

Eliminating Time:

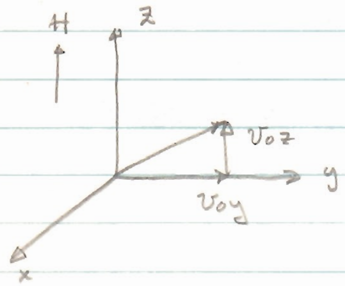
$$\left(x - \frac{v_{oy}}{\omega}\right)^2 + y^2 = \left(\frac{v_{oy}}{\omega}\right)^2 \quad , \quad R = \frac{v_{oy}}{\omega}$$

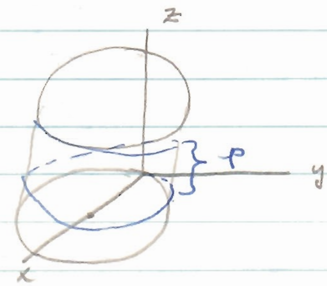


LECTURE II   2-9-61

Charged Particle in Homogeneous Magnetic Field:
Result:

$$\left(x - \frac{v_{oy}}{\omega}\right)^2 + y^2 = \left(\frac{v_{oy}}{\omega}\right)^2$$



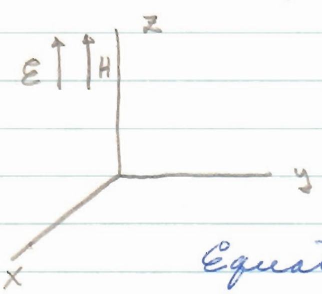


Motion is helical. The pitch is:

$$p = v_{oz}\left(\frac{2\pi}{\omega}\right) = \left(\frac{2\pi\, v_{oz}\, mc}{eH}\right)$$

Electric and Magnetic Fields present Together:

a) $\mathcal{E}$ and H parallel



Given: v_{oy}, v_{oz}
$v_{ox} = 0$
$x = y = z = 0$ initially

Equations of motion:

$\bar{\mathcal{E}} = \bar{I}_z \mathcal{E}$
$\bar{H} = \bar{I}_z H$

$$m\frac{dv_z}{dt} = e\mathcal{E}$$

$$m\frac{dv_x}{dt} = e v_y \frac{H}{c}$$

$$m\frac{dv_y}{dt} = -e v_x \frac{H}{c}$$

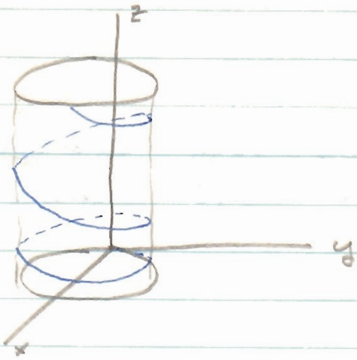
Solutions are:

$$z = v_{oz} t + \frac{1}{2}\frac{e\mathcal{E}}{m} t^2$$

$$x = -\frac{v_{oy}}{\omega}\cos\omega t + \frac{v_{oy}}{\omega}$$

$$y = \frac{v_{oy}}{\omega}\sin\omega t$$

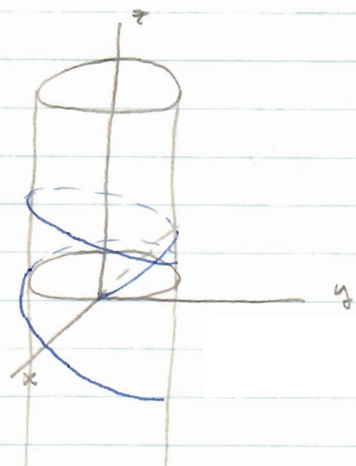
The motion is seen to be:



The pitch now increases with z because of the $\mathcal{E}$ field. We define the pitch as in terms of the number of cycles:

$$p = v_{0z}\left(\frac{2\pi}{\omega}\right)n + \frac{1}{2}\frac{e\mathcal{E}}{m}\left(\frac{2\pi}{\omega}\right)^2 n^2$$

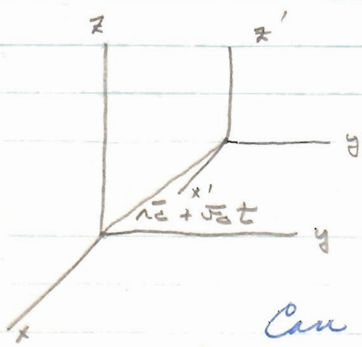
For a negatively charged particle:



If initial velocity is $+z$, there will be turning point and particle accelerates downward

b) $\mathcal{E}$ and H $\perp$ to each other.

$$m\frac{d\bar{v}}{dt} = e\bar{\mathcal{E}} + \frac{e\bar{v}\times\bar{H}}{c}$$



$$\bar{r} = \bar{r}_c + \bar{v}_c t + \bar{r}'$$

$$\bar{v} = \bar{v}_c + \bar{v}'$$

$$m\left(\frac{d\bar{v}'}{dt}\right) = e\mathcal{E} + \frac{e(\bar{v}_c + \bar{v}')\times H}{c}$$

Can we have $\boxed{\frac{v_c \times H}{c} = -\mathcal{E}}$

thus eliminating $\mathcal{E}$? Only when $\mathcal{E}$ and H are $\perp$.

The vector solution to this equation is:

$$\bar{v}_c = \frac{c\,(\bar{\mathcal{E}} \times \bar{H})}{H^2}$$

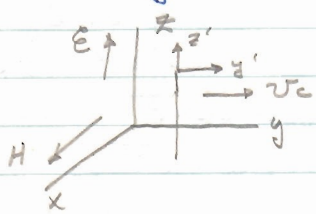
Thus choosing a proper moving coordinate system with velocity v_c, we can eliminate $\mathcal{E}$.

$$m\left(\frac{dv'}{dt}\right) = \frac{e\,v' \times H}{c}$$

Initial Conditions: A. v_0 along y direction, $0 \leq v_0$

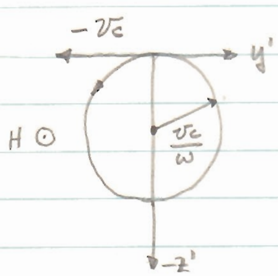
Consider first $v_0 = 0$

Stationary system:



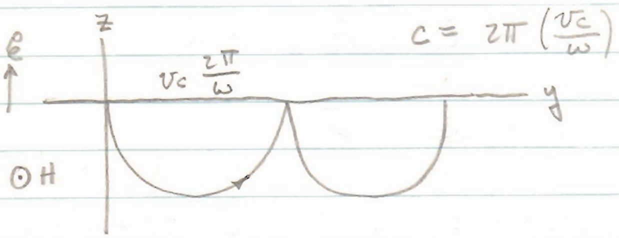
$$\bar{v}_c = 1_y\left(\frac{c\,\mathcal{E}}{H}\right)$$

Moving system:



negative charge:

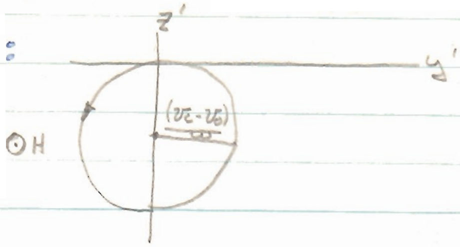
Stationary system:



Path is that of point on rim of rolling wheel.

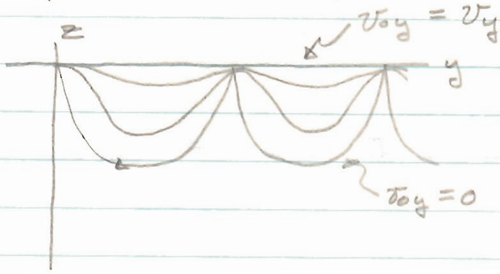
B. $0 < v_{oy} < v_c$

MCS:



$(v_c - v_{oy})$

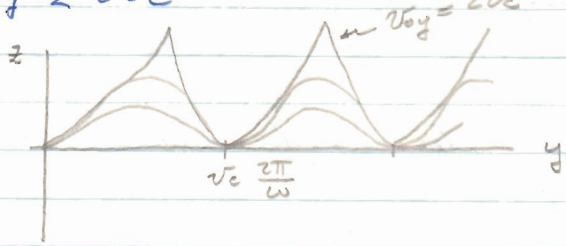
SCS:

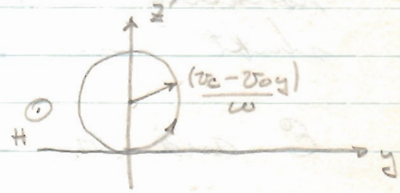


We see that if proper initial velocity is chosen, particle will go thru undeflected. Principle of velocity selector.

C. $v_c < v_{oy} < 2v_c$

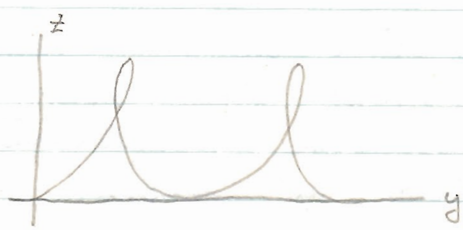
SCS:





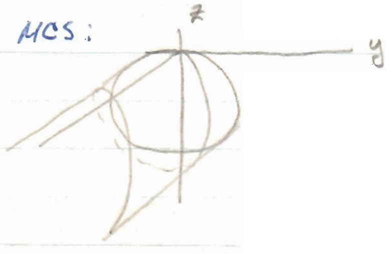
D. $2v_c < v_{oy} <$

SCS:

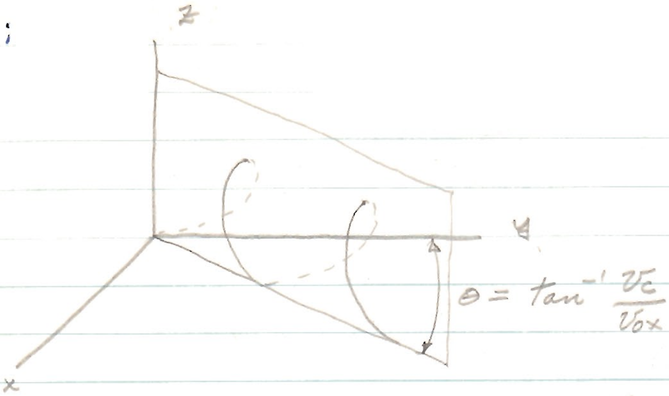


d) velocity component along x-direction.

MCS:

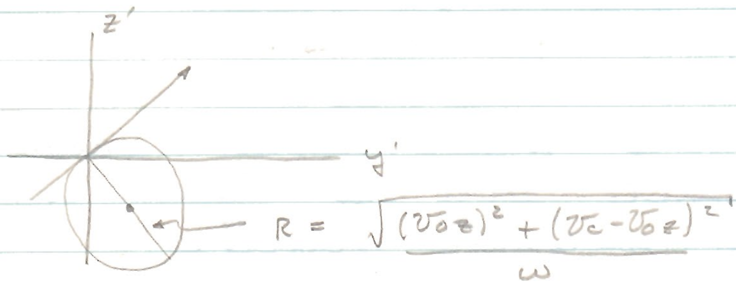


SCS:



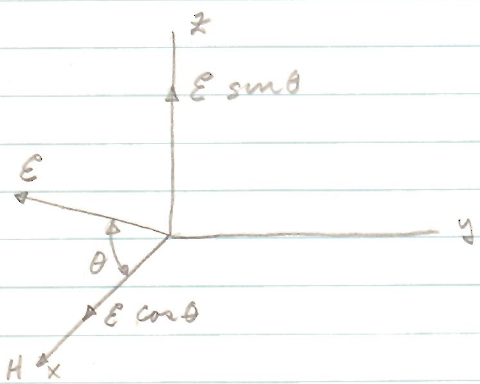
e) Initial velocity component in z direction:

MCS:



Nothing essentially new occurs expect a shift in maximum of circle in SCS:

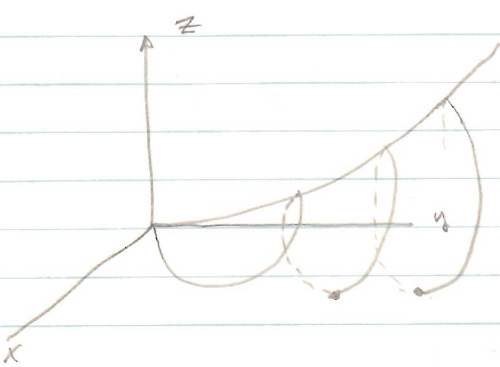
$\mathcal{E}$ and H not $\perp$ to each other



$$v_d = \frac{1_y(\mathcal{E}\sin\theta)c}{H}$$

Motion in MCS same as before except that now we have uniform acceleration along arc of helix.

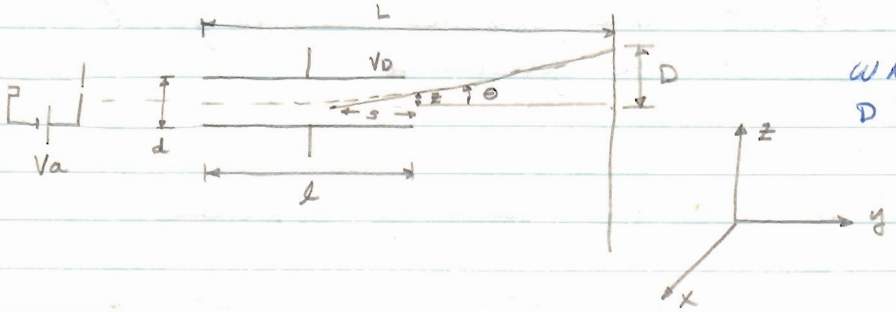
Motion:



LECTURE III 2-11-61

Applications of the Motion of Charged Particles in Uniform Electric and Magnetic Fields.

1. Motion in Uniform $\mathcal{E}$ field: Cathode Ray Oscilloscope.



what is the deflection D?

The motion between the plates is given by:

$$z = \frac{e\,\mathcal{E}_z}{2m\,v_{oy}^2}, \quad \mathcal{E}_z = \frac{V_D}{d}, \quad v_{oy}^2 = \frac{2eV_A}{m}$$

now: $\frac{z}{s} = \tan\theta$, $s = \frac{z(l)}{\tan\theta} = \frac{e\,\mathcal{E}_z\,l^2}{2m\,v_{oy}^2} \cdot \frac{1}{\frac{e\,\mathcal{E}_z\,l}{m\,v_{oy}^2}}$

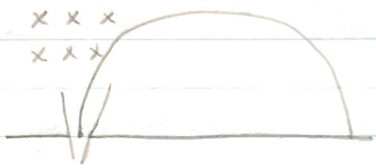
$= \frac{l}{2}$ so that ray has virtual source at center of plates.

now: $D = (L - \frac{l}{2})\tan\theta = (L - \frac{l}{2})\frac{e V_D l}{md\left(\frac{2eV_A}{m}\right)}$

$$= \left(L - \frac{l}{2}\right)\frac{l}{2d}\left(\frac{V_D}{V_A}\right)$$

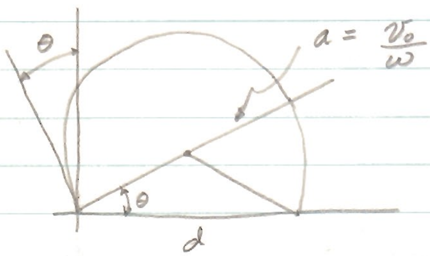
note that the higher V_A, the less the deflection. Usually corrected by post-deflection acceleration.

2. Motion in Uniform Magnetic Field: Danysz Focussing 180° Beta Ray Spectrometer.



For well-defined beam $d = \frac{2v}{\omega}$ so we have velocity selector.

However, in practise we have divergent pencil which must be focused.



By law of cosines:

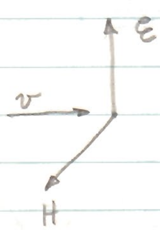
$$d^2 = 2a^2 - 2a^2 \cos(180 - 2\theta)$$

$$= 2a^2 (1 + \cos 2\theta)$$

$$\approx 2a^2 \left(1 + 1 - \frac{4\theta^2}{2}\right) = 4a^2 (1 - \theta^2)$$

Thus $d \approx 2a(1 - \frac{1}{2}\theta^2)$ and for small θ, the diameter is only affected in the second order. Thus the spectrometer is self-focusing. One can also see this by tilting the circle through small angles.

3. Crossed $\mathcal{E}$ and H Fields: Thompson e/m Measurement.



For no deflection, $v = \frac{c\mathcal{E}}{H}$

from $(F\uparrow)_{el} = e\mathcal{E}$, $(F\downarrow)_m = \frac{evH}{c}$

and $F\uparrow = F\downarrow$

Procedure is to get to this critical point and then turn off electric field.

Then radius of curvature is: $R = \frac{v_c}{\omega}$

or $R = \dfrac{v_c}{\left(\frac{eH_c}{mc}\right)} = \dfrac{mc^2 \mathcal{E}_c}{eH_c^2}$

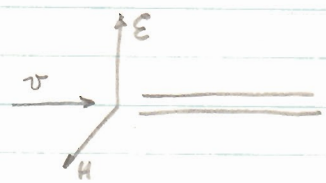
or $\dfrac{e}{m} = \dfrac{c^2 \mathcal{E}_c}{R H_c^2}$

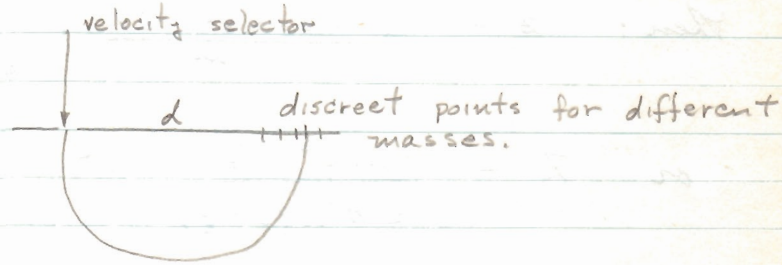
Thompson found in 1897, $\frac{e}{m} = 7.7 \times 10^6$ emu/gm

Present data: $1.758 \cdot 10^7$ emu/gm, off by 2, not bad.

4. Bainbridge Mass Spectrometer

First use velocity selector to get particles of different mass but constant velocity.

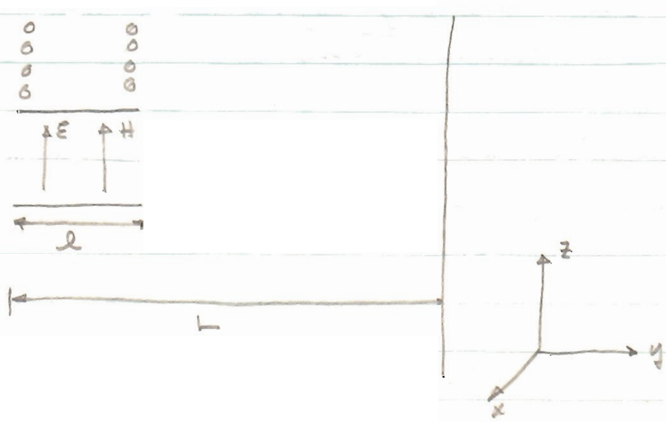




now: $v = \dfrac{c\,\mathcal{E}_s}{H_s}$

$d = \dfrac{2v}{\omega} = \left(\dfrac{2c\,\mathcal{E}_c}{H_s\, e\, H_d}\right) m$ or $\boxed{d = \left(\dfrac{2c^2 \mathcal{E}}{e\, H_s H_d}\right) m}$

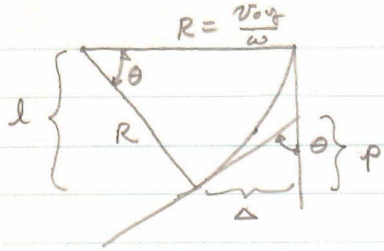
5. Parallel EM Fields: Thompson's Mass Spectrometer.



now:

$z = \dfrac{e\,\mathcal{E}_z l}{m v_{0y}^2}\left(L - \dfrac{l}{2}\right)$

recalling that now particles enter with different velocities.



$$\frac{\Delta}{p} = \tan\theta , \quad p = \frac{\Delta}{\tan\theta}$$

$$p = \frac{(R - R\cos\theta)}{\tan\theta} \approx \frac{R^2(1-\cos\theta)}{l}$$

for θ small.

$$p \approx \frac{R^2}{l}\left(1 - 1 + \frac{\theta^2}{2}\right) = \frac{R^2\theta^2}{2l} = \frac{R^2}{2l}\left(\frac{l}{R}\right)^2 \approx \frac{l}{2}$$

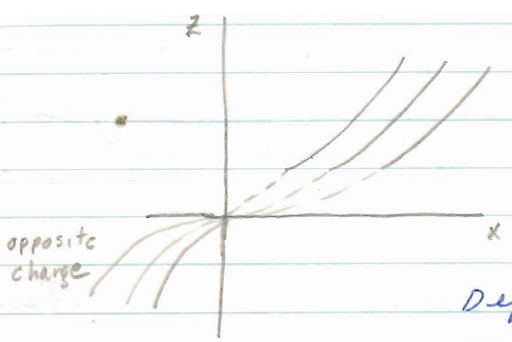
so we have the center as a virtual source for θ small.

Now: $x = (L - \frac{l}{2})\tan\theta = (L - \frac{l}{2})\frac{l}{R} = (L - \frac{l}{2})\frac{leH}{v_{oy}mc}$

or $\left(\frac{1}{v_{oy}}\right)^2 = \left[\frac{x\,mc}{eH(L-\frac{l}{2})l}\right]^2$

Then: $z = \frac{e\,\mathcal{E}_z\,l(L-\frac{l}{2})}{m} \cdot \frac{m^2c^2x^2}{eH(L-\frac{l}{2})^2l^2}$

or $Z = \frac{mc^2\mathcal{E}_z}{eH^2l(L-\frac{l}{2})}x^2$



Thus we have a different curve from each mass whose curvature gives $\frac{e}{m}$

Deflection in $\mathcal{E}$ field depends on velocity squared, in H field on momentum.

LECTURE IV 2-14-61

Relativistic Motion of Charged Particles in Electric and Magnetic Fields.

Special Relativity:

It turns out that motion of electrons thru potential drops of 3-4 kilovolts.

Basic Assumptions:

1) Laws of physics are the same in all coordinate systems that move at constant velocity relative to each other (relativistic invariance).
2) Velocity of light is the same for all coordinate systems moving with uniform velocity relative to each other.

Lorentz transformation arises from (2).

Let ① and ② represent two c.s. moving at constant velocity relative to each other. Then:

In ①: $(dx_1^1)^2 + (dx_1^2)^2 + (dx_1^3)^2 = c^2(dt_1)^2$

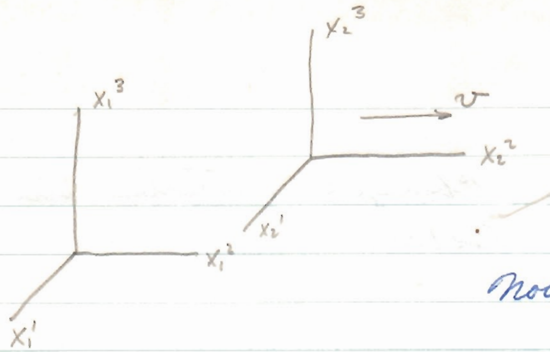
In ②: $(dx_2^1)^2 + (dx_2^2)^2 + (dx_2^3)^2 = c^2(dt_2)^2$

This can be written as a 4 vector:

$$\sum_{\mu=1}^{4} (dx_1^\mu)^2 = \sum_{\gamma} (dx_2^\gamma)^2 \quad \text{(length preserving)}$$

The transformation connecting ① and ② must preserve arc length in the 4 space. That is:

$$\begin{pmatrix} dx_1^1 \\ dx_1^2 \\ dx_1^3 \\ ic\,dt_1 \end{pmatrix} = L \begin{pmatrix} dx_2^1 \\ dx_2^2 \\ dx_2^3 \\ ic\,dt_2 \end{pmatrix}$$



now: $x_1^1 = x_2^1$

$x_1^3 = x_2^3$

and $x_2^2 = \dfrac{x_1^2 - vt}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$, $\left(\frac{v}{c}\right)^2 \equiv \beta^2$

$$t_2 = \frac{t_1 - \frac{v x_1^2}{c^2}}{\sqrt{1-\beta^2}}$$

Now all physical laws must remain invariant under the Lorentz transformation, just as Newton's Laws are invariant under the Galilian. Thus we must formulate the physical laws as 4 dimensional quantities:

Examples of 4 vectors and scalars:

① Scalar (Proper time scalar)

$$(d\tau)^2 = -\frac{1}{c^2}\sum_{\mu}(dx^\mu)^2$$

$$= -\frac{1}{c^2}\left((dx^1)^2 + (dx^2)^2 + (dx^3)^2 - c^2dt^2\right)$$

If all dx's are zero (proper choice of c.s.) then $d\tau^2 = dt^2$ (c.s moving with particle)

Now differentiate by t:

$$\left(\frac{d\tau}{dt}\right)^2 = -\frac{1}{c^2}\underbrace{\left[\left(\frac{dx^1}{dt}\right)^2 + \left(\frac{dx^2}{dt}\right)^2 + \left(\frac{dx^3}{dt}\right)^2\right]}_{v} + 1$$

$$= 1 - \left(\frac{v}{c}\right)^2$$

$$\therefore \quad \frac{d\tau}{dt} = \sqrt{1-\beta^2}$$

② 4 vector (World Velocity)

$$u_\nu = \left.\frac{d\bar{r}}{d\tau}\right|_{\text{for space comp.}} ; \quad ic\left(\frac{dt}{d\tau}\right)_{\text{time}}$$

$$u_\nu = \left.\frac{v_i}{\sqrt{1-\beta^2}}\right|_{i=1,2,3} \qquad \left.\frac{ic}{\sqrt{1-\beta^2}}\right|_{\nu=4}$$

$$\boxed{u_\nu u_\nu = \frac{v^2}{1-\beta^2} - \frac{c^2}{1-\beta^2} = \frac{v^2-c^2}{\left(\frac{1}{c^2}\right)(c^2-v^2)} = -c^2}$$

Generalize Newton's Laws and Lorentz Force

$\frac{d}{dt}(m v_i) = F_i$ (classical)

$\frac{d}{d\tau}(m u_\mu) = F_\mu$

Now at low velocities $F_i = e E_i + \frac{e(v \times H)_i}{c}$

We deal with the vector and scalar potentials:

$$\bar{E} = -\nabla\varphi - \frac{1}{c}\frac{\partial \bar{A}}{\partial t}, \quad H = \nabla \times \bar{A}$$

These are direct consequences of Maxwell's equations.

$$\nabla \cdot \bar{H} = 0 \quad \text{and} \quad \nabla \times E = -\frac{1}{c}\frac{\partial H}{\partial t}$$

$$\nabla \times E = -\frac{1}{c}\frac{\partial}{\partial t}(\nabla \times A), \quad \nabla \times \underbrace{\left(E + \frac{1}{c}\frac{\partial A}{\partial t}\right)}_{\nabla\varphi} = 0$$

$$\therefore E = -\nabla\varphi - \frac{1}{c}\frac{\partial A}{\partial t}$$

thus we write:

$$\bar{F} = e\left\{-\nabla\varphi - \frac{1}{c}\frac{\partial A}{\partial t} + \frac{v \times (\nabla \times A)}{c}\right\}$$

or $$F_x = e\left\{-\frac{\partial\varphi}{\partial x} - \frac{1}{c}\frac{\partial A_x}{\partial t} + \frac{1}{c}\left(\bar{v}\cdot\frac{\partial}{\partial x}\bar{A} - (\bar{v}\cdot\nabla)A_x\right)\right\}$$

now $\frac{dA_x}{dt} = \left(\frac{\partial A_x}{\partial t}\right) + \bar{v} \cdot \nabla A_x$

which is the proper form of a total time derivative. Plugging in:

$$F_x = e\left\{ -\frac{\partial \varphi}{\partial x} + \frac{1}{c}\,\bar{v} \cdot \frac{\partial \bar{A}}{\partial x} - \frac{1}{c}\left(\frac{dA_x}{dt}\right) \right\}$$

Introduce the notation:

$A_\mu = \; A_x \; A_y \; A_z \; i\varphi$

$V_\mu = \; v_x \; v_y \; v_z \; ic$

then: $$\boxed{F_x = e\left\{ \frac{1}{c} \sum_\mu V_\mu \frac{\partial A_\mu}{\partial x} - \frac{1}{c}\left(\frac{dA_x}{dt}\right) \right\}}$$

which is still classical. To go to relativistic, make $t \to \tau$, $v_\nu \to u_\nu$, then:

$$\boxed{F_\nu = \frac{e}{c}\left\{ \sum_{\mu=1}^{4} u_\mu \frac{\partial A_\mu}{\partial x^\nu} - \frac{dA_\nu}{d\tau} \right\} = \frac{d}{d\tau}(m u_\nu)}$$

we have time-like component that we did not have classically.

Examine the space-like components:

$$\frac{1}{\sqrt{1-\beta^2}} \frac{d}{dt}\left(\frac{m v_i}{\sqrt{1-\beta^2}}\right) = \frac{e}{c}\frac{1}{\sqrt{1-\beta^2}}\left\{ \sum_\mu V_\mu \frac{\partial A_\mu}{\partial x_i} - \frac{d}{dt} A_i \right\}$$

$$= \frac{e}{\sqrt{1-\beta^2}}\left(\mathcal{E} + \frac{v \times H}{c}\right)$$

or $$\boxed{\frac{d}{dt}\left(\frac{m v_i}{\sqrt{1-\beta^2}}\right) = e\left(\mathcal{E}_i + \frac{(v \times H)_i}{c}\right)}$$

which is the relativistically correct equation for the space like components.

Examine the time-like component:

$$\mathcal{F}_4 = \frac{e}{c}\left\{\sum_\nu u_\nu \frac{\partial A_\nu}{\partial x_4} - \frac{dA_4}{d\tau}\right\}$$

$$= \frac{1}{\sqrt{1-\beta^2}}\frac{e}{c}\left\{\frac{\bar{v}}{ic}\cdot\frac{\partial A}{\partial t} + \frac{ic}{ic}\frac{\partial(\varphi i)}{\partial t} - \frac{d(i\varphi)}{dt}\right\}$$

$$= \frac{1}{\sqrt{1-\beta^2}}\frac{e}{c}\left\{\frac{v}{ic}\cdot\frac{\partial A}{\partial t} - i\bar{v}\cdot\nabla\varphi\right\}$$

$$= \frac{i\bar{v}e}{c\sqrt{1-\beta^2}}\cdot\underbrace{\left\{-\nabla\varphi - \frac{1}{c}\frac{\partial A}{\partial t}\right\}}_{\mathcal{E}}$$

or $$\mathcal{F}_4 = \frac{ie\bar{v}\cdot\bar{E}}{c\sqrt{1-\beta^2}} = \frac{i}{c\sqrt{1-\beta^2}}\frac{dT}{dt} = \frac{d}{d\tau}(m u_4)$$

$$= \frac{1}{\sqrt{1-\beta^2}}\frac{d}{dt}\left(\frac{mic}{\sqrt{1-\beta^2}}\right)$$

or $$\frac{d}{dt}\left(\frac{mc^2}{\sqrt{1-\beta^2}}\right) = \frac{dT}{dt}, \quad T = \text{kinetic energy}$$

Integrating, setting the constant of integration zero (arbitrary, but useful):

$$\boxed{T = \frac{mc^2}{\sqrt{1-\beta^2}}}$$

which is the usual equation for the relativistic kinetic energy.

LECTURE V   2-16-61

We have shown:

$$\frac{dT}{dt} = \frac{d}{dt}\left(\frac{mc^2}{\sqrt{1-\beta^2}}\right)$$

$$T = \frac{mc^2}{\sqrt{1-\beta^2}} + \text{constant}$$

since energy is relative, the constant is arbitrary because only changes in energy are important.

Now, the momentum 4 vector is:

$$P_\mu = \left(\frac{m\bar{v}}{\sqrt{1-\beta^2}}\right), \underbrace{\frac{icm}{\sqrt{1-\beta^2}}}_{i\frac{T}{c}}$$

Notice that $T \to mc^2$ as $v \to 0$.
now: $T = mc^2\left[1-\left(\frac{v}{c}\right)^2\right]^{-1/2}$

$$= mc^2\left[1 + \frac{1}{2}\left(\frac{v}{c}\right)^2 + \frac{\frac{1}{2}\frac{3}{2}}{2!}\left(\frac{v}{c}\right)^4 + \cdots\right]$$

$$= mc^2 + \frac{1}{2}mv^2 + \cdots$$

Consider a fission process such that no external work is done on the system, that is, $\Delta T = 0$. Then:

$$\Delta mc^2 \cong \frac{1}{2}m\left(v_2^2 - v_1^2\right)$$

thus the mass goes over into velocity after the process is complete.

Consider pair production:

① γ ray → ② yielding 2 mass particles.

The energy of the γ ray must go into the creation of the mass:

$$h\nu = 2mc^2\left(1 + \frac{1}{2}\left(\frac{v}{c}\right)^2 + \cdots\right)$$

Order of magnitude for electrons:

$$(mc^2)_{electron} = \frac{9.1 \cdot 10^{-28} \cdot 9 \cdot 10^{20}}{1.6 \cdot 10^{-12}} \simeq .51\ MEV$$

$$(mc^2)_{proton} = 936\ MEV$$

Note that γ ray must have energy to create <u>two</u> particles. We need high energy accelerator to produce pair creation of heavy particles.

Expression of Magnitude of momentum 4 Vector:

$$\sum_\mu P_\mu P_\mu = \frac{(mv)^2}{\sqrt{1-\beta^2}} - \frac{m^2c^2}{\sqrt{1-\beta^2}} = -m^2c^2 = p^2 - \frac{\Gamma^2}{c^2}$$

We take $\vec{p}$ to be the space component of relativistic momentum, that is:

$$\vec{p} = \frac{m\vec{v}}{\sqrt{1-\beta^2}}$$

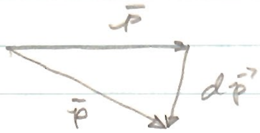
Then: $p^2c^2 = \Gamma^2 - m^2c^4$

or: $\boxed{\Gamma^2 = p^2c^2 + m^2c^4}$

1) Constant Magnetic Field Case: Consider velocity $\perp$ H.

That is:

$$\frac{d}{dt}\left(\frac{mv}{\sqrt{1-\beta^2}}\right) = \frac{e\,v\times H}{c} \longrightarrow \frac{evH}{c} = \frac{d\vec{p}}{dt}$$



$$\frac{dp}{p} = d\theta\ , \quad \frac{dp}{dt} = p\,\frac{d\theta}{dt}$$

$$p\omega = \frac{evH}{c} = \frac{e\rho\omega H}{c}\ , \quad v = \rho\omega$$

ρ radius of curvature

or $\boxed{p = \frac{e}{c} H \rho}$ so that $p \sim \rho$ even in the relativistic case.

We now relate the relativistic kinetic energy to the radius of curvature, viz:

$$T^2 - m^2c^4 = e^2H^2\rho^2$$

We write $T = mc^2 + \Delta E$ where ΔE is the motional change in KE.

Then $\rho = \frac{1}{eH}\left[T^2 - m^2c^4\right]^{1/2} = \frac{\Delta E}{eH}\left[1 + \left(\frac{2mc^2}{\Delta E}\right)\right]^{1/2}$

Properties:

High energy limit, $\rho \sim \frac{\Delta E}{eH}$ and becomes independent of the mass.

Consider an applied field of 10 kilogauss:

ρ (meters)

$\frac{\Delta E}{\text{mev}}$	electron	proton	deuterons
10	.035	.457	6.46
100	.334	1.48	2.06
1000	3.33	5.64	7.25
10000	33.33	36.3	39.0

We see that ρ becomes independent of mass for high energies.

Equations of Motion: Transverse and Longitudinal Mass.

Consider Space Components:

$$\vec{F} = \frac{d}{dt}\left(\frac{m\vec{v}}{\sqrt{1-\beta^2}}\right) = \frac{m}{\sqrt{1-\beta^2}}\frac{d\vec{v}}{dt} + \frac{m\vec{v}}{(1-\beta^2)^{3/2}}\frac{1}{c^2}\vec{v}\cdot\frac{d\vec{v}}{dt}$$

or $$\left[\frac{m}{\sqrt{1-\beta^2}} + \frac{m}{c^2}\frac{\vec{v}\vec{v}}{(\sqrt{1-\beta^2})^3}\right]\cdot\frac{d\vec{v}}{dt} = \vec{F}$$

Now the change in velocity need not be in the direction of the force in view of the fact that the mass is now a tensor m^*.

Case of Acceleration and $\vec{F}$ in same direction:
$\frac{d\vec{v}}{dt} \parallel \vec{F}$.

1) $\vec{v} \parallel \vec{F}$:

$$\left\{\begin{array}{l} \hat{1}_F: \quad \frac{m}{\sqrt{\quad}}\frac{d\vec{v}_\perp}{dt} + \frac{m v^2\, \hat{1}_F\hat{1}_F\cdot\frac{dv_I}{dt}}{(\quad)^{3/2}} = F\,\hat{1}_F \\ \text{which proves that } \left(\frac{dv}{dt}\right)_{\perp \text{ to } F} = 0 \end{array}\right\}$$

Now: $$\left[\frac{(1-(\frac{v}{c})^2)\,m}{(\quad)^{3/2}} + \frac{\frac{mv^2}{c^2}}{(\quad)^{3/2}}\right]\frac{d\vec{v}}{dt} = \vec{F}$$

or: $$\frac{m}{(1-(\frac{v}{c})^2)^{3/2}}\frac{d\vec{v}}{dt} = \vec{F}, \quad \text{then: } m^* = \frac{m}{(1-\beta^2)^{3/2}}$$

(longitudinal mass)

2) $\vec{v} \perp \vec{F}$: dot $\vec{v}$ into original equation.
Then: $\vec{v}\cdot\vec{F} = 0$

or: $$\left[\frac{m}{\sqrt{\quad}} + \frac{mv^2}{(\quad)^{3/2}}\right]\vec{v}\cdot\frac{d\vec{v}}{dt} = 0 \quad \text{or} \quad \vec{v}\cdot\frac{d\vec{v}}{dt} = 0$$

Thus, returning to original equation, putting $\bar{v} \cdot \frac{d\bar{v}}{dt} = 0$:

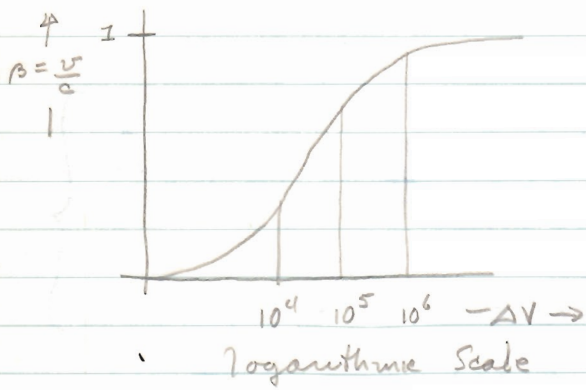
$$\frac{m}{\sqrt{1-\left(\frac{v}{c}\right)^2}} \frac{d\vec{v}}{dt} = \vec{F}_1 \quad \text{so} \quad \vec{F}_1 \parallel \frac{d\vec{v}}{dt}$$

and $m^* = \dfrac{m}{\sqrt{1-\beta^2}}$ (transverse mass)

Velocity after Acceleration Through ΔV:

$T + V \neq$ constant (conservative)

then $e\Delta V = \Delta T = \left(\dfrac{mc^2}{\sqrt{1-\beta^2}} - mc^2 \right)$



start from rest.

LECTURE VI 2-18-61

Cyclotron Frequency:

Recall: $\dfrac{m}{\sqrt{1-\beta^2}} \dfrac{d\vec{v}}{dt} = \dfrac{e\vec{v} \times \vec{H}}{c}$

From analogy with non-relativistic case:

$$\omega = \frac{eHc}{\left(\frac{mc^2}{\sqrt{1-\beta^2}}\right)} = \frac{eHc}{T} = \frac{eHc}{(mc^2 + \Delta E)}$$

or $\omega = \left(\frac{eH}{mc}\right) \frac{1}{1+\left(\frac{\Delta E}{mc^2}\right)} = \frac{\omega_0}{\left(1+\frac{\Delta E}{mc^2}\right)}$

For large ΔE: $\omega = \frac{\omega_0}{\left(\frac{\Delta E}{mc^2}\right)}$

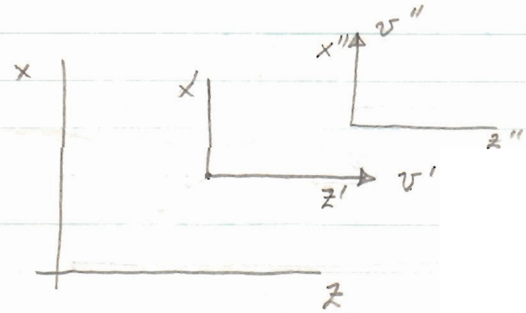
In synchrotron, we synchronize the magnetic field and electric field to counteract the slowing down of ω. Example:

ΔE (mev)	f (MC) electrons	f (MC) protons	$H = 10$ kgauss
10	1635	15.2	
100	143	13.8	
1000	14	7.46	
10000	1.4	1.32	

Thomas Precession: An axis fixed on a body whose center of mass moves with velocity $\vec{v}(t)$ undergoes a frequency of precession ω:

$$\vec{\omega} = \frac{1}{2c^2} \dot{\vec{v}} \times \vec{v}$$

Relativistically, acceleration of a body must involve rotation as two Lorentz transformation cannot reduce to one. Gives rise to spin and modification of Landé g factor.



The difference between x and x' correspond to velocity. x', x'' correspond to acceleration. x' centered on body. x is the observing coordinate system.

We know: $x' = x$

$y' = y$

$z' = (z - v't)\beta(v')$, $\beta(v') = \frac{1}{\sqrt{1-(\frac{v'}{c})^2}}$

and $t' = \left(t - \frac{v'z}{c^2}\right) \beta(v')$

To connect ' and '':

$$x'' = (x' - v''t') \beta(v'')$$

$$y'' = y', \quad z'' = z'$$

$$t'' = \left(t' - \frac{v''x'}{c^2}\right) \beta(v'')$$

Relate to xyz:

$$x'' = \left(x - v''\left\{t - \frac{v'z}{c^2}\right\} \beta(v')\right) \beta(v'')$$

$$y'' = y$$

$$z'' = (z - v't) \beta(v')$$

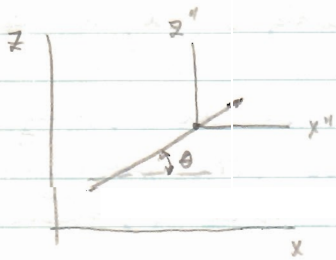
$$t'' = \left[\left(t - \frac{v'z}{c^2}\right) \beta(v') - \frac{v''x}{c^2}\right] \beta(v'')$$

We will now show that the '' coordinate system is skew from the unprimed system or the '' system is rotating with respect to xyz: We examine the origins:

$$x'' = z'' = 0$$

$$x = v''\left(t - \frac{v'z}{c^2}\right)\beta(v') = v''t\left(1 - \frac{v'^2}{c^2}\right)\beta(v')$$
$$= v''t\left(1 - \frac{v'^2}{c^2}\right)^{1/2}$$

$$z = v't$$



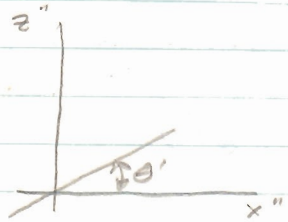
$$\boxed{\tan\theta = \frac{v''}{v'}\sqrt{1 - \frac{v'^2}{c^2}}}$$

Moving to '' as reference:

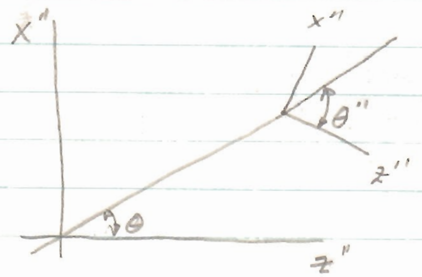
Take $x = z = 0$

$$x'' = -v'' t \, \beta(v') \, \beta(v'')$$

$$z'' = -v' t \, \beta(v')$$



$$\boxed{\begin{aligned} \tan\theta' &= \frac{x''}{z''} = \frac{v''}{v'}\,\beta(v'') \\ &= \frac{v''}{v'}\,\frac{1}{\sqrt{1-\left(\frac{v''}{c}\right)^2}} \end{aligned}}$$



We take $v'' \ll v'$ as velocity $\perp$ to v' is assumed to be small.

to find change in angle:

$$\tan\theta' - \tan\theta \cong \theta' - \theta = \frac{v''}{v'}\left[\frac{1}{\sqrt{1-\left(\frac{v''}{c}\right)^2}} - \sqrt{1-\left(\frac{v'}{c}\right)^2}\right]$$

$$= \frac{v''}{v'}\left[1 + \frac{1}{2}\left(\frac{v''}{c}\right)^2 - \left(1 - \frac{1}{2}\left(\frac{v'}{c}\right)^2\right)\right]$$

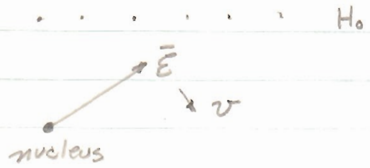
or: $$\Delta\theta = \frac{1}{2c^2}\,\frac{v''}{v'}\left[v''^2 + v'^2\right]$$

now: $$v'' = \left(\frac{dv'}{dt}\right) dt$$

or $$\frac{d\theta}{dt} = \frac{1}{2c^2}\left(\frac{dv'}{dt}\right) v'$$

or $$\vec{\omega} = \frac{1}{2c^2}\,\vec{\dot{v}} \times \vec{v}$$

Spin - Orbit Coupling:



H_0 is a reference field.

Define: $\vec{A} = \hbar \vec{S}$ (angular momentum)

and $\vec{\mu} = \left(\frac{-e}{mc}\right) \hbar \vec{S}$

Recall $H = H_0 + \frac{\mathcal{E} \times v}{c}$ and from dynamics:

$$\left(\frac{d\vec{A}}{dt}\right)_{dyn.} = \mathcal{J} = \vec{\mu} \times \vec{H} = \mu \times \left(H_0 + \frac{\mathcal{E} \times v}{c}\right)$$

Now: $\vec{\mathcal{E}} = -\left(\frac{1}{r} \frac{d\varphi}{dr}\right) \vec{r}$

$$= \vec{\mu} \times \left(H_0 - \left(\frac{1}{r} \frac{\partial \varphi}{\partial r}\right) \frac{\vec{r} \times m\vec{v}}{mc}\right)$$

$$= \vec{\mu} \times \left(H_0 - \frac{1}{r} \frac{\partial \varphi}{\partial r} \frac{\hbar l}{mc}\right)$$

From mechanics: $\vec{\mu} \times \vec{H} = \vec{\mathcal{J}}$ and $U = -\mu \cdot H$

then: $U_{s.o.} = -\mu \cdot \left(-\frac{1}{r} \frac{\partial \varphi}{\partial r} \frac{\hbar l}{mc}\right)$

or $U_{s.o} = -\frac{e\hbar^2}{(mc)^2} \left(\frac{1}{r} \frac{\partial \varphi}{\partial r}\right) \vec{S} \cdot \vec{l}$

which gives only 2 times the right answer. The reason is that we have not included Thomas precession:

$$\left(\frac{d\vec{A}}{dt}\right)_{km} = \vec{\omega} \times \vec{A}$$

$$\left(\frac{d\vec{A}}{dt}\right)_{Tot} = \left(\frac{d\vec{A}}{dt}\right)_{km} + \left(\frac{d\vec{A}}{dt}\right)_{dyn} = \mu \times \left(H_0 + \frac{\mathcal{E} \times v}{c}\right) + \underbrace{\omega \times A}_{\frac{\omega \times \mu}{\alpha}}$$

or $\left(\frac{d\vec{A}}{dt}\right)_{TOT} = \mu \times \left(H_0 + \frac{\mathcal{E} \times v}{c} - \frac{\omega}{\alpha} \right)$

Now: $\omega = \frac{1}{2c^2} \dot{\vec{v}} \times \vec{v} = -\frac{1}{2c^2} \frac{e}{m} \vec{\mathcal{E}} \times \vec{v}$

$(m\dot{v} = -e\mathcal{E})$

$= \mu \times \left(H_0 + \frac{1}{2} \frac{\mathcal{E} \times v}{c} \right)$

Then: $U_{so} = \left(-\frac{1}{2}\right)\left(\frac{1}{r}\frac{\partial \varphi}{\partial r}\right) \frac{e\hbar}{m^2c^2} \vec{s}\cdot\vec{l}$

LECTURE VII   2-21-61

Motion of Particles in Inhomogeneous Fields:

1. Inhomogeneous Electric Fields.
   a) Methods of determining the field
      i.) Electrolytic Tank, map out equi-potential lines with probe.
   b) Properties of Potential:
      The electrostatic potential distribution set up by an arbitrary charge distribution cannot have a maximum or minimum at any point in free space (Earnshaw's Theorem).

Proof: Want to show $\overline{V}^{\text{sphere}} = V_0$

$$\overline{V} = \frac{1}{4\pi r^2} \iint_{\theta\,\varphi} V(r,\theta,\varphi)\, r^2 \sin\theta \, d\theta \, d\varphi$$

$$\frac{\partial \overline{V}}{\partial r} = \frac{1}{4\pi} \iint \left(\frac{\partial V}{\partial r}\right) \sin\theta \, d\theta \, d\varphi$$

$$= \frac{1}{4\pi r^2} \iint \nabla V \cdot d\vec{S} = \frac{1}{4\pi r^2} \int_\tau \underbrace{\nabla \cdot \nabla V}_{0} \, d\tau$$

We now shrink the sphere and find in the limit:

$$\overline{V} = V_0$$

Since this is so, any assumed maximum or minimum is seen not to exist.

Second Proof: If $V =$ constant over a surface S then $V =$ constant inside S from the uniqueness of solutions of Laplace's equation.

Suppose maximum in field, then potential falls off as we leave the equipotential surface, but V must be constant inside the surface, so max. or min. cannot occur. Consequences of Earnshaw's Theorem:

i. Stable atom cannot exist under Coulomb's law.
ii. Charged particles cannot find equilibrium point in free space.
iii. Stable lattice cannot exist under Coulomb's law

Given two particles: $\frac{e_1}{m_1}$ moves in field $V(x, y, z)$

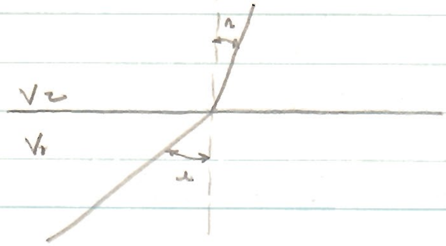
$\frac{e_2}{m_2}$ " " " $K\,V(x, y, z)$

If we start the particles from rest $(\vec{v} = 0)$ at same point, the trajectories are identical: Proof:

$$\left.\begin{aligned} m_1 \frac{d^2 r}{dt^2} &= e_1 \nabla V \\ m_2 \frac{d^2 r}{dt^2} &= e_2 K \nabla V \end{aligned}\right\} \quad \frac{d^2 r}{d\left(\frac{e}{m}\right)_1 t^2} = \nabla V = \frac{d^2 r}{d\left(\frac{e}{m}\right)_2 K t^2}$$

Condition of initial velocity $= 0$ very important.

c. Electron Optics Analogue of Snell's Law:



$$\sin i = \frac{v_{\parallel}}{v_{TOT_1}}$$

$$\sin r = \frac{v_{\parallel}}{v_{TOT_2}}$$

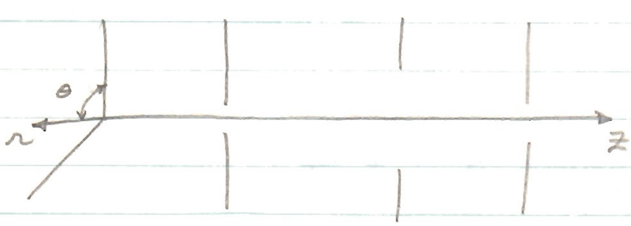
Assuming zero velocity at start:

$$v_{1\,TOT} = \sqrt{\frac{2eV_1}{m}} \quad ; \quad v_{2\,TOT} = \sqrt{\frac{2eV_2}{m}}$$

$$\therefore \quad \frac{\sin i}{\sin r} = \sqrt{\frac{V_2}{V_1}}$$

However, in practice it is diffecult to construct a potential with a simple discontinuity. The usual way to approach the problem is thru inhomogeneous fields analogous to variable index of refraction. Technique very empirical.

Lens Equation for Electrons Moving near the Axis of an Axially symmetric electric field:



Assume a Taylor series expansion of $V(z,r)$:

$$V(z,r) = V_0(z) + c_1(z)r + c_2(z)r^2 + c_3(z)r^3 + \ldots$$

We assume that we know $V_0(z)$ and we will find that this is sufficient to determine $V(z,r)$.

Use Laplace's Equation: $\frac{\partial^2 V}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) = 0$

or:

$$V_0'' + C_1''(z)\,r + C_2''(z)\,r^2 + \cdots$$

$$+ \frac{C_1}{r} + 4\,C_2(z) + 9\,C_3(z)\,r + 16\,C_4(z)\,r^2 + \cdots = 0$$

Equating terms:

$\frac{1}{r}\ \Big\{\ \frac{C_1}{r} = 0\ ,\ C_1 = 0$

$r^0\ \Big\{\ \frac{\partial^2 V_0(z)}{\partial z^2} = -4\,C_2(z)$

$r^1\ \Big\{\ C_1'' = -9\,C_3 = 0\ ,\ C_3 = 0$

$r^2\ \Big\{\ 16\,C_4 = -C_2''(z) = -\frac{1}{16}\cdot -\frac{1}{4}\frac{\partial^4 V}{\partial z^4} = \frac{1}{64}\frac{\partial^4 V}{\partial z^4}$

Therefore:

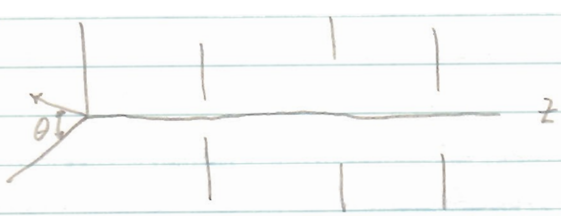
$$V(z,r) = V_0(z) - \frac{1}{4}\left(\frac{\partial^2 V_0}{\partial z^2}\right) r^2 + \frac{1}{64}\left(\frac{\partial^4 V_0}{\partial z^4}\right) r^4 + \cdots$$

The direction of deflection is dependent upon the sign of $\frac{\partial^2 V_0}{\partial z^2}$

LECTURE VIII   2-23-61

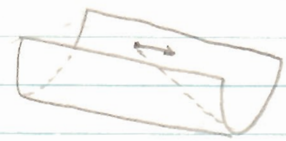
Continuation of Inhomogeneous Fields:

$$V(z,r) = V_0(z) - \frac{1}{4}\left(\frac{\partial^2 V_0}{\partial z^2}\right) r^2 + \frac{1}{64}\left(\frac{\partial^4 V_0}{\partial z^4}\right) r^4 + \cdots$$

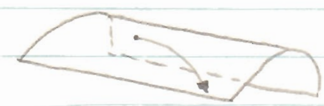


now: $m \dfrac{d^2 r}{dt^2} = -e\,\mathcal{E}_r = +e \dfrac{\partial V}{\partial r} = -\dfrac{e}{2}\left(\dfrac{\partial^2 V_0}{\partial z^2}\right) r$

If $V_0''(z) > 0$:



$V_0''(z) < 0$:



From definition of derivative:

$$\frac{df}{dt} = v_z \frac{\partial f}{\partial z} + v_\theta \frac{\partial f}{\partial \theta} + v_r \frac{\partial f}{\partial r} = v_z \frac{\partial f}{\partial z}$$

since v_θ, v_r are negligible.

$$\therefore \quad \frac{dr}{dt} = v_z \frac{dr}{dz}$$

then: $v_z \dfrac{d}{dz}\left(v_z \dfrac{dr}{dz}\right) = -\dfrac{e}{2m}\left(\dfrac{\partial^2 V_0(z)}{\partial z^2}\right) r$

Assume: $v_z = \sqrt{\dfrac{2e\,V(r,z)}{m}} \approx \sqrt{\dfrac{2e\,V_0(z)}{m}}$

taking initial velocity to be small. We then get:

$$\boxed{\sqrt{V_0(z)}\,\frac{d}{dz}\left(\sqrt{V_0(z)}\,\frac{dr}{dz}\right) \cong -\frac{1}{4}\left(\frac{\partial^2 V_0(z)}{\partial z^2}\right) r}$$

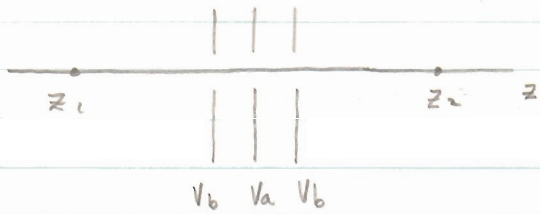
which is essentially the lens equation.

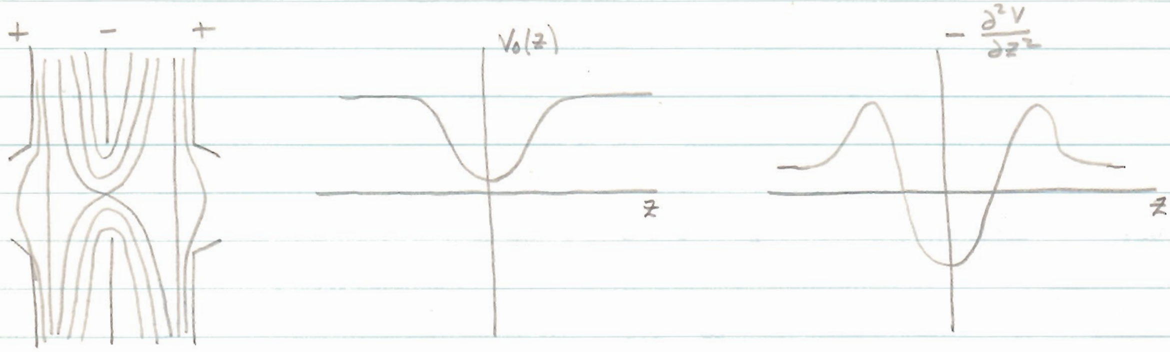
Properties: (1) mass and charge do not enter. shows that just $\mathcal{E}$ field will not determine e/m, need H field too.

(2) Homogeneous in V, that is, $V \to KV$ gives same equation. Thus can use AC.

(3) Homogeneous in r: if $r(z)$ is a solution so is $c\,r(z)$. This means that zero crossings of electron will be at constant points.

e) Example: The Focal Length of a Short Symmetrical Lense.





We proceed to integrate the differential equation by method of successive approximations. First, consider r on RHS constant, then resubstitute and converge on proper answer. Thus, take $r = r_1$, $\left(\frac{dr}{dz}\right) = \left(\frac{dr}{dz}\right)_{z_1}$, $\left(\frac{dr}{dz}\right)_{z_2}$ and get:

$$\sqrt{V_0(z)}\;\frac{dr}{dz}\Big]_{z_1}^{z_2} = -\frac{r_1}{4}\int \frac{V_0''(z)\;dz}{\sqrt{V_0(z)}}$$

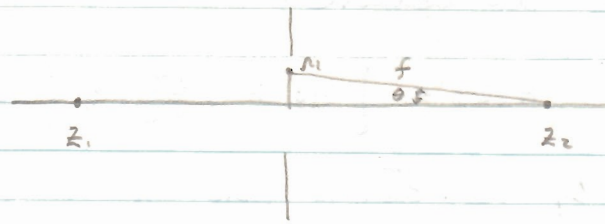
$$\left.\frac{dr}{dz}\right|_{z_2} - \left.\frac{dr}{dz}\right|_{z_1} = \frac{-r_1}{4\sqrt{V_b}}\int_{z_1}^{z_2}\frac{V_0''(z)}{\sqrt{V_0(z)}}\,dz$$

$$= \frac{-r_1}{4\sqrt{V_b}}\int_{-\infty}^{\infty}\frac{d\left(\frac{dV_0(z)}{dz}\right)}{\sqrt{V_0(z)}}$$, since integrand vanishes outside z_1, z_2 anyway.

Integrating by parts:

$$\int_{-\infty}^{\infty} = \underbrace{\frac{1}{\sqrt{V_0}}\;\frac{dV}{dz}\Big|_{-\infty}^{\infty}}_{=\,0} + \frac{1}{2}\int_{-\infty}^{\infty}\frac{\left(\frac{dV_0}{dz}\right)\left(\frac{dV_0}{dz}\right)}{V_0^{3/2}(z)}\,dz$$

Then: $$\left(\frac{dr}{dz}\right)_{z_2} - \left(\frac{dr}{dz}\right)_{z_1} = -\frac{r_1}{8 V_b^{1/2}} \int_{-\infty}^{\infty} \left(\frac{dV_0}{dz}\right)^2 \frac{dz}{V_0^{3/2}(z)}$$



If beam enters parallel to z, $\left(\frac{dr}{dz}\right)_{z_1} = 0$ and the focal length is as shown.

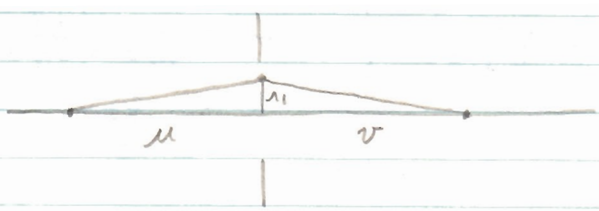
Then: $$\frac{r_1}{f} = -\tan\theta = -\left(\frac{dr}{dz}\right)_{z_2}$$

and we have:

$$\boxed{\frac{1}{f} = \frac{1}{8 V_b^{1/2}} \int_{-\infty}^{\infty} \left(\frac{dV}{dz}\right)^2 \frac{dz}{V_0^{3/2}(z)}}$$

We take r_1 as the average position of the particle in the lens.
We see that the beam will always converge.

Non-parallel Entrant Beam:



$$\frac{r_1}{u} = \left(\frac{dr}{dz}\right)_{z_1}, \quad -\frac{r_1}{v} = \left(\frac{dr}{dz}\right)_{z_2}$$

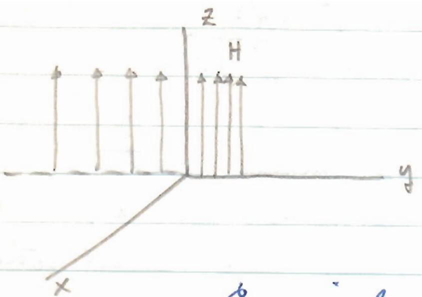
or $$-\frac{r_1}{v} - \frac{r_1}{u} = -\frac{r_1}{f}$$

$$\boxed{\frac{1}{f} = \frac{1}{v} + \frac{1}{u}}$$

which appears the usual equation of geometrical optics.

Inhomogeneous Magnetic Fields:

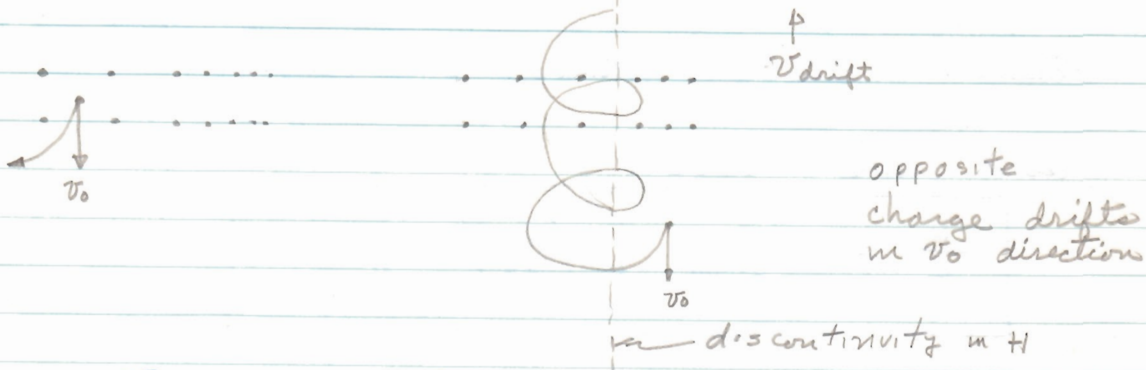
1) Weakly Inhomogeneous Magnetic Fields: assume that field changes small over the orbit compared to the field inside the orbit. Can apply sort of perturbation method to these cases.



Field increases in density along y direction:

$$H(z) = \left[H_0 + \left(\frac{\partial H}{\partial y}\right) y \right] \hat{1}_z$$

Consider only motions in x-y plane with $\vec{v}_0 = v_0 1_x$



Initial equations of motion:

$$m \frac{d\vec{v}}{dt} = \frac{e \vec{v} \times \vec{H}}{c}$$

$$m \left(\frac{dV_x}{dt}\right) = \frac{e}{c} v_y \left(H_0 + \frac{\partial H}{\partial y} y\right)$$

$$m \left(\frac{dv_y}{dt}\right) = \frac{e}{c} v_x \left(H_0 + \frac{\partial H}{\partial y} y\right)$$

Define:

$$\omega_0 = \frac{e H_0}{mc}$$

$$\omega_1 = \frac{e}{mc} \left(\frac{\partial H}{\partial y}\right) R$$

$$R = \frac{v_0}{\omega_0}$$

Thus:

$$\frac{dV_x}{dt} = \left(\omega_0 + \frac{\omega_1 y}{R}\right) v_y$$

$$\frac{dV_y}{dt} = \left(\omega_0 + \frac{\omega_1 y}{R}\right) v_x$$

LECTURE IX   2-25-61

Inhomogeneous Magnetic Fields:

We now write:

$$\frac{d}{dt}(v_x + i v_y) = -i\left(\omega_0 + \frac{\omega_1 y}{R}\right)(v_x + i v_y)$$

Now assume $\frac{\omega_1}{\omega_0} \ll 1$ or that the gradient of the field is small. In the first approximation, use y for the case of homogeneous fields. That is:

$$y = -R(1 - \cos\omega_0 t)$$

The perturbation of the inhomogeneous field is small in the y direction only. Then:

$$\frac{d}{dt}(v_x + i v_y) = -i\left\{\omega_0 - \omega_1(1 - \cos\omega_0 t)\right\}(v_x + i v_y)$$

or $$\int \frac{d(v_x + i v_y)}{(v_x + i v_y)} = \ln(v_x + i v_y) = -i\int\left[(\omega_0 - \omega_1) + \omega_1\cos\omega_0 t\right]dt$$

$$= -i(\omega_0 - \omega_1)t + \frac{\omega_1}{\omega_0}\sin\omega_0 t$$

$$(v_x + i v_y) = \underset{0}{\underset{\shortparallel}{(v_x + i v_y)_0}}\ \exp\left\{-i\left[(\omega_0 - \omega_1)t + \frac{\omega_1}{\omega_0}\sin\omega_0 t\right]\right\}$$

Now:

$$v_x = v_0\cos\left[(\omega_0 - \omega_1)t + \frac{\omega_1}{\omega_0}\sin\omega_0 t\right]$$

$$= v_0\left[\cos(\omega_0 - \omega_1)t\ \underbrace{\cos\left(\frac{\omega_1}{\omega_0}\sin\omega_0 t\right)}_{1} - \sin(\omega_0 - \omega_1)t\ \underbrace{\sin\left(\frac{\omega_1}{\omega_0}\sin\omega_0 t\right)}_{\left(\frac{\omega_1}{\omega_0}\right)\sin\omega_0 t}\right]$$

using the fact that $\frac{\omega_1}{\omega_0} \ll 1$

Then: $v_x = v_0 \left[\underbrace{\overline{\cos \omega_0 t}}_{0} - \underbrace{\overline{(\sin \omega_0 t) \frac{\omega_1}{\omega_0} \sin \omega_0 t}}_{1/2 \frac{\omega_1}{\omega_0}} \right]$

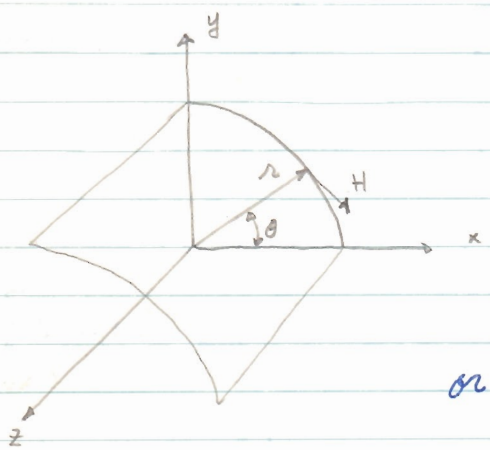
$$\boxed{\bar{v}_x = -\frac{1}{2} \frac{\omega_1}{\omega_0} v_0}$$

We say that the drift velocity is $\bar{v}_x = v_d$ so:

$$v_d = -\frac{1}{2} \frac{\partial H / \partial y}{H} \frac{v_0^2}{\underset{\left(\frac{e}{m}\right)\frac{H}{c}}{\omega_0}}$$

It is seen that the direction of drift is dependent on charge of particle, and inversely to magnitude. v_d is directly proportional to mass of particle.

2) Motion in Curving Field



We have a constant field such that:

$$\vec{H} = -\hat{1}_\theta H_\theta$$

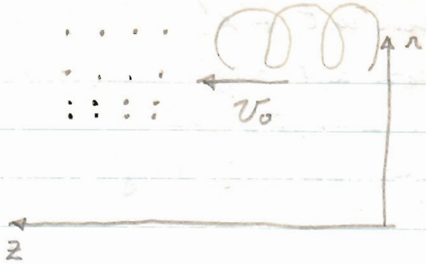
The field must satisfy curl $\vec{H} = 0$

or:
$$\begin{vmatrix} \frac{1_r}{r} & 1_\theta & \frac{1_z}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ 0 & -rH_\theta & 0 \end{vmatrix} = 0$$

We obtain: $\frac{1_r}{r} = 0$, $\frac{1_z}{r} \frac{\partial}{\partial r}(r H_\theta) = 0$

or $r \frac{dH_\theta}{dr} + H_\theta = 0$, $\left(\frac{\partial H_\theta}{\partial r}\right)_{r_0} = -\left(\frac{H}{r}\right)_{r_0}$

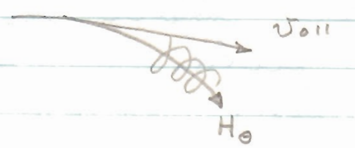
Thus the field gradient goes at $\frac{1}{r}$ in r direction a)



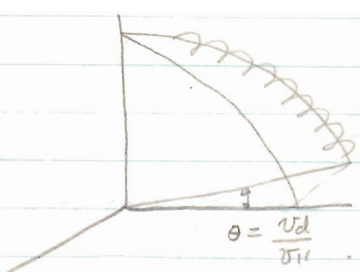
We get immediately get:

$$\boxed{\left|(v_d)_{v_0 \perp H}\right| = \frac{\frac{1}{2}\left(\frac{\partial H}{\partial y}\right) v_{0\perp}^2}{H_0\, \omega_0} = \frac{\frac{1}{2} v_{0\perp}^2}{r_0\, \omega_0}}$$

b) $(v_{0\parallel})$ Parallel to Field line:



The circular path creates a centripetal force to push particle outward which causes drift in or out of paper depending on charge. Has effect of fictitious electric field.



$H_x = H(r) \sin\theta$

$H_y = -H(r) \cos\theta$

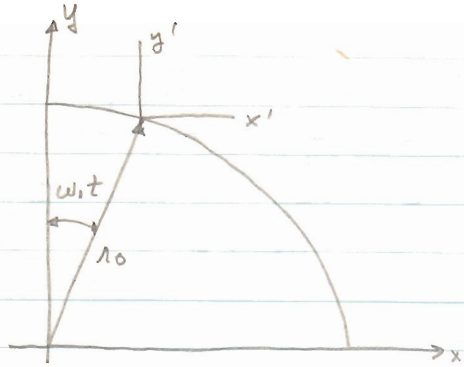
$$m\dot{\vec{v}} = \frac{e\, \vec{v} \times \vec{H}}{c}$$

$$\frac{dv_x}{dt} = v_z\, \omega_0 \cos\theta$$

$$\frac{dv_y}{dt} = v_z\, \omega_0 \sin\theta$$

$$\frac{dv_z}{dt} = -\omega_0 (v_x \cos\theta + v_y \sin\theta), \quad \omega_0 = \frac{e H(r_0)}{mc}$$

The proper moving coordinate system to go to is one in which one of the axis is always in the radial direction. However, the algebra is difficult and we will take the new system near y where it is nearly alway parallel to r. Correct result is obtained.



$$x = r_0 \sin \omega_1 t + x'$$
$$y = r_0 \cos \omega_1 t + y'$$
$$v_x = r_0 \omega_1 \cos \omega_1 t + v_x'$$
$$v_y = - r_0 \omega_1 \sin \omega_1 t + v_y'$$
$$v_z = v_z'$$
$$\frac{dv_x}{dt} = -\omega_1^2 r_0 \sin \omega_1 t + \dot{v}_x'$$
$$\frac{dv_y}{dt} = -\omega_1^2 r_0 \cos \omega_1 t + \dot{v}_y'$$

We assume that the radius of the orbit must be much smaller than r_0. That is, the gradient of the field is small; it curves only slowly. Then:

$$\sin \omega_1 t \cong \cos \theta$$
$$\cos \omega_1 t \approx \sin \theta$$

$$\cos \theta = \frac{x}{r} = \frac{r_0}{r} \sin \omega_1 t + \frac{x'}{r}$$

Then:
$$\frac{dv_x'}{dt} = r_0 \omega_1^2 \sin \omega_1 t + \omega_0 v_z' \sin \omega_1 t$$
$$\frac{dv_y'}{dt} = r_0 \omega_1^2 \cos \omega_1 t + \omega_0 v_z' \cos \omega_1 t$$
$$\frac{dv_{z'}}{dt} = -\omega_0 v_{x'} \sin \omega_1 t - \omega_0 v_{y'} \cos \omega_1 t$$

We now inject the assumption of $y' \parallel r_0$ and write $\omega_1 t \ll 1$ and get:

$$\frac{dv_{x'}}{dt} \simeq 0 \quad , \quad \boxed{\begin{aligned} \frac{dv_{y'}}{dt} &= r_0 \omega_1^2 + \omega_0 v_{z'} \\ \frac{dv_{z'}}{dt} &= -\omega_0 v_{y'} \end{aligned}}$$

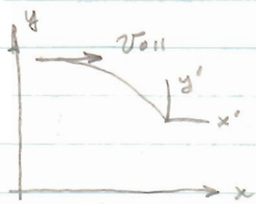
LECTURE X 2-28-61

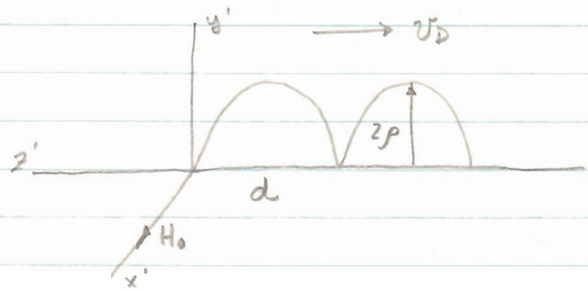
The previous equation denote (almost) ordinary motion in uniform H in x' direction except for $r_0 \omega_1^2$. This constant force term can be represented as an equivalent electric field $\mathcal{E}$:

$$e\mathcal{E} = m r_0 \omega_1^2 \qquad \text{with } \omega_1 = \frac{v_{0\parallel}}{r}$$

Then the drift velocity is:

$$v_D = \frac{c\mathcal{E}}{H} = \frac{c}{He} m r_0 \omega_1^2 = \frac{c\, m\, v_{0\parallel}^2}{e H r_0} = \frac{v_{0\parallel}^2}{\omega_0 r_0}$$

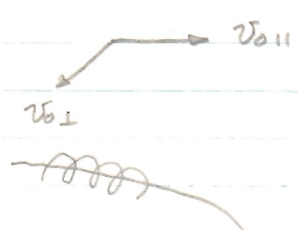




$$d = 2\pi\rho = \frac{2\pi v_{0\parallel}^2}{\omega_0^2 r_0}, \quad \text{since } \frac{v_D}{\omega_0} = \rho = \frac{v_{0\parallel}^2}{\omega_0^2 r_0}$$

with $\dfrac{\rho}{r_0} = \dfrac{v_{0\parallel}^2}{\omega_0^2}\dfrac{1}{r_0^2} \ll 1$ as per assumption.

c) Synthesis of (a) and (b):

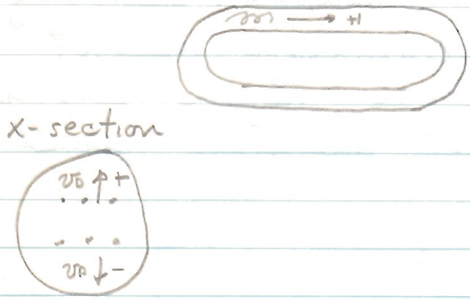


$$\text{Then: } (v_D)_{Total} = \frac{v_{0\perp}^2}{2 r_0 \omega_0} + \frac{v_{0\parallel}^2}{\omega_0 r_0}$$

$$= \frac{1}{r_0 \omega_0}\left[\frac{1}{2} v_{0\perp}^2 + v_{0\parallel}^2\right]$$

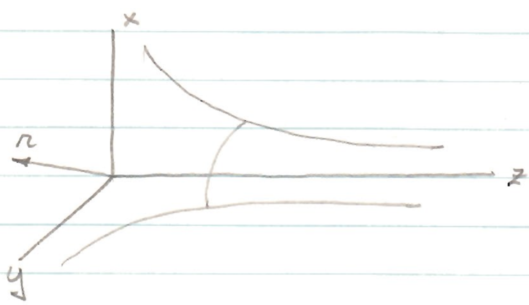
Reference: Spitzer "Ionized Gases"

Plasma Confinement :



when going around corner, particles of opposite sign separate due to curvature of field as above equation have shown.

3. Motion of Charged Particles Moving in Slowly Converging or Diverging Magnetic Field.



$$H = \hat{1}_r H_r(r,z) + \hat{1}_z H_z(r,z)$$

Now: $\nabla \times H = 0$, $\nabla \cdot H = 0$

$\nabla \cdot H$ gives :

$$\frac{1}{r}\frac{\partial}{\partial r}(r H_r) + \frac{\partial H_z}{\partial z} = 0$$

or $$\boxed{\frac{\partial}{\partial r}(r H_r) = - r \frac{\partial H_z}{\partial z}}$$

$$\begin{vmatrix} 1_r & 1_\theta & \frac{1_z}{r} \\ \frac{\partial}{\partial r} & \frac{1}{r}\frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ H_r & 0 & H_z \end{vmatrix} = 0 \quad \text{gives} \quad \frac{\partial H_z}{\partial r} = \frac{\partial H_r}{\partial z}$$

Assume : $H_z(r,z) = H_z(z)$, $H_R(r,z) = H_r(r)$

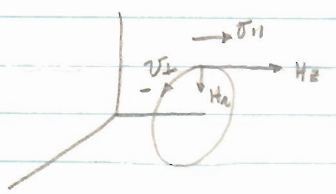
Then : $r H_r = -\frac{r^2}{2}\left(\frac{\partial H}{\partial z}\right)$; $H_r = -\frac{r}{2}\left(\frac{\partial H}{\partial z}\right)$

we see that for the assumption, $\frac{\partial H}{\partial z}$ must be constant.

Physically, the assumption holds for slow gradual convergence as the gradient is constant over the period of the orbit.

We then re-write the field as:

$$H = \hat{1}_r \left(-\frac{r}{2}\frac{\partial H}{\partial z}\right) + 1_z\, H_z(z)$$



Equations of Motion:

$$m\left(\frac{dv_\parallel}{dt}\right) = -\frac{e\,(v\times H)}{c} = -\frac{e}{c}\, v_\perp H_r$$

$v_\perp = r\omega$ since in this direction we have cyclotronic motion. Then:

$$m\left(\frac{dv_\parallel}{dt}\right) = -\frac{e}{c}\, v_\perp \left(\frac{r}{2}\right)\frac{\partial H}{\partial z} = \underbrace{-\frac{1}{2}\,\frac{m v_\perp^2}{H_z}}_{\text{magnetic moment } \mu}\,\frac{\partial H_z}{\partial z}$$

We will show μ is invariant to H_z, then $m\left(\frac{dv_\parallel}{dt}\right)$ is a constant of the motion.

Definition of μ: $\mu = \frac{iA}{c}$

Examine μ dimensionally: $\mu = \frac{\pi r^2}{c}\left(\frac{e}{\tau}\right) = \frac{\pi r^2}{c}\frac{e}{2\pi}\frac{eH_z}{mc}$

Then:

$$\mu = \frac{e^2}{2\pi m c^2}\cdot \pi r^2 H_z = \frac{e^2}{2\pi m c^2}\,\Phi$$ — flux

Use: $r = \frac{v_\perp}{\omega}$ and get: $\boxed{\mu = \frac{m v_\perp^2}{2 H_z}}$

The concept of μ only holds strictly for circular orbit. Thus, we must have slowly converging field; so particle makes many orbits before field changes appreciably: Conditions on time τ:

$$\left.\begin{aligned} &\frac{1}{\omega\tau} \ll 1 \quad \text{or} \quad \omega\tau \gg 1 \\ &\text{and} \quad v_\parallel\, \tau \left(\frac{\partial H}{\partial z}\right) \ll H_z \end{aligned}\right\} \quad \frac{1}{\omega} \ll \tau \ll \frac{H_z}{v_\parallel \frac{\partial H}{\partial z}}$$

or $\boxed{\frac{\omega H_z}{v_\parallel\left(\frac{\partial H}{\partial z}\right)} \gg 1}$ (Adiabatic Invariance Condition)

The basic assumption for independence of field components can be written:

$$\frac{\left(\frac{\partial H}{\partial z}\right) \rho}{H_z} << 1$$

LECTURE XI 3-2-61

Recall that KE cannot be changed by static magnetic field. Then,

$$T = \tfrac{1}{2} m v_\perp^2 + \tfrac{1}{2} m v_\parallel^2$$

$$\frac{dT}{dt} = 0 = \frac{d}{dt}\left(\tfrac{1}{2} m v_\perp^2\right) + \frac{d}{dt}\left(\tfrac{1}{2} m v_\parallel^2\right)$$

Now: $\frac{d}{dt}\left(\tfrac{1}{2} m v_\parallel^2\right) = m v_\parallel \frac{dv_\parallel}{dt} = -\mu v_\parallel \frac{\partial H_z}{\partial z}$

$$= -\frac{d}{dt}(\mu H_z)$$

$$-\mu v_\parallel \frac{\partial H_z}{\partial z} = -\mu \frac{dH_z}{dt} = -\frac{d}{dt}(\mu H_z)$$

Then: $-\mu \frac{dH_z}{dt} = -\mu\left(\frac{dH_z}{dt}\right) + H_z \frac{d\mu}{dt}$

and $\boxed{\frac{d\mu}{dt} = 0}$, therefore μ is a constant of the motion.

Consequences:

1) Φ constant as $H\uparrow$ as $r\downarrow$, $\pi r^2 H =$ constant

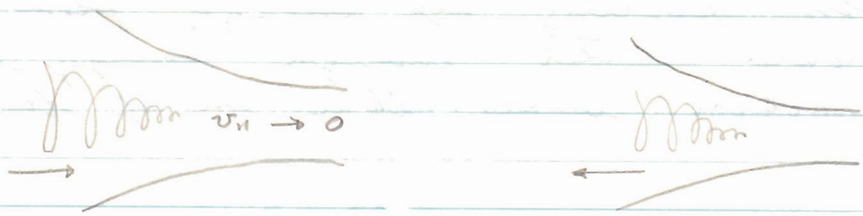
$$r \sim \frac{1}{\sqrt{H}}$$

2) $\frac{v_\perp^2}{H} =$ constant : as $H\uparrow$ $v_\perp\uparrow$, $v_\perp \sim \sqrt{H}$

3) since sum of velocity squares must be constant, as $H\uparrow$ $v_\parallel\downarrow$. $m\frac{dv_\parallel}{dt} = -\mu\left(\frac{\partial H_z}{\partial z}\right)$

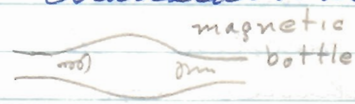
so $v_\parallel$ decreases to zero, then turns and goes back.

Thus net motions are:

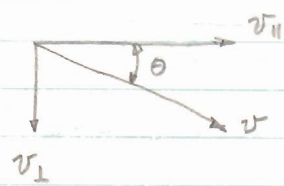


Before $v_{\parallel} \to 0$ After $v_{\parallel} \to 0$ and particle turns.

Converging field transfers translational KE into rotational KE.



This magnetic mirror is used to contain plasmas. However, if initial velocity is great enough, we can penetrate mirror since field cannot converge indefinitely. We now consider these conditions.



$$v_{\parallel} = v \cos\theta \quad , \quad v_{\perp} = v \sin\theta$$

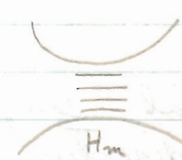
$$\frac{1}{2}\,\frac{m v_{\perp}^2}{H} = \text{constant} = \frac{1}{2}\,\frac{m v_{\perp 0}^2}{H_0}$$

$$\frac{v_{\perp}^2}{H} = \frac{v_{\perp 0}^2}{H_0}$$

which implies: $\boxed{\dfrac{\sin^2\theta_0}{H_0} = \dfrac{\sin^2\theta}{H} = \dfrac{1}{H}}$

Turning point field $= H_m = \dfrac{H_0}{\sin^2\theta}$

For $H_m < \dfrac{H_0}{\sin^2\theta_0}$ particle emerges



Thus, the critical angle is:

$$\boxed{\sin^2\theta_0\Big|_c = \frac{H_0}{H_m}}$$

$\theta_c < \theta$: reflected

$\theta < \theta_c$: emerges

This principle is also used for a cosmic ray acceleration theory (Fermi). If two clouds of ionized gases are approaching each other with relatively velocity v. Particles will bounce back and forth picking up energy from the approaching clould until enough energy is reached that particle leaks thru one of the clouds.

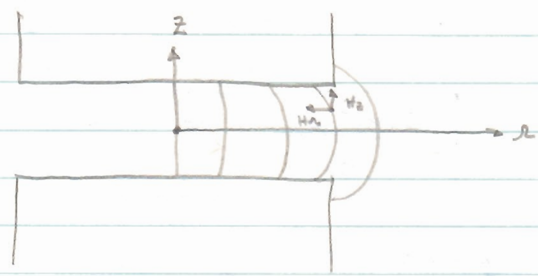
Stability of Particle Orbits in High Energy Accelerators:

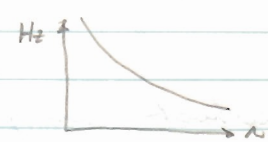
In usual cyclotron, some focusing field is needed to keep particles in plane of machine. In early machines, natural inhomogeniety in magnetic fields did this. Recall:

$$\rho = \frac{\Delta E}{eH}\left[1 + \frac{2mc^2}{\Delta E}\right]^{1/2}$$

$$T = mc^2 + \Delta E \quad , \quad \omega = \frac{\omega_0}{1 + \left(\frac{\Delta E}{mc^2}\right)}$$

Reference: Livingston "H.E. Accelerators"





Radial component of H produces restoring force in z-direction.

Restoring force in radial direction: suppose particle deflected off orbit r_0 to r:

$$\frac{mv^2}{r_0} \rightarrow \frac{mv^2}{r} \quad \text{suffers force} \quad \frac{e H_z(r) v}{c}$$

For stability, H_z must fall off more slowly than $\frac{1}{r}$

We take $H_z(r) = H_0\left(\frac{r}{r_0}\right)^{-n}$

Equations of motion:

$$\vec{H} = \hat{I}_r H_r(z) + I_z H_z(r)$$

$$\vec{v} = \dot{r} \hat{I}_r + r\dot{\theta} I_\theta + \dot{z} I_z$$

$$\frac{d\vec{p}}{dt} = \frac{e\vec{v} \times \vec{H}}{c}$$

$$\left(\frac{dp_\theta}{dt}\right) = -\frac{e}{c}\left(\dot{r} H_z - \dot{z} H_r\right)$$

$$\left(\frac{dp}{dt}\right)_r = \frac{e}{c} r\dot{\theta} H_z \quad , \quad \left(\frac{dp}{dt}\right)_z = \frac{e}{c} r\dot{\theta} H_r$$

$$\frac{d\vec{p}}{dt} = m\left[(\ddot{r} - r\dot{\theta}) I_r + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) I_\theta + \ddot{z} I_z\right]$$

LECTURE XII 3-4-61

Continuation of HE Accelerator:

$$m(\ddot{r} - r\dot{\theta}^2) = \frac{e}{c} r\dot{\theta} H_z(r) \quad , \quad H_z(r) = H_0\left(\frac{r}{r_0}\right)^{-n}$$

For unperturbed orbit:

$$\dot{\theta} = -\frac{e H_z(r)}{mc} \quad , \; r = r_0 \, , \; \dot{\theta} = \dot{\theta}_0 = \omega_0 = \frac{e H_0}{mc}$$

Assumption of no energy change for perturbed orbit:

$$\omega_0 r_0 = \omega r = \dot{\theta} r$$

now: $$\ddot{r} = \frac{e H_z(r)}{mc} r\dot{\theta} + r\dot{\theta}^2$$

and: $$\ddot{r} = -\omega_0^2 r_0 \left(\frac{r}{r_0}\right)^n + \frac{(r_0\omega_0)^2}{r}$$

$$(r\dot{\theta})_{r=r_0} = \frac{-e H r_0}{mc}$$

Then: $\ddot{r} = \omega_0^2 r_0 \left[-\left(\frac{r}{r_0}\right)^{-n} + \left(\frac{r_0}{r}\right) \right]$

Take: $r = r_0 + \rho$ where ρ is the perturbation.

$$\ddot{\rho} = \omega_0^2 r_0 \left[\frac{r_0}{r_0 + \rho} - \left(\frac{r_0 + \rho}{r_0}\right)^{-n} \right]$$

$$= \omega_0^2 r_0 \left[1 - \frac{\rho}{r_0} - 1 + \frac{n\rho}{r_0} + \dots \right]$$

or $\boxed{\ddot{\rho} = -(1-n)\,\omega_0^2 \rho}$ radial oscillation

For z:

$$m\dot{z} = \frac{e}{c} r \dot{\theta} H_r$$

$$\frac{\partial H_r}{\partial z} = \frac{\partial H_z}{\partial r}, \quad H_r = \left(\frac{\partial H_z}{\partial r}\right) z = \frac{-n H_0 z}{r_0}$$

Then: $\boxed{\ddot{z} = -\omega_0^2 n z}$ z oscillation

using $r\dot{\theta} = \omega_0 r_0$

For stability for both oscillations, n must have condition:

$$\boxed{0 < n < 1}$$

Period of oscillation greater than cyclotron period. Certain values are disallowed since coupling will occur which is not shown above. Will get resonance effect. Disallowed values are $n = 0, .25, .50, .75, .80, 1.0$

usually n is in this region.

Now the energy of a harmonic oscillator is:

$$E = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \omega^2 x^2 \quad , \quad x = \begin{cases} z \\ \rho \end{cases}$$

$$\omega_r = \sqrt{1-n}\ \omega_0 \qquad \rho = P \cos \omega_r t$$
$$\omega_z = \sqrt{n}\ \omega_0 \qquad z = Z \cos \omega_z t$$

Now, if $x = A \cos \omega t$: $E = \frac{1}{2} m \omega^2 A^2 \left(\cos 2\omega t + \sin^2 \omega t \right)$

$$= \frac{1}{2} m \omega^2 A^2$$

$$P = \sqrt{\frac{2E_r}{m\omega_r^2}} = \sqrt{\frac{2E_r}{m(1-n)\omega_0^2}}$$

$$Z = \sqrt{\frac{2E_z}{m\omega_z^2}} = \sqrt{\frac{2E_z}{m(n)\omega_0^2}}$$

which gives use the amplitude of the oscillations as a function of n. Time change of field and relativistic motion has not been included. These effects lead to damping. Trick is to consider now is that ω_0 is a function of time. Consider then the equation:

$$\ddot{X} = -\omega^2(t)\, x$$

The time change of the field is due to the acceleration of the particle.

Solution of d.e.:

$$x = A(t)\, e^{i\int^t \omega(t')\,dt'}$$

which gives:

$$\frac{d^2A}{dt^2} + 2i\omega \frac{dA}{dt} + i \frac{d\omega}{dt} A(t) = 0$$

Make adiabatic approximation (field changes slowly during orbit of particle). This gets rid of $\frac{d^2A}{dt^2}$.

Thus: $\frac{dA}{A} = -\frac{1}{2}\frac{d\omega}{\omega}$ or $A(t) = \frac{\text{constant}}{\sqrt{\omega(t)}}$

which would have been found rigourously also for the damping. But the energy and amplitude are related: $E = \frac{1}{2} m\omega^2 A^2$

Then:

$$\boxed{E(t) = \frac{1}{2}\frac{m\omega^2}{\omega}\,\text{constant} = J\,\omega(t)}$$

Note that this is of form of QM harmonic oscillator. J is called the <u>adiabatic invariant</u> and is equal to $(n+\frac{1}{2})\hbar$. Problems like these were of great importance in the old quantum theory. This is related to WKB approximation:

$$\frac{d^2 u}{dx^2} + k^2(x)\,u = 0 \quad , \quad k = \sqrt{\frac{2m}{\hbar^2}\left[E - V(x)\right]}$$

$V(x)$ changes very little over the wavelength of the particle. The solution is:

$$u(x) = \frac{\text{constant}}{\sqrt{k(x)}}\, e^{\pm \int^x k(x')\,dx'}$$

so we see that this is connected mathematically to what we have been doing.

Finally:

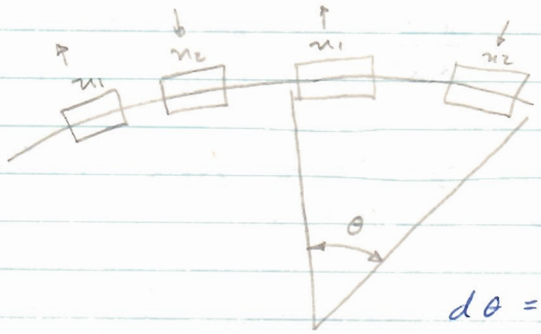
$$\boxed{A \sim \left(\frac{1}{\sqrt{1-n}} \text{ or } \frac{1}{\sqrt{n}}\right)\frac{1}{\sqrt{\omega}}}$$ non-relativistic

Relativistically: $\frac{A(t)}{A(0)} = \sqrt{\frac{H(0)}{H(t)}}$ thus amplitude goes as $\frac{1}{\sqrt{H(t)}}$

A typical oscillation amplitude $\pm 5\%$ of orbit radius. Can show that limit exists on size of machine.

Breakthrough made by: Courant, Livingston, Snyder
<u>PR 88</u> 1190 (1952)

Developed alternate gradient focusing, discovered by accident.



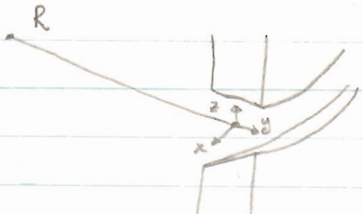
The new equations of motion are:

$\ddot{\rho} = -(1-n_1)\,\omega_0^2\,\rho$

$\ddot{z} = -n_1\,\omega_0^2\,z$

$d\theta = \omega_0\,dt$

For odd sections: $\left(\frac{d^2\rho}{d\theta^2}\right) = -(1-n_1)\,\rho_1$, $\frac{d^2 z}{d\theta^2} = -n_1 z$



$x = R\theta$

$\frac{d^2 z}{d\theta^2} = -n z$, $\frac{d^2\rho}{d\theta^2} + \rho - n\rho = 0$

$$n = -\frac{\partial \ln H}{\partial \ln R} = \frac{\frac{1}{H}\frac{\partial H}{\partial R}}{R} \cdot \frac{\frac{1}{H}\frac{\partial H}{\partial y}}{R}$$

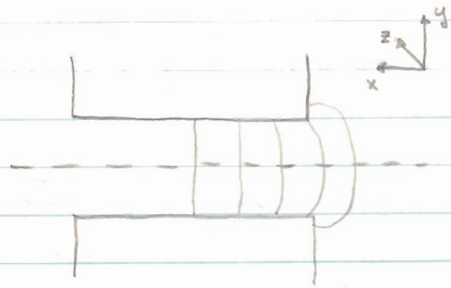
Then: $\frac{d^2 z}{dx^2} - Kz = 0$, $K = \frac{\partial H_z}{\partial y}\cdot\frac{1}{H_z R}$

$$\frac{d^2 y}{dx^2} + Ky + \underbrace{\frac{y}{R^2}}_{\substack{\| \\ 0 \\ \text{for large } R}} = 0$$

Two equations: one stable and one unstable.

LECTURE XIII 3-7-61

Lecture by Snyder: Accelerating Machines

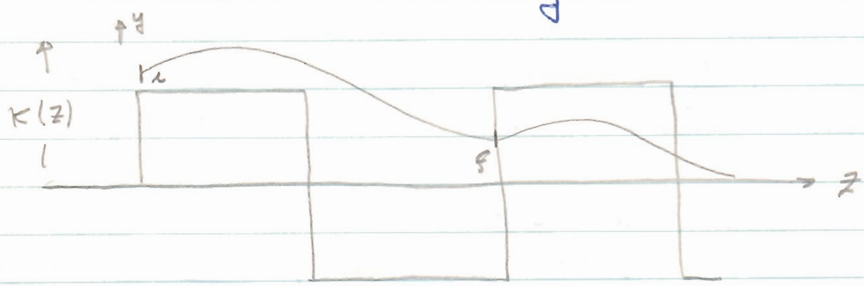


$$\frac{d^2y}{dz^2} = -\frac{n}{\rho^2}\,y$$

$$\frac{d^2x}{dz^2} = -\frac{(1-n)}{\rho^2}\,x$$

These equations give stability for $0 < n < 1$.

Alternate Field Focusing: Consider $\frac{d^2y}{dz^2} = -K(z)\,y$



We assert: $\left.\begin{aligned} y_f &= a\,y_I + b\,y_I' \\ y_f' &= c\,y_I + d\,y_I' \end{aligned}\right\}$ $ad - bc = 1$

Assume: $y_f = \lambda\,y_I$
$y_f' = \lambda\,y_I'$

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0, \quad \lambda^2 - (a+d) + 1 = 0$$

$$\lambda = \frac{+(a+d) \pm \sqrt{(a+d)^2 - 4}}{2},$$ can be real or imaginary (complex)

Will get allowed and forbidden regions.

Examine solutions in constant region:
$y'' + Ky = 0$
$y = A\cos\sqrt{K}\,z + B\sin\sqrt{K}\,z$

We get: $y = y_I \cos\sqrt{K}\, z + \dfrac{y_I'}{\sqrt{K}} \sin\sqrt{K}\, z$

$$y' = -y_I \sqrt{K} \sin\sqrt{K}\, z + y_I' \cos\sqrt{K}\, z$$

$y'' - Ky = 0$ gives hyperbolic functions.

The matrix which takes us across a whole period is the product of those that go across each + and − half-period.

$$\begin{pmatrix} y_F \\ y_F' \end{pmatrix} = \begin{pmatrix} \cosh\sqrt{K}L & \frac{1}{\sqrt{K}}\sinh\sqrt{K}L \\ \sqrt{K}\sinh\sqrt{K}L & \cosh\sqrt{K}L \end{pmatrix} \begin{pmatrix} \cos\sqrt{K}L & \frac{1}{\sqrt{K}}\sin\sqrt{K}L \\ -\sqrt{K}\sin\sqrt{K}L & \cos\sqrt{K}L \end{pmatrix}$$

We get: $\frac{1}{2}(a+d) = \cosh\sqrt{K}L \cos\sqrt{K}L$

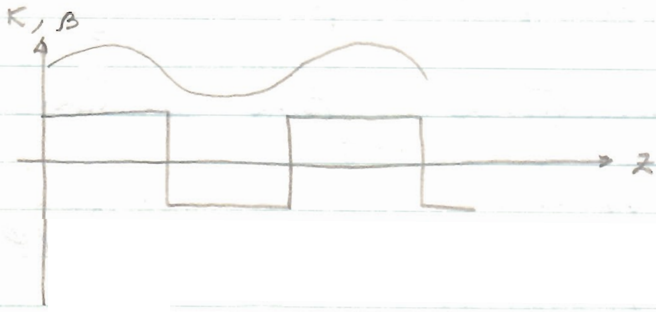
If $\sqrt{K}L < 1$, calling $\sqrt{K}L = \theta$:

$\frac{1}{2}(a+d) = 1 - \frac{1}{6}\theta^4 + \cdots$ which is less than 1 and thus λ can be complex.

Consider: $y'' + K(z)y = 0$

We assert that a solution can be written in the form:

$$y = a\,\beta^{1/2}(z)\cos\varphi(z), \quad \varphi'(z) = \frac{1}{\beta(z)}$$



The term "wavelength of a machine" means the number of cycles a particle makes about its equilibrium line in an orbit around the machine.

Define: $\alpha(z) = -\frac{1}{2}\frac{d}{dz}\beta(z)$

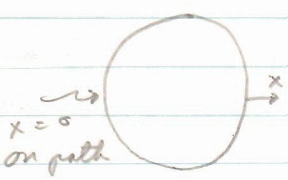
Get: $\cos^2\varphi + \sin^2\varphi = 1 = \frac{1}{a^2\beta(z)}\left\{y^2 + (\alpha y + \beta y')^2\right\}$

which is the equation of an ellipse. What this means is that the area of the ellipse is constant in y, y' space even though it wobbles during a transit through the orbit.

LECTURE XIV 3-9-61

In previously results, we assumed that momentum was such that orbit was closed path. In this case, the equation must be of the structure:

$$\frac{d^2x}{dz^2} + K(z)\,x = \frac{\Delta B}{B\rho}$$



Take for solution: $x = \beta^{1/2}(z)\,\eta(z)$

Define a new independent variable: $\varphi = \frac{1}{\nu}\int_0^z \frac{dz}{\beta(z)}$ such that $\varphi(C) - \varphi(0) = 2\pi$ and get for new equation:

$$\frac{d^2\eta}{d\varphi^2} + \nu^2\eta = f(\phi) = \beta^{3/2}(z)\,\frac{\Delta B}{\rho B}$$

Solve by Green's functions: take solution for which η is periodic in 2π:

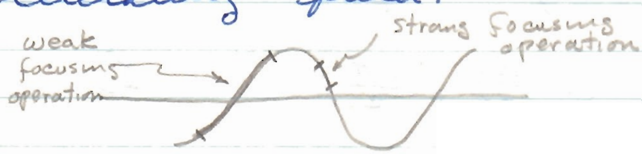
$$\eta(\varphi) = \frac{\nu}{2\sin\pi\nu}\int_{\varphi}^{\varphi+2\pi} f(\psi)\cos\left\{\nu(\pi+\varphi-\psi)\right\}d\psi$$

If ν is an integer, no orbits exists, and we have resonance phenomena. At any other values of ν we are OK. ν is the number of wavelengths in orbit.

another problem is that the machine must be able to accept particle with spread in momentum and keep them in the machine. In this case, the RHS becomes:

$$\frac{1}{\rho}\left(\frac{\Delta p}{p}\right)$$

where ρ is the radius of curvature of the orbit at a point. As the p is increased, the closed orbits expand. In most machines we can tolerate spreads in momentum of about 2%. However, all particles are not accelerated the same. Also the periods of the orbit depend upon the energy and path length. This will cause in one cause the particle will slip out of sync with the accelerating field.



The strong points of the strong focusing machines are that they are able to accept wide variations in momentum and undergo cyclotron oscillations and still keep stable orbits.

LECTURE XV 3-11-61

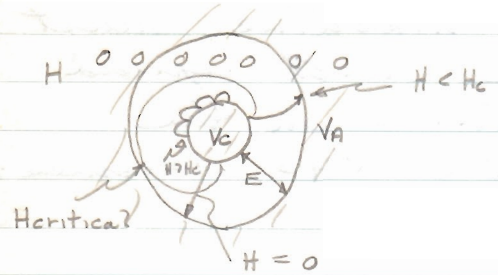
Magnetron Oscillator:

Converts DC to microwave:

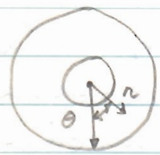
600 mc — 60,000 mc

50 cm — 1/2 cm.

DC Magnetron (A.W. Hull, PR18, 31 (1921))



We see that this acts as a switch on the electric current.



$$V(r) \sim \ln r$$

$$E(r) = \frac{-V_{A-c}\,\hat{1}_r}{r\ln\left(\frac{r_a}{r_c}\right)}$$

Hence: in r direction

$$m\left\{\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right\} = |e\,\mathcal{E}(r)| - r\left(\frac{d\theta}{dt}\right)\frac{He}{c}\quad \hat{1}_r$$

In θ direction, use torque equation:

$$\frac{d}{dt}\left\{mr^2\frac{d\theta}{dt}\right\} = r\frac{dr}{dt}\left(\frac{eH}{c}\right)\qquad \hat{1}_\theta$$

This can be integrated: $\frac{d}{dt}(r^2\theta') = \frac{\omega_c}{2}\frac{d}{dt}r^2$

$\omega_c = \frac{eH}{mc}$

$$\therefore\quad \dot\theta = \frac{1}{r^2}\left\{\frac{\omega_c}{2}r^2 + c\right\},\quad \dot\theta = 0,\ r = r_c$$

$$\dot\theta = \frac{1}{r^2}\left\{\frac{\omega_c}{2}r^2 - \frac{\omega_c r_c^2}{2}\right\}$$

or $\dot\theta = \frac{\omega_c}{c}\left\{1 - \left(\frac{r_c}{r}\right)^2\right\}$

Recall: $\frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + \frac{1}{2}m\left(r\frac{d\theta}{dt}\right)^2 = e\left\{V(r) - V_c\right\}$

Then:

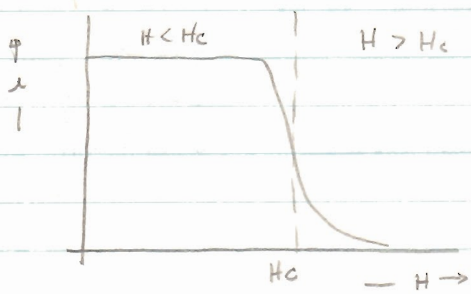
$$\boxed{\left(\frac{dr}{dt}\right)^2 + \frac{\omega_c r^2}{4}\left\{1 - \left(\frac{r_c}{r}\right)\right\}^2 = \frac{2e}{m}\left\{V(r) - V_c\right\}}$$

Consider the cut-off case: $\frac{dr}{dt} = 0,\ r = r_a$
then:

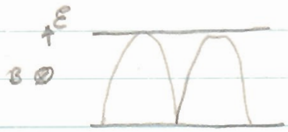
$$\frac{\omega_c^2 r_a^2}{4}\left\{1-\left(\frac{r_c}{r_a}\right)^2\right\}^2 = \frac{2e}{m}\left\{V_A - V_c\right\}$$

which gives the relation between V_{AC} and H for cut-off:

$$\boxed{V_{AC} = \frac{e}{m}\,\frac{H_c^2}{c^2}\,\frac{r_a^2}{8}\left\{1-\left(\frac{r_c}{r_a}\right)^2\right\}^2}$$



What is condition for cutoff in plane parallel case:



limit of high radius of curvature.

$$d = 2\,\frac{v_c}{\omega} = \frac{2c\mathcal{E}}{\omega H} = \frac{2c\,V_{AC}}{\frac{eH^2}{mc}\,d}$$

or $V_{AC} = \frac{e H_c^2}{m c^2}\left(\frac{d^2}{2}\right)$

Compare with $V_{AC} = \frac{e H_c^2}{m c^2}\,\frac{1}{8}\,r_a^2\left\{1-\left(\frac{r_c}{r_a}\right)^2\right\}^2$

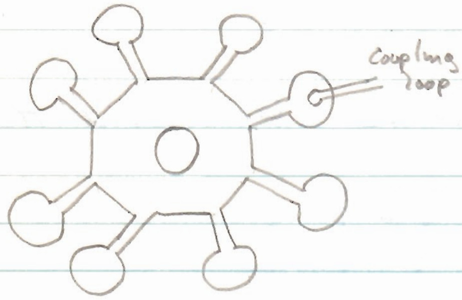
$r_c = r_a - d$, $\frac{d}{r_a}$ or $\frac{d}{r_c} \ll 1$, then:

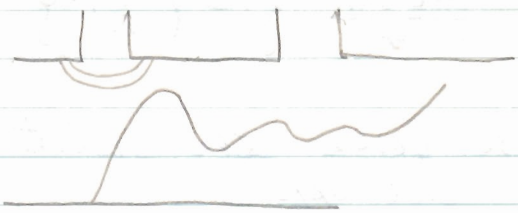
$$V_{AC} = \frac{e H_c^2}{m c^2}\left(\frac{1}{8}\right) r_a^2\left\{1-\left(\frac{r_a - d}{r_a}\right)^2\right\}^2$$

Consider $f = \frac{r_a^2}{8}\left(\frac{2d}{r_a}\right)^2$ expanding in series.

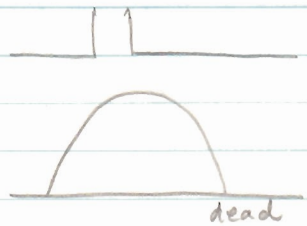
$= \frac{d^2}{2}$ which shows reduction to the plane parallel case.

Oscillator:

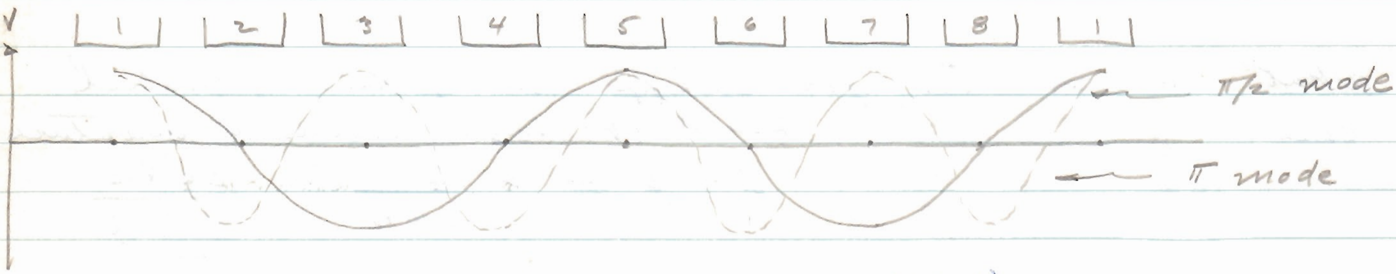




electron pushes against cavity field



field gives energy to electron which is self-defeating



If $\Delta\varphi$ = phase advance, $e^{i(\Delta\varphi N)} = 1$

then $\Delta\varphi N = 2\pi n$, $n = 1, 2, 3, \ldots$

In π mode, $n = N/2$

$\pi/2$ mode, $n = N/4$

$\Delta\varphi = \pi$

field shifts to next slot in time $\frac{T}{2}$ where T is period, for electron to be there, or $(p+\frac{1}{2})T$

Suppose kT is time for electron to make complete circuit. $kT = \frac{2\pi}{\omega_{avg}}$, $\left(\frac{kT}{N}\right) = (p+\frac{1}{2})T$, π mode

In general:

$$\left(\frac{kT}{N}\right) = \left(p + \frac{n}{N}\right)T$$

$$\boxed{k = (Np + n)}$$

$$(Np+n)\,T = \frac{2\pi}{\omega_{avg}} = (Np+n)\frac{1}{f_{avg}}$$

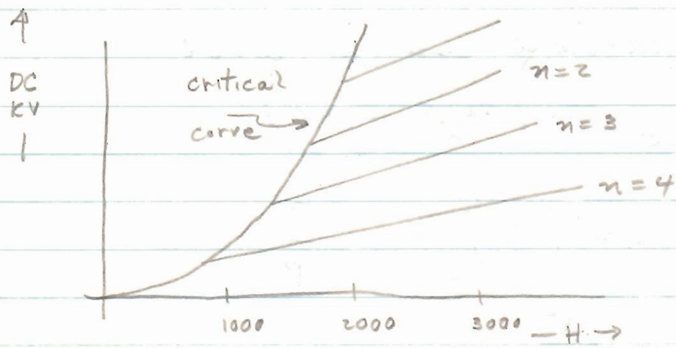
p, n integers, $N = 8$

$$\frac{2\pi}{\left(\frac{d\theta}{dt}\right)_{avg}} = \frac{2\pi}{\dfrac{v_{avg}}{\left(\frac{r_a+r_c}{2}\right)}} = \frac{2\pi}{\dfrac{c\,\mathcal{E}}{H\left(\frac{r_a+r_c}{2}\right)}}$$

or

$$\boxed{(Np+n)\frac{1}{f_{avg}} = \frac{2\pi}{\dfrac{c\,V_{AC}}{H(r_a-r_c)\left(\frac{r_a+r_c}{2}\right)}} = \frac{2\pi}{\dfrac{2c\,V_{AC}}{H\,(r_a^2-r_c^2)}}}$$

$$V_{AC} = \frac{\pi f_{avg} H}{(Np+n)\,c}\, r_a^2 \left\{1-\left(\frac{r_c}{r_a}\right)^2\right\}$$



$p=0$ fundamental mode

π mode

$f \sim 2800$ MC/sec

Reference: B.S.T.J., Fisk, et al <u>25</u> 167 (1946)

LECTURE XVI   3-14-61

Wave Picture of Electrons

De Broglie's Approach:

Light: $E = h\nu$, $p = \frac{h\nu}{c} = h/\lambda$

Look at momentum 4-vector:

$$p = \left(\frac{m\vec{v}}{\sqrt{1-\beta^2}} , \frac{iT}{c} \right)$$

	Light	Particles
E	$h\nu$	$h\nu = T$
p	h/λ	$p = h/\lambda$

$$\therefore \quad p = \left(\frac{h}{\lambda} \hat{1}_x ; \frac{ih\nu}{c} \right) = \left(h\vec{k} ; \frac{ih\nu}{c} \right)$$

where $\vec{k} = \frac{1}{\lambda} \vec{1}_x$

Associated with the particle is a wave:

$$e^{-2\pi i (k \cdot r - \nu t)} = e^{-2\pi i \left(\frac{1_x \cdot r}{\lambda} - \nu t \right)}$$

Now: $V_{phase} = \nu \lambda = (h\nu)\left(\frac{\lambda}{h}\right) = \frac{T}{p}$

$$= \frac{mc^2}{\sqrt{1-\beta^2} \left(\frac{mv}{\sqrt{1-\beta^2}} \right)} = \frac{c^2}{v_{particle}}$$

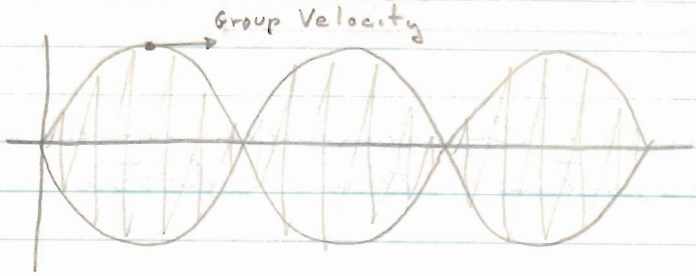
so we cannot use the phase velocity of the wave as the velocity of the particle. Try group velocity:

Form a packet:

$$\psi_1 + \psi_2 = e^{-2\pi i (k_1 x - \nu_1 t)} + e^{-2\pi i (k_2 x - \nu_2 t)}$$

$$= e^{-2\pi i \left\{ \left(\frac{k_1 + k_2}{2} \right) x - \left(\frac{\nu_1 + \nu_2}{2} \right) t \right\}} \left\{ \begin{array}{l} e^{2\pi i \left[\frac{(k_2 - k_1)}{2} x + \frac{(\nu_1 - \nu_2)}{2} t \right]} \\ + e^{-2\pi i \left[\frac{(k_2 - k_1)}{2} x + \frac{(\nu_1 - \nu_2)}{2} t \right]} \end{array} \right\}$$

or $\psi_1 + \psi_2 = 2 \cos 2\pi \left\{ \left(\frac{k_2 - k_1}{2} \right) x + \left(\frac{\nu_1 - \nu_2}{2} \right) t \right\} e^{\{\ \}}$



$(k_2 - k_1) x = (\nu_2 - \nu_1) t$ (constant point on envelope)

$$\frac{dx}{dt} = \frac{\nu_2 - \nu_1}{k_2 - k_1} = \frac{d\nu}{dk} = v_g$$

We must know the relation between ν and k, $\nu = \nu(k)$ which is the dispersion relation.

$$\boxed{\begin{aligned} v_{phase} &= \nu \lambda \\ v_{group} &= \frac{d\nu}{dk} = \frac{d\nu}{d\left(\frac{1}{\lambda}\right)} \end{aligned}}$$

What is v_{group} for particle? Want $\frac{d\nu}{dk}$:

$$\frac{d\nu}{dk} = \frac{d(h\nu)}{d(hk)} = \frac{dT}{dp} = \frac{\left(\frac{dT}{d\beta}\right)}{\left(\frac{dp}{d\beta}\right)}, \quad \beta = \left(\frac{v}{c}\right)$$

$$v_g = \frac{\frac{d}{d\beta} \frac{mc^2}{\sqrt{1-\beta^2}}}{\frac{d}{d\beta}\left(\frac{mc\beta}{\sqrt{1-\beta^2}}\right)}$$

Use: $T^2 = p^2 c^2 + m^2 c^4$

and find $T\,dT = c^2 p\,dp$

with result $v_g = \frac{dT}{dp} = \frac{c^2 p}{T} = \frac{m v c^2}{\sqrt{1-\beta^2}} \frac{1}{\frac{mc^2}{\sqrt{1-\beta^2}}}$

$= v$ which is the velocity of the particle

Therefore:
$$\boxed{p = \frac{h}{\lambda} = \frac{mv}{\sqrt{1-\beta^2}}}$$

Davisson and Germer: Experimental evidence of electron wave properties thru electron diffraction.

LECTURE $\underline{\text{XVII}}$ 3-16-61

<u>Fermi Statistics for Electrons in a Metal:</u>

We hold that the free electron wave functions are modulated by the cell-periodic atomic wave functions:

$$\psi = u_k(r)\, e^{i(k\cdot r - \frac{E}{\hbar}t)} \quad , \quad k = \frac{2\pi}{\lambda}$$

$$u_k(r) = u_k(r+R) \quad ; \quad R = \text{translation vector}$$

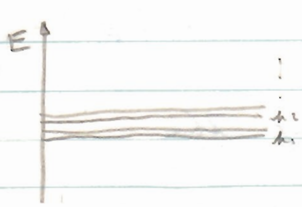
Recall: $v_g = \frac{\partial \omega}{\partial k}$ which leads to $\bar{v}_g = \frac{1}{\hbar}\bar{\nabla}_k E$

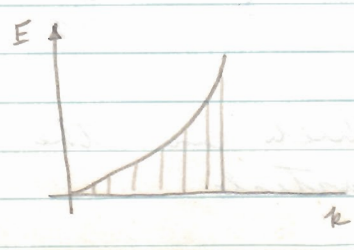
from $\omega = E/\hbar$. Thus we must have $E = E(k)$ or a dispersion relation between E and k.

For free electron: $E = \frac{1}{2}mv^2 = \frac{1}{2m}(mv)^2 = \frac{1}{2m}\left(\frac{h}{\lambda}\right)^2$

then
$$\boxed{E = \frac{(\hbar k)^2}{2m}}$$

Allowed values of k are determined by boundary conditions. Because of spin, there are two electrons per state.





The question of concern is how do electrons fill up these states? If use Boltzmann statistics, get all electrons in lower level at $T=0$.

$$\underset{E_1}{||||||} \quad \cdots\cdots \quad \underset{E_s}{||||||}$$

$\frac{N_s}{2}$ energy levels for N_s states

We have n_s electrons to put into N_s states. How can they be arranged?

$$\overset{z_1\ z_2}{|\ |}\ |\ |\ |\ \cdots\cdots\ |\ \overset{z_{N_s}}{|}$$

$$\begin{array}{cc} z_1 \cdots z_n & z_k \cdots z_k \\ |||||| & 0000 \\ n_s & (N_s - n_s) \end{array}$$

The number of possible ways to arrange these is given by the binomial factor:

$$\Omega(n_s) = \frac{N_s!}{(N_s - n_s)!\,(n_s)!}$$

; number of filled states around E_s

Now we ask for probability of n_1 in N_1, n_2 in N_2, etc.

$$\Omega(n_1, n_2, \ldots n_s \ldots) = \prod_s \frac{N_s!}{(N_s - n_s)!\,(n_s)!}$$

$$\ln \Omega = \sum_s \left[\ln N_s! - \ln (N_s - n_s)! - \ln n_s! \right]$$

Using Stirling's approximation: $\ln N! = N \ln N - N$

$$\ln \Omega = \sum_s N_s \ln N_s - (N_s - n_s) \ln (N_s - n_s) - n_s \ln n_s$$

We now want to maximize with respect to n_s:

$$\frac{\partial \ln \Omega}{\partial n_s} = \ln \left[\frac{(N_s - n_s)}{n_s} \right]$$

This is not zero as there are constraints on the problem because the number of electrons is constant and so is the total energy.

$$\sum_s n_s = N \quad , \quad \sum_s n_s E_s = E_{tot}$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$\alpha \qquad\qquad\qquad -\beta$$

$$\sum_s \left(\frac{\partial \ln \Omega}{\partial n_s}\right) \delta n_s + \sum_s \left[\alpha - \beta E_s\right] \delta n_s = 0$$

since $\delta \Omega = \sum \left(\frac{\partial \ln \Omega}{\partial n_s}\right) \delta n_s$

$$\sum_s \left\{ \ln\left(\frac{N_s - n_s}{n_s}\right) + \alpha - \beta E_s \right\} \delta n = 0$$

$$\ln\left(\frac{N_s - n_s}{n_s}\right) = \left(\beta E_s - \alpha\right) \quad , \quad \frac{N_s - n_s}{n_s} = e^{\beta E_s - \alpha}$$

and get:

$$\left(\frac{n_s}{N_s}\right) = \frac{1}{\left[e^{\beta E_s - \alpha} + 1\right]} = f_0(E_s) \quad \{\text{Fermi Function}\}$$

What are α, β? Use statistical mechanics and Thermodynamics:

$$S = k \ln \Omega_{max}$$

chemical potential

$$dQ = T dS = dE + P dV + \mu dN$$

$$dS = \left(\frac{\partial S}{\partial E}\right)_{V,N} dE + \left(\frac{\partial S}{\partial V}\right) dV + \left(\frac{\partial S}{\partial N}\right) dN$$

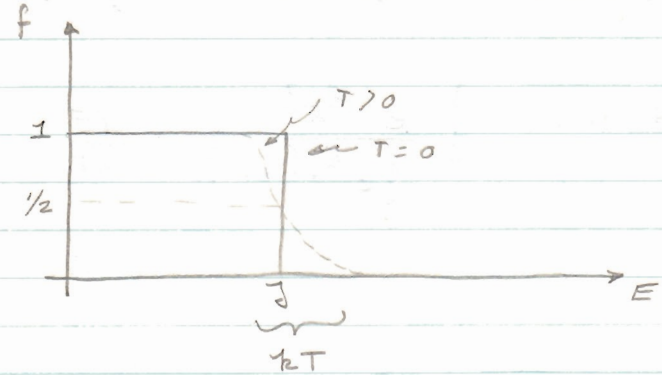
$$\left(\frac{\partial S}{\partial E}\right)_{V,N} = \frac{1}{T} \quad ; \quad \left(\frac{\partial \ln \Omega_{max}}{\partial E}\right)_{V,N} = \frac{1}{kT}$$

$$\delta \ln \Omega_{max} = \sum (\beta E_s - \alpha) \delta n_s = \sum_s \beta E_s \delta n_s = \beta \delta E$$

$$\left(\frac{\delta \ln \Omega_{max}}{\delta E}\right)_{V,N} = \beta = \frac{1}{kT}$$

which gives:

$$\boxed{f_0 = \frac{1}{e^{\frac{E-\zeta}{kT}}+1}} \qquad \zeta = \alpha k T$$



LECTURE XVIII 3-18-61

Continuation of Fermi Function:

These functions appear frequently in integrals of the type:

$$I = \int_0^\infty f_0(E)\, g(E)\, dE$$

Lemma on integrals involving $f_0(E)$:

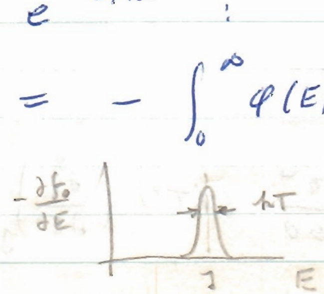
$$I = \int_0^\infty f_0 \left(\frac{\partial \varphi}{\partial E}\right) dE = f_0 \varphi \Big|_0^\infty - \int_0^\infty \varphi(E) \left(\frac{\partial f_0}{\partial E}\right) dE$$

Now: $\varphi = \int_0^E g(E')\, dE'$, $E = 0,\ f_0 = 1,\ \varphi = 0$

$E = \infty,\ \varphi \to \infty,\ f_0 \to 0$ exponentially

Thus, for φ's which go to infinity less rapidly than $e^{E/kT}$:

$$I = -\int_0^\infty \varphi(E) \left(\frac{\partial f_0}{\partial E}\right) dE \longrightarrow \varphi(\zeta) \quad \text{as } T \to 0$$



now $\dfrac{\partial f_0}{\partial E} = \dfrac{-\frac{1}{kT}\, e^{\frac{E-\zeta}{kT}}}{\left\{e^{\frac{E-\zeta}{kT}}+1\right\}^2}$

It is reasonable to use a series expansion of $\varphi(E)$ about ζ because of above graph.

$$I = \int_0^\infty \left[\varphi(\zeta) + \left(\frac{\partial\varphi}{\partial E}\right)_\zeta (E-\zeta) + \dots\right] \frac{e^{\frac{E-\zeta}{kT}}}{\left\{e^{\frac{E-\zeta}{kT}}+1\right\}^2} \frac{dE}{kT}$$

Put $x = \dfrac{E-\zeta}{kT}$

$$I = \int_{-\frac{\zeta}{kT}}^{\infty} \left\{ \varphi(0) + \left(\frac{\partial\varphi}{\partial x}\right)_0 x + \varphi''(0)\frac{x^2}{2} + \dots\right\} \frac{e^x\, dx}{(e^x+1)^2}$$

If $kT \ll \zeta$, as usual in most metals, lower limit $\to -\infty$

note $\dfrac{e^x}{(e^x+1)^2} = \dfrac{1}{(e^x+1)(e^{-x}+1)}$ which is an even function, thus odd terms in series vanish:

$$I = \int_{-\infty}^{\infty} \frac{\varphi(0)\, e^{-x}\, dx}{(e^{-x}+1)^2} + \frac{1}{2}\int_{-\infty}^{\infty} \frac{\varphi''(0)\, x^2 e^{-x}}{(e^{-x}+1)^2}\, dx$$

$$= \varphi(0) + \frac{\pi^2}{6}\varphi''(0) + \dots\ O(\varphi^{iv})$$

Using original notation:

$$I = \varphi(\zeta) + \frac{\pi^2}{6}(kT)^2 \frac{\partial^2\varphi(\zeta)}{\partial E^2}$$

$$= \int_0^\zeta g(E')\, dE' + \frac{\pi^2}{6}(kT)^2\left(\frac{\partial g}{\partial E}\right)_\zeta + \frac{7\pi^4}{360}(kT^4)\left(\frac{\partial^3 g}{\partial E^3}\right)_\zeta + \dots$$

Calculation of Density of States:

Recall Bloch functions: $\psi_k(r) = u_k(r)\, e^{i(k\cdot r - \omega t)}$

Use BVK Boundary Conditions:

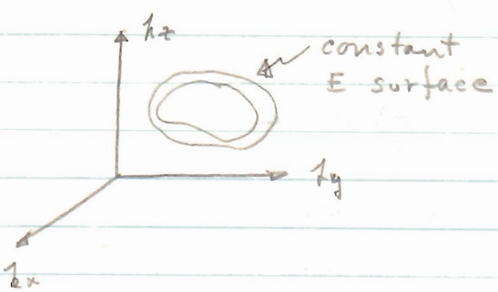
$$\psi_k(r) = \psi_k(r + L)$$

Will get as usual: $k_x L_x = 2\pi n_1$
$k_y L_y = 2\pi n_2$
$k_z L_z = 2\pi n_3$

$$dn = dn_1\, dn_2\, dn_3 = \frac{L_x L_y L_z}{(2\pi)^3}\,(dk_x\, dk_y\, dk_z) = \frac{V}{8\pi^3}\, dk$$

$$= \rho(k)\, dk = \frac{V}{4\pi^3}\, dk \quad \text{(including spin)}$$

We want relation between $\rho(k)\,dk$ and $N(E)\,dE$:



In general, would get

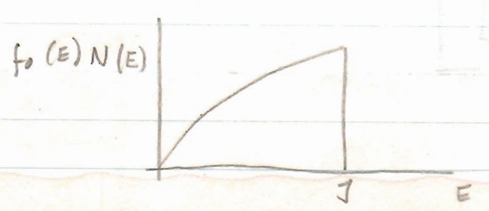
$$N(E)\, dE = \int \frac{dS}{|\nabla_k E|}\, dE$$

For metals use effective mass approximation:
$E = \dfrac{\hbar^2 k^2}{2m^*}$. Constant E surfaces are spheres,
$\therefore\ dk = 4\pi k^2 dk = 4\pi k\,(k\,dk)$

Then $dE = \dfrac{\hbar^2}{m^*}\, k\, dk$. Now use $k = \left(\dfrac{2m^* E}{\hbar^2}\right)^{1/2}$

and get:

$$\boxed{N(E)\, dE = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\sqrt{E}\; dE}$$



Calculation of Fermi Energy:

The total number of electrons is:

$$N = \int_0^\infty n(E)\,dE = \int_0^\infty N(E)\, f_0(E)\, dE$$

$$= \frac{V}{2\pi^2}\left(\frac{2m^*}{\hbar^2}\right)^{3/2} \underbrace{\int_0^\infty \sqrt{E}\, f_0(E)\, dE}_{\left\{ \frac{2}{3}\zeta^{3/2} + \frac{\pi^2}{6}(kT)^2 \cdot \frac{1}{2}\zeta^{-1/2} + \dots \right\}}$$

Consider first $T=0$:

$$N = \frac{V}{3\pi^2}\left(\frac{2m^*}{\hbar^2}\right)^{3/2} \zeta^{3/2}(0)$$

$$\zeta(0) = (3\pi^2)^{2/3}\left(\frac{N}{V}\right)^{2/3}\left(\frac{\hbar^2}{2m^*}\right)$$

Metal	$\zeta(0)$ eV	$\frac{\zeta(0)}{k}$ °K	M. P. °K
Li	4.76	55,400	150
Na	3.15	36,000	-100
Rb	1.84	21,000	39°
Cu	7.1	82,500	1083

It is seen that $\frac{(kT)^2}{\zeta}$ is such that higher order terms can be dropped. Now consider:

$$N = \frac{2}{3}\left(\frac{V}{2\pi^2}\right)\left(\frac{2m^*}{\hbar^2}\right)^{3/2}\zeta^{3/2}(T) + \frac{\pi^2}{6}(kT)^2\frac{V}{4\pi^2}\left(\frac{2m^*}{\hbar^2}\right)^{3/2}\frac{1}{\sqrt{\zeta(T)}}$$

Using definition of $\zeta(0)$; $\zeta(T) = \dfrac{\zeta(0)}{\left[1+\frac{1}{8}\pi^2\left(\frac{kT}{\zeta(T)}\right)^2\right]^{2/3}}$

Then:

$$\boxed{\zeta(T) = \zeta(0)\left[1 - \frac{\pi^2}{12}\left(\frac{kT}{\zeta(0)}\right)^2\right]}$$

LECTURE XVIX 3-21-61

Transition to Boltzmann Statistics:

Fermi: $n_s = \dfrac{N_s}{e^{\frac{E-\zeta}{kT}}+1} = \dfrac{N_s}{e^{\beta E_s - \alpha}+1}$

Boltzmann: $\dfrac{n_s}{N} = \dfrac{N_s e^{-E_s/kT}}{\sum_s N_s e^{-E_s/kT}}$

Fermi to Boltzmann: $e^{-\alpha} >> 1$

then $\dfrac{n_s}{N_s} = e^{-\beta E_s} e^{\alpha} << 1$, low density of electrons.

now: $\dfrac{n_s}{N_s} = e^{-\beta E_s} e^{\alpha} = \dfrac{N e^{-E_s/kT}}{\sum_s N_s e^{-E_s/kT}}$

Thus $e^{-\alpha} = \dfrac{\sum_s N_s e^{-E_s/kT}}{N} >> 1$

Free Electrons Case: N_s is given by density of states for quasi-free electrons:

$$e^{-\alpha} = \frac{\frac{V}{2\pi^2}(kT)^{3/2}\int_0^\infty \left(\frac{2m^*}{\hbar^2}\right)^{3/2} \frac{\sqrt{E}}{(kT)^{3/2}} e^{-E/kT} dE}{N}$$

$$= \left(\frac{V}{N}\right)\frac{1}{2\pi^2}\left(\frac{2m^* kT}{\hbar^2}\right)^{3/2}\int_0^\infty \sqrt{x}\, e^{-x} dx$$

$$\therefore\ e^{-\alpha} = 2\left(\frac{V}{N}\right)\left(\frac{2\pi m^* kT}{h^2}\right)^{3/2} >> 1$$

is the criteria for the Boltzmann limit.

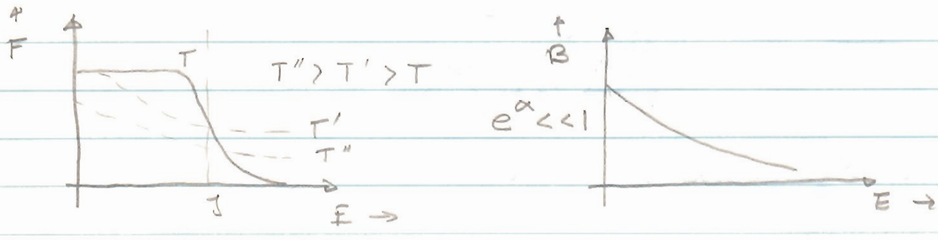
Take the de Broglie wavelength:

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mE}}$$

Now $E \sim \frac{3}{2} kT$ (approximately)

$$\therefore \quad \lambda = \frac{h}{\sqrt{2mE}} = \frac{h}{\sqrt{3mkT}}$$

Then $e^{-\alpha} = \left(\frac{2\pi}{3}\right)^{3/2} \frac{V}{N} \left(\frac{1}{\lambda}\right)^3 \gg 1$



Degeneration of Fermi to Boltzmann as temperature increases.

Relation of Fermi Distribution to Thermodynamics and Statistical Mechanics.

Recall: $\quad S = k \ln \Omega_{max}$

$$\ln \Omega = \sum_s' \left\{ N_s \ln N_s - n_s \ln n_s - (N_s - n_s) \ln (N_s - n_s) \right\}$$

$$= \sum_s' \left\{ N_s \ln \frac{N_s}{N_s - n_s} - n_s \ln \frac{n_s}{N_s - n_s} \right\}$$

Using: $\quad \frac{n_s}{N_s} = \frac{1}{e^{\frac{E_s - \zeta}{kT}} + 1} \; ; \quad e^{E_s - \zeta/kT} = \frac{N_s}{n_s} - 1$

$$\therefore \quad \sum' n_s \ln \frac{n_s}{N_s - n_s} = \sum' n_s \ln \left(\frac{N_s}{n_s} - 1 \right)$$

$$= \sum' n_s \left(\frac{E_s - \zeta}{kT} \right) = \frac{E - N\zeta}{kT}$$

Now: $\quad \sum_s' N_s \ln \frac{N_s}{N_s - n_s} = \sum' - N_s \ln \left(1 - \frac{n_s}{N_s} \right)$

$$= - \sum' N_s \ln \frac{e^{(\,)}}{e^{(\,)} + 1} = \sum' N_s \ln \left(1 + e^{-(\,)} \right)$$

Then:

$$\ln \Omega_{max} = \sum_s N_s \ln\left(1 + e^{-\frac{(E_s - \mathcal{J})}{kT}}\right) + \left(\frac{E - N\mathcal{J}}{kT}\right)$$

and:

$$S = k\frac{(E - N\mathcal{J})}{kT} + k \sum N_s \ln\left(1 + e^{-\frac{(E_s - \mathcal{J})}{kT}}\right)$$

Now: $E = \sum \dfrac{N_s E}{e^{\frac{E_s - \mathcal{J}}{kT}} + 1}$

Then the Helmholtz Free Energy:

$$F = (E - TS) = N\mathcal{J} - kT \sum_s N_s \ln \cancel{N_s}\left(1 + e^{\frac{-(E_s - \mathcal{J})}{kT}}\right)$$

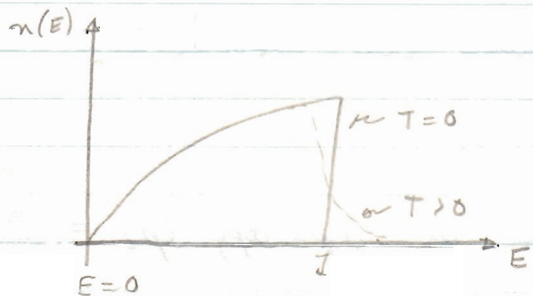
not similar to classical $F = -kT \ln Z$
where $Z = \sum N_s e^{-E_s/kT}$.

Distribution Functions:

(1) $n(E)\,dE$ (Energy Distribution Function)
= number of particles with energy between
E and $E + dE$ = $N(E)\, f_0(E)\, dE$

For free particle:

$$n(E) = \frac{V}{2\pi^2}\left(\frac{2m^*}{\hbar^2}\right)^{3/2} \frac{\sqrt{E}}{e^{E - \mathcal{J}/kT} + 1}$$



Can show at
$T = 0$, $E = \frac{3}{5}\mathcal{J}$
(because of exclusion)
while classically
$E = 0$, when $T = 0$

LECTURE XX   3-28-61

Distribution Functions:

1) Energy Distribution

$$n(E)\,dE = \frac{V}{2\pi^2}\left(\frac{2m^*}{\hbar^2}\right)^{3/2}\frac{\sqrt{E}}{e^{\frac{E-\zeta}{kT}}+1}$$

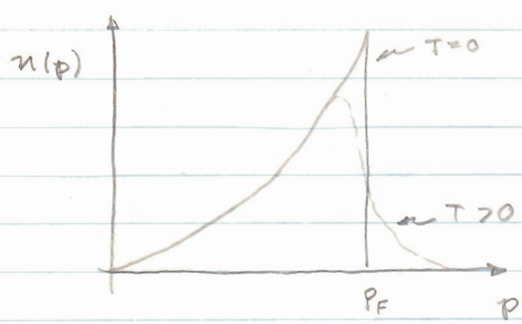
2) Momentum Distribution Function:

$$E = \frac{p^2}{2m^*}$$

$$n(E)\,dE = n(p)\,dp$$

$$n(p) = n(E)\frac{dE}{dp} = \frac{V}{2\pi^2}\left(\frac{2m^*}{\hbar^2}\right)^{3/2}\frac{p\,\frac{p}{2m^*}}{\sqrt{2m^*}}\,\frac{1}{e^{\frac{\frac{p^2}{2m^*}-\zeta}{kT}}+1}$$

$$= \frac{V}{\pi^2\hbar^3}\,\frac{p^2}{\left\{e^{\frac{\frac{p^2}{2m^*}-\zeta}{kT}}+1\right\}}$$



3) Distribution in Momentum space: p_x, p_x+dp_x, etc.

Density of electrons in momentum space

$$= \frac{n(p)\,dp}{4\pi p^2\,dp}$$

Number of electrons in $dp_x\,dp_y\,dp_z = \frac{n(p)}{4\pi p^2}\,dp_x\,dp_y\,dp_z$

$$n(p_x p_y p_z) = \frac{V}{4\pi^3\hbar^3}\,\frac{1}{\left\{e^{\frac{p_x^2+p_y^2+p_z^2}{2m^*}-\zeta}+1\right\}}$$

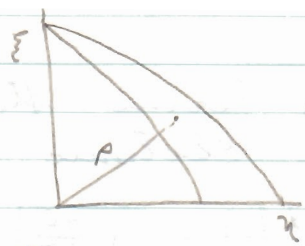
4) How many electrons have momentum between p_x and p_x+dp_x regardless of p_y, p_z ?

$$n(p_x)\,dp_x = \frac{V\,dp_x}{4\pi^3\hbar^2}\iint_{-\infty}^{\infty} dp_y\,dp_z \,\frac{1}{\left[e^{\frac{p_y^2+p_z^2}{2mkT}}\,e^{\frac{\frac{p_x^2}{2m}-\zeta}{kT}}+1\right]}$$

Make the substitution ; $\xi^2 = \dfrac{p_y^2}{2mkT}$; $\eta^2 = \dfrac{p_z^2}{2mkT}$

Then $$n(p_x) = \frac{V(2mkT)}{4\pi^3\hbar^3}\iint_{-\infty} \frac{d\xi\,d\eta}{e^{\xi^2+\eta^2}e^{a}+1}$$

$$a = \left(\frac{p_x^2}{2mkT} - \frac{\zeta}{kT}\right)$$



$2\pi\rho\,d\rho = d\xi\,d\eta$

$\xi^2+\eta^2 = \rho^2$

Then: $$n(p_x) = \frac{2mkTV\,2\pi}{4\pi^3\hbar^3}\int_0^\infty \frac{\rho\,d\rho}{e^{\rho^2}e^{a}+1}$$

$$= \frac{2mkTV\,2\pi}{4\pi^3\hbar^3}\underbrace{\int_0^\infty \frac{\rho\,d\rho\,e^{-\rho^2}}{e^{a}+e^{-\rho^2}}}_{= -\frac{1}{2}\ln\left(e^{a}+e^{-\rho^2}\right)\Big|_0^\infty}$$

$$= \frac{1}{2}\ln\frac{e^{a}+1}{e^{a}} = \frac{1}{2}\ln\left(1+e^{-a}\right)$$

Finally :

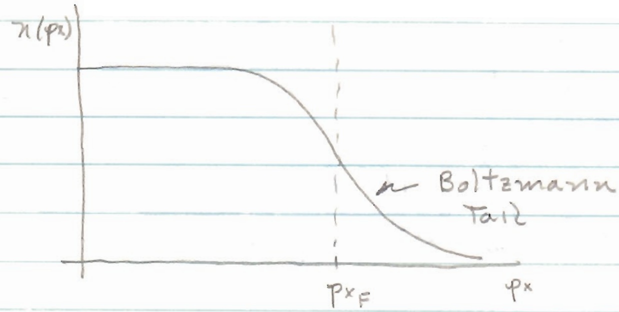
$$\boxed{n(p_x) = \frac{V m^* kT}{2\pi^2\hbar^3}\ln\left\{1+e^{-\frac{\left(\frac{p_x^2}{2m}-\zeta\right)}{kT}}\right\}}$$

a) low energy $\dfrac{p_x^2}{2m} \ll \zeta$

$$\boxed{n(p_x) = \frac{m^* V \zeta}{2\pi^2\hbar^3}}$$

b) $\frac{p_x^2}{2m} \gg \mathcal{J}$, use $\ln(1+e^{-x}) \sim e^{-x}$

$$\boxed{n(p_x) = \frac{mkTV}{2\pi^2\hbar^3} e^{\frac{-\frac{p_x^2}{2m}+\mathcal{J}}{kT}}}$$



To go from $n(p_x)$ to $n(v_x)$: $n(p_x)\,dp_x = n(v_x)\,dv_x$

or $n(v_x) = m^* n(p_x)$

Applications of Fermi Statistics

a) Calculate Electronic Specific Heat

$$C_v = \left(\frac{dQ}{dT}\right)_v = \left(\frac{\partial E}{\partial T}\right)_v \quad \text{since } dQ = dE + PdV$$

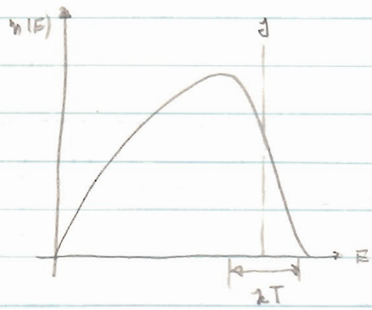
We have for lattice vibration contribution:

$C_v = 3NkT$, with N atoms

Classically for electrons, $C_v = \frac{3}{2}Nn kT$, so total specific heat should be:

$$\boxed{C_v = 3N\left(1+\frac{n}{2}\right)k}$$

where n is the number of conduction electrons per atom.



$$\langle E\rangle_{\text{electron}} = \left(\frac{3}{2}kT\right) Nn \left(\frac{kT}{\mathcal{J}}\right)$$

$$= \frac{3}{2}\frac{(kT)^2}{\mathcal{J}} Nn$$

with $C_v = 3kT\frac{k}{\mathcal{J}} Nn$

$$= (3Nnk)\left(\frac{kT}{\mathcal{J}}\right) = 3Nk\left(\frac{T}{\frac{\mathcal{J}}{k}}\right)$$

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Electronic Specific Heat:

Total energy: $E_T = \int_0^\infty n(E)\, E\, dE = \int_0^\infty N(E)\, E\, f_0(E)\, dE$

$$= \underbrace{\left[\int_0^{J} E\, N(E)\, dE \right]}_{A} + \frac{\pi^2}{6} (kT)^2 \left(\frac{\partial \{E N(E)\}}{\partial E} \right)_{E=J} + \cdots$$

where $N = \frac{V}{2\pi^2} \left(\frac{2m^*}{\hbar^2} \right)^{3/2} \sqrt{E}$, N is total # of electrons

$$N(E) = \frac{3}{2} \frac{N \sqrt{E}}{J^{3/2}(0)}$$

because: $\int_0^{J(0)} N(E)\, dE = N \overset{?}{=} \frac{3N}{2 J^{3/2}(0)} \int_0^{J(0)} \sqrt{E}\, dE = N$

Then we can write $E\, N(E) = \frac{3}{2} N \left(\frac{E}{J(0)} \right)^{3/2}$

and $\int_0^{J} E\, N(E)\, dE = \frac{3}{5} N\, J(0)$

or $\boxed{E(0) = \frac{3}{5} N\, J(0)}$

or in general, for $T \neq 0$: $\boxed{E_A = \frac{3}{5} N\, J(0) \left(\frac{J(T)}{J(0)} \right)^{5/2}}$

so the average energy of an electron at $T = 0$ is $3/5\ J(0)$, where classically it would be zero.

now: $\left[\frac{d(E N(E))}{dE} \right]_{E=J} = \frac{9}{4} \frac{N E^{1/2}}{J^{3/2}(0)} \Bigg|_{E=J} = \frac{9}{4} \frac{N}{J(0)} \left(\frac{E}{J(0)} \right)^{1/2} \Bigg|_{E=J}$

or

$$E_T(T) = \frac{3}{5} N\, J(0) \left(\frac{J(T)}{J(0)} \right)^{5/2} + \frac{\pi^2}{6} (kT)^2 \frac{9}{4} \frac{N}{J(0)} \left(\frac{J(T)}{J(0)} \right)^{1/2}$$

Recalling: $\dfrac{\mathcal{J}(T)}{\mathcal{J}(0)} = 1 - \dfrac{\pi^2}{12}\dfrac{(kT)^2}{\mathcal{J}^2(0)}$

which gives, keeping to order T^2:

$$E_T(T) = \frac{3}{5} N \mathcal{J}(0)\left[1 - \frac{\pi^2}{12}\left(\frac{kT}{\mathcal{J}(0)}\right)^2\right]^{5/2} + \frac{\pi^2}{6}(kT)^2 \frac{9}{4}\frac{N}{\mathcal{J}(0)}$$

or:

$$\boxed{E_T(T) = \frac{3}{5} N \mathcal{J}(0)\left[1 + \frac{5\pi^2}{12}\left(\frac{kT}{\mathcal{J}(0)}\right)^2\right]}$$

thus,
$$C_V = \left(\frac{\partial E}{\partial T}\right)_V = \frac{3}{5} N \mathcal{J}(0)\,\frac{5\pi^2}{12}\, 2\left(\frac{kT}{\mathcal{J}(0)}\right)\frac{k}{\mathcal{J}(0)}$$
$$= \frac{\pi^2}{2} N k \left(\frac{kT}{\mathcal{J}(0)}\right)$$

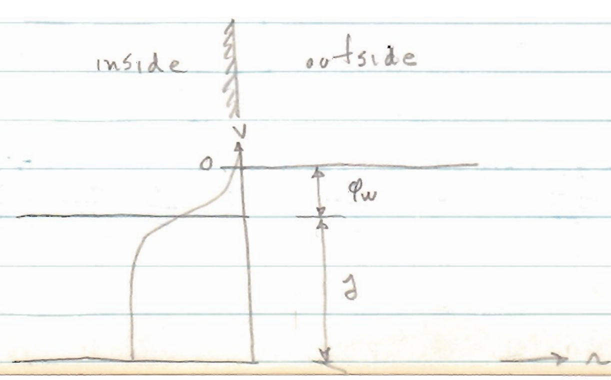
or
$$\boxed{C_V = \frac{\pi^2}{2} N k \left(\frac{T}{T_F(0)}\right)}$$

Measurement: $C_V\,(\text{lattice}) \cong Nk\left(\frac{T}{\Theta}\right)^3$

$C_V\,(\text{electrons}) \sim Nk\left(\frac{T}{T_F(0)}\right)$

so that at low temperatures can see C_V due to electrons. However, at high temperatures $C_V\,(\text{lattice}) \sim 3Nk$ and $C_V\,(\text{electron}) \sim T$ so could measure gradual increase in slope to get C_V (electron)

b) Thermionic Emission:



See that enough tail exists so emission takes place.

No. of electrons striking surface in time dt and area dA $= \frac{n(p_x)}{V} (v_x\, dt)\, dA$

Define Reflection coefficient $= r(p_x)$ = probability that particle with momentum p_x will be reflected. Then, number of electrons escaping

$$= \int_{p_x = \sqrt{2m(J+\varphi_w)}}^{\infty} \frac{n(p_x)}{V} \frac{p_x}{m} \left(1 - r(p_x)\right) dp_x\, dt\, dA$$

Except for tunneling, we must have

$$\frac{p_x^2}{2m} > J + \varphi_w$$

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Thermionic Emmission:

The current density is then:

$$J = \frac{e}{m^*} \int_{p_{x\,min}}^{\infty} \frac{N}{V}(p_x)\, p_x\, dp_x \left(1 - r(p_x)\right)$$

Now: $$\frac{N}{V}(p_x) = \frac{1}{4\pi^3} \iint_{-\infty}^{\infty} \frac{dp_y\, dp_z}{e^{\left(\frac{p^2}{2m} - J\right)/kT} + 1}$$

For $\frac{p_x^2}{2m} > (J + \varphi_w)$, $\frac{\varphi_w}{kT} >> 1$

Then:
$$\frac{N}{V}(p_x) = \frac{1}{2\pi^2}\left(\frac{m^* kT}{\hbar^3}\right) e^{J/kT} e^{\frac{-p_x^2}{2m^* kT}}$$

We use an average $r(p_x)$ so we can remove it from the integral.

$$\therefore \quad J = \overline{(1-r)} \; \frac{e}{m^*} \; \frac{1}{2\pi^2} \; \frac{m^* kT}{\hbar^3} \; e^{\mathcal{J}/kT} \int_{p_{x\,min}}^{\infty} \frac{p_x \, e^{-\frac{p_x^2}{2m^* kT}} \, dp_x}{(2m^* kT)}$$

Let $y = \dfrac{p_x}{\sqrt{2m^* kT}}$, $\quad y_{min} = \sqrt{\dfrac{2m^*(\mathcal{J} + \varphi_w)}{2m^* kT}}$

$$J = \overline{(1-r)} \; \frac{e}{m^*} \; \frac{4(m^* kT)^2}{2\pi^2} \; \frac{e^{\mathcal{J}/kT}}{2\hbar^3} \underbrace{\int_{y_{min}}^{\infty} y \, e^{-y^2} \, dy}_{-\frac{1}{2} e^{-y^2} \Big|_{y_{min}}^{\infty} = \frac{1}{2} e^{-\frac{\mathcal{J} + \varphi_w}{kT}}}$$

Therefore:

$$\boxed{J = \overline{(1-r)} \; \frac{e \, m^* (kT)^2}{2\pi \, \hbar^3} \; e^{-\frac{\varphi_w}{kT}}}$$

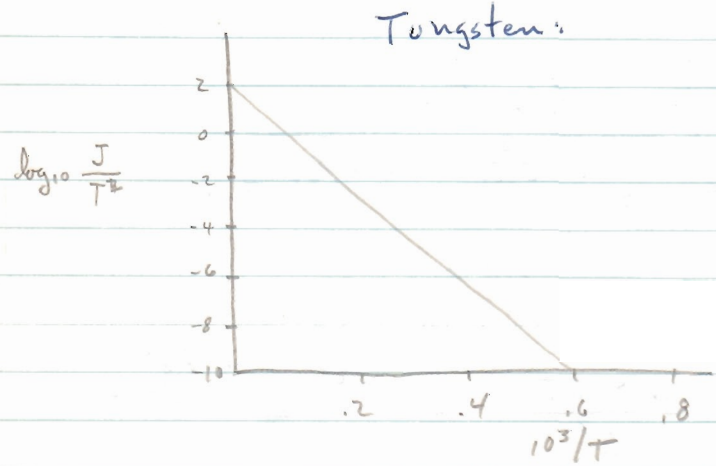
or $\quad J = A \, \overline{(1-r)} \, T^2 \, e^{-\varphi_w/kT}$

where $A = 120$ amp/cm^2 (K°)^2

Called Dushman - Richardson equation.
To show strong exponential dependence, examine:
$\ln\left(\frac{J}{T^2}\right)$ vs $\frac{1}{T}$

using $\quad \ln\left(\frac{J}{T^2}\right) = \ln\left(A\,\overline{(1-r)}\right) - \frac{\varphi_w}{kT}$

Tungsten:



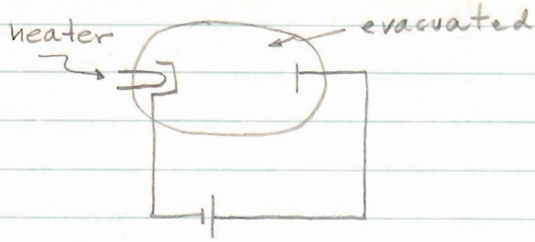
$\varphi_w = 4.54$ ev

$\frac{\varphi_w}{k} = 52{,}600$ °K

For $T = 1000$, $e^{-52.6}$

$T = 2000$, $e^{-26.3}$

Experimental Determination:



Conditions:

1) All emitted electrons must be drawn away from surface – no space charge
2) Not too big voltage – field emission
3) High vacuum with no residual gas.

4) Interpretation
   a) include effect of thermal expansion on q_w
   b) polycrystalline emitter gives an average q_w
   c) Difference between surface area measured geometrically and actual case.

Typical Values:

	q_w (eV)
Al	4.20
Ag	4.89
Ba	2.51
Co	4.41
Cs	1.93
K	2.22
W	4.54

What is distribution of electrons outside Metal?

Energy Distribution in the Emitted Electrons:

Number of electrons in velocity range $|d\vec{v}|$ striking surface

$$= \frac{1}{4\pi^3\hbar^3}\left\{ \frac{v_{xi}\, dv_{xi}\, dv_{yi}\, dv_{zi}}{e^{\frac{\frac{1}{2}m(v_{xi}^2 + v_{yi}^2 + v_{zi}^2) - J}{kT}} + 1} \right\}$$

$$\tfrac{1}{2} m v_{xi}^2 \geq J + q_w \qquad v_{y0}^2 = v_{iy}^2$$

$$\tfrac{1}{2} m v_{xi}^2 - (J + q_w) = \tfrac{1}{2} m v_{x0}^2 \qquad v_{z0} = v_{zi}$$

Now the distribution outside is:

$$\frac{N}{V}(v_{x0})\, v_{x0}\, dv_{x0}\, dv_{y0}\, dv_{z0}$$

Using the above relations:

$$\frac{N}{V}(v_{x0}) = \frac{1}{4\pi^3\hbar^3}\left\{\frac{1}{e^{\frac{\frac{1}{2}m(v_{x0}^2+v_{y0}^2+v_{z0}^2)+\varphi W}{kT}}+1}\right\}$$

no. outside moving away $= \frac{N}{V}(v_{x0})\, v_{x0}\, dv_{x0}\, dv_{y0}\, dv_{z0}$

$$0 < v_{x0} < \infty$$
$$-\infty < v_{y0} < \infty$$
$$-\infty < v_{z0} < 0$$

$$\bar{E}_{kin} = \frac{\int \frac{1}{2}m(v_x^2+v_y^2+v_z^2)\left(\frac{dN}{dt}\right) dv_x\, dv_y\, dv_z}{\int\left(\frac{dN}{dt}\right) dv_x\, dv_y\, dv_z}$$

$$= \frac{\int_0^\infty v_x\, dv_x \int_{-\infty}^{\infty} dv_y \int_{-\infty}^{\infty} dv_z\ \frac{1}{2}m(v_x^2+v_y^2+v_z^2)\, e^{-\frac{\frac{1}{2}m(v_x^2+v_y^2+v_z^2)}{kT}}\, e^{-\frac{\varphi W}{kT}}}{\int_0^\infty v_x\, dv_x \iint_{-\infty}^{+\infty} e^{-\frac{1}{2}m(\ )/kT}\, e^{-\varphi W/kT}}$$

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Evaluate $\bar{E}_{kin}$ by putting: $\frac{1}{2}\frac{m v_x^2}{kT} = \chi^2$, etc.

and get $\bar{E}_{kin} = 2kT$ which is energy that electron takes from metal and results in a cooling process. Get extra degree of freedom from v_x term.

If effective mass is included:

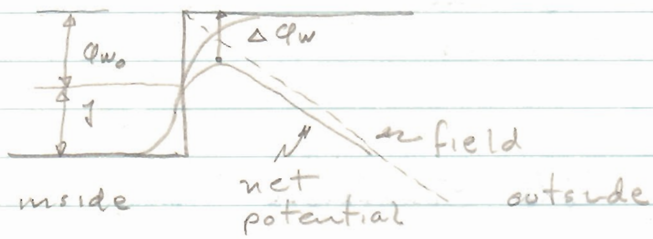
$$\frac{1}{2}m^* v_{xi}^2 - (J + \varphi W) = \frac{1}{2}m v_{x0}^2$$

$$m^* v_{yi} = m v_{y0}$$
$$m^* v_{zi} = m v_{z0}$$

Then distribution becomes:

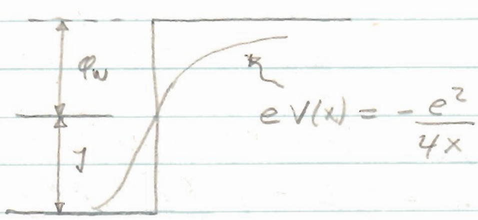
$$N(v_0) = \frac{\left(\frac{m}{m^*}\right)^3}{e^{\frac{\frac{1}{2}m\left[v_{x0}^2 + \frac{m}{m^*}v_{y0}^2 + \frac{m}{m^*}v_{z0}^2\right] - \varphi_W}{kT}} + 1}$$

Field Enhanced Emission: Schottky Effect
(Lowering of φ_W by field strength)



$\varphi_W = \varphi_{W_0} - \Delta\varphi_W$

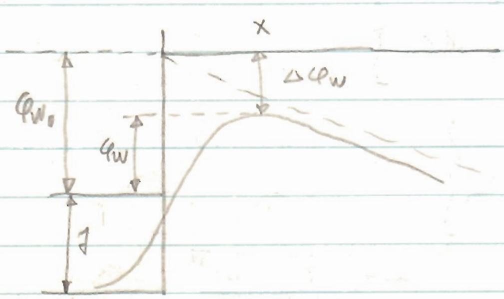
ⓐ What is the P.E. of an electron vs x as electron approaches metal surface?



Use mirror image

$\mathcal{F} = \frac{e^2}{4x^2}$

For classical approach to hold, $eV \gg \varphi_W$, find that $x > 10$ Å



Now the total field is:

$$eV_{TOT} = \frac{-e^2}{4x} - eEx$$

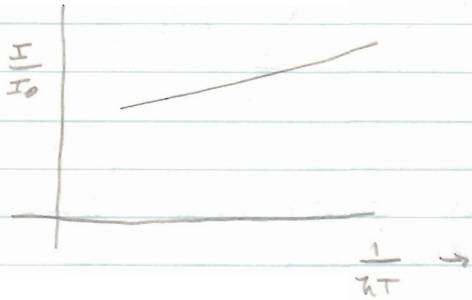
Minimize to get x_0:

$$x_0 = \frac{1}{2}\sqrt{\frac{e}{E}}$$

$$\therefore \Delta \varphi_W = \frac{-e^2}{4 \cdot \frac{1}{2}\sqrt{\frac{e}{E}}} - \frac{eE}{2}\sqrt{\frac{e}{E}} = -e^{3/2}\sqrt{E}$$

and $\varphi_W = \varphi_{W_0} - e^{3/2}\sqrt{E}$

and $J = J_0 \, e^{\frac{e^{3/2}\sqrt{E}}{kT}}$

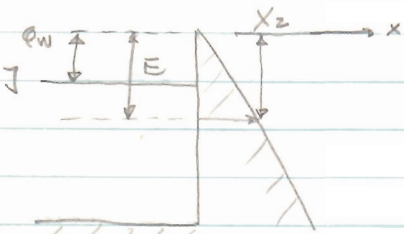


Plotting $\ln\left(\frac{I}{I_0}\right) = \frac{e^{3/2}\sqrt{E}}{kT}$

Typical Values: $E = 10^3 \, v/cm$
$x_0 = 10^{-5} \, cm$
$\Delta\varphi \simeq .01 \, eV$

High Field Emission: Cold Emission

Reference: Good & Muller, Hand. der Phys., Vol XXI



Tunneling occurs due to thiness of barrier.

Consider one-dimensional case:
Energy parameter is: $W = E - \left(\frac{p_y^2 + p_z^2}{2m}\right)$

Probability that electron with "x directed energy" is given by WKB approximation is:

$$D(W) = \exp\left\{-\int_{x_1}^{x_2}\sqrt{\frac{8m}{\hbar^2}\left(V(x) - W\right)}\,dx\right\}$$

now $D(W) = \exp\left\{-\int_0^{\frac{W}{eF}}\sqrt{\frac{-eFx + |W|}{\hbar^2/8m}}\,dx\right\}$

$$-\ln D(W) = \frac{2}{3}\sqrt{\frac{8m}{\hbar^2}}\,\frac{|W|^{3/2}}{eF}\;;\quad \therefore\; D(W) = e^{-\frac{4\sqrt{2m|W|^3}}{3\hbar eF}}$$

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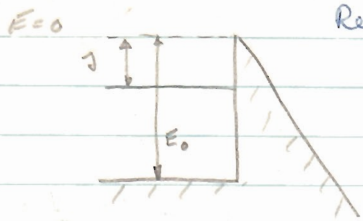


Recall:

$$|\vec{J}| = \rho w$$

and: $$E = \underbrace{E_0 + \frac{p_x^2}{2m^*}}_{W} + \left(\frac{p_y^2 + p_z^2}{2m^*}\right)$$

and we found: $$D(W) = \exp\left\{\frac{-\frac{4}{3}\sqrt{2m|W|^3}}{\hbar e F}\right\}$$

Now the emission current is:

$$J = e \cdot 2 \iiint \frac{N(p_x p_y p_z)}{V} \frac{p_x}{2m^*} dp_x\, dp_y\, dp_z\, D(W)$$

$$\frac{p_x\, dp_x}{m^*} = dW; \quad J_0 = e \int_{W=-|E_0|}^{\infty} dW\, D(W) \underbrace{\iint \frac{N}{V}(p_x p_y p_z)\, dp_y\, dp_z}_{I}$$

where $$I = \frac{1}{4\pi^3\hbar^3} \iint \frac{dp_y\, dp_z}{e^{\frac{E-J}{kT}} + 1}$$

$$= \frac{1}{4\pi^3\hbar^3} \iint_{-\infty}^{\infty} \frac{dp_y\, dp_z}{e^{\frac{W-J}{kT}}\, e^{\frac{(p_y^2+p_z^2)}{2m^*kT}} + 1}$$

since $$E = W + \frac{p_y^2 + p_z^2}{2m^*}$$

Change to $\begin{matrix} p_x = \rho \cos\theta \\ p_y = \rho \sin\theta \end{matrix}$: Then:

$$I = 2\pi \int \frac{\rho\, d\rho}{e^{\frac{W-J}{kT}}\, e^{\frac{\rho^2}{2m^*kT}} + 1} = 2\pi \int \frac{\rho\, d\rho\, e^{\frac{-\rho^2}{2m^*kT}}}{e^{\frac{W-J}{kT}} + e^{-\frac{\rho^2}{2m^*kT}}}$$

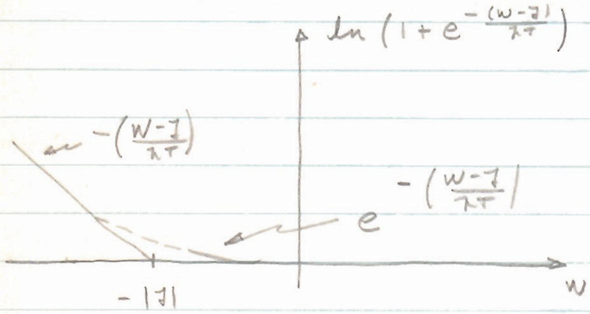
$$= 2\pi m^* kT \ln\left(e^{\frac{W-J}{kT}} + e^{\frac{-\rho^2}{2m^*kT}}\right)\Bigg|_{\infty}^{0}$$

or $$I = 2\pi m^* kT \ln\left[1 + e^{-\frac{(W-J)}{kT}}\right] \frac{1}{4\pi^3\hbar^3}$$

and: $$J = \frac{e\, m^* kT}{2\pi^2 \hbar^3} \int_{-|E_0|}^{\infty} dW\, D(W)\, \ln\left[1 + e^{-\left(\frac{W-J}{kT}\right)}\right]$$

We have two cases: $-|J| < W < \infty$

or $-|E_0| < W < -|J|$



$-|J| < W < +\infty$; $\ln(\,) \to e^{-\left(\frac{W-J}{kT}\right)}$

$-|E_0| < W < -|J|$; $\ln(\,) \to -\left(\frac{W-J}{kT}\right)$

However, $D(W)$ behaves so strongly below J that most of contribution is from around J. Physically, above J, not many available electrons. Below J; wall is too thick. Therefore: Expand $|W|^{3/2}$ around J:

$$|W|^{3/2} = |J|^{3/2} - \frac{3}{2}|J|^{1/2}(W - J) + \cdots \qquad J = -|J|$$

$$\therefore D(W) = e^{-c + \frac{(W-J)}{d}}$$

where $c = \frac{4}{3}\frac{\sqrt{2m^* |J|^3}}{\hbar e F}$, $d = \frac{\hbar e F}{2(2m^*|J|)^{1/2}}$

Then:

$$J = \frac{e\, m^* kT}{2\pi^2 \hbar^3}\, \frac{(-1)\, e^{-c}}{kT} \int_{-|E_0|}^{-|J|} (W-J)\, dW\, e^{-\frac{(W-J)}{d}}$$

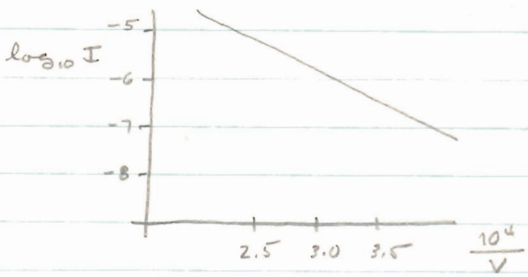
Change variables: $x = -\frac{(W-J)}{d}$

$$\therefore J = \frac{e m^* (-1)\, e^{-c}\, d^2}{2\pi^2 \hbar^3} \int_{\infty}^{0} (-x)\, e^{-x}\, (-dx)$$

or $$\boxed{J = \frac{e^3 F^2}{8\pi h\, \phi_W}\, e^{-\frac{4}{3}\frac{\sqrt{2 m^* \phi_W^3}}{\hbar e F}}}$$

First done by Fowler and Nordheim, Proc. Roy. Soc A119 173 (1928)

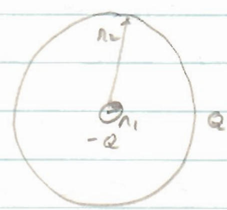
Plot $\ln\left(\frac{I}{F^2}\right) = C - \frac{1}{F}\,\frac{4}{3}\,\frac{\sqrt{2m^*\varphi_w^3}}{\hbar e}$



Experiment done by:

Dyke and Trolan PR 89 799, (1953)

Field at sharp Tip:



$$V(r) = Q\left(\frac{1}{r_2} - \frac{1}{r}\right)$$

$$V(A \to C) = Q\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$$

$$V(r) = \frac{\left(\frac{1}{r_2} - \frac{1}{r}\right) V_{AC}}{\left(\frac{1}{r_2} - \frac{1}{r_1}\right)}$$

$$\frac{\partial V}{\partial r} = \frac{-\frac{1}{r^2}\,V_{AC}}{\left(\frac{1}{r_2} - \frac{1}{r_1}\right)} \sim \frac{V_{AC}}{\left(\frac{r^2}{r_1}\right)} \to \frac{V_{AC}}{r_1}$$

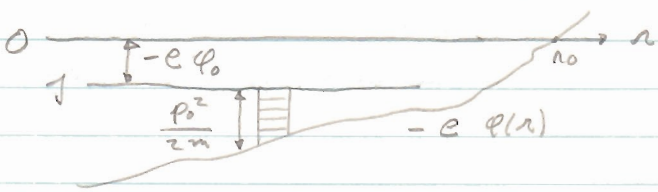
For $V_{AC} \sim 100\,V$, $r \sim 10^{-5}$, $E \sim 10^7\ v/cm$

Tip radius	φ_W	F : v/cm	J amp/cm²
$2 \cdot 10^{-5}$ cm	4.5	$2.6 \cdot 10^7$	1
$2 \cdot 10^{-5}$ cm	4.5	$6.4 \cdot 10^7$	10^7

LECTURE XXV   4-18-61

Fermi-Thomas Statistical Model of Atom:

We assume that the electrons are moving in some potential:



$T = 0$

Variation of potential is small over one de Broglie wavelength. That is, in a small volume, electrons are distributed as if they were free. ∴

$$-e\,\varphi(r) + \frac{p_0^2}{2m} = -e\,\varphi_0$$

and the state density is given by:

$$n(r) = \frac{8\pi}{3}\left(\frac{2m}{h^2}\right)^{3/2} \zeta_0^{3/2}(r) \quad , \quad \zeta_0 = \frac{p_0^2}{2m}$$

Then:

$$\boxed{\frac{p_0^2}{2m} = \left(\frac{3}{8\pi}\right)^{2/3} \frac{h^2}{2m}\, n(r)^{2/3}}$$

From $\frac{p_0^2}{2m} = e\left(\varphi(r) - \varphi_0\right)$ and the fact that $\frac{p_0^2}{2m}$ cannot be negative, thus r_0 denotes the boundary of the atom. We now write:

$$\left(\frac{3}{8\pi}\right)^{2/3} \frac{h^2}{2m}\, n(r)^{2/3} = e\left\{\varphi(r) - \varphi_0\right\}$$

From Poisson's Equation:

$$\boxed{\nabla^2\left\{\varphi(r) - \varphi_0\right\} = -4\pi\rho = 4\pi e\, n(r)}$$

We will strive for self-consistency in potential.

Therefore:

$$\boxed{\nabla^2\{\varphi(r)-\varphi_0\} = 4\pi e\left(\frac{8\pi}{3}\right)\left(\frac{2m}{h^2}\right)^{3/2}\{e\varphi(r)-e\varphi_0\}^{3/2}}$$

This is the Thomas - Fermi equation

a) Distribution of charge in neutral atom, r_0 at ∞, and $\varphi_0 = 0$.

Boundary Conditions:

Write $\varphi(r) = \frac{Ze}{r}\chi(r)$

As $r \to 0$, $\varphi \to \frac{Ze}{r}$, $\chi(r) \to 1$

As $r \to \infty$, $r\varphi(r) \to 0$, or $\chi(r) \to 0$ because at ∞, nucleus is completely shielded.

Now: $\nabla^2\varphi(r) = \frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{d}{dr}\left(\frac{Ze}{r}\chi(r)\right)\right]$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\left(\frac{Ze}{r}\chi' - \frac{Ze}{r^2}\chi\right)\right]$$

$$= \frac{Ze\chi''}{r}$$

Then: $\frac{Ze\chi''}{r} = \frac{32\pi^2}{3}e^{5/2}\left\{\frac{Ze}{r}\chi\right\}^{3/2}\left(\frac{2m}{h^2}\right)^{3/2}$

Make the substitutions $r = bx$

$b = \left(\frac{3\pi}{4}\right)^{2/3}\left(\frac{\hbar^2}{2me^2}\right)\frac{1}{Z^{1/3}}$

and get:

$$x^{1/2}\frac{d^2\chi}{dx^2} = \chi^{3/2} \quad ; \quad \chi(0) = 1, \ \chi(\infty) = 0$$

Numerically integrated by Bush and Caldwell PR 38, 1898 (1931)

x	0	.1	.2	.417	.500	1.00	3.90	10	30
χ	1	.882	.793	.650	.607	.125	.110	.0244	.0022

therefore: $n(r) = \frac{8\pi}{3}\left(\frac{2me^2 Z}{b}\right)^{3/2}\left\{\frac{\chi(x)}{x}\right\}$

Calculation of the Effective Radius of Atom:

Define r_{eff} as radius where half of electrons lie within:

$$\int_0^{r_{eff}} dr\, 4\pi r^2\, n(r) = \frac{Z}{2}$$

Use $r = bx$:

$$4\pi\,(\text{constant})\int_0^{x_{eff}} b\,dx\, b^2 x^2 \left(\frac{Z}{b}\right)^{3/2}\left(\frac{\chi(x)}{x}\right)^{3/2}$$

$$= \frac{Z}{2} = \text{const} \int \frac{1}{Z} Z^2 \left(\frac{\chi(x)}{x}\right)^{3/2} dx\, x^2$$

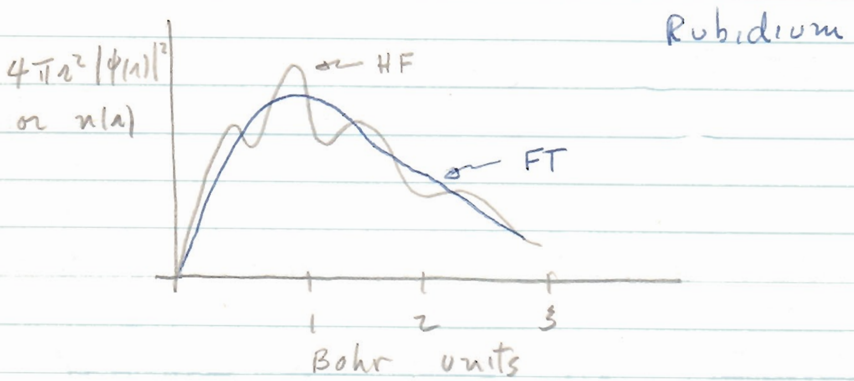
$$\text{const.}\int_0^{x_{eff}} x^2\,dx\left(\frac{\chi(x)}{x}\right)^{3/2} = 1$$

x_{eff} is same for all atoms.

$$r_{eff} = x_{eff}\, b = \frac{A}{Z^{1/3}} = \frac{1.33\ \text{Bohr units}}{Z^{1/3}}$$

Bohr unit $= .5292$ Å

As Z increases, r_{eff} decreases, seems paradoxical, but is not since ½ of electrons involved. Comparison with Hartree-Fock results.



LECTURE XXVI 4-20-61

Calculation of Potential at $r=0$; set up by electrons:

$$\varphi(r) = \frac{Ze}{r}\chi(r)$$

Recall: $x^{1/2}\chi'' = \chi^{3/2}$, so we cannot expand around zero in the usual manner. From the fact that $\chi \to 1$ as $x \to 0$, then:

$$\chi'' \Rightarrow \frac{1}{x^{1/2}} \quad \text{as} \quad x \to 0$$

$$\therefore \quad \chi = \frac{4}{3}x^{3/2} + Bx + c \qquad x \to 0$$

$$\chi = 1, \quad x \to 0, \quad c = 1$$

From table: $\frac{\partial \chi}{\partial x} = -1.59$, $x \to 0$, $\therefore B = 1.59$

$$\therefore \quad \boxed{\chi(x)_{x\to 0} = \frac{4}{3}x^{3/2} - 1.59x + 1}$$

Thus: $x = 0$
$\chi'' \to \infty$
$\chi' \to -1.59$
$\chi \to 1$

what about $\varphi(r)$?

$$\varphi(r) = \frac{Ze}{r} - \frac{Ze}{r}(1.59x) + \frac{4}{3}x^{3/2}\frac{Ze}{r}$$

$r = bx$:

$$\varphi(r) = \frac{Ze}{r} - \frac{Ze(1.59)}{b} + \frac{4}{3}\frac{x^{3/2}Ze}{bx}$$

$$eV_{el}(0) = \frac{-(1.59)Ze^2}{b}, \quad b = \left(\frac{3\pi}{4}\right)^{2/3}\frac{\hbar^2}{2mc^2}\frac{1}{Z^{1/3}} = \frac{.885}{Z^{1/3}}\left(\frac{\hbar^2}{me^2}\right)$$

In atomic units: $\frac{\hbar^2}{me^2}$ = 1 bohr unit = .5292 Å

= 1 atomic unit of length

$\frac{me^4}{\hbar^2}$ = 1 atomic unit of energy = 2 rydbergs

1 rydberg = 13.605 ev

$$\boxed{\begin{aligned} eV(0) &= -1.80\ Z^{4/3} \text{ atomic units} \\ &= -3.60\ Z^{4/3} \text{ Rydbergs} \end{aligned}}$$

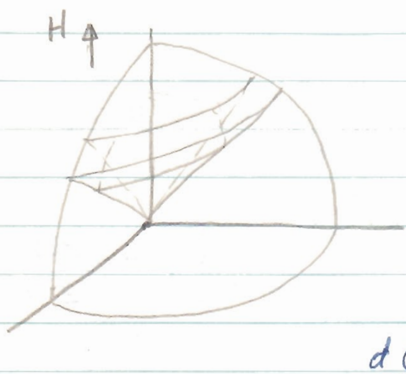
Comparison with Hartree Treatment:
Dickenson, PR80, 563 (1950)
Foldy, PR83, 393 (1951)

$eV(0) = -2.4\ Z^{1.4}$ rydbergs (Hartree)
$= -3.60\ Z^{1.33}$ (FT)

Criteria of validity for F-T Model Application:
$Z^{2/3} \gg 1$
Fails at two points: ① near nucleus
② large distance from atom because statistics cannot be applied.

Calculation of Diagmagnetic Shielding of Nucleus

Shielding is due to electrons. Consider spherical distribution of charge in a magnetic field:



Larmour precession:

$$\omega_L = \frac{eH}{2mc}$$

$$dA = r\,dr\,d\theta$$

$$\frac{dQ}{dt} = di = \frac{e\,n(r)\,dA\,(v\,dt)}{dt}$$

Use Biot-Savart law: $dH_z = \dfrac{i\,dl \times r}{c r^3}$

$$= \frac{2\pi r \sin\theta \; r\sin\theta}{c r^3}\, di = \frac{2\pi \sin^2\theta \, di}{c r}$$

Then:

$$H_z = -\hat{I}_z \int_{\theta=0}^{\frac{2\pi}{2}}\int_{r=0}^{\infty} \frac{2\pi \sin^2\theta}{c r}\, e\, n(r)\, r^2\, dr\, d\theta\, \sin\theta\, \frac{eH}{2mc}$$

using $v = r\omega\sin\theta$

$$H_z = -\frac{I_z}{6}\left(\frac{e}{mc^2}\right) H \underbrace{\int_0^\infty \frac{e\, n(r)}{r}\, 4\pi r^2\, dr}_{eV(0) = \int_0^\infty \frac{\rho(r)\, d\tau}{r}}$$

$\therefore$ $H_{nucleus} = H(1-\sigma)$, where:

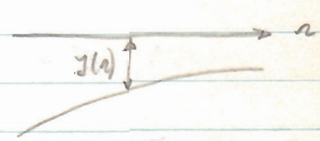
$$\boxed{\sigma = \frac{1}{6}\frac{e}{mc^2}\int_0^\infty \frac{\rho(r)}{r}\, d\tau = \frac{1}{6}\frac{e}{mc^2}\, V(0)}$$

$$\boxed{\sigma = .159 \cdot 10^{-4}\, Z^{4/3}}$$

Lamb, <u>PR60</u>, 817 (1941)

For $Z^{4/3} \sim 50$, $\sigma \sim 6 \cdot 10^{-4}$, must be taken into account when measuring nuclear magnetic moment.

Ionization Energy of a Neutral Atom



$IE = KE + PE$ or:

$$IE = \underbrace{\int \frac{3}{4}\, I(r)\, n(r)\, d\tau}_{KE} - \underbrace{\int \frac{Ze}{r}\, e\, n(r)\, d\tau}_{\text{electrons bound to nucleus}} + \underbrace{\frac{1}{2}\int e\, n(r)\, d\tau \int_0^\infty \frac{\rho(n(r'))\, d\tau'}{|r'-r|}}_{\text{electron repulsion of each other}}$$

If we use Virial theorem for coulombic field:

$$\overline{T} = -\tfrac{1}{2}\overline{V}$$

we need only to calculate T:

$$T = \frac{3}{5}\int \underbrace{J(r)}_{e\varphi(r)}\, n(r)\, d\tau$$

$$\varphi(r) = \frac{Ze}{r}\chi(r)$$

$$T = \frac{32\pi^2}{5}e\left(\frac{2me}{\hbar^2}\right)^{3/2} b^{1/2}(Ze)^{5/2}\int_0^\infty \frac{\chi^{5/2}}{x^{1/2}}\,dx$$

since $n(r) = \frac{8\pi}{3}\left(\frac{2me}{\hbar^2}\right)^{3/2}\varphi^{3/2}(r)$

Milne has shown: $\int_0^\infty \frac{\chi^{5/2}}{x^{1/2}}\,dx = -\frac{5}{7}\chi(0)\left(\frac{d\chi}{dx}\right)_{x=0}$

LECTURE XXVII 4-22-61

Recall; $\int_0^\infty \frac{\chi^{5/2}}{x^{1/2}}\,dx = -\frac{5}{7}\chi(0)\left(\frac{d\chi}{dx}\right)_0$

$$T = \frac{8\sqrt{2}}{\pi}(.885)^{1/2}(1.59)\,Z^{7/3} = .77\,Z^{7/3} \text{ atomic units}$$

$$= 1.54\,Z^{7/3} \text{ Rydbergs}$$

First done by G. Allard, J. Phys & Rad. 9 (225) (1948)

Now $E = T + V$, $T = -\frac{1}{2}V$

$$\therefore E = -1.54\,Z^{7/3}$$

Now the ionization energy on the T-F model to remove all the electrons:

$$\left.\begin{aligned}(IE)_{exp.} &= 1.13\,Z^{7/3} \text{ Ryd.}\\ (IE)_{TF} &= 1.54\,Z^{7/3}\end{aligned}\right\}\quad (IE)_{QM} \sim Z^{2.4}$$

Proof of Milne's Theorem:

$$\int_0^\infty \frac{\chi^{5/2}}{x^{1/2}}\,dx = I = \frac{4}{2}\int_0^\infty \chi^{5/2}\,d\left(x^{1/2}\right)$$

$$= \underbrace{\frac{1}{2}\chi^{5/2}x^{1/2}\Big|_0^\infty}_{0} - 2\,\frac{5}{2}\int x^{1/2}\left(\frac{d\chi}{dx}\right)\chi^{3/2}\,dx$$

$$= -5\int x\underbrace{\left(\frac{\chi^{3/2}}{x^{1/2}}\right)}_{\chi''}\left(\frac{d\chi}{dx}\right)dx = -5\int x\,\chi''\chi'\,dx$$

$$= -\frac{5}{2}\int x\,\frac{d}{dx}\left(\frac{d\chi}{dx}\right)^2 dx = -\frac{5}{2}\int x\,d\left(\frac{d\chi}{dx}\right)^2$$

$$= -\frac{5}{2}\left[x\left(\frac{d\chi}{dx}\right)^2\Big|_0^\infty - \int_0^\infty\left(\frac{d\chi}{dx}\right)^2 dx\right]$$

or $$I = \frac{5}{2}\int_0^\infty\left(\frac{d\chi}{dx}\right)^2 dx$$

We can also write I as:

$$I = \int_0^\infty \frac{\chi^{3/2}}{x^{1/2}}\,\chi\,dx = \int_0^\infty\left(\frac{d^2\chi}{dx^2}\right)\chi\,dx = \int_0^\infty \chi\,\frac{d}{dx}\left(\frac{d\chi}{dx}\right)dx$$

or $$I = \int_0^\infty \chi\,d\left(\frac{d\chi}{dx}\right) = \chi\,\frac{d\chi}{dx}\Big|_0^\infty - \int_0^\infty\left(\frac{d\chi}{dx}\right)^2 dx$$

Then:

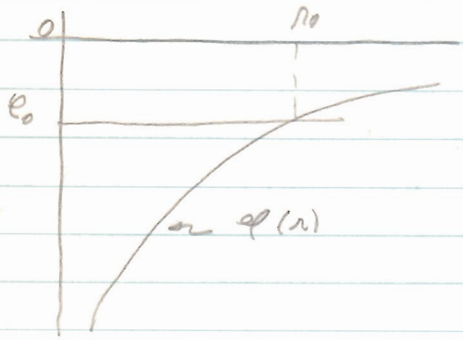
$$I = -\chi(0)\left(\frac{d\chi}{dx}\right)_0 - \frac{2}{5}I$$

or $$\boxed{I = \int_0^\infty \frac{\chi^{5/2}}{x^{1/2}}\,dx = -\frac{5}{7}\,\chi(0)\left(\frac{d\chi}{dx}\right)_0}$$

Reference: N. H. March, Advances in Physics 6 (1957)

Electronic Charge Distribution in Atoms: Ionized Atoms:

Fermi - Thomas Model:



φ_0 = total energy of most energetic electron.
∴ there now exists a finite radius r_0 outside which electrons cannot be found.

Recall: $\nabla^2\left\{\varphi(r)-\varphi_0\right\} = \frac{32\pi^2}{3}\left(\frac{2me}{h^2}\right)^{3/2} e\left\{\varphi(r)-\varphi_0\right\}^{3/2}$

Let $\frac{Ze}{r}\chi(r) = \varphi(r)-\varphi_0$ and get:

$$x^{1/2}\frac{d^2\chi}{dx^2} = \chi^{3/2} \quad ; \quad x\to 0, \chi \to 1$$

What is B.C. at ∞? Better yet what is BC at r_0?

Z charges on nucleus; $(Z-z)$ electrons
$\Rightarrow$ net + charge $+z|e|$
We must have:

$$\int_0^{r_0} n(r)\, d\tau = z$$

and $\nabla^2\left\{\varphi(r)-\varphi_0\right\} = -4\pi\rho = -4\pi|e|n(r)$

Then:

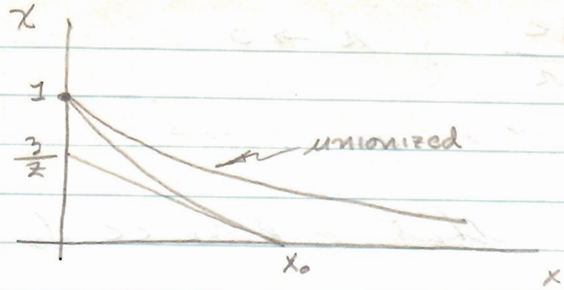
$$-\frac{1}{4\pi e}\int_0^{r_0}\nabla^2\left\{\varphi(r)-\varphi_0\right\}d\tau = z = -\frac{1}{4\pi e}\int_S dS\cdot\nabla\frac{Ze}{r}\chi(r)$$

and:

$$z = -\frac{1}{4\pi e}\,4\pi r_0^2\,\frac{Ze}{r_0}\left(\frac{\partial\chi}{\partial r}\right)_{r_0}$$

or: $$\boxed{-\frac{z}{Z} = \left(\frac{\partial\chi}{\partial r}\right)_{r_0} r_0 = \left(\frac{\partial\chi}{\partial x}\right)_{x_0} x_0}$$

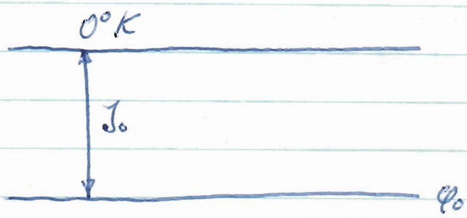
which makes the BC at r_0.



Can associate certain degree of ionization with each slope.

LECTURE XXVIII 4-29-61

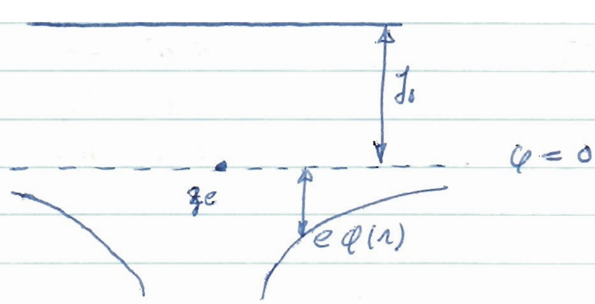
Electrostatic Field of an Ionized Impurity Atom in a Metal



Since we want charge neutrality: $\nabla^2 \varphi = -4\pi\rho = 0$ and $n_+(r) = n_-(r)$ at all points.

$$\boxed{n_-(r) = \left(\frac{8\pi}{3}\right)\left(\frac{2m}{h^2}\right)^{3/2} J_0^{3/2}}$$

Now consider a Cu host lattice with Al (3 valence electrons). 2 electrons and 2 positive charges in vicinity of impurity. Z = excess no. of valence electrons donated by impurity.



$$n_-(r) = \frac{8\pi}{3}\left(\frac{2m}{h^2}\right)\left(J_0 + e\,\varphi(r)\right)^{3/2}$$

$$n_+(r) = \frac{8\pi}{3}\left(\frac{2m}{h^2}\right)\left(J_0\right)^{3/2}$$

Does not include + core Ze.

Now use Poissons equation: $\nabla^2 \varphi(r) = 4\pi|e|\left(n_-(r) - n_+(r)\right)$

$$\boxed{\nabla^2 \varphi = \frac{32\pi^2 |e|}{3}\left(\frac{2m}{h^2}\right)^{3/2}\left[\left(J_0 + e\,\varphi(r)\right)^{3/2} - J_0^{3/2}\right]}$$

Boundary Conditions: $\varphi \to \frac{ze}{r}$; $r \to 0$

$r\varphi(r) \to 0$ as $r \to \infty$

Now assume r is such that $e\varphi(r) << \mathcal{J}_0$, then:

$$[\ \ ] = \mathcal{J}_0^{3/2}\left(1 + \frac{e\varphi(r)}{\mathcal{J}_0}\right)^{3/2} - \mathcal{J}_0^{3/2}$$

$$= \frac{3}{2} e\varphi(r)\mathcal{J}_0^{1/2}$$

Then: $\nabla^2\varphi(r) = \frac{32\pi^2}{3} e^2 \left(\frac{2m}{h^2}\right)^{3/2} \varphi(r)\mathcal{J}_0^{1/2}$

where $\mathcal{J}_0^{1/2} = \left(\frac{3}{8\pi} n_0\right)^{1/3} \frac{1}{\left(\frac{2m}{h^2}\right)^{1/2}}$, $n_0 = n_- = n_+$ with no impurities added

$$\boxed{\nabla^2\varphi(r) = 16\pi^2 e^2 \left(\frac{2m}{h^2}\right)\left(\frac{3}{8\pi} n_0\right)^{1/3} \varphi(r)}$$

or $\nabla^2\varphi(r) = \frac{4}{\left(\frac{\hbar^2}{me^2}\right)} \left(\frac{3n_0}{\pi}\right)^{1/3} \varphi = \frac{\varphi}{\lambda^2}$

$$\lambda^2 = \frac{a_0}{4}\left(\frac{\pi}{3n_0}\right)^{1/3} \quad ; \quad a_0 = \frac{-\hbar^2}{me^2}$$

The solution is: $\boxed{\varphi(r) = \frac{ze}{r} e^{-r/\lambda}}$

λ is Mott shielding length: $\lambda = .55$ Å Cu

$= .58$ Å Ag

$.58$ Å Au

The following should hold:

$$\int (n_- - n_+)\, d\tau \overset{?}{=} z = \int_0^\infty \frac{1}{4\pi e} \nabla^2\varphi(r)\, 4\pi r^2 dr$$

$$= \frac{ze}{e}\int_0^\infty \nabla^2\left(\frac{e^{-r/\lambda}}{r}\right) r^2 dr = \frac{z}{\lambda^2}\int_0^\infty e^{-r/\lambda} r\, dr = z$$

so holds even though not good at $r = 0$

Region of Validity of Solution: We must have:

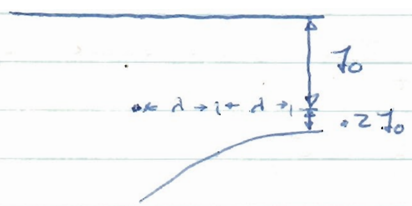
$$\frac{e\varphi(r)}{I_0} \ll 1 \quad , \quad x = \frac{r}{\lambda}$$

$$\frac{e\varphi(r)}{I_0} = \frac{e^2}{\lambda}\,\frac{e^{-x}}{x}\,\frac{1}{\left(\frac{3}{8\pi}n_0\right)^{2/3}\left(\frac{h^2}{2m}\right)} = \left(\frac{32}{\pi^2}\right)\left(\frac{\lambda^3}{a_0^3}\right)\frac{e^{-x}}{x}$$

Choose arbitrarily: $\frac{e\varphi(r_0)}{I_0} = .2$; for $r < r_0$, assumption fails

Then $\frac{e^{-x_0}}{x_0} = .0617$; $\therefore$ $x_0 \cong 2 = \frac{r}{\lambda}$

So can consider good for r past 2λ



The Physics of Fully Ionized Gases

Plasma Densities: Metal 10^{23}/cc
Gas $10^{12}-10^{13}$ /cc

We have considered in the course, very low density electron systems (ballistic) and high density systems (metals). Now we do intermediate.

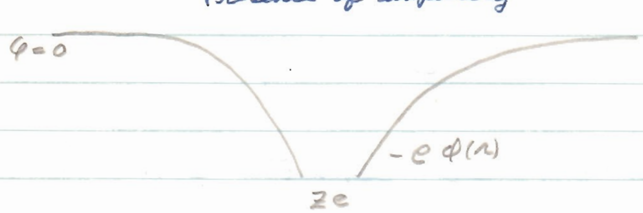
1) Shielding Properties of a Plasma:
Shielding length depends on temperature.

LECTURE XXIX 5-2-61

Physics of Fully Ionized Gases:

1) Debye Shielding Length:

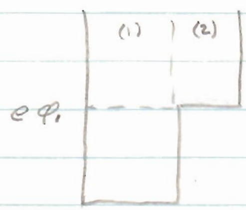
We proceed much as in the Thomas-Fermi treatment of the atom. We suppose a gas of electrons which obeys MB statistics.



$n_+ = n_-$ in unperturbed problem and $\varphi = 0$

We take for the density in the presence of impurity: $n_-(r) = \underbrace{n_0}_{\text{unperturbed density.}} e^{e\varphi(r)/kT}$

Consider:



For equilibrium exchange between (1) and (2), requires a matching of normalization constants or $\left(\frac{N}{V}\right)$ = density.

Now, # of electrons between u and du velocities is:

$$n(u) = \left(\frac{N}{V}\right)\left(\frac{m}{2\pi kT}\right)^{1/2} e^{-\frac{1}{2}\frac{mu^2}{kT}}$$

number of electrons/sec going from $1 \to 2 = \displaystyle\int_{u_1 = \sqrt{\frac{2e\varphi_1}{m}}}^{\infty} u_1\, n(u_1)\, du_1$

$$= \left(\frac{N}{V}\right)_1 \left(\frac{m}{2\pi kT}\right)_1^{1/2} \int_{\sqrt{\frac{2e\varphi_1}{m}}}^{\infty} u_1\, e^{-\frac{1}{2}\frac{mu_1^2}{kT}}\, du_1$$

number of electrons/sec going from $2 \to 1 = \displaystyle\int_0^{-\infty} u_2\, n(u_2)\, du_2$

$$= \left(\frac{N}{V}\right)_2 \left(\frac{m}{2\pi kT}\right)_2^{1/2} \int_0^{-\infty} u_2\, e^{-\frac{1}{2} m u_2^2/kT}\, du_2$$

Thus: $\left(\frac{N}{V}\right)_1 = \left(\frac{N}{V}\right)_2 \dfrac{\int_0^\infty u_2 e^{(\,)_2} du_2}{\int_{\sqrt{\,}}^\infty u_1 e^{(\,)_1} du_1}$ $\left\} \begin{array}{l} y^2 = \frac{1}{2}\frac{mu^2}{kT} \\ y^2_{min} = \frac{1}{2}\frac{m}{kT}\frac{2e\varphi_1}{m} = \frac{e\varphi_1}{kT} \end{array} \right.$

$$= \left(\frac{N}{V}\right)_2 \frac{\int_0^\infty y e^{-y^2} dy}{\int_{y_m}^\infty y e^{-y^2} dy} = \left(\frac{N}{V}\right)_2 e^{\frac{e\varphi_1}{kT}}$$

which proves our assertion $n_-(r) = n_0 e^{e\varphi(r)/kT}$

We have for Poisson's equation:

$$\nabla^2 \varphi(r) = 4\pi\rho = -4\pi e\,(n_+ - n_-)$$

$$= -4\pi e \left\{ Z\,\delta(r) + n_0 - n_- \right\}$$

where $(n_0 - n_-(r)) = n_0 - n_0 e^{e\varphi(r)/kT}$

Assume: $\frac{e\varphi(r)}{kT} \ll 1$, then:

$$(n_0 - n_-(r)) \cong -\frac{n_0 e\varphi(r)}{kT}$$

Then:

$$\nabla^2 \varphi(r) = \frac{4\pi n_0 e^2}{kT}\varphi(r) - 4\pi e Z\,\delta(r)$$

$$= \frac{\varphi(r)}{\lambda_D^2} - 4\pi e Z\,\delta(r)$$

where $\boxed{\lambda_D^2 = \frac{kT}{4\pi n_0 e^2}}$

and the solution is asserted to be:

$$\boxed{\varphi(r) = \frac{Ze}{r} e^{-r/\lambda_D}}$$

which is actually a shielded coulomb potential where λ_D is the <u>debye shielding length</u>.

λ_D can be expressed in term of the mean square of the thermal velocity.

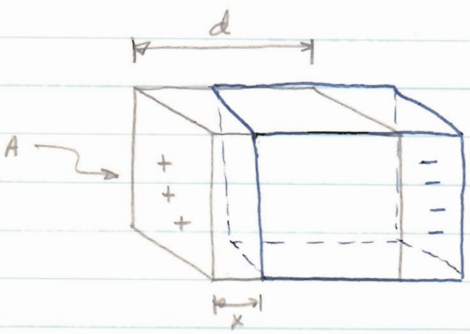
$$\lambda_D^2 = \left(\frac{kT}{4\pi n_0 e^2} \right) = \frac{m \langle v^2 \rangle}{12 \pi n_0 e^2}$$

Compare with d^2_{mott};

$$d^2_{mott} = \frac{\hbar^2 \#}{m e^2} \left(\frac{\pi}{3 n_0} \right)^{1/3} = \left(\frac{\frac{8}{3} J_0}{4\pi n_0 e^2} \right)$$

or $\frac{8}{3} J_0 \rightarrow kT$

2) Plasma Oscillation Frequency:



We displace the negative charges with respect to the positive charges

Now: The energy density of the displacement is:

$$W = \frac{E^2}{8\pi} V$$

Now: $E = 4\pi\sigma = \frac{4\pi Q}{A} = \frac{4\pi e n_0 A x}{A}$

so: $W(x) = \frac{(4\pi)^2 e^2 n_0^2 x^2}{8\pi} Ad$

The force due to displacement is:

$$F_x = -\left(\frac{\partial W}{\partial x} \right) = M \frac{d^2 x}{dt^2} = -4\pi e^2 n_0^2 \, x \, A d$$

This can be written: $\frac{d^2 x}{dt^2} = -\omega_p^2 x$

so that: $\boxed{\omega_p^2 = \frac{4\pi e^2 n_0}{m}}$ for absolute zero and all wavelengths

Some Orders of Magnitude:

$d_D \sim 10^{-3}$ cm , $n_0 = 10^{12}/cc$, $kT \sim 3$ ev or $36{,}000$ °K

$$\nu_p = 9 \cdot 10^3 n_0^{1/2} \sim 2.7 \cdot 10^{10} \text{ cps}$$

Now for $T \neq 0$ we have dispersion of the plasma waves and the relation is stated to be:

$$\sum_i \frac{1}{(\omega - k \cdot v_i)^2} = \frac{m}{4\pi e^2} = \int \frac{n(v)\,dv}{(\omega - k \cdot v)^2}$$

The longwavelength waves will have the plasma oscillation frequency and collective motion while the short wavelength will not and individual motion must be considered.

At what values of wave length can plasma oscillation be used?

when $k^2 \langle v^2 \rangle \sim \omega_p^2$; $\boxed{k^2 \langle v^2 \rangle \simeq 3\omega_p^2}$

$$k^2 = \frac{3\omega_p^2}{\langle v^2 \rangle} = \frac{1}{d_D^2}$$

so dividing point is about the debye length.

To worry about collective actions, see if any electrons with energy around $\hbar\omega_p$ are around:

$$\frac{\mathcal{J}}{\hbar\omega_p} = \frac{\frac{h^2}{2m}\left(\frac{3}{8\pi}\right)^{2/3} n_0^{2/3}}{\frac{h}{2\pi}\frac{2\sqrt{\pi}}{\sqrt{m}}\, e\, n_0^{1/2}} = \sqrt{\frac{a_0}{r_s}}\;\frac{3^{5/6}}{4^{7/6}} < 1$$

r_s = radius of Wigner Seitz Cell. Thus plasma oscillations are not usually excited.

LECTURE XXX 5-4-61

Propagation of Electromagnetic Waves in a Plasma:

Maxwell's Equations:

$$\nabla\cdot\vec{E} = 4\pi\rho \qquad \nabla\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$$

$$\nabla\cdot\vec{B} = 0 \qquad \nabla\times\vec{B} = \frac{4\pi\vec{J}}{c} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t}$$

Now: $\rho = e\,(n_i z - n_e)$: n_i = # of ions/cc
n_e = # of electron/cc
z = charge on each ion

$\vec{J} = e\,(z\, n_i \vec{v}_i - n_e \vec{v}_e)$: v_i - average velocity of ion at point r, time t.
v_e = same for electron

and $$\vec{v} = \frac{n_i m_i \vec{v}_i + n_e m_e \vec{v}_e}{n_i m_i + n_e m_e}$$

with $\mu\vec{v} = \vec{p}\,(r,t)$; μ = reduced mass.

Maxwell's Wave Equation:

$$\nabla\times(\nabla\times E) = -\frac{1}{c}\frac{\partial}{\partial t}(\nabla\times B) = -\frac{1}{c}\frac{4\pi}{c}\frac{\partial J}{\partial t} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}$$

$$= \nabla\underbrace{(\nabla\cdot E)}_{4\pi\rho} - \nabla^2 E$$

or $$\boxed{\nabla^2 E - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = 4\pi\nabla\rho + \frac{4\pi}{c^2}\frac{\partial J}{\partial t}}$$

For no ρ or J, $E = E_0\, e^{i(k\cdot r - \omega t)}$, $\omega = ck$

Relations between ρ, J, $\vec{v}$, $\vec{E}$, $\vec{H}$ called equations of motion

Let $f\,(r,w,t)\,|dr|\,|dw|$ = no of $\left(\begin{matrix}\text{electrons}\\ \text{or}\\ \text{ions}\end{matrix}\right)$ with velocities between w & $w+dw$ at position between r, $r+dr$ at t.

We write the Boltzmann Equation:

$$\left(\frac{df}{dt}\right)_{field} = \left(\frac{df}{dt}\right)_{collision}$$

From definition of total derivative:

$$\left(\frac{df}{dt}\right)_{field} = \frac{\partial f}{\partial t} + (\nabla_r f)\cdot\frac{d\vec{r}}{dt} + (\nabla_w f)\cdot\frac{d\vec{w}}{dt}$$

since $m\frac{dW}{dt} = \mathcal{F}$, then:

$$\left(\frac{df}{dt}\right)_{field} = \frac{\partial f}{\partial t} + (\nabla_r f)\cdot\frac{d\vec{r}}{dt} + \frac{(\nabla_w f)\cdot\mathcal{F}}{m}$$

now the density of electrons at r and t is:

$$n(r,t) = \int_{-\infty}^{\infty} f(r,w,t)\,|dW|$$

then:

$$v(r,t) = \frac{\int_{-\infty}^{\infty}\vec{w}\,f(r,w,t)\,|dw|}{\int_{-\infty}^{\infty} f(r,w,t)\,|dw|}\,;\quad n(r,t)\,v(r,t) = \int_{-\infty}^{\infty} w\,f\,|dw|$$

We now have to look for time rate of change of mean momentum:

$$m\,n\,v(r,t) = \int m\,f(r,w,t)\,w\,dW$$

Multiply Boltzmann equation by mw and integrate:

$$\int_{-\infty}^{\infty}\Big[\underset{1}{m\vec{w}\frac{df}{dt}} + \underset{2}{m\vec{w}\,\nabla_r f\cdot\vec{w}} + \underset{3}{\vec{w}\,\nabla_w f\cdot\vec{\mathcal{F}}}\Big]dw = \int_{-\infty}^{\infty} mW\left(\frac{df}{dt}\right)_{coll} dW$$

Consider 1: $m\frac{\partial}{\partial t}\int_{-\infty}^{\infty} w\,f(r,w,t)\,|dw| = \frac{\partial}{\partial t}\left[m\,n\,\vec{v}(r,t)\right]$

Consider ②: $m \int_{-\infty}^{\infty} \vec{w} \left(\frac{\partial f}{\partial x} w_x + \frac{\partial f}{\partial y} w_y + \frac{\partial f}{\partial z} w_z \right) dw$

$$= m \left[\frac{\partial}{\partial x} \int \vec{w} f w_x dw + \frac{\partial}{\partial y} \int \vec{w} f w_y dw + \frac{\partial}{\partial z} \int w f w_z dw \right]$$

$$= m \left(\frac{\partial}{\partial x} n \overline{w w_x} + \frac{\partial}{\partial y} n \overline{w w_y} + \frac{\partial}{\partial z} n \overline{w w_z} \right)$$

Consider ③: $\int_{-\infty}^{\infty} \vec{w}\, \nabla_w f \cdot \mathcal{F}\, dw$

Identity: $\vec{\nabla}_w \cdot (f \vec{\mathcal{F}} \underset{\sim}{\vec{w}}) = (\nabla_w \cdot f \vec{\mathcal{F}}) \vec{w} + (f \vec{\mathcal{F}} \cdot \nabla_w) \vec{w}$

now: $\nabla_w \cdot f \mathcal{F} = \frac{\partial}{\partial w_x} f F_x + \frac{\partial}{\partial w_y} f F_y + \cdots = \vec{\mathcal{F}} \cdot \vec{\nabla}_w f$

since $\mathcal{F} = \frac{w \times H}{c}$ or F_x independent of w_x, etc.
Then:

$$\nabla_w \cdot (f \underset{\sim}{\vec{\mathcal{F}} \vec{w}}) = (\vec{\mathcal{F}} \cdot \nabla_w f) \vec{w} + (f \vec{\mathcal{F}} \cdot \nabla_w) \vec{w}$$

$$= (\vec{\mathcal{F}} \cdot \vec{\nabla}_w f) \vec{w} + f \vec{\mathcal{F}}$$

Integrating $\nabla_w \cdot (f \vec{\mathcal{F}} \vec{w})$ by Gauss' theorem gives 0 since $f \to 0$ faster than $w \to \infty$ on the surface of integration. Then ③ becomes:

$$- \int f \vec{\mathcal{F}} \, |dw| = n(r,t) \overline{\vec{\mathcal{F}}}$$

Thus the Boltzmann equation becomes:

$$\frac{\partial}{\partial t} \left(m\, n(r,t)\, v(r,t) \right) + m \sum_{j=1,2,3} n \frac{\partial}{\partial x_j} \overline{\vec{w} w_j} - n(r,t) \vec{\mathcal{F}} = \int m w \left(\frac{\partial f}{\partial t} \right)_{coll.} dw$$

We split up the motion into average motion and motion about this average.

$$w = \vec{v}(r,t) + \vec{u}(r,t) = \vec{v} + \vec{u}$$

$$\vec{v} = \langle w \rangle = \frac{\int w f dw}{\int f dw}, \quad \langle u \rangle = 0$$

Then the middle term becomes:

$$\sum \frac{\partial}{\partial x_j} m n \langle \vec{w} w_j \rangle = \sum \frac{\partial}{\partial x_j} m n \langle (v+u)(v_j+u_j) \rangle$$

$$= \sum \frac{\partial}{\partial x_j} m n \left[v v_j + \langle u u_j \rangle \right]$$

Assume isotropy: $\langle u u_x \rangle = \langle u_x u_x \rangle + \underbrace{\langle u_y u_x \rangle}_{0} + \underbrace{\langle u_z u_x \rangle}_{0}$

$$= \langle u_x^2 \rangle = \frac{1}{3} \langle u^2 \rangle$$

Then: $\sum \frac{\partial}{\partial x_j} m n \langle u u_j \rangle = \sum_j \frac{\partial}{\partial x_j} \left(\frac{1}{3} m n \langle u_j^2 \rangle \right)$

$$= \nabla \left(\frac{1}{3} m n \langle u^2 \rangle \right) = \nabla (n k T) = \nabla p \; ; \; p = \text{pressure}$$

because: $\frac{1}{2} m \langle u^2 \rangle = \frac{3}{2} kT$ and $p = n k T$

Now: $m \sum \frac{\partial}{\partial x_j} (n v v_j) = m \vec{v} (\nabla \cdot n \vec{v}) + (n \vec{v} \cdot \nabla) \vec{v}$

Finally:

$$n m \frac{\partial v}{\partial t} + n m (v \cdot \nabla) \vec{v} + \nabla p(r,t) + m \vec{v} \left(\frac{\partial n}{\partial t} + \nabla \cdot n \vec{v} \right)$$
$$= n \vec{F} + \int dw \, m w \left(\frac{df}{dt} \right)_{coll.}$$

LECTURE XXXI 5-6-61

Reading Period Assignment:
A. J. Dekker, Ch. 12 & 14
Amasa Bishop "Project Sherwood", pp 1-64.

Recall:

$$\underbrace{n m \left(\frac{\partial}{\partial t} + \vec{v} \cdot \nabla \right)}_{d/dt} \vec{v} + \nabla p(r,t) + m \vec{v} \left[\frac{\partial n}{\partial t} + \nabla \cdot n \vec{v} \right]$$
$$= n \vec{F} + \int dw \, m w \left(\frac{\partial f}{\partial t} \right)_{coll}$$

now: $\vec{F}_i = q\left(\vec{E} + \frac{\vec{v}\times\vec{B}}{c}\right)$

then: $$\boxed{n\, m \frac{d\vec{v}}{dt} = n\, q\left(\vec{E} + \frac{\vec{v}\times\vec{B}}{c}\right) - \nabla_i p + P} \quad (A)$$

where P is momentum transferred by collisions

Linearize equations of motion: Assumptions:

1) Neglect terms quadratic in v: $\vec{v}\cdot\nabla\vec{v} \to 0$
2) Charge neutrality $n_i e z_i - n_e e = 0$
3) Pressure is scalar: isotropy about center of mass.

Equation (A) is for ions and we can write another as (B) for electrons.

(A) + (B) gives on LHS: $n_i m_i \frac{\partial v_i}{\partial t} + n_e m_e \frac{\partial v_e}{\partial t}$

Recall: $$\vec{v} = \frac{n_i m_i v_i + n_e m_e v_e}{\underbrace{n_i m_i + n_e m_e}_{\mu}}$$

and $\frac{\partial n}{\partial t} + \nabla\cdot(n\vec{v}) = 0$

We can neglect terms in $\frac{\partial n}{\partial t}$ since they include velocity and then give terms quadratic in v. Thus:

$$\boxed{\mu\left(\frac{\partial v}{\partial t}\right) = \frac{J\times B}{c} - \nabla p + P_i + P_e}$$

where $J = n_i z_i e\, v_i - n_e e\, v_e$; $P = p_i + p_e$

Since overall collision momentum transfer between ions and electrons is same: $P_i + P_e = 0$

Now consider (A) − (B): We can write:

$$\frac{\partial J}{\partial t} = n_i z e \frac{\partial v_i}{\partial t} - n_e e \frac{\partial v_e}{\partial t}$$

We consistently neglect terms of order $\frac{m_e}{m_i}$ and $\frac{z e \nabla p_i}{m_i} << \frac{e \nabla p_e}{m_e}$

Then we get:

$$\frac{\partial J}{\partial t} = \frac{n_e e^2}{m_e} E + \frac{n_e e^2 v_e}{m_e} \times \frac{B}{c} + \frac{e}{m_e} \nabla P_e - \frac{e P_e}{m_e}$$

note: $v_e \simeq \left(v - \frac{J}{e n_e} \right)$

$$= \frac{n_i m_i v_i + n_e m_e v_e}{\underset{\substack{\downarrow \\ n_i m_i}}{M}} - \frac{n_i Z e v_i - n_e e v_e}{e n_e} \simeq v_e$$

Thus we have:

$$\boxed{\frac{m_e}{n_e e^2}\left(\frac{\partial \vec{J}}{\partial t}\right) \equiv \vec{E} + \frac{\vec{v} \times \vec{B}}{c} + \frac{1}{n_e e} \nabla P_e + \frac{\vec{J} \times \vec{B}}{n_e e c} - \frac{\vec{P}_e}{n_e e}}$$

should be able to see $\vec{P}_e = |\vec{P}_e| \frac{\vec{J}}{|\vec{J}|}$ and then we can write $\frac{\vec{P}_e}{n_e e} = \eta \vec{J}$

which is actually the resistivity or damping term.

Application to E.M Wave Propagation

model: (1) Consider only E field
(2) Neglect pressure gradient
(3) At first, assume $\eta = 0$ or plasma has infinite conductivity.

Thus we have: $\frac{\partial \vec{J}}{\partial t} = \frac{n_e e^2}{m_e} \vec{E}$

with: $\nabla^2 E = \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} + \frac{4\pi}{c^2} \frac{\partial J}{\partial t} + 4\pi \nabla \rho$

or $\left(\nabla^2 - \frac{4\pi n_e e^2}{m_e c^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \vec{E} = 4\pi \nabla \rho$

(4) Now consider only transverse wave, since longitudinal will change density.

Can see from: $\nabla \cdot \vec{E} = 4\pi\rho$

$$\vec{E} = 1_y E_y e^{i(kx - \omega t)}$$

then $\nabla \cdot E = \frac{\partial}{\partial y} E_y e^{ikx - i\omega t} = 0$, $\therefore \rho = 0$

Assuming: $\vec{E} = 1_y E_y e^{i(kx-\omega t)}$?

$$\left(-k^2 - \frac{\omega_p^2}{c^2} + \frac{\omega^2}{c^2}\right) = 0$$

or $(ck)^2 = (\omega_p^2 - \omega^2)$

or $$\boxed{k^2 = -\left(\frac{\omega_p^2 - \omega^2}{c^2}\right)}$$

For phase velocity;

$$\frac{\omega}{k} = c\sqrt{1 + \frac{\omega_p^2}{\omega^2 + \omega_p^2}}$$

or $$\boxed{v_{ph} = \frac{c}{\sqrt{1 - \left(\frac{\omega_p}{\omega}\right)^2}}}$$

For $\omega > \omega_p$, get propagation

For $\omega < \omega_p$: $k = \frac{i\omega_p}{c}\sqrt{1 - \left(\frac{\omega}{\omega_p}\right)^2}$

and
$$\vec{E} = 1_y E_y e^{-x \frac{\omega_p}{c}\sqrt{1-\left(\frac{\omega}{\omega_p}\right)^2}} e^{-i\omega t}$$

$$= 1_y E_y e^{-x/\lambda} e^{-i\omega t}$$

so damping occurs and we have a penetration depth:

$$\boxed{\lambda = \frac{c}{\omega_p\sqrt{1 - \left(\frac{\omega}{\omega_p}\right)^2}}}$$

For $\omega < \omega_p$, the material current and displacement current buck each other out.

If we include damping, we get skin depth:

$$\lambda = c\sqrt{\frac{\eta}{2\pi\omega}}$$

References: Spitzer's little book.

Applied Physics 231

March, 1961

Problems

Due: 2 1/2 weeks

i.e. March ~~15~~ 21

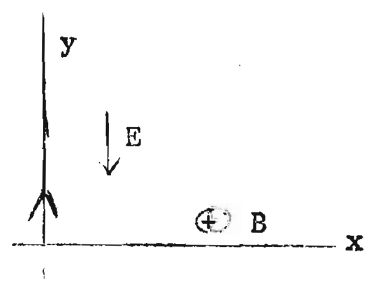
1. Imagine 3 electrons fixed at the corners of an equilateral triangle 10^{-7} cm. on a side. If the electrons are suddenly released and allowed to fly apart under their mutual repulsion, what will be the final velocity of each? Express the final kinetic energy of each electron in electron-volts.

2. (a) Find the radius of curvature of the path of an electron whose kinetic energy is 25 electron volts, moving in a plane perpendicular to the earth's magnetic field. The strength of the earth's field may be taken as 0.7 gauss.

   (b) If the radius of the outermost ion orbit in a cyclotron is 50 cm., and the magnetic field at that position is 15000 gauss, what is the final energy in million electron volts (MEV) of doubly charged helium ions accelerated in this machine?

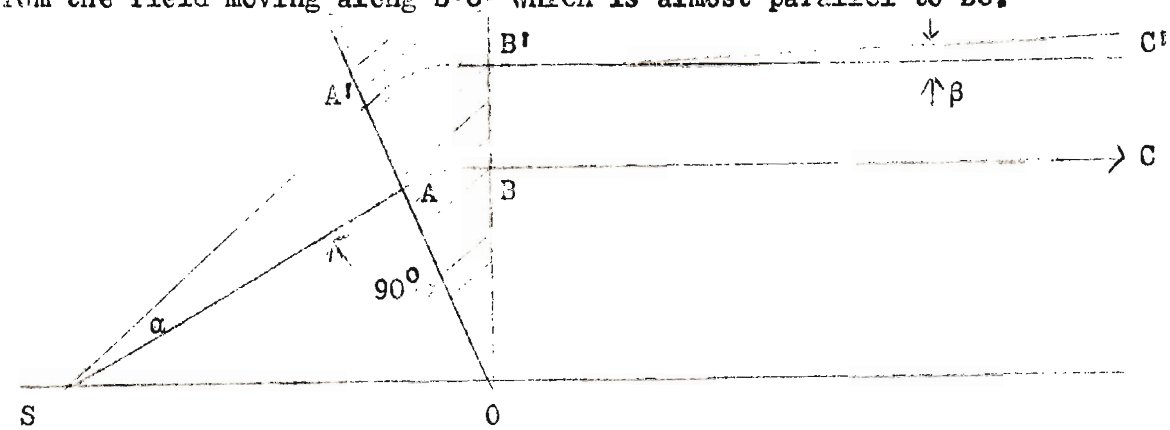
   Mass of helium ion = 6.66×10^{-24} gm.

3. The stationary coordinate system shown is located in a region in which there is a uniform electric field, E, in the negative y direction, of 900 volts/cm., and a uniform magnetic field B of 75 gauss directed downward perpendicular to the paper. An electron leaves the origin moving the positive y direction with a velocity of 9×10^8 cm/sec. By transferring the problem to a suitably moving coordinate system and then back again trace the trajectory of the electron in the x, y plane in sufficient detail to



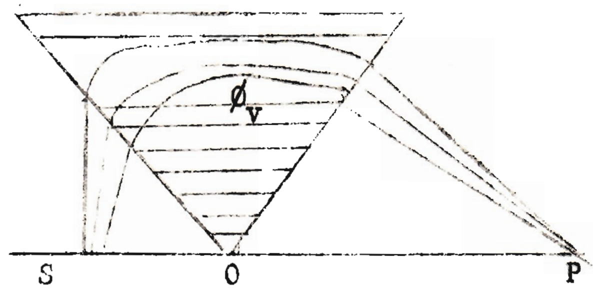
satisfy yourself that your analysis of the motion is complete. In particular locate the points x_1 and x_2 where the electron crosses the x axis for the first and second time after it leaves the origin. Show that any electron leaving the origin, no matter what its initial speed and direction, must pass through the point $x = x_2$, $y = 0$.

4. A uniform magnetic field B, perpendicular to the paper, extends through the shaded sector in the figure below. S is a source of electrons of velocity v so that an electron entering the field at A moves along an arc AB the center of which is at O. Thus BC is parallel to SO. Show that an electron of the same velocity moving initially along OA', which differs from OA by the small angle α, will emerge from the field moving along B'C' which is almost parallel to BC.



That is, show that β is of the order α^2, for small α. (a corollary to this result is that parallel trajectories entering the sector from the right (and with the sign of H reversed) would be approximately focused at S). Combining both results one obtains the basic principle of the magnetic sector spectrograph, which is that electrons of suitable velocity leaving S, below, will be refocused at P, on SO extended, to the first order;

there is no restriction on θ or ϕ.



5. The velocity analyzer used in some magnetic spectrographs consists of two slits S_1 and S_2, separated by a region of uniform electric and magnetic field strength. E,B, and the line S_1S_2 are mutually perpendicular. Ions of velocity Ec/B can travel without deflection along the line S_1S_2. Moreover, by a suitable choice of E and B a <u>focusing</u> property can be achieved for ions of a particular e/m. Such ions leaving S_1 at any small angle with the line S_1S_2 will pass through S_2, if they have the correct speed Ec/B. Find the condition for focusing. Let d be the distance between the slits.

✓

6. Compute the focal length of a symmetrical electrostatic lens in which the potential distribution along the axis is given by the following formula:

$$V = V_0(1 + a + az/b) \text{ for } -b < z < 0$$

$$V = V_0(1 + a - az/b) \text{ for } 0 < z < b$$

$$V = V_0 \text{ for } z < -b \text{ and } z > b$$

✓

7. Between two cylindrical electrodes (radii 1 cm. and 2 cm. repectively) exists a potential difference of 500 volts. Calculate the tangential speed an electron must have to describe a circular orbit with radius 1.5 cm. midway between the electrodes. Show that the electrons starting at a point A with small deviations from the tangential

direction are refocused at a point B, 127° from A ($127^\circ = \pi/\sqrt{2}$ radians). There is no magnetic field.

8. Work out an approximate numerical relation between the energy of an electron expressed in MEV and the corresponding value of Br in gauss-cm., for the extreme relativistic case, $(1 - \beta^2)^{+1/2} \ll 1$. Find the energy above which the value of Br given by this approximate formula is in error by less than 1%. What is the orbit diameter in a synchrotron designed to produce one-billion-volt electrons if the magnetic field at the orbit reaches a final value of 10,000 gauss?

9. Start with the parametric equations of motion (i.e. $x = x(t)$, $y = y(t)$) for a + charged particle in the field configuration of problem 4. Show that:

   1) If the particle has zero initial velocity, then the trajectory has cusps at $x = \frac{2\pi v_c}{\omega}$.

   2) If the particle has initial velocity $0 < v_o \hat{1}_x < v_c \hat{1}_x$ that the trajectory has zero slope $(\frac{dy}{dx} = 0)$ at $x = \frac{n}{2} (\frac{2\pi v_c}{\omega}) = x_n$, $n = 0,1,2..$

   3) If $v_c < v_o \hat{1}_x < 2v_c$, $(\frac{dy}{dx}) = 0$ at x_n.

   4) If $v_o \hat{1}_x = v_o \hat{1}_x$, that $(\frac{dy}{dx}) = \infty$ at x_{2n+1}.

   5) If $v_o \hat{1}_x$ $v_c \hat{1}_x$; the orbit loops over to form a curve with two vertical tangents symmetrically located around x_{2n+1}.

   6) If $v_o \hat{1}_x$ is negative that the trajectory loops as well, and has vertical tangents symmetric about x_{2n}.

10. Use the Thomas precession to show that (to order $(\frac{v}{c})^2$) the cyclotron resonance frequency is equal to the electron spin resonance frequency for an electron moving in a uniform magnetic field.

11. Show that the Lorentz transformation preserves length in 4 dimensional space.

12. Obtain the relativistically correct trajectory, for a particle starting from rest in perpendicularly crossed electric and magnetic fields.

13. Show that in a synchrotron the rotation frequency of the charged particle approaches a constant value as the energy increases.

14. Show that U_ν, A_ν transform like components of a 4 vector under a Lorentz transformation while v_ν does not.

v_ν

ASSIGNMENT #1

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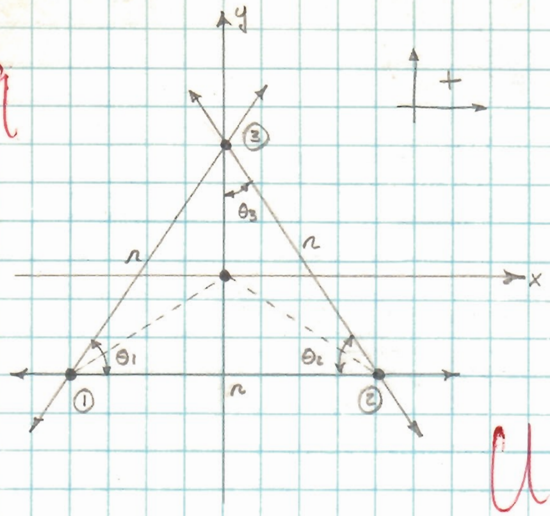
Would you write on one side only please — Thank you

1. 9



The coordinate system is centered on the 3-fold axis of the equilateral triangle.

$\theta_1 = \theta_2 = 60°$

$\theta_3 = 30°$

$\sin\theta_3 = 1/2$ $\quad 3\frac{e^2}{r}$

$\cos\theta_3 = \sqrt{3}/2$

$U = \frac{1}{2}\sum \frac{q_i q_j}{r}$

$r^2 = 2y^2 - 2y^2 \cos 120°$

$= 2y^2(1 - \cos 120°) = 3y^2$

①	9
②	10
③	9½
④	10
⑤	10
⑥	10
⑦	10
⑧	10
⑨	10
⑩	10
⑪	10
⑫	6
⑬	10
⑭	4
	129½
	140

(1) As there is complete 3-fold symmetry about our chosen axis, we need only consider the motion of the electron at ③ since it is the most convenient and the motions of the other two will be similar. $3 \cdot \frac{1}{2}mv^2$

At $t=0$, $v_0 = 0$, $y_0 = \frac{r_0}{\sqrt{3}}$

(2) Coulomb's Law: $F = \frac{c^2e^2}{r^2}$ (emu), or $F = \frac{c^2e^2}{3y^2}$

(3) At ③: $m\frac{d^2x}{dt^2} = 0$

$$m\frac{d^2y}{dt^2} = \frac{c^2e^2}{3}\left\{\frac{\sqrt{3}}{2y^2} + \frac{\sqrt{3}}{2y^2}\right\} = \frac{c^2e^2}{\sqrt{3}\,y^2}$$

Note that the force is as if there were a charge of $e/3^{1/4}$ at the origin.

(4) $\frac{d^2y}{dt^2} = \frac{c^2e^2}{m\sqrt{3}}\frac{1}{y^2}$; $v = \frac{dy}{dt}$; $\frac{d^2y}{dt^2} = v\frac{dv}{dy}$

$\therefore\; v\,dv = \frac{c^2e^2}{\sqrt{3}\,m}\frac{dy}{y^2}$; $v^2 = -\frac{c^2e^2}{\sqrt{3}\,my} + C$

(5) $$v = \left\{\frac{c^2e^2}{m r_0} - \frac{c^2e^2}{\sqrt{3}\,my}\right\}^{1/2} = \frac{ce}{\sqrt{m}}\left\{\frac{1}{r_0} - \frac{1}{\sqrt{3}\,y}\right\}^{1/2}$$

2 somewhere

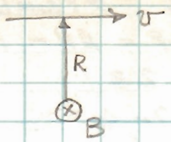
(6) $$v_\infty = \frac{ce}{\sqrt{m r_0}} = \frac{(3\cdot10^{10})(1.6\cdot10^{-20})}{\{9.1\cdot10^{-28}\cdot10^{-7}\}} = 5\cdot10^7 \text{ cm/sec}$$ 7.1

dropped a factor $\frac{1}{2}$ 9

(7) $$E_\infty = \frac{1}{2}mv_\infty^2 = \frac{c^2e^2}{2r_0} = \frac{1}{2}(9.1\cdot10^{-28})(25\cdot10^{14})$$

$= 1.14\cdot10^{-12}$ ergs $= .72$ ev $\quad$ 1.44 eV

2.a.



10

(1) $F = \dfrac{mv^2}{R} = evB$

(2) $E = \frac{1}{2} evBR$; $v = \sqrt{\dfrac{2E}{m}}$

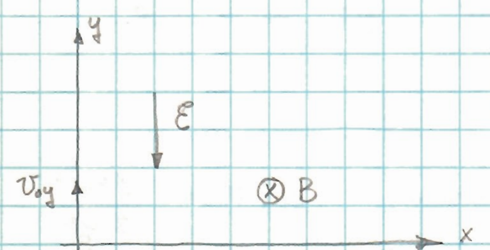
(3) $E = e\sqrt{\dfrac{E}{2m}}\, BR$; $R = \dfrac{\sqrt{2mE}}{eB}$

(4) $R = \dfrac{\{2 \cdot 9.1 \cdot 10^{-28} \cdot 25 \cdot 1.6 \cdot 10^{-12}\}^{1/2}}{1.6 \cdot 10^{-20} \cdot 7 \cdot 10^{-1}} = 24$ cm

b. (1) $E = \dfrac{e^2 B^2 R^2}{2m} = \dfrac{\{3.2 \cdot 10^{-20} \cdot 1.5 \cdot 10^4 \cdot 5 \cdot 10^1\}^2}{(6.66 \cdot 10^{-24}) \cdot 2}$

$= 43 \cdot 10^{-6}$ ergs $= 27$ Mev.

3.



$\mathcal{E} = 900$ v/cm $= 9 \cdot 10^{10}$ abv/cm $= 3$ stv/cm
$B = 75$ gauss
$v_{oy} = 9 \cdot 10^8$ cm/sec

Use ESU !!

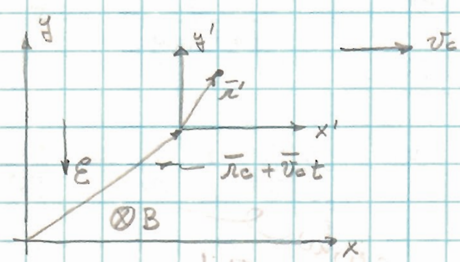
(1) The equation of motion is: $m\dfrac{d\bar{v}}{dt} = e\bar{\mathcal{E}} + \dfrac{e\bar{v} \times \bar{B}}{c}$

(2) We can eliminate $\mathcal{E}$ by transferring to a moving coordinate system such that:

$\dfrac{\bar{v}_c \times \bar{B}}{c} = -\bar{\mathcal{E}}$ which can be done since $\bar{\mathcal{E}} \perp \bar{B}$

$\therefore \quad v_c = \dfrac{\mathcal{E} c}{B}$ in the $+x$ direction.

(3) We set up a moving co-ordinate system:



The position of the particle is then:
$\bar{r} = \bar{r}_c + \bar{v}_c t + \bar{r}'$
and then: $\bar{v} = \bar{v}_c + \bar{v}'$
where $\bar{v}'$ is the velocity of the particle in the moving system

Thus the equation of motion is: $m\dfrac{d\bar{v}'}{dt} = e\bar{\mathcal{E}} + \dfrac{e(\bar{v}_c + \bar{v}') \times B}{c}$

We now eliminate the electric field as above and get:

$$m\frac{d\bar{v}'}{dt} = \frac{e\,\bar{v}' \times B}{c}$$

Assignment #1
Continued

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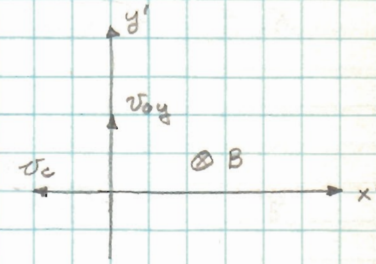
9½

Problem 3
Continued:

(4) In the moving c.s., $\overline{v_0}' = \overline{v_{0y}} - \overline{v_c}$



$$\left.\begin{aligned} m\frac{dv_y'}{dt} &= \frac{e}{c}\, v_x' B \\ m\frac{dv_x'}{dt} &= -\frac{e}{c}\, v_y' B \end{aligned}\right\} \quad \omega = \frac{eB}{mc}$$

$$\therefore \quad \frac{d}{dt}(v_x' + i v_y') = i\omega(v_x' + i v_y')$$

$$(v_x' + i v_y') = (v_x' + i v_y')\, e^{i\omega t} = (-v_c + i v_{0y})\, e^{i\omega t}$$

$$= -v_c \cos\omega t - i v_c \sin\omega t + i v_{0y}\cos\omega t - v_{0y}\sin\omega t$$

$$v_x' = -(v_c \cos\omega t + v_{0y}\sin\omega t)$$

$$v_y' = v_{0y}\cos\omega t - v_c \sin\omega t$$

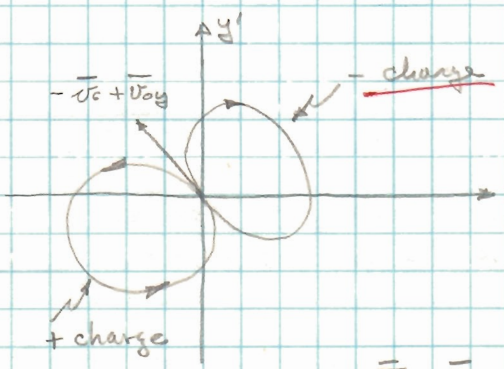
$$x' = -\left(\frac{v_c}{\omega}\sin\omega t - \frac{v_{0y}}{\omega}\cos\omega t + \frac{v_{0y}}{\omega}\right)$$

$$y' = \frac{v_{0y}}{\omega}\sin\omega t + \frac{v_c}{\omega}\cos\omega t - \frac{v_c}{\omega}$$

$$\left(x' + \frac{v_{0y}}{\omega}\right)^2 + \left(y' + \frac{v_c}{\omega}\right)^2 = \left\{\left(\frac{v_c}{\omega}\right)^2 + \left(\frac{v_{0y}}{\omega}\right)^2\right\} \sin^2\left\{\omega t - \tan^{-1}\frac{v_{0y}}{v_c}\right\}$$

$$+ \left\{\left(\frac{v_{0y}}{\omega}\right)^2 + \left(\frac{v_c}{\omega}\right)^2\right\} \cos^2\left\{\omega t - \tan^{-1}\frac{v_{0y}}{v_c}\right\}$$

$$= \left(\frac{v_{0y}}{\omega}\right)^2 + \left(\frac{v_c}{\omega}\right)^2 \text{ ; equation of circle.}$$



(5) $v_{0y} = 9 \cdot 10^8$ cm/sec

$$v_c = \frac{3 \cdot 3 \cdot 10^{10}}{\frac{3}{4} \cdot 10^2} = 12 \cdot 10^8 \text{ cm/sec}$$

$$\omega = \frac{(-4.8 \cdot 10^{-10})(\frac{3}{4} \cdot 10^2)}{(9.11 \cdot 10^{-28})(3 \cdot 10^{10})}$$

$$= -.132 \cdot 10^{10} = -1.32 \cdot 10^9 \text{ /sec}$$

$$\frac{v_{0y}}{\omega} = -6.8 \cdot 10^{-1}, \quad \left(\frac{v_{0y}}{\omega}\right)^2 = 4.6 \cdot 10^{-1}; \quad \frac{v_c}{\omega} = -9.1 \cdot 10^{-1}, \quad \left(\frac{v_c}{\omega}\right)^2 = 8.3 \cdot 10^{-1}$$

$$\therefore \quad (x' - 6.8 \cdot 10^{-1})^2 + (y' - 9.1 \cdot 10^{-1})^2 = 12.9 \cdot 10^{-1} = \{11.4 \cdot 10^{-1}\}^2$$

(6) Now $x = v_c t + x' = \dfrac{\mathcal{E} c}{B} t + x'$

$y = y'$

(7) Rewrite: $x' = -\left(\dfrac{v_c}{\omega} \sin \omega t - \dfrac{v_{0y}}{\omega} \cos \omega t \right) - \dfrac{v_{0y}}{\omega}$

$$y' = \left(\frac{v_{0y}}{\omega} \sin \omega t + \frac{v_c}{\omega} \cos \omega t \right) - \frac{v_c}{\omega}$$

$$x' = - \underbrace{\frac{\sqrt{v_c^2 + v_{0y}^2}}{\omega}}_{A} \sin \underbrace{\left\{ \omega t - \underbrace{\tan^{-1} \frac{v_{0y}}{v_c}}_{\gamma} \right\}}_{\theta} - \frac{v_{0y}}{\omega}$$

$$y' = \frac{\sqrt{v_c^2 + v_{0y}^2}}{\omega} \cos \left\{ \omega t - \tan^{-1} \frac{v_{0y}}{v_c} \right\} - \frac{v_c}{\omega}$$

(8)
$$\left. \begin{aligned} x &= -A \sin \theta - \frac{v_{0y}}{\omega} + v_c t \\ y &= A \cos \theta - \frac{v_c}{\omega} \end{aligned} \right\} \quad \theta = \omega t - \gamma, \; t = \frac{\theta - \gamma}{\omega}$$

$$\therefore \quad x = -A \sin \theta + \frac{v_c}{\omega} \theta + \left\{ \frac{v_c}{\omega} \gamma - \frac{v_{0y}}{\omega} \right\}$$

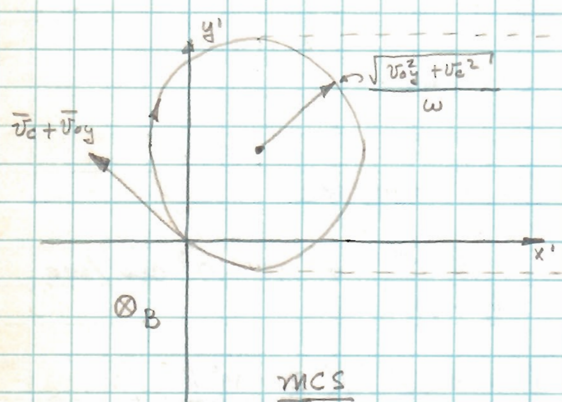
(9) $\theta = \cos^{-1} \dfrac{y - \frac{v_c}{\omega}}{A} = \cos^{-1} \dfrac{\omega y - v_c}{\sqrt{v_c^2 + v_{0y}^2}}$

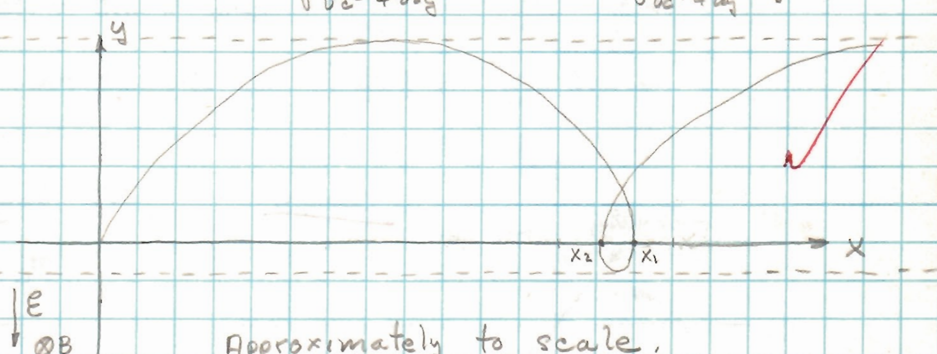
$$\sin \theta = \frac{\left\{ A^2 - y^2 + 2 \frac{v_c}{\omega} y - \left(\frac{v_c}{\omega} \right)^2 \right\}^{1/2}}{A}$$

$$-A \sin \theta = -\left\{ -y^2 + 2 \frac{v_c}{\omega} y + \left(\frac{v_{0y}}{\omega} \right)^2 \right\}^{1/2}$$

(10) $x = -\left\{ -y^2 + 2 \dfrac{v_c}{\omega} y + \left(\dfrac{v_{0y}}{\omega} \right)^2 \right\}^{1/2} + \dfrac{v_c}{\omega} \left\{ \cos^{-1} \dfrac{\omega y - v_c}{\sqrt{v_c^2 + v_{0y}^2}} + \tan^{-1} \dfrac{v_{0y}}{-v_c} \right\} - \dfrac{v_{0y}}{\omega}$

$$x = -\left\{ \left(\frac{v_{0y}}{\omega} \right)^2 + 2 \frac{v_c}{\omega} y - y^2 \right\}^{1/2} + \frac{v_c}{\omega} \left\{ \cos^{-1} \frac{\omega y - v_c}{\sqrt{v_c^2 + v_{0y}^2}} - \cos^{-1} \frac{v_c}{\sqrt{v_c^2 + v_{0y}^2}} \right\} - \frac{v_{0y}}{\omega}$$





Approximately to scale.
See below for x_1, x_2.

SCS

(11) The period of one orbit is the circumference divided by the orbital velocity:

$$T = \frac{\dfrac{2\pi \sqrt{v_{0y}^2 + v_c^2}}{\omega}}{\sqrt{v_{0y}^2 + v_c^2}} = \frac{2\pi}{\omega}$$

so that the period of the orbit in the MCS is independent of the velocities.

Assignment #1
Continued

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Problem 3
Continued:

(12) Look again at the parametric equations of motion:

$$x = -\left(\frac{v_c}{\omega}\sin\omega t - \frac{v_{oy}}{\omega}\cos\omega t\right) - \frac{v_{oy}}{\omega} + v_c t$$

$$y = y' = \left(\frac{v_{oy}}{\omega}\sin\omega t + \frac{v_c}{\omega}\cos\omega t\right) - \frac{v_c}{\omega}$$

(13) We note that when the orbit is completing its <u>first</u> cycle, it is crossing the x-axis for the <u>second</u> time. That is, at $t = T = \frac{2\pi}{\omega}$, $y = 0$

and $x_2 = 2\pi\frac{v_c}{\omega} = (6.28)(9.1\cdot 10^{-1}) = \underline{5.7\ \text{cm}}$

It is easily seen that x_2 is completely independent of the initial velocity regardless of its speed and direction. The period T is independent of any velocities while the only non-vanishing term in equations (12) is $v_c t$ in the x equation. Of course the initial velocity must lie in the x-y plane. The crux of the matter is that the particle in the MCS always passes thru the origin of the MCS, completing one period, corresponding to the second intersection with the x-axis, regardless of the initial velocity.

(14) For electron motion, we change the sign of ω everywhere:

$$x = -\frac{v_c}{\omega}\sin\omega t - \frac{v_{oy}}{\omega}\cos\omega t + \frac{v_{oy}}{\omega} + v_c t$$

$$y = \frac{v_{oy}}{\omega}\sin\omega t - \frac{v_c}{\omega}\cos\omega t + \frac{v_c}{\omega}$$

Equation of circle in MCS:

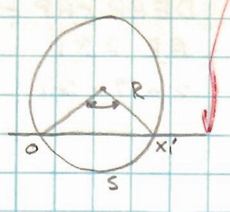
$$\left(x' - \frac{v_{oy}}{\omega}\right)^2 + \left(y' - \frac{v_c}{\omega}\right)^2 = \left(\frac{v_{oy}}{\omega}\right)^2 + \left(\frac{v_c}{\omega}\right)^2$$

(15) When the particle cuts the x-axis for the first time:

$$y' = 0; \quad (x')^2 - 2x'\frac{v_{oy}}{\omega} + \left(\frac{v_{oy}}{\omega}\right)^2 = \left(\frac{v_{oy}}{\omega}\right)^2 + \left(\frac{v_c}{\omega}\right)^2 - \left(\frac{v_c}{\omega}\right)^2$$

$$x'^2 - 2x'\frac{v_{oy}}{\omega} = 0; \quad x'\left(x' - 2\frac{v_{oy}}{\omega}\right) = 0$$

$$x' = 0, \quad x' = \frac{2v_{oy}}{\omega}$$

(16)  The time to the first intersection is:

$$t_1 = \frac{2\pi R - s}{v} \quad ; \quad s = R\theta \; ; \; x_1' = 2R\sin\tfrac{1}{2}\theta$$

$$t_1 = \frac{2R}{v}\left\{\pi - \sin^{-1}\frac{x_1'}{2R}\right\}$$

$$= \frac{2}{\omega}\left\{\pi - \sin^{-1}\frac{x_1'\,\omega}{2\sqrt{v_c^2 + v_{0y}^2}}\right\}$$

(17) $$x_1' = \frac{2\,v_{0y}}{\omega}$$

$$\therefore\; t_1 = \frac{2}{\omega}\left\{\pi - \sin^{-1}\left[\frac{v_{0y}}{\sqrt{v_c^2 + v_{0y}^2}}\right]\right\}$$

(18) $$x_1 = -\frac{v_c}{\omega}\sin\left\{2\pi - 2\sin^{-1}\left[\frac{v_{0y}}{\sqrt{v_c^2+v_{0y}^2}}\right]\right\}$$

$$-\frac{v_{0y}}{\omega}\cos\left\{2\pi - \sin^{-1}\left[\frac{v_{0y}}{\sqrt{v_c^2+v_{0y}^2}}\right]\right\} + \frac{v_{0y}}{\omega} + \frac{v_c}{\omega}\left\{2\pi - 2\sin^{-1}\left[\frac{v_{0y}}{\sqrt{v_{0y}^2+v_c^2}}\right]\right\}$$

or $$x_1 = 2\left\{\frac{v_{0y}}{\omega} + \frac{v_c}{\omega}\left(\pi - \sin^{-1}\left[\frac{v_{0y}}{\sqrt{v_{0y}^2+v_c^2}}\right]\right)\right\}$$

$$= 2\left\{(.68) + (.91)\left(\pi - \sin^{-1}.6\right)\right\}$$

$$= 2\left\{(.68) + (.91)(2.49)\right\} = 2(.68 + 2.26) = 2(2.94)$$

$$= 5.82 \text{ cm.} \quad 5.91$$

Assignment #1
Continued

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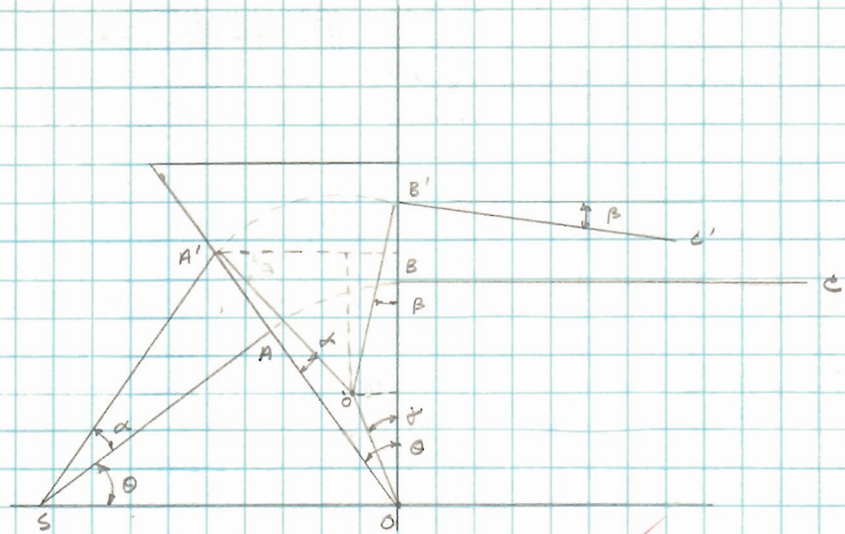
4.



(1) Since the velocities are the same for each trajectory, the radii of curvature will be the same for each path, viz:

$$OA' = O'B' = OA = OB$$

(2) Now: $\frac{OO'}{\sin\beta} = \frac{B'O'}{\sin\gamma}$; $\frac{O'O}{\sin\alpha} = \frac{A'O'}{\sin(\theta-\gamma)}$

(3) $OO'\sin\gamma = (OA + AA')\sin\theta - O'A'\sin(\theta+\alpha)$

or $\sin\beta = \left(1 + \frac{AA'}{O'B'}\right)\sin\theta - \sin(\theta+\alpha)$

(4) $\frac{A'A}{SA}\cdot\frac{SA}{O'B'} = \frac{A'A}{SA}\cdot\frac{SA}{OA} = \tan\alpha\cot\theta$

(5) $\sin\beta = \sin\theta + \tan\alpha\cos\theta - \sin\theta\cos\alpha - \sin\alpha\cos\theta$

$= \sin\theta\{1-\cos\alpha\} + \{\tan\alpha - \sin\alpha\}\cos\theta$

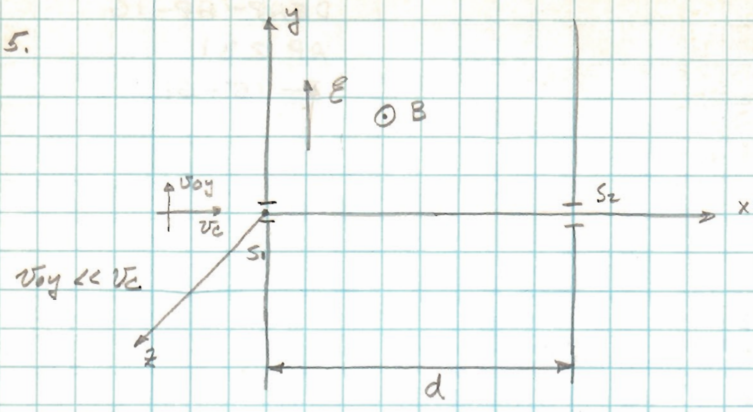
(6) For α small, $\sin\beta \sim \beta$, $\tan\alpha \simeq \sin\alpha$,

$\cos\alpha = 1 - \frac{\alpha^2}{2} + \cdots$

(7) $\therefore\ \beta = \frac{\alpha^2}{2}\sin\theta$ QED.

(8) As stated in the problem, a corollary to this result leads to the basic principle of the magnetic sector spectrograph, that is, particles entering from CC' are focused at S, so a magnetic "prism" can be constucted with two focal points.

5.



$v_{oy} \ll v_c$

(1) The equation of motion is:

(2) $$m \frac{d}{dt}\left(v_x \bar{i} + v_y \bar{j}\right) = e\mathcal{E}\bar{j} + \frac{e}{c}\left(-v_x B \bar{j} + v_y B \bar{i}\right)$$

(3) choose the usual moving coordinate system to eliminate $\mathcal{E}$: $v_x = v_x' + v_c$

$v_y = v_y'$

$\frac{v_c B}{c} = \mathcal{E}$

(4) $$\dot{v}_x' = \omega v_y' \qquad \dot{v}_y' = -\omega v_x'$$

$$\dot{v}_x + i\dot{v}_y = -i\omega\left\{v_x' + i v_y'\right\} \quad ; \quad v_x' + i v_y' = \left(v_{x0}' + i v_{oy}'\right) e^{-i\omega t}$$

(5) Take $v_{x0}' = 0$, $v_{y0}' = v_{y0} \ll v_c$

$$v_x' + i v_y' = i v_{oy}\left(\cos \omega t - i \sin \omega t\right)$$

$$v_x' = v_{oy} \sin \omega t \quad , \quad v_y' = v_{oy} \cos \omega t$$

(6) $$y' = y = \frac{v_{oy}}{\omega} \sin \omega t ,$$

$$v_x = v_c + v_{oy} \sin \omega t \simeq v_c \quad \text{as } v_c \gg v_{oy},$$

assuming the particle is at $x, y = 0$ at $t = 0$.

$\therefore \quad x = v_c t$

(7) The transit time is: $\tau = \frac{d}{v_c}$. We stipulate that the particle must have $y = 0$ at $t = \tau$ in order that it be focused at S_2.

$\therefore \quad \sin \omega\tau = 0 \; ; \quad \omega\tau = n\pi$, $n = 1$ because particle will orbit otherwise.

spare me!

(8) $$\omega\tau = \frac{eB}{mc} \cdot \frac{dB}{\mathcal{E}c} = \pi$$

or $$\left(\frac{e}{m}\right)\frac{B^2}{c^2\mathcal{E}} = \frac{\pi}{d}$$

more important if you choose 2π then all particles would end up at S_2 as you showed in prob 3

Thus providing a criteria for choosing B and $\mathcal{E}$ such that focusing of partices of specific charge e/m at S_2 is enabled.

Assignment # 1
Continued

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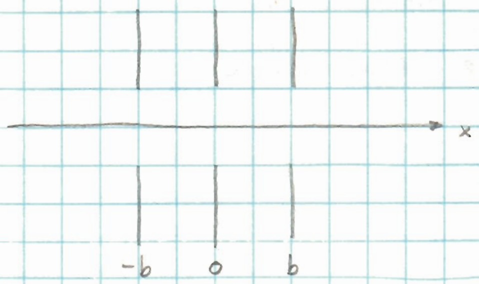
6. We have a symmetrical electrostatic lens:



$$V = V_0\left(1 + a + \frac{az}{b}\right), \quad -b < z < 0$$

$$V = V_0\left(1 + a - \frac{az}{b}\right), \quad 0 < z < b$$

$$V = V_0, \quad z < -b, \quad z > b$$

(1) The equation derived in class for the focal length of this lens is:

$$\frac{1}{f} = \frac{1}{8 V_b^{1/2}} \int_{-\infty}^{\infty} \left(\frac{dV}{dz}\right)^2 \frac{dz}{V^{3/2}}$$

(2) $V_b = V_0$, $\frac{dV}{dz} = \frac{aV_0}{b}$, $-b < z < 0$; $-\frac{aV_0}{b}$, $0 < z < b$; 0 otherwise

(3)
$$\frac{1}{f} = \frac{a^2}{8b^2}\left\{\int_{-b}^{0} \frac{dz}{\left(1 + a + \frac{az}{b}\right)^{3/2}} + \int_0^b \frac{dz}{\left(1 + a - \frac{az}{b}\right)^{3/2}}\right\}$$

$$= \frac{a}{8b}\left\{\int_{-a}^{0} \frac{du}{\left[(1+a) + u\right]^{3/2}} + \int_0^a \frac{du}{\left[(1+a) - u\right]^{3/2}}\right\}$$

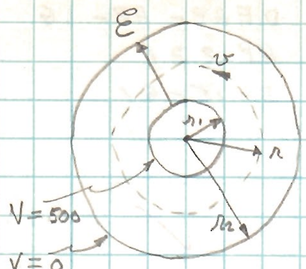
$$= \frac{a}{8b}\left\{\int_{1}^{1+a} \frac{du}{u^{3/2}} - \int_{1+a}^{1} \frac{du}{u^{3/2}}\right\} = \frac{a}{4b}\int_1^{(1+a)} \frac{du}{u^{3/2}}$$

$$= \frac{a}{4b}\left\{-2u^{-1/2}\right\}_1^{1+a} = \frac{a}{4b}\left\{2 - \frac{2}{(1+a)^{1/2}}\right\}$$

$$= \frac{a}{2b}\left\{\frac{-1}{(1+a)^{1/2}} + 1\right\}$$

(4) $\therefore$
$$f = \frac{2b}{a}\left\{1 - \frac{1}{(1+a)^{1/2}}\right\}^{-1}$$

7. a.



$r_1 = 1$ cm
$r_2 = 2$ cm

(1) We must have for equilibrium orbit:

$$F_r = -e\,\mathcal{E}_r\,, \qquad \frac{mv^2}{r} = -e\,\mathcal{E}_r$$

$$\therefore\ v^2 = -\frac{e\,\mathcal{E}_r r}{m} = \frac{e r}{m}\frac{dV}{dr}$$

(2) Laplace's equation: $\nabla^2 V = 0$

or, since there is no θ or z dependence:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dV}{dr}\right) = 0$$

subject to $V = V_0\,,\ r = r_1$
$V = 0\,,\ r = r_2$

(3) $$r\frac{dV}{dr} = A\ ; \qquad V = A\ln r + B$$

$$\left.\begin{aligned} V_0 &= A\ln r_1 + B \\ 0 &= A\ln r_2 + B \end{aligned}\right\} \quad A = \frac{V_0}{\ln r_1/r_2}\,, \qquad B = -\frac{V_0}{\ln r_1/r_2}\ln r_2$$

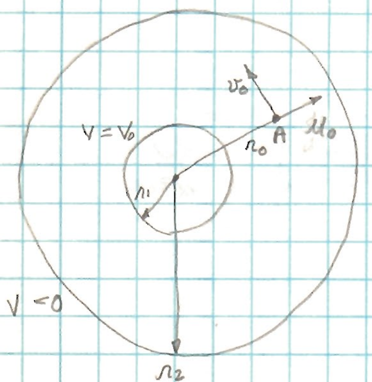
$$V = \frac{V_0}{\ln r_1/r_2}\ln r/r_2 = V_0\,\frac{\ln r_2/r}{\ln r_2/r_1}$$

$$\frac{dV}{dr} = \frac{V_0}{r\ln r_1/r_2} = \frac{-V_0}{r\ln r_2/r_1}$$

(4) $$v^2 = \frac{-e V_0}{m\ln r_2/r_1}\ ; \qquad v = \left\{\frac{-eV_0}{m\ln r_2/r_1}\right\}^{1/2}$$

$$v = \left\{\frac{(1.6\cdot 10^{-20})(5\cdot 10^{10})}{(9.11\cdot 10^{-28})\ln 2}\right\}^{1/2} = 1.13\cdot 10^{9}\ \text{cm/sec} \sim 4\%\text{ of } c$$

b.



(1) The general equations of motion are:

radial: $$m\frac{d^2 r}{dt^2} = mr\left(\frac{d\theta}{dt}\right)^2 - e\,\mathcal{E}_r$$

angular: $$\vec{L} = \frac{d\vec{M}}{dt}\ ; \qquad M = mvr$$

or $$re\,\mathcal{E}_\theta = \frac{d}{dt}\left\{mr^2\frac{d\theta}{dt}\right\}$$

Since $\mathcal{E}_\theta = 0$,

$$mr^2\frac{d\theta}{dt} = \text{constant}$$, or $\vec{M}$ is conserved throughout the motion.

(2) The initial conditions are that at $r = r_0$, $\omega_0 = \frac{v_0}{r_0}$,

$$\left(\frac{dr}{dt}\right)_0 = u_0\,, \qquad u_0 \ll v_0$$

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7. Continued:

b. (3) It was shown in part (a) that the velocity necessary for a stable orbit is independent of its position in the field. From equation a.(1):

$$\mathcal{E}_r = \frac{m v_0^2}{e r}$$

(4) Since angular momentum is conserved, the angular momentum throughout ~~is~~ equal to the initial angular momentum:

$$\frac{dM}{dt} = 0 \quad ; \quad M = \text{constant} \quad ; \quad M_0 = m v_0 r_0 = \text{constant}$$

$$\therefore \quad m r^2 \frac{d\theta}{dt} = m v_0 r_0$$

(5) $$\therefore \quad m \frac{d^2 r}{dt^2} = \frac{m^2 v_0^2 r_0^2}{m^2 r^4} \cdot m r - \frac{m v_0^2}{r}$$

or $$\frac{d^2 r}{dt^2} = \frac{v_0^2 r_0^2}{r^3} - \frac{v_0^2}{r}$$

It is seen that a restoring force is always in effect.

(6) $$\frac{d^2 r}{dt^2} = u \frac{du}{dr}$$

$$\therefore \int u\,du = v_0^2 r_0^2 \int \frac{dr}{r^3} - v_0^2 \int \frac{dr}{r}$$

or $$\frac{u^2}{2} = -\frac{v_0^2 r_0^2}{2 r^2} - v_0^2 \ln r + C$$

$$\frac{u_0^2}{2} = -\frac{v_0^2}{2} - v_0^2 \ln r_0 + C$$

$$C = \frac{v_0^2}{2} + v_0^2 \ln r_0 \quad , \text{ since } u_0 \ll v_0$$

$$\therefore \quad \frac{u^2}{2} = -\frac{v_0^2 r_0^2}{2 r^2} - v_0^2 \ln r + \frac{v_0^2}{2} + v_0^2 \ln r_0$$

or $$u^2 = \dot{r}^2 = v_0^2 \left\{ 1 - \left(\frac{r_0}{r}\right)^2 - 2 \ln \frac{r}{r_0} \right\}$$

(7) Now transfer the co-ordinate system to r_0 via:

$r = r_0 + \rho$ where ρ is the displacement from r_0 and it is assumed $\rho \ll r_0$; and $\rho = 0$, at $t = 0$

$$\therefore \quad \dot{\rho}^2 = v_0^2 \left\{ 1 - \left(1 + \frac{\rho}{r_0}\right)^{-2} - 2 \ln\left(1 + \frac{\rho}{r_0}\right) \right\}$$

Now:

$$\left(1 + \frac{\rho}{r_0}\right)^{-2} = 1 - 2\frac{\rho}{r_0} + 3\left(\frac{\rho}{r_0}\right)^2 - 4\left(\frac{\rho}{r_0}\right)^3 + \cdots$$

$$\ln\left(1 + \frac{\rho}{r_0}\right) = \frac{\rho}{r_0} - \frac{1}{2}\left(\frac{\rho}{r_0}\right)^2 + \frac{1}{3}\left(\frac{\rho}{r_0}\right)^3 - \cdots$$

$$\dot{\rho}^2 = v_0^2 \left\{ 2\frac{\rho}{r_0} - 3\left(\frac{\rho}{r_0}\right)^2 + 4\left(\frac{\rho}{r_0}\right)^3 - 2\frac{\rho}{r_0} + \left(\frac{\rho}{r_0}\right)^2 - \frac{2}{3}\left(\frac{\rho}{r_0}\right)^3 + \cdots \right\}$$

$$\simeq -2 v_0^2 \left(\frac{\rho}{r_0}\right)^2$$

(8) $$\therefore \quad \frac{d\rho}{dt} = \pm i \sqrt{2}\, v_0 \frac{\rho}{r_0} \quad ; \quad \rho = A \cos \sqrt{2}\frac{v_0}{r_0} t + B \sin \sqrt{2}\frac{v_0}{r_0} t$$

(9) $$\frac{d\rho}{dt} = u_0 \quad , \quad t = 0$$

$$0 = A$$
$$u_0 = \sqrt{2}\,\frac{v_0}{r_0} B$$

(10) $$\therefore \quad \rho = \frac{u_0}{\sqrt{2}\, v_0} r_0 \sin \sqrt{2}\frac{v_0}{r_0} t$$

Now the particle crosses r_0 at the zeros of:

$$\sin \sqrt{2}\frac{v_0}{r_0} t = 0$$

Now $\theta = \omega t$ and under our approximation of small ρ: $\omega = \frac{v_0}{r_0}$ or $\theta = \frac{v_0}{r_0} t$

$$\therefore \quad \sin \sqrt{2}\,\theta = 0 \quad \text{or} \quad \sqrt{2}\,\theta = n\pi$$

$\theta = \dfrac{n\pi}{\sqrt{2}}$. The first crossing or focusing point is:

$$\theta = \pi/\sqrt{2} = 127^\circ$$

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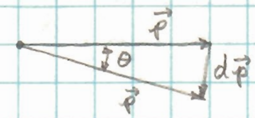
8. (1) $\dfrac{d\bar{p}}{dt} = e\bar{v} \times \bar{B}$ (emu)



$\theta = \dfrac{dp}{p}$; $\dfrac{dp}{dt} = p\omega$

(2) $\bar{v} \perp \bar{B}$: $\dfrac{dp}{dt} = evB = p\omega = e\rho\omega B$

$\therefore \quad p = e\rho B$

(3) $T^2 = p^2c^2 + m^2c^4$

Also: $T = \dfrac{mc^2}{\sqrt{1-\beta^2}}$

Let us split up this total KE into rest KE and motional KE, called E_k, $\therefore$

$$T = mc^2 + E_k$$

(4) $\therefore \quad m^2c^4 + 2mc^2E_k + E_k^2 = e^2\rho^2B^2c^2 + m^2c^4$

$$E_k(2mc^2 + E_k) = e^2c^2B^2\rho^2 = E_k^2\left(1 + \frac{2mc^2}{E_k}\right)$$

(5) $\therefore \quad B\rho = \dfrac{E_k}{ec}\left\{1 + \dfrac{2mc^2}{E_k}\right\}^{1/2}$ (exact expression)

(6) Now: $T = mc^2 + E_k = \dfrac{mc^2}{\sqrt{1-\beta^2}}$

or $\dfrac{E_k}{mc^2} = \dfrac{1}{\sqrt{1-\beta^2}} - 1$

In the extreme relativistic case: $\sqrt{1-\beta^2} \ll 1$
or $\dfrac{1}{\sqrt{1-\beta^2}} \gg 1$

$\therefore \quad \dfrac{E_k}{mc^2} \gg 1$ or $E_k \gg mc^2$

(7) $\therefore$ In the extreme relativistic case, (5) becomes:

$$B\rho = \frac{E_k}{ec}$$

(8) Numerically: $ec = \dfrac{(1.6\cdot10^{-20})(3\cdot10^{10})}{1.6\cdot10^{-12}} = 300 \dfrac{\text{ev}}{\text{gauss}\cdot\text{cm}}$

$= 3\cdot10^{-4}$ Mev / gauss - cm ✓

(9) $\therefore \quad E_k = 3 \cdot 10^{-4} \; B\rho$ ✓

(10) % error in $B\rho = \epsilon = \dfrac{(B\rho)_{correct} - (B\rho)_{approx}}{(B\rho)_{correct}}$

$$= \frac{\frac{E_k}{ec}\left\{\;\right\}^{1/2} - \frac{E_k}{ec}}{\frac{E_k}{ec}\left\{\;\right\}^{1/2}} = 1 - \frac{1}{\left\{1 + \frac{2mc^2}{E_k}\right\}^{1/2}}$$

(11) $1 + \dfrac{2mc^2}{E_k} = \dfrac{1}{(1-\epsilon)^2}$; $\dfrac{2mc^2}{E_k} = \dfrac{1 - 1 + 2\epsilon - \epsilon^2}{(1-\epsilon)^2}$

$$\therefore \quad E_k = 2mc^2 \left\{ \frac{(1-\epsilon)^2}{\epsilon(2-\epsilon)} \right\}$$

(12) For 1%, $\epsilon = .01 \; \therefore E_k = mc^2 \dfrac{1.98}{(10^{-2})(1.99)}$

$\cong 100 \; mc^2 = 100 \cdot .51 \text{ Mev} = 51 \text{ Mev}$ ✓

b. (1) $E_k = 3 \cdot 10^{-4} B\rho$

(2) $\rho = \dfrac{E_k}{3 \cdot 10^{-4} B} = \dfrac{10^3 \text{ Mev}}{3 \text{ Mev/cm}} = 3.33$ meters ✓

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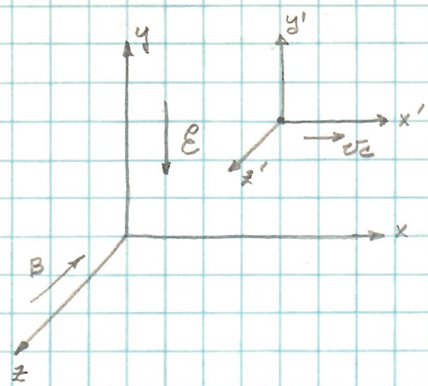
9. a.



(1) We take a positive charge q. The equation of motion is:

$$m\frac{d\bar{v}}{dt} = q\bar{\mathcal{E}} + \frac{q\,\bar{v}\times\bar{B}}{c} \quad \text{(esu!)}$$

(2) We postulate a moving co-ordinate system: position: $\bar{r} = \bar{r}_c + \bar{v}_c t + \bar{r}'$
velocity: $\bar{v} = \bar{v}_c + \bar{v}'$

(3) $\therefore \quad m\frac{d\bar{v}'}{dt} = q\bar{\mathcal{E}} + q\frac{(\bar{v}_c+\bar{v}')\times\bar{B}}{c}$

Since $\bar{\mathcal{E}} \perp \bar{B}$, we can choose $\bar{v}_c$, such that

$\frac{q\,\bar{v}_c\times\bar{B}}{c} = -\bar{\mathcal{E}}q$, which makes v_c as shown.

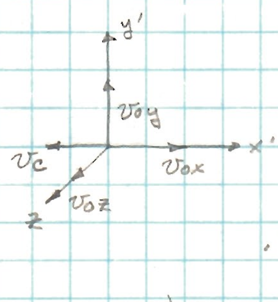
$$\therefore \quad m\frac{d\bar{v}'}{dt} = \frac{q\,\bar{v}'\times\bar{B}}{c}$$

or: $m(\dot{v}_x'\bar{i} + \dot{v}_y'\bar{j} + \dot{v}_z'\bar{k}) = \frac{q}{c}(v_x'\bar{i} + v_y'\bar{j} + v_z'\bar{k})\times -B\bar{k}$

$= \frac{q}{c}\{v_x'B\bar{j} - v_y'B\bar{i}\}$; $\dot{v}_z' = 0$

or $\left.\begin{aligned} m\dot{v}_x' &= -\frac{q}{c}Bv_y' \\ m\dot{v}_y' &= \frac{q}{c}Bv_x' \end{aligned}\right\}$ $\left.\begin{aligned} \dot{v}_x' &= -\omega v_y' \\ \dot{v}_y' &= \omega v_x' \end{aligned}\right\}$ $\omega \equiv \frac{qB}{mc}$

(4) $\frac{d}{dt}(v_x' + iv_y') = i\omega(v_x' + iv_y')$



$\bar{v} = \bar{v}_c + \bar{v}'$
$v_{ox} = v_c + v_{ox}'$
$v_{oy} = v_{oy}'$
$v_{oz} = v_{oz}'$; $v_z' = v_{oz}$

$$\therefore \quad (v_x' + iv_y') = (v_{ox} - v_c + iv_{oy})e^{i\omega t}$$

$= (v_{ox} - v_c)\cos\omega t + i(v_{ox} - v_c)\sin\omega t + iv_{oy}\cos\omega t - v_{oy}\sin\omega t$

or $v_x' = (v_{ox} - v_c)\cos\omega t - v_{oy}\sin\omega t$

$v_y' = (v_{ox} - v_c)\sin\omega t + v_{oy}\cos\omega t$

$v_z' = v_{oz}$

(5) Choose $r_c = 0$ at $t = 0$: $x', y', z' = 0$, $t = 0$

$$x' = \frac{(v_{ox} - v_c)}{\omega} \sin \omega t + \frac{v_{oy}}{\omega} \cos \omega t - \frac{v_{oy}}{\omega}$$

$$y' = -\frac{(v_{ox} - v_c)}{\omega} \cos \omega t + \frac{v_{oy}}{\omega} \sin \omega t + \frac{(v_{ox} - v_c)}{\omega}$$

$$z' = v_{oz} t$$

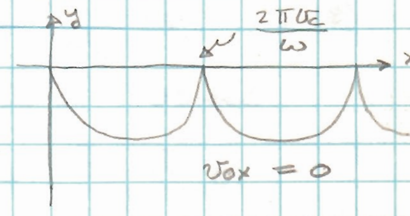
(6)

$$x = v_c t + x'$$
$$y = y'$$
$$z = z'$$

$$x = v_c t + \frac{(v_{ox} - v_c)}{\omega} \sin \omega t + \frac{v_{oy}}{\omega} \cos \omega t - \frac{v_{oy}}{\omega}$$

$$y = -\frac{(v_{ox} - v_c)}{\omega} \cos \omega t + \frac{v_{oy}}{\omega} \sin \omega t + \frac{(v_{ox} - v_c)}{\omega}$$

$$z = v_{oz}\, t$$



(7) Consider $v_{oy}, v_{oz} = 0$:

$$x = v_c t + \frac{(v_{ox} - v_c)}{\omega} \sin \omega t$$

$$y = -\frac{(v_{ox} - v_c)}{\omega} \cos \omega t + \frac{(v_{ox} - v_c)}{\omega}, \quad z = 0$$

$$\cos \omega t = \frac{(v_{ox} - v_c) - \omega y}{v_{ox} - v_c}; \quad \sin \omega t = \left\{(v_{ox} - v_c)^2 - (v_{ox} - v_c)^2 + 2\omega (v_{ox} - v_c) y - \omega^2 y^2\right\}^{1/2} \Big/ v_{ox} - v_c$$

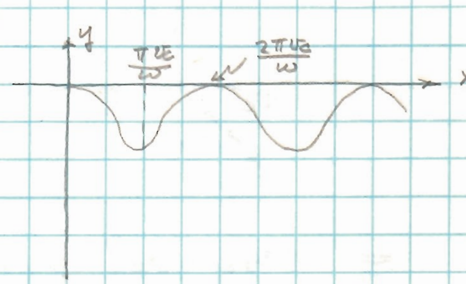
$$\therefore \quad x = \frac{v_c}{\omega} \cos^{-1} \left\{ \frac{(v_{ox} - v_c) - \omega y}{v_{ox} - v_c} \right\} + \frac{\left\{2\omega (v_{ox} - v_c) y - \omega^2 y^2\right\}^{1/2}}{\omega}$$

(8) Take $V_{ox} = 0$: $\quad x = \frac{v_c}{\omega} \cos^{-1} \left\{ \frac{v_c + \omega y}{v_c} \right\} + \frac{\left\{-2\omega v_c y - \omega^2 y^2\right\}^{1/2}}{\omega}$

when $y = 0$: $\quad x_n = \frac{v_c}{\omega} \cos^{-1} 1 = 2\pi n \frac{v_c}{\omega}, \quad n = 0, 1, 2, 3, \ldots$

Could also be done by finding the vertical tangents!

b. (1) Take $0 < v_{ox} < v_c$:



(2)

$$\frac{dx}{dt} = v_c + (v_{ox} - v_c) \cos \omega t$$

$$\frac{dy}{dt} = (v_{ox} - v_c) \sin \omega t$$

(3)

$$\frac{dy}{dx} = \frac{(v_{ox} - v_c) \sin \omega t}{v_c + (v_{ox} - v_c) \cos \omega t} = 0$$

$$\therefore \quad \sin \omega t = 0, \quad \omega t = n\pi; \quad n = 0, 1, 2, 3, \ldots$$

(4) $x = v_c t + \frac{(v_{ox} - v_c)}{\omega} \sin \omega t; \quad x_n = n\pi \frac{v_c}{\omega} = \frac{n}{2}\left(\frac{2\pi v_c}{\omega}\right), \quad n = 0, 1, 2, \ldots$

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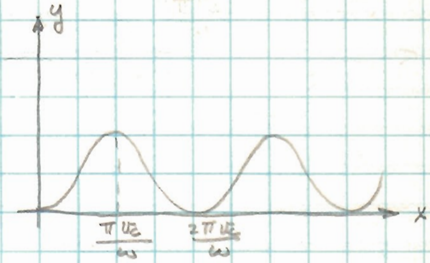
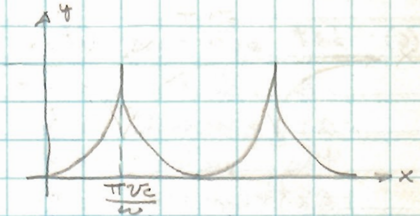
9. Continued



c. (1) Take $v_c < v_{ox} < 2v_c$

(2) $$\frac{dy}{dx} = \frac{(v_{ox} - v_c)\sin\omega t}{v_c + (v_{ox} - v_c)\cos\omega t} = 0$$

which leads to $x_n = \frac{n}{2}\left(\frac{2\pi v_c}{\omega}\right)$ as before in b.



d. (1) Take $v_{ox} = 2v_c$:

(2) $$\frac{dy}{dx} = \frac{v_c \sin\omega t}{v_c(1+\cos\omega t)} = \infty$$

(3) $\therefore\ (1+\cos\omega t) = 0$; $\cos\omega t = -1$; $\omega t = (2n+1)\pi$, $n = 0, 1, 2, \cdots$

(4) However, this is also the zeroes of $\sin\omega t$, so we check: ✓

$$\lim_{\theta\to(2n+1)\pi} \frac{\sin\omega t}{1+\cos\omega t} = \lim_{\theta\to(2n+1)\pi} \cot\theta = \infty \text{ , ok.}$$

(5) $\therefore\ x_n = (2n+1)\pi\frac{v_c}{\omega} = \frac{(2n+1)}{2}\left(\frac{2\pi v_c}{\omega}\right)$

e. (1) Take $v_{ox} > 2v_c$: $v_{ox} = 2v_c + \delta v_{ox}$

(2) $$\frac{dy}{dx} = \frac{(2v_c + \delta v_{ox} - v_c)\sin\omega t}{v_c + (2v_c + \delta v_{ox} - v_c)\cos\omega t}$$

(3) $v_c + (v_c + \delta v_{ox})\cos\omega t = 0$, $\cos\omega t = \frac{-v_c}{v_c + \delta v_{ox}}$

$$\sin\omega t = \pm\frac{\left\{v_c\,\delta v_{ox} + (\delta v_{ox})^2\right\}^{1/2}}{v_c + \delta v_{ox}}$$

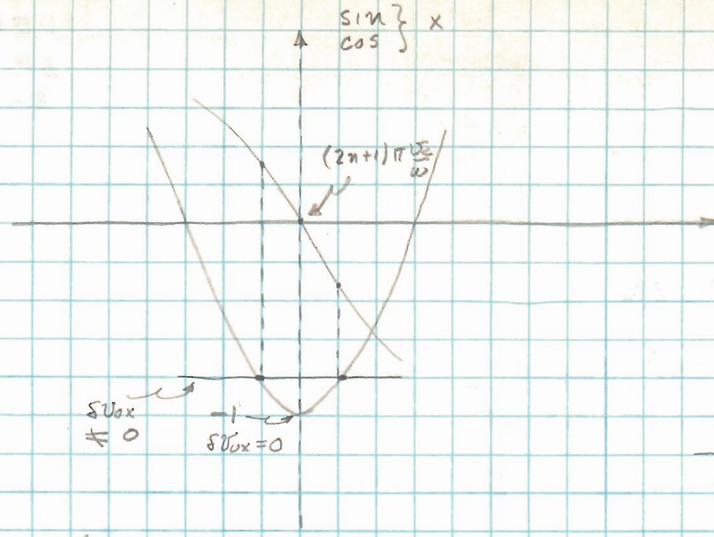
(4) $$x = v_c t + \frac{(v_c + \delta v_{ox})}{\omega}\sin\omega t$$

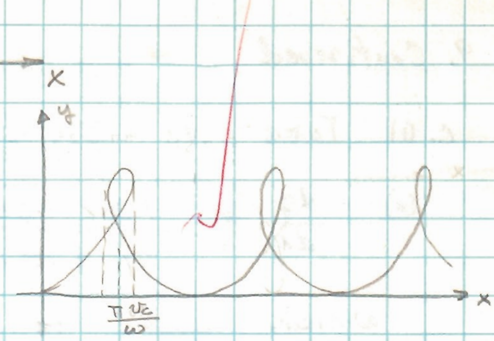
$$= \frac{v_c}{\omega}\cos^{-1}\frac{-v_c}{v_c+\delta v_{ox}} \pm \frac{\left\{v_c\,\delta v_{ox} + (\delta v_{ox})^2\right\}^{1/2}}{\omega}$$

(5) We see that when $\delta v_{ox} = 0$, we get:

$x_n = (2n+1)\pi\frac{v_c}{\omega}$ as before.

The symmetry of the vertical tangents can be best shown by a diagram.



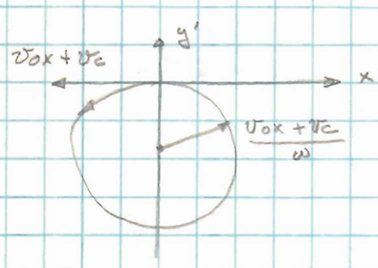


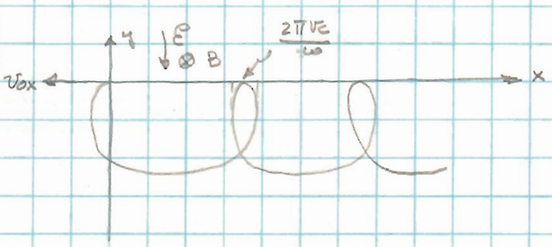
(6) Let us consider $\delta v_{ox} << v_c$, then for the $\cos^{-1}$ term in e (4), the addition of δv_{ox} will cause the argument of $\cos^{-1}$ to decrease slightly from -1 to the value shown. However, where there was only one extremum at $x_n = (2n+1)\pi \frac{v_c}{\omega}$, there is now two symmetrically positioned about the old one. In order to determine the sign of the second term that goes with each of the new extrema, we project to the sine curve as shown. Thus the negative sign goes with the upper value of $\cos^{-1}$ and (+) with the lower, so symmetry is still preserved.

f. (1) $v_{ox} < 0$: $$\frac{dy}{dx} = \frac{-(v_{ox}+v_c)\sin\omega t}{v_c - (v_{ox}+v_c)\cos\omega t}$$

(2) $$x = v_c t - \frac{(v_{ox}+v_c)}{\omega}\sin\omega t$$

$$y = \frac{(v_{ox}+v_c)}{\omega}\cos\omega t - \frac{(v_{ox}+v_c)}{\omega}$$



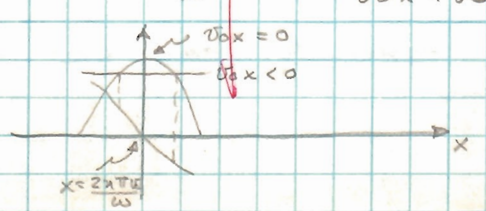


(3) Suppose $v_{ox} = 0$; $\therefore$ when $\frac{dy}{dx} = \infty$, $\cos\omega t = 1$, $\omega t = 2n\pi$

$x_n = 2n\pi \frac{v_c}{\omega}$, $n = 0, 1, 2, 3, \cdots$

(4) $v_{ox} < 0$; $\frac{dy}{dx} = \infty$ when : $\cos\omega t = \frac{v_c}{v_{ox}+v_c}$, $\sin\omega t = \pm\frac{\{v_{ox}^2 + v_{ox}v_c\}^{1/2}}{v_{ox}+v_c}$

(5) $$x = \frac{v_c}{\omega}\cos^{-1}\frac{v_c}{v_{ox}+v_c} \pm \frac{\{v_{ox}^2 + v_{ox}v_c\}^{1/2}}{\omega}$$



Using an analysis similar to that above, we immediately see the symmetry and determine which value of $\cos^{-1}$ belongs to the proper sign of the second term.

Assignment #1
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10. (1) $$\frac{d\vec{p}}{dt} = \frac{e\vec{v}\times\vec{B}}{c}$$

(2) $$\frac{d}{dt}\left\{\frac{m\vec{v}}{\sqrt{1-\frac{\bar{v}\cdot\bar{v}}{c^2}}}\right\} = \frac{m}{\sqrt{1-\beta^2}}\frac{d\vec{v}}{dt} - (m\bar{v})\frac{1}{2}\{1-\beta^2\}^{-3/2}$$

$$\cdot -\frac{2}{c^2}\bar{v}\cdot\frac{d\bar{v}}{dt} = \frac{m}{\sqrt{1-\beta^2}}\frac{d\bar{v}}{dt} + \frac{m/c^2}{\{1-\beta^2\}^{3/2}}\bar{v}\left(\bar{v}\cdot\frac{d\bar{v}}{dt}\right)$$

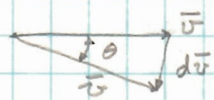
$$= \frac{e\bar{v}\times\bar{B}}{c}$$

(3) Since $\bar{v}\times\bar{B}\perp\bar{v}$, $\bar{v}\cdot\frac{d\bar{v}}{dt} = 0$, $\therefore \bar{v}\perp\frac{d\bar{v}}{dt}$

and

$$\frac{m}{\sqrt{1-\beta^2}}\frac{d\bar{v}}{dt} = \frac{e\,\bar{v}\times\bar{B}}{c}$$

(4) Take $\bar{v}\perp\bar{B}$:

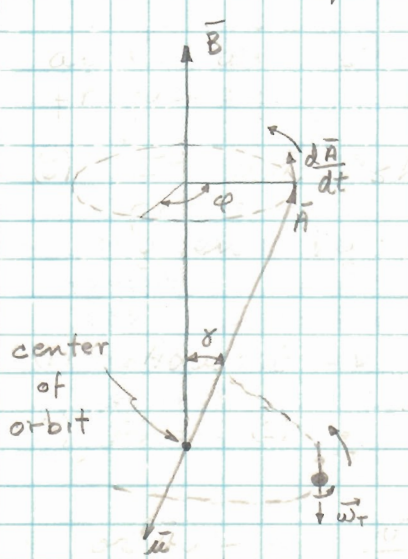


$$\theta = \frac{d\bar{v}}{\bar{v}}$$

$$\bar{v}\,\omega = \frac{d\bar{v}}{dt}$$

(5) $$\therefore \frac{mv\omega}{\sqrt{1-\beta^2}} = \frac{evB}{c}\ ; \quad \omega_c = \frac{eB}{mc}\sqrt{1-\beta^2} \simeq \frac{eB}{mc}\left\{1-\frac{v^2}{2c^2}\right\}$$ ✓

(6) Consider the precession of an electron orbit including Thomas precession: We consider velocities of the electron in the direction of B to be negligible. The angular momentum equation is:



(7) $$\frac{d\bar{A}}{dt} = \vec{\mu}\times\bar{B} + \vec{\omega}_T\times\bar{A}$$

$$= \left(-\frac{e}{mc}\vec{A}\times\vec{B} + \frac{1}{2c}\dot{\vec{v}}\times\vec{v}\times\vec{A}\right)$$

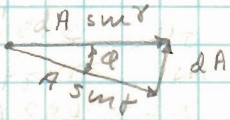
(8) We take as + the direction of $d\vec{A}/dt$:

$$\therefore \frac{dA}{dt} = \left\{\frac{eB}{mc}\sin\gamma - \frac{1}{2c}v\frac{dv}{dt}\sin\gamma\right\}A$$ ✓

$$= \left\{\frac{eB}{mc} - \frac{1}{2c}v\left(\frac{evB}{mc}\right)\right\}A\sin\gamma$$

$$= \frac{eB}{mc}\left\{1-\frac{v^2}{2c^2}\right\}A\sin\gamma$$

(9)

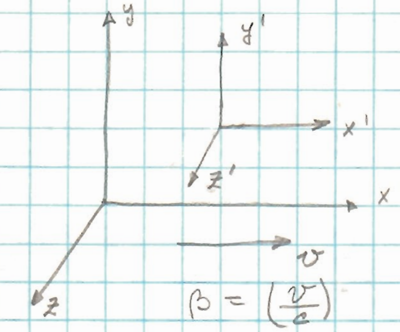


$$\therefore \varphi = \frac{dA}{A\sin\gamma}, \text{ or } \frac{dA}{dt} = \frac{d\varphi}{dt}A\sin\gamma$$

(10) Define: $\omega_s = \frac{d\varphi}{dt}$: $\therefore \omega_s = \frac{eB}{mc}\left\{1-\frac{v^2}{2c^2}\right\}$ ✓, which is seen to equal the cyclotron resonance to $\frac{v^2}{2c^2}$

11. (1) Given an arbitrary direction of velocity, it is always possible to perform a similitude transformation To bring a principle axis of the coordinate system parallel to the direction of velocity. We assume our system has been so prepared:

(2)



The equations of transformation are:

$$dx' = \frac{dx - \beta(c\,dt)}{\sqrt{1-\beta^2}}$$

$$dy' = dy \quad , \quad dz' = dz$$

$$(\imath c\,dt') = \imath\left\{\frac{(c\,dt) - \beta\,dx}{\sqrt{1-\beta^2}}\right\}$$

(3) We define the differential element of length in 4-space via the Pythogorean theorem, viz:

$$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 - (c\,dt)^2$$

(4) Now, in the moving system:

$$(ds')^2 = (dx')^2 + (dy')^2 + (dz')^2 - (c\,dt')^2$$

$$= \frac{(dx)^2 - 2\beta(c\,dt)(dx) + \beta^2(c\,dt)^2}{1-\beta^2} + (dy)^2 + (dz)^2$$

$$- \frac{(c\,dt)^2 + 2\beta(c\,dt)(dx) - \beta^2(dx)^2}{1-\beta^2}$$

$$= \frac{(dx)^2(1-\beta^2) - (c\,dt)^2(1-\beta^2)}{1-\beta^2} + (dy)^2 + (dz)^2$$

$$= (dx)^2 + (dy)^2 + (dz)^2 - (c\,dt)^2 = (ds)^2$$

(5) Thus, the length of a vector in 4-space is invariant under the Lorentz transformation. Although the velocity was taken in the x direction, the result is perfectly general as a principle axis transformation is always possible. ✓

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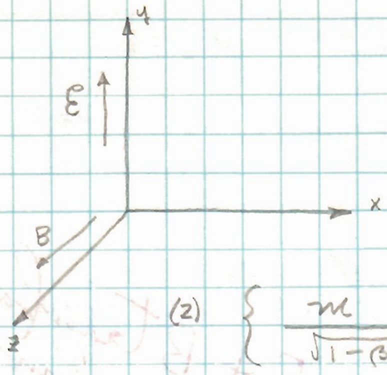
12.



(1) $v_{0x} = v_{0y} = v_{0z} = 0$:
The relativistically correct equation is:

$$\frac{d}{dt}\left\{\frac{m\bar{v}}{\sqrt{1-\beta^2}}\right\} = e\bar{\mathcal{E}} + \frac{e\bar{v}\times\bar{B}}{c}$$

(2) $$\left\{\frac{m}{\sqrt{1-\beta^2}} + \frac{m}{c^2}\frac{\bar{v}\bar{v}}{(\sqrt{1-\beta^2})^3}\right\}\cdot\frac{d\bar{v}}{dt} = e\bar{\mathcal{E}} + \frac{e\bar{v}\times\bar{B}}{c}$$

(3) $$\frac{m}{\sqrt{1-\beta^2}}\left\{\dot{v}_x\bar{i} + \dot{v}_y\bar{j} + \dot{v}_z\bar{k}\right\} + \frac{m/c^2}{\{1-\beta^2\}^{3/2}}\left\{\dot{v}_x v_x + \dot{v}_y v_y + \dot{v}_z v_z\right\}$$

$$\left\{v_x\bar{i} + v_y\bar{j} + v_z\bar{k}\right\} = e\mathcal{E}\bar{j} + \frac{e}{c}\left\{v_x\bar{i} + v_y\bar{j} + v_z\bar{k}\right\}\times B\bar{k}$$

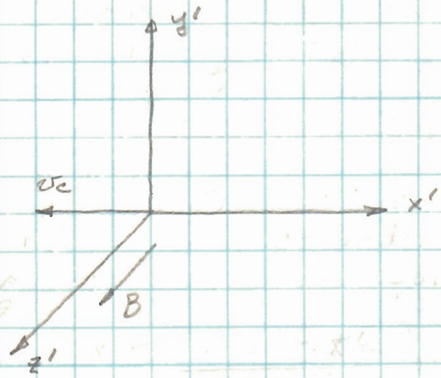
$$= e\mathcal{E}\bar{j} + \frac{e}{c}\left\{-v_x B\bar{j} + v_y B\bar{i}\right\}$$

(4) $$\frac{m\dot{v}_x}{\sqrt{1-\beta^2}} + \frac{m/c^2}{\{1-\beta^2\}^{3/2}}\left\{\dot{v}_x v_x + \dot{v}_y v_y + \dot{v}_z v_z\right\}v_x = \frac{e v_y B}{c}$$

$$\frac{m\dot{v}_y}{\sqrt{1-\beta^2}} + \frac{m/c^2}{\{1-\beta^2\}^{3/2}}\left\{\dot{v}_x v_x + \dot{v}_y v_y + \dot{v}_z v_z\right\}v_y = e\mathcal{E} - \frac{e v_x B}{c}$$

$$\frac{m\dot{v}_z}{\sqrt{1-\beta^2}} + \frac{m/c^2}{\{1-\beta^2\}^{3/2}}\left\{\dot{v}_x v_x + \dot{v}_y v_y + \dot{v}_z v_z\right\}v_z = 0$$

(5) We shall try a new approach! Consider the relativistic motion of a particle in a constant magnetic field with an initial velocity as shown:



(6) We choose $v_c = \frac{\mathcal{E}c}{B}$

where $\mathcal{E}$ is (right now) a fictitious electric field.

(7) The equation of motion is:

$$\frac{d}{dt'}\left\{\frac{m\bar{v}'}{\sqrt{1-\beta'^2}}\right\} = \frac{e\bar{v}'\times\bar{B}}{c}$$

where $\beta' = \frac{|v'|}{c}$

(8) Since the velocity is always ⊥ to the force, the longitudinal mass is zero and:

$$\frac{m}{\sqrt{1-\beta'^2}}\frac{d\bar{v}'}{dt'} = \frac{e\bar{v}'\times\bar{B}}{c}$$

(9) $$\frac{m}{\sqrt{1-\beta'^2}}\left(\dot{v}_x'\,\bar{\imath} + \dot{v}_y'\,\bar{\jmath}\right) = \frac{e}{c}\left(v_x'\bar{\imath} + v_y'\bar{\jmath}\right)\times B\,\bar{k}$$

$$= \frac{e}{c}\left(-v_x' B\,\bar{\jmath} + v_y' B\,\bar{\imath}\right)$$

(10) $$\left.\begin{aligned} \frac{m}{\sqrt{1-\beta'^2}}\frac{dv_x'}{dt'} &= \frac{e}{c}v_y' B \\ \frac{m}{\sqrt{1-\beta'^2}}\frac{dv_y'}{dt'} &= -\frac{e}{c}v_x' B \end{aligned}\right\} \quad \left.\begin{aligned} \dot{v}_x' &= \omega_0\sqrt{1-\beta'^2}\,v_y' \\ \dot{v}_y' &= -\omega_0\sqrt{1-\beta'^2}\,v_x' \end{aligned}\right\} \quad \omega_0 = \frac{eB}{mc}$$

(11) By virtue of the velocity $\perp$ force, the velocity magnitude, hence β'^2, is independent of the time.
Define $\omega = \omega_0\sqrt{1-\beta'^2}$:

$$\frac{d}{dt'}\left(v_x' + i v_y'\right) = -i\omega\left(v_x' + i v_y'\right)$$

$$v_x' + i v_y' = \left(v_{x0}' + i v_{y0}'\right)e^{-i\omega t'}$$

E & B transform to moving system they are not imaginary see posted solutions

(12) But $v_{x0}' = -v_c$, $v_{y0}' = 0$

$$\therefore\quad v_x' + i v_y' = -v_c\, e^{-i\omega t'} = -v_c\cos\omega t' + i v_c \sin\omega t'$$

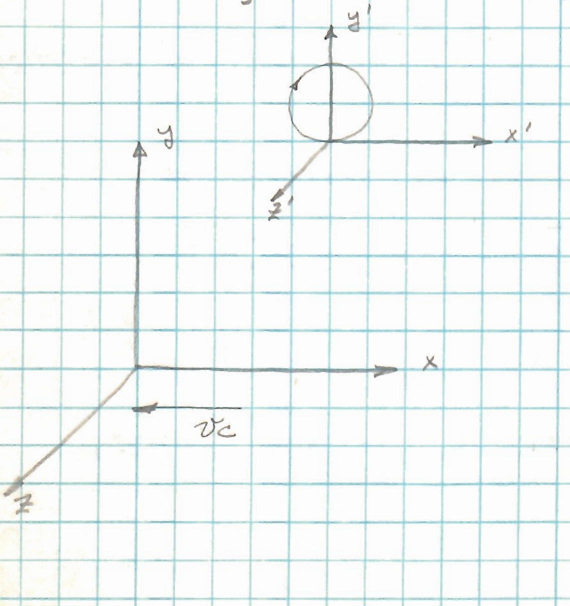
or $$v_x' = -v_c\cos\omega t'$$

$$v_y' = v_c \sin\omega t'$$

(13) Take $x' = y' = 0$ at $t' = 0$

$$\left.\begin{aligned} x' &= -\frac{v_c}{\omega}\sin\omega t' \\ y' &= -\frac{v_c}{\omega}\cos\omega t' + \frac{v_c}{\omega} \end{aligned}\right\} \quad (x')^2 + \left(y' - \frac{v_c}{\omega}\right)^2 = \left(\frac{v_c}{\omega}\right)^2$$

(14) Now suppose we consider a new co-ordinate system moving at velocity v_c in the minus x' direction:



$$x' = \frac{x - \beta_c(ct)}{\sqrt{1-\beta_c^2}} \quad ; \quad \beta_c = \frac{v_c}{c}$$

$$y' = y$$
$$z' = z$$

$$(ct') = \frac{ct - \beta_c x}{\sqrt{1-\beta_c^2}}$$

$$dx' = dx - \beta_c(c\,dt)$$

$$c\,dt' = c\,dt - \beta_c\,dx$$

$$dy' = dy$$

Assignment # 1
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Problem 12
Continued:

(15) Now note: $\beta'^2 = \dfrac{v_x'^2 + v_y'^2}{c^2} = \dfrac{v_c^2}{c^2} = \beta_c^2$

from (12).

(16) $\therefore \quad \omega = \omega_0 \sqrt{1-\beta_c^2}$

(17) $$\omega t' = \omega_0 \sqrt{1-\beta_c^2} \cdot \frac{t - \frac{\beta_c x}{c}}{\sqrt{1-\beta_c^2}} = \omega_0 \left(t - \frac{v_c x}{c^2}\right)$$

(18) $$\frac{x - v_c t}{\sqrt{1-\beta_c^2}} = \frac{-v_c}{\omega_0 \sqrt{1-\beta_c^2}} \sin \omega_0 \left(t - \frac{v_c x}{c^2}\right)$$

or $$x = v_c t - \frac{v_c}{\omega_0} \sin \omega_0 \left(t - \frac{v_c x}{c^2}\right)$$

(19) $$y = \frac{v_c}{\omega_0 \sqrt{1-\beta_c^2}} \left\{ 1 - \cos \omega_0 \left(t - \frac{v_c x}{c^2}\right) \right\}$$

These are the parametric equations for the trajectory of relativistic motion. If we choose $v_c = \dfrac{\mathcal{E}c}{B}$, we see that we have the relativistic equations for motion in crossed $\mathcal{E}$ and B fields in the x y z system. Note that as $c \to \infty$, we get the classical result.

(20) At $t = 0$: $\quad x = \dfrac{v_c}{\omega_0} \sin \dfrac{\omega_0 v_c x}{c^2}$

If the slope of the RHS of this equation is less than 1 at $x = 0$, the equation has only one root; that at $x = 0$.

$$\left\{ \frac{d}{dx} (RHS) \right\}_{x=0} = \frac{v_c}{\omega_0} \cdot \frac{\omega_0 v_c}{c^2} = \left(\frac{v_c}{c}\right)^2 < 1$$, so all right

on the initial conditions. The parameter time can be eliminated, but not much will be gained. We will work with the parametric equations. OVER:

(21) $y = 0: \quad \cos \omega_0\left(t - \frac{v_c x}{c^2}\right) = 1$

$$\text{or} \quad \omega_0\left(t - \frac{v_c x}{c^2}\right) = 2n\pi \quad , n = 0, 1, 2, 3, \ldots$$

$$\therefore \quad x_n = (2n\pi + \omega_0 t)\frac{c^2}{v_c}$$

Already we see that the points on the x-axis are time dependent or there is a "phase shift" on the x-axis intercept.

(22) $x_n = v_c\left\{\frac{2n\pi}{\omega_0} + \frac{v_c x_n}{c^2}\right\}$

$$\text{or} \quad x_n\left(1 - \frac{v_c^2}{c^2}\right) = 2n\pi\frac{v_c}{\omega_0}$$

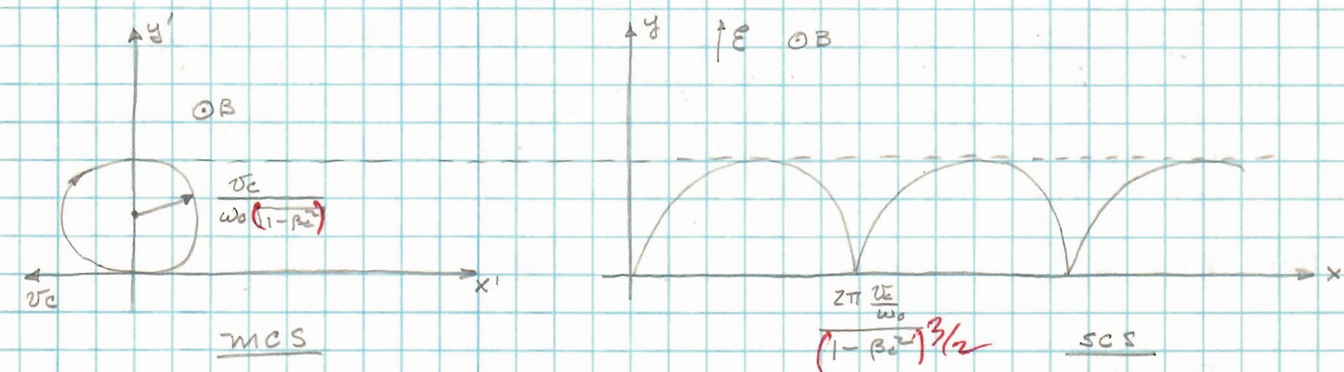
$$\text{or} \quad x_n = \frac{2n\pi\frac{v_c}{\omega_0}}{1 - \beta_c^2}$$

Thus we see that the effect of the "phase shift" is to contract the distance between the intercept points over the classical case. Eliminating time:

(23) $x(1 - \beta_c^2) = \frac{v_c}{\omega_0}\cos^{-1}\left\{\frac{v_c - \omega_0 y\sqrt{1-\beta_c^2}}{v_c}\right\}$

$$\pm\left\{2\omega_0 y\sqrt{1-\beta_c^2} - \omega_0^2 y^2(1-\beta_c^2)\right\}^{1/2}$$

We see that a Lorentz-Fitzgerald contraction occurs. The only thing one would not expect is the "double" contraction on along the x-direction.



OK but you have wrong B so wrong ω and R

$$H_+' = \frac{H_+ - \frac{v_c \times E}{c}}{\sqrt{1-\beta^2}} = \frac{H\left(1 - \frac{E^2}{H^2}\right)}{\sqrt{1-\beta^2}} = H\sqrt{1-\beta^2}$$

$$\frac{v_c}{c} = \frac{E}{H} = \beta \qquad \frac{E}{H} = \beta$$

So heres where you get extra $\sqrt{1-\beta^2}$

Assignment #1
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13. (1) It was shown in problem 8 that the following relation holds for a relativistically moving charged particle in a magnetic field:

$$B\rho = \frac{E_k}{ec}\left\{1 + \frac{2mc^2}{E_k}\right\}^{1/2} \quad (\text{emu})$$

where B is the perpendicular field to the motion of the particle.

(2) Now:

$$T = \frac{mc^2}{\sqrt{1-\beta^2}} = mc^2 + E_k \,; \quad \sqrt{1-\beta^2} = \frac{mc^2}{mc^2 + E_k}$$

(3) Also, for $v \perp B$:

$$\frac{dp}{dt} = \frac{m}{\sqrt{1-\beta^2}}\frac{dv}{dt} = evB \quad (\text{emu})$$

Now $\frac{dv}{dt} = v\omega$, as shown in problem 10.

$$\therefore \quad \omega = \frac{eB}{m}\sqrt{1-\beta^2} = \frac{eB}{m}\left(\frac{mc^2}{mc^2+E_k}\right)$$

(4) Finally: $\omega = \frac{e}{m} \cdot \frac{E_k}{ec\rho}\left\{\frac{m^2c^4}{m^2c^4+2mc^2E_k+E_k^2} \cdot \frac{E_k+2mc^2}{E_k}\right\}^{1/2}$

$$m^2c^4+2mc^2E_k+E_k^2 \,\Big/\, \frac{1/E_k}{\begin{matrix} E_k + 2mc^2 \\ E_k + 2mc^2 + \frac{m^2c^4}{E_k}\end{matrix}} \qquad = \frac{1}{E_k}\left\{1 - \left(\frac{mc^2}{E_k+mc^2}\right)^2\right\}$$

or $$\omega = \frac{c}{\rho}\left\{1 - \left(\frac{mc^2}{E_k+mc^2}\right)^2\right\}^{1/2} \quad (\text{emu})$$

(5) As $E_k \to \infty$, $\omega \to \frac{c}{\rho}$ where ρ is the radius of the particle orbit, so that for high energies, the frequency becomes independent of the energy and inversely proportional to the radius.

14\. a. (1) Consider the World velocity 4-vector:

$$u_\nu = \frac{v_\nu}{\sqrt{1-\beta^2}}\Bigg|_{\nu=1,2,3} \; ; \; \frac{ic}{\sqrt{1-\beta^2}}\Bigg|_{\nu=4}$$

(2) $$\sum_{\nu=1}^{4} (u_\nu)^2 = \frac{v^2}{1-\beta^2} - \frac{c^2}{1-\beta^2} = -c^2$$

Thus the (world velocity)^2 is invariant under the Lorentz transformation and it thus transforms as a 4-vector.

not a proof $\frac{1}{2}$

b. (1) Take as components of a 4-vector, the potentials

$$A_\nu = (A_x, A_y, A_z, i\varphi)$$

which would transform as:

$$A_x' = \frac{A_x - \beta\varphi}{\sqrt{1-\beta^2}}, \quad A_y' = A_y, \quad A_z' = A_z, \quad i\varphi' = i\left(\frac{\varphi - \beta A_x}{\sqrt{1-\beta^2}}\right)$$

assuming that the coordinate systems have been suitably prepared so that their axes are parallel and the velocity is in the x direction.

(2) $$\sum_{\nu=1}^{4} (A_\nu)^2 = A_x'^2 + A_y'^2 + A_z'^2 - \varphi'^2$$

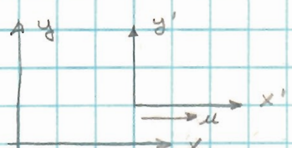
$$= \frac{A_x^2 - 2\beta\varphi A_x + \beta^2\varphi^2 - \varphi^2 + 2\beta A_x\varphi - \beta^2 A_x^2 + (1-\beta^2)(A_y^2 + A_z^2)}{1-\beta^2}$$

$$= A_x^2 + A_y^2 + A_z^2 - \varphi^2$$

No proves nothing try this for any A_4 and you get same result 0

Therefore A_ν transforms like a 4-vector.

c. (1) Consider:



A particle moves in the ' system with velocity v' and in the unprimed system with velocity v.

(2) $$x = \frac{x' + \beta(ct')}{\sqrt{1-\beta^2}}\; ; \; y = y', \; z = z', \quad ct = \frac{(ct') + \beta x'}{\sqrt{1-\beta^2}}$$

(3) $$dx = \frac{dx' + \beta(c\,dt')}{\sqrt{1-\beta^2}}\; ; \; dy = dy'; \; dz = dz'; \; c\,dt = \frac{(c\,dt') + \beta\,dx'}{\sqrt{1-\beta^2}}$$

(4) $$\frac{dx}{dt} = c\left\{\frac{dx' + \beta(c\,dt')}{c\,dt' + \beta\,dx'}\right\} = c\left\{\frac{dx'}{c\,dt' + \beta\,dx'} + \frac{\beta\,(dt'\,c)}{c\,dt' + \beta\,dx'}\right\}$$

$$= c\left\{\frac{1}{\frac{c}{v_x'} + \beta} + \frac{\beta}{1 + \frac{u v_x'}{c^2}}\right\} = \frac{v_x' + u}{1 + \frac{u v_x'}{c^2}} = v_x$$

Assignment #1
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14. Continued:

c. (5) $$v_y = \frac{c\,dy'\sqrt{\quad}}{c\,dt' + \beta\,dx'} = \frac{\sqrt{\quad}}{\frac{1}{v_y'} + \frac{\beta}{c}\frac{dx'}{dy'}}$$

$$\frac{dx'}{dy'} = \frac{dx'}{dt'}\frac{dt'}{dy'} = \frac{v_x'}{v_y'}$$

$$\therefore\ v_y = \frac{\sqrt{\quad}}{\frac{1}{v_y'} + \frac{\beta}{c}\frac{v_x'}{v_y'}} = \frac{v_y'\sqrt{1-\beta^2}}{1 + \frac{u v_x'}{c^2}}$$

(6) Similarly: $$v_z = \frac{v_z'\sqrt{1-\beta^2}}{1 + \frac{u v_x'}{c^2}}$$

(7) $$\frac{d\,(ct)}{dt} = c$$

(8) We propose the 4-vector: $v_\nu = (v_x, v_y, v_z, ic)$

$$\therefore \sum_{\nu=1}^{4} (v_\nu)^2 = v_x^2 + v_y^2 + v_z^2 - c^2$$

$$= \frac{v_x'^2 + 2u v_x' + u^2 + v_y'^2(1-\beta^2) + v_z'^2(1-\beta^2) - c^2\left(1 + \frac{2u v_x'}{c^2} + \frac{u^2 v_x'^2}{c^4}\right)}{\left(1 + \frac{u v_x'}{c^2}\right)^2}$$

$$= \frac{v_x'^2(1-\beta^2) + v_y'^2(1-\beta^2) + v_z'^2(1-\beta^2) + u^2 - c^2}{\left(1 + \frac{u v_x'}{c^2}\right)^2}$$

$$= \left\{\frac{(1-\beta^2)}{\left(1 + \frac{u v_x'}{c^2}\right)^2}\right\}\left(v_x'^2 + v_y'^2 + v_z'^2 - c^2\right)$$

(9) The length of this vector is not invariant under the Lorentz transformation and it thus does not form a true 4-vector. 3

Applied Physics 231

Due Sat 5-6-61 Problem Set II

1. a) By carrying the Fermi statistics to the classical limit, obtain the Maxwell Boltzmann energy distribution for a classical gas

i.e.

$$n(E) = \frac{N}{V}\sqrt{\frac{4E}{\pi (kT)^3}}\, e^{-E/kT}$$

b) From the result in (a), obtain the distribution function for

(1) momentum
(2) a component of momentum
(3) speed
(4) a component of the speed

c) Sketch the form of all these distributions and compare with the Fermi statistics.

2. Determine the average energy and average speed of a particle obeying the classical statistics.

3. a) Gas molecules obeying the classical statistics are confined to a large container which has a small hole of area A. Obtain an expression for the number of molecules emerging per second from the hole.

b) What is the energy distribution of the emergent molecules? What is their average energy and speed? Compare these with the average energy and speed of the molecules inside the box.

4. Obtain expressions for the momentum, and energy of the average electron, and the electron at the Fermi level, as a function of temperature using the Fermi statistics.

5. Determine the pressure exerted by a free electron gas at $T = 0^{o}K$. Calculate this pressure in atmospheres for an electron gas of the same density as that in sodium. Show how the pressure is modified for $T \neq 0$.

6. For an electron gas with the same density as the conduction electrons in Na, what is the temperature which must be exceeded in order that the statistics go over to the Maxwell Boltzman case.

7. Calculate the wavelength of an electron at the Fermi surface in metallic lithium. Compare this number with the lattice spacing.

8. Derive an expression for the total potential energy of a cloud of electrons imagined to be at rest and distributed throughout a spherical volume of radiums $\underline{a}$ with a uniform density of $\underline{n}$ electrons/cm^3. In other words calculate the work done in assembling such a cloud, the electrons being originally infinitely far apart. Now let the cloud fly apart and consider the kinetic energy ultimately acquired by each electron, which will depend on the position it occupied in the cloud. Derive an energy-distribution function $F(E)$ so the $F(E)\,dE$ is the fraction of the total number of electrons having final energies between E and $E + dE$. What fraction of the electrons have final energies less than half the maximum energy? relative to E_{min}.

9. A cylindrical diode has a tungsten cathode of diameter 0.4 mm. surrounded by an anode 1 cm. in diameter. Calculate the current flow to the anode, per cm, length, under each of the following conditions. The cathode temperature is $2000^{\circ}K$ in each case and the thermionic constants for tungsten may be taken as $A = 60$ amp/$cm^2(^{\circ}k)^2$

$$\frac{e\phi_w}{k} = 52500 \text{ deg.}$$

a) Anode 0.5 volts negative with respect to cathode.

b) Anode 300 volts positive (neglect space charge).

c) Anode 5000 volts positive.

$e\varphi$

Problem 10

Show that for an assembly of electrons in equilibrium the Fermi level is a constant throughout the system.

ASSIGNMENT #2

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① 10
② 10
③ $9\frac{1}{2}$
④ $9\frac{1}{2}$
⑤ 10
⑥ 10
⑦ 10
⑧ 7
⑨ 10
⑩ 10

96
100

1\. a. (1) Definitions from lecture:

$$n(E)\,dE = N(E)\,f_0(E)\,dE$$

For free electron gas: $N(E) = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\sqrt{E}$

Fermi to Boltzmann: $e^{-\alpha} \gg 1$; $\alpha = J/kT$

with $f_0(E) = e^{-E/kT}\,e^{\alpha}$

(2) In terms of discrete levels:

Fermi: $f_0(E_s) = \dfrac{1}{e^{\beta E_s - \alpha} + 1} = \dfrac{n_s}{N_s}$

Boltzmann: $\dfrac{n_s}{N} = \dfrac{N_s\,e^{-E_s/kT}}{\sum_s N_s\,e^{-E_s/kT}}$

In the limit, $e^{-\alpha} \gg 1$: $f_0(E_s) = e^{-\beta E_s}e^{\alpha} = \dfrac{N\,e^{-E_s/kT}}{\sum_s N_s\,e^{-E_s/kT}}$

or $e^{-\alpha} = \dfrac{\sum_s N_s\,e^{-E_s/kT}}{N}$

(3) Going back to continuum: $e^{-\alpha} = \frac{1}{N}\int_0^\infty N(E)\,e^{-E/kT}\,dE$

(4) $\therefore$ $n(E)\,dE = \dfrac{N}{V}\dfrac{N(E)\,e^{-E/kT}\,dE}{\int_0^\infty N(E')\,e^{-E'/kT}\,dE'}$; dividing by V to form density.

(5) $\therefore$ $\int_0^\infty N(E')\,e^{-E'/kT}\,dE' = \dfrac{V}{2\pi^2}\left(\dfrac{2m^* kT}{\hbar^2}\right)^{3/2}\int_0^\infty \sqrt{x}\,e^{-x}\,dx$

$$= \frac{V\sqrt{\pi}}{4\pi^2}\left(\frac{2m^* kT}{\hbar^2}\right)^{3/2}$$

(6) $\therefore$ $n(E)\,dE = \dfrac{2N}{V}\cdot\dfrac{1}{\sqrt{\pi}}\cdot\dfrac{\sqrt{E}}{(kT)^{3/2}}\,e^{-E/kT}\,dE$

$$\boxed{n(E)\,dE = \frac{2N}{V}\sqrt{\frac{E}{\pi(kT)^3}}\,e^{-E/kT}\,dE}$$

b. (1) In a free gas: $E = \dfrac{p^2}{2m} = \dfrac{p_x^2 + p_y^2 + p_z^2}{2m}$

(2) $n(p) = n\left(E = \frac{p^2}{2m}\right)|J|$

(3) $J = \left(\frac{dE}{dp}\right)_{E=\frac{p^2}{2m}} = \frac{p\,dp}{m}$

(4) $$\boxed{n(p) = \frac{2N}{V} \frac{p^2}{\sqrt{2\pi (mkT)^3}} e^{-\frac{p^2}{2mkT}}}$$

(5) $n(p_x p_y p_z) = n\left(p = \sqrt{p_x^2 + p_y^2 + p_z^2}\right) |J|$

The volume transformation is: $\frac{4}{3}\pi p^3 = \Omega$

$\therefore \quad J = \frac{dp}{d\Omega} = \left(\frac{d\Omega}{dp}\right)^{-1} = \frac{1}{4\pi p^2}$

$\therefore \quad n(p_x p_y p_z) = \frac{N}{V} (2\pi mkT)^{-3/2} e^{-\frac{(p_x^2+p_y^2+p_z^2)}{2mkT}}$

$$n(p_x) = \frac{N}{V} (2\pi mkT)^{-3/2} \iint_{-\infty}^{\infty} e^{\frac{-(p_x^2+p_y^2+p_z^2)}{2mkT}} dp_y\, dp_z$$
$$= \frac{N}{V} (2\pi mkT)^{-1/2} e^{-p_x^2/2mkT}$$

or $$\boxed{n(p_x) = \frac{N}{V} (2\pi mkT)^{-1/2} e^{-p_x^2/2mkT}}$$

(6) $n(v) = n(p)\,|J|$

$p = mv \quad ; \quad J = \frac{dp}{dv} = m$

$$\therefore \quad \boxed{n(v) = \frac{4\pi N}{V} \left\{\sqrt{\frac{m}{2\pi kT}}\right\}^3 v^2 e^{-\frac{m}{2kT}v^2}}$$

(7) Similarly:

$$\boxed{n(v_x) = \frac{N}{V} \sqrt{\frac{m}{2\pi kT}}\, e^{-\frac{m}{2kT}v_x^2}}$$

Assignment #2
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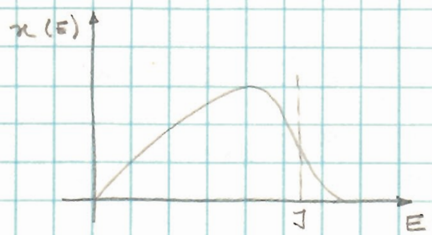
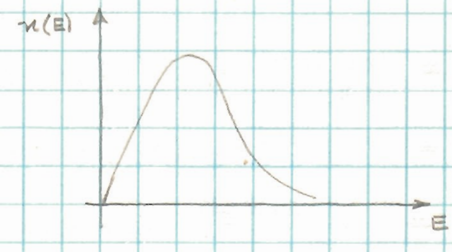
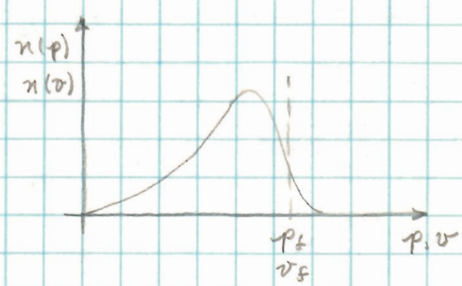
Problem 1
Continued

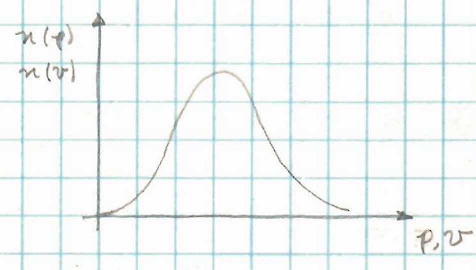
1. c. (1) Fermi Boltzmann



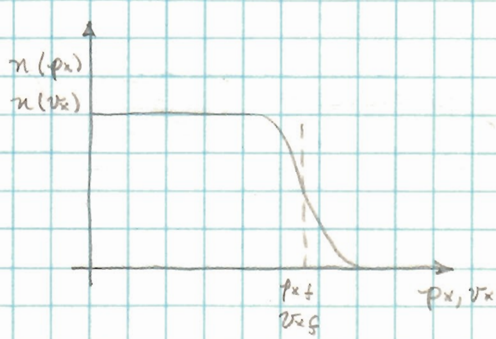


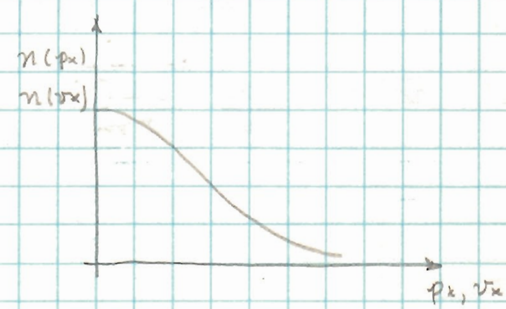
(2)





(3)





In above, scales are different for fermi and boltzmann distributions; only shapes are indicated as requested.

2. a. (1) $\bar{E} = \int_0^\infty E\, n(E)\, dE = \frac{2N}{V\sqrt{\pi}} \int_0^\infty \left(\frac{E}{kT}\right)^{3/2} e^{-E/kT} dE$

$= \frac{2N\,kT}{V\sqrt{\pi}} \int_0^\infty x^{3/2} e^{-x} dx$

(2) $\int_0^\infty x^{n/2} e^{-x} dx = -x^{n/2} e^{-x}\Big]_0^\infty + \int_0^\infty \frac{n}{2} x^{n/2-1} e^{-x} dx$

$= \left(\frac{n}{2}\right)\left(\frac{n}{2}-1\right)\left(\frac{n}{2}-2\right)\cdots\left(\frac{n}{2}-m+1\right) \int_0^\infty x^{\frac{n}{2}-m} e^{-x} dx$

$= 2^{-\left(\frac{n-1}{2}\right)} n(n-2)(n-4)\cdots(1) \int_0^\infty x^{1/2} e^{-x} dx$

$= 2^{-\left(\frac{n+1}{2}\right)} n(n-2)(n-4)\cdots(1)\sqrt{\pi}$

(3) $\therefore \bar{E} = \frac{3}{2}\frac{N}{V} kT$; $\bar{E}$/particle $= \frac{3}{2} kT$; since $\int_0^\infty n(E)\, dE = \frac{N}{V}$ ✓

b. (1) $\bar{v} = \int_0^\infty v\, n(v)\, dv = \frac{4\pi N}{V\sqrt{\pi}} \int_0^\infty \left\{\frac{v^2 m}{2kT}\right\}^{3/2} e^{-\frac{m}{2kT}v^2} dv$

$= \frac{4\pi N}{V\sqrt{\pi}\,\pi} \cdot \frac{1}{2}\sqrt{\frac{2kT}{m}} \int_0^\infty x\, e^{-x} dx$

$= \frac{4\pi N}{V\sqrt{\pi}\,\pi} \cdot \frac{1}{2}\sqrt{\frac{2kT}{m}}$

(2) $\therefore \bar{v} = \frac{2N}{V}\sqrt{\frac{2kT}{\pi m}}$; $\bar{v}$/particle $= \sqrt{\frac{8kT}{\pi m}}$ ✓

Assignment #2
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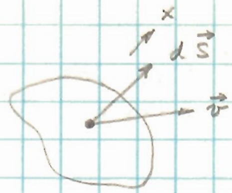
3. a.



(1) $$\frac{dn(v)}{dt} = \int_A n(v)\, \vec{v}\cdot d\vec{S}$$

$$= A\, n(v)\, v \cos\vartheta = A\, n(v)\, v_x$$

(2) $$\frac{dN}{dt} = \int \frac{dn(v)}{dt}\, dv = A\int n(v)\, v_x\, dv$$

$$= \frac{AN}{V}\left\{\frac{m}{2\pi kT}\right\}^{3/2}\int_0^\infty v_x\, e^{-\frac{m}{2kT}v_x^2}\, dv_x \iint_{-\infty}^{\infty} e^{\frac{-m}{2kT}(v_y^2+v_z^2)}\, dv_y\, dv_z$$

$$= \frac{N}{V\pi^{3/2}}\sqrt{\frac{2kT}{m}}\int_0^\infty x\, e^{-x^2}\, dx \int_{-\infty}^{\infty} e^{-y^2}\, dy \int_{-\infty}^{\infty} e^{-z^2}\, dz$$

$$= \frac{AN}{V}\sqrt{\frac{kT}{2\pi m}}$$

(3) $\therefore$ $$\boxed{\frac{dN}{dt} = \frac{AN}{V}\sqrt{\frac{kT}{2\pi m}}}$$ ✓

(4) Another way: transfer to spherical co-ordinates:

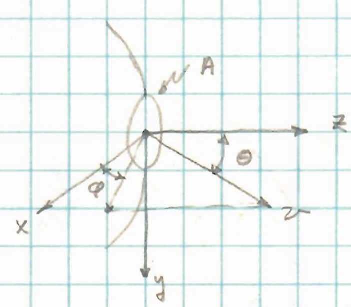
$$\therefore \quad n(v) = \frac{N}{V}\left\{\frac{m}{2\pi kT}\right\}^{3/2} e^{-\frac{m}{2kT}v^2}\, v^2 \sin\theta$$

(5) $$\frac{dN}{dt} = \frac{AN}{V}\int_0^\infty\int_0^{\pi/2}\int_0^{2\pi}\left\{\frac{m}{2\pi kT}\right\}^{3/2} e^{-\frac{m}{2kT}v^2}\, v^3\cos\theta\sin\theta\, dv\, d\theta\, d\phi$$

$$= \frac{AN}{V\pi^{3/2}}\sqrt{\frac{2kT}{m}}\cdot 2\pi\int_0^\infty x^3 e^{-x^2}\, dx\int_0^{\pi/2}\cos\theta\sin\theta\, d\theta$$

$$= \frac{AN}{V\pi^{1/2}}\sqrt{\frac{2kT}{m}}\int_0^\infty z\, e^{-z}\, dz\int_0^1 u\, du = \frac{AN}{V\pi^{1/2}}\sqrt{\frac{2kT}{m}}\cdot\frac{1}{2}$$

$$= \frac{AN}{V}\sqrt{\frac{kT}{2\pi m}}$$



b. (1) $$\frac{dn(E)}{dt} = \frac{dn(v)}{dt}\,|J| \quad ; \quad E = \tfrac{1}{2}mv^2 \; ; \; v = \sqrt{\frac{2E}{m}}$$

$$|J| = \frac{dv}{dE} = \frac{1}{\sqrt{2Em}}$$

(2) $$\frac{dn(E)}{dt} = \frac{AN}{V}\left\{\frac{m}{2\pi kT}\right\}^{3/2} e^{-E/kT}\left\{\frac{2E}{m}\right\}^{3/2}\frac{1}{\sqrt{2Em}}\int_0^{2\pi}\!\!\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta\,d\varphi$$

$$= \frac{AN}{V}\cdot 2\pi\cdot\frac{1}{\pi^{3/2}}\left\{\frac{1}{kT}\right\}^{3/2}\frac{E}{\sqrt{2m}}$$

(3) $$\boxed{\frac{dn(E)}{dt} = \frac{AN}{V}\left\{\frac{\cancel{2}}{\cancel{2}\pi m(kT)^3}\right\}^{1/2} E\, e^{-E/kT}}$$

(4) $$\bar{E} = \frac{\int_0^\infty E\,\frac{dn(E)}{dt}\,dE}{\int_0^\infty \frac{dn(E)}{dt}\,dE} = \frac{\int_0^\infty E^2 e^{-E/kT}\,dE}{\int_0^\infty E\,e^{-E/kT}\,dE}$$

$$= \frac{(kT)^3}{(kT)^2}\,\frac{\int_0^\infty z^2 e^{-z}\,dz}{\int_0^\infty z\,e^{-z}\,dz} = 2kT$$

$$\therefore \boxed{\bar{E} = 2kT}$$ ✓ as expected, compared with $\frac{3}{2}kT$ inside as shown in class; $\frac{\bar{E}_o}{\bar{E}_i} = \frac{4}{3}$ ✓

(5) $$\frac{dn(v)}{dt} = \frac{2\pi AN}{V}\left\{\frac{m}{2\pi kT}\right\}^{3/2} v^3\, e^{-\frac{m}{2kT}v^2}$$

(6) $$\bar{v} = \frac{\int_0^\infty v\,\frac{dn(v)}{dt}\,dv}{\int_0^\infty \frac{dn(v)}{dt}\,dv} = \frac{\int_0^\infty v^4 e^{-\frac{m}{2kT}v^2}\,dv}{\int_0^\infty v^3 e^{-\frac{m}{2kT}v^2}\,dv}$$

$$= \sqrt{\frac{2kT}{m}}\,\frac{\int_0^\infty z^4 e^{-z^2}\,dz}{\int_0^\infty z^3 e^{-z^2}\,dz} = 2\sqrt{\frac{2kT}{m}}\,\frac{\int_0^\infty z^4 e^{-z^2}\,dz}{\int_0^\infty z\,e^{-z}\,dz}$$

$$= 2\sqrt{\frac{2kT}{m}}\;\frac{3\sqrt{\pi}}{8}$$

(7) $$\therefore \boxed{\bar{v} = \frac{3}{4}\sqrt{\frac{2\pi kT}{m}}}$$ ✓ $$;\ \therefore \frac{\bar{v}_o}{\bar{v}_i} = \frac{3\pi}{8}$$ ✓

(8) Inside: $$\bar{v} = \frac{\int_0^\infty v^3 e^{-m/2kT\,v^2}\,dv}{\int_0^\infty v^2 e^{-m/2kT\,v^2}\,dv} = \sqrt{\frac{kT}{2m}}\,\frac{4}{\sqrt{\pi}} = \sqrt{\frac{8kT}{\pi m}}$$

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4. (1) $$\bar{E} = \frac{\int_0^\infty E\, N(E)\, f_0(E)\, dE}{\int_0^\infty N(E)\, f_0(E)\, dE} = \frac{1}{N}\int_0^\infty E\, N(E)\, f_0(E)\, dE$$

(2) $$N(E) = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\sqrt{E}$$

Since: $$J(0) = \left\{3\pi^2 \frac{N}{V}\right\}^{2/3}\frac{\hbar^2}{2m} \; ; \; \left(\frac{2m}{\hbar^2}\right)^{3/2} = \frac{3\pi^2 \frac{N}{V}}{J_0^{3/2}}$$

$$N(E) = \frac{3}{2} N \frac{\sqrt{E}}{J_0^{3/2}}$$

(3) $$\therefore \bar{E} = \frac{3}{2 J_0^{3/2}}\int_0^\infty E^{3/2} f_0(E)\, dE$$

$$= \frac{3}{2 J_0^{3/2}}\left\{\int_0^J E^{3/2}\, dE + \frac{\pi^2}{6}(kT)^2\left(\frac{3}{2}E^{1/2}\right)_J + \frac{7\pi^4}{360}(kT)^4\left(-\frac{3}{8}E^{-3/2}\right)_J + \cdots\right\}$$

$$= \frac{3}{2 J_0^{3/2}}\left\{\frac{2}{5}J^{5/2} + \frac{\pi^2}{4}(kT)^2 J^{1/2} - \frac{7\pi^4}{960}(kT)^4 J^{-3/2} + \cdots\right\}$$

$$= \frac{3}{5}J_0\left(\frac{J}{J_0}\right)^{5/2} + \frac{3\pi^2}{8 J_0}\left(\frac{J}{J_0}\right)^{1/2}(kT)^2 + \cdots$$

(4) It was shown in lecture that: $$\frac{J}{J_0} = 1 - \frac{\pi^2}{12}\frac{(kT)^2}{J_0^2}$$

$$\therefore \bar{E} = \frac{3}{5}J_0\left\{1 - \frac{\pi^2}{12}\left(\frac{kT}{J_0}\right)^2\right\}^{5/2} + \frac{3\pi^2}{8 J_0}\left\{1 - \frac{\pi^2}{12}\left(\frac{kT}{J_0}\right)^2\right\}^{1/2}(kT)^2 + \cdots$$

$$= \frac{3}{5}J_0\left\{1 - \frac{5}{2}\frac{\pi^2}{12}\left(\frac{kT}{J_0}\right)^2\right\} + \frac{3\pi^2}{8 J_0}(kT)^2 + \cdots$$

(5) Keeping to quadratic terms in T; taking $kT < J_0$:

$$\boxed{\bar{E} = \frac{3}{5}J_0\left\{1 + \frac{5\pi^2}{12}\left(\frac{kT}{J_0}\right)^2\right\}}$$

(6) $$\bar{p} = \frac{\int_0^\infty p\, N(p)\, f_0(p)\, dp}{\int_0^\infty N(p)\, f_0(p)\, dp} = \frac{1}{N}\int_0^\infty p\, N(p)\, f_0(p)\, dp$$

$$= \frac{1}{N}\int_0^\infty p\, n(p)\, dp$$

(7) From Lecture: $n(p) = \dfrac{V}{\pi^2\hbar^3}\,\dfrac{p^2}{e^{\frac{p^2}{2mkT}-\frac{\zeta}{kT}}+1}$

$$\zeta_0 = \left\{3\pi^2\frac{N}{V}\right\}^{2/3}\frac{\hbar^2}{2m}$$

(8) Define: $p_{f_0} \equiv \sqrt{2m\zeta_0} = \hbar\left\{3\pi^2\frac{N}{V}\right\}^{1/3}$; $\dfrac{V}{\pi^2\hbar^3} = \dfrac{3N}{p_{f_0}^3}$

Also: $p_f \equiv \sqrt{2m\zeta}$

(9) $\therefore \quad \bar{p} = \dfrac{3}{p_{f_0}^3}\displaystyle\int_0^\infty \frac{p^3\,dp}{e^{\frac{p^2-p_f^2}{2mkT}}+1}$

$$\text{or}\quad \bar{p} = \frac{3}{(2m)^{3/2}\,\zeta_0^{3/2}}\int_0^\infty \frac{(2m)^{3/2}E^{3/2}\cdot\frac{m}{\sqrt{2mE}}\,dE}{e^{(E-\zeta)/kT}+1}$$

$$= 3\sqrt{\frac{m}{2}}\,\frac{1}{\zeta_0^{3/2}}\int_0^\infty E\,f_0(E)\,dE$$

$$= 3\sqrt{\frac{m}{2}}\,\frac{1}{\zeta_0^{3/2}}\left\{\int_0^\zeta E\,dE + \frac{\pi^2}{6}(kT)^2\right\} = 3\sqrt{\frac{m}{2}}\,\frac{1}{\zeta_0^{3/2}}\left\{\frac{\zeta^2}{2}+\frac{\pi^2}{6}(kT)^2\right\}$$

(10) $\bar{p} = \dfrac{3}{2}\sqrt{\dfrac{m}{2}}\,\dfrac{1}{\zeta_0^{1/2}}\left\{\zeta_0\left(\dfrac{\zeta}{\zeta_0}\right)^2+\dfrac{\pi^2}{3}\,\dfrac{(kT)^2}{\zeta_0^2}\,\zeta_0\right\} = \dfrac{3}{2}\sqrt{\dfrac{m}{2}}\,\zeta_0^{1/2}\left\{\left(\dfrac{\zeta}{\zeta_0}\right)^2+\dfrac{\pi^2}{3}\left(\dfrac{kT}{\zeta_0}\right)^2\right\}$

Using: $\dfrac{\zeta}{\zeta_0} = 1-\dfrac{\pi^2}{12}\left(\dfrac{kT}{\zeta_0}\right)^2$

$$\bar{p} = \frac{3}{2}\sqrt{\frac{m}{2}}\,\zeta_0^{1/2}\left\{1-\frac{\pi^2}{12}\left(\frac{kT}{\zeta_0}\right)^2+\frac{\pi^2}{3}\left(\frac{kT}{\zeta_0}\right)^2\right\}$$

(11) $\therefore \quad \boxed{\bar{p} = 3\sqrt{\dfrac{m}{8}}\,\zeta_0^{1/2}\left\{1+\dfrac{\pi^2}{6}\left(\dfrac{kT}{\zeta_0}\right)^2\right\}}$

(12) The energy of an electron at the fermi level is ζ:

$$N = \int_0^\infty n(E)\,dE = \frac{3N}{2\,\zeta_0^{3/2}}\int_0^\infty \sqrt{E}\,f_0(E)\,dE = \frac{3N}{2\,\zeta_0^{3/2}}\left\{\frac{2}{3}\zeta^{3/2}+\frac{\pi^2}{12}(kT)^2\zeta^{-1/2}\right\}$$

keeping quadratic terms in T; $\therefore \quad \left(\dfrac{\zeta}{\zeta_0}\right) = \left\{1+\dfrac{\pi^2}{8}\left(\dfrac{kT}{\zeta}\right)^2\right\}^{-2/3}$

(13) Expand to T^2: $\dfrac{\zeta}{\zeta_0} = 1-\dfrac{\pi^2}{12}\left(\dfrac{kT}{\zeta}\right)^2$

(14) Since $kT \ll \zeta$, we can write $\dfrac{kT}{\zeta}\rightarrow\dfrac{kT}{\zeta_0}$:

$\therefore \quad \boxed{\zeta = \zeta_0\left\{1-\dfrac{\pi^2}{12}\left(\dfrac{kT}{\zeta_0}\right)^2\right\}} \quad \boxed{p_f = \sqrt{2m\zeta} = p_{f_0}\left\{1-\dfrac{\pi^2}{24}\left(\dfrac{2mkT}{p_{f_0}^2}\right)^2\right\}}$

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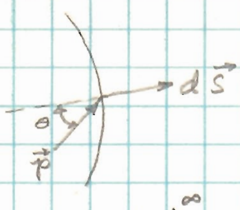
5.



(1) The pressure by definition is the net $(\text{momentum})_\perp$ per unit time incident upon a unit area

(2) $$P = \frac{1}{A}\int_0^\infty p \, \frac{dn(p)}{dt} \cos\theta \, dp$$

(3) $$\frac{dn(p)}{dt} = \frac{A}{m} n(p) \, p\cos\theta = \frac{AN}{mV}(2\pi m kT)^{-3/2} p_z \, e^{-\frac{(p_x^2+p_y^2+p_z^2)}{2mkT}}$$

$$= \frac{AN}{mV}(2\pi m kT)^{-3/2} p^3 \cos\theta \, e^{-\frac{p^2}{2mkT}} \sin\theta$$

(4) $$P = \frac{2\pi N}{mV}(2\pi m kT)^{-3/2}\int_0^\infty p^4 e^{-\frac{p^2}{2mkT}} dp \int_0^\pi \cos^2\theta \sin\theta \, d\theta$$

$$= \frac{2\pi N}{mV} \, \frac{2mkT}{\pi^{3/2}} \int_0^\infty z^4 e^{-z^2} dz \int_{-1}^1 u^2 \, du$$

$$= \frac{8NmkT}{V\pi^{1/2}\cdot 3m}\cdot\frac{3}{8}\sqrt{\pi} = \frac{N}{V}kT$$; this is the pressure under M-B statistics

(5) Fermi - Dirac: $$n(p) = \frac{1}{4\pi^3\hbar^3}\,\frac{p^2\sin\theta}{e^{(\frac{p^2}{2m}-J)/kT}+1}$$ (Lecture)

(6) $$\boxed{P = \frac{1}{3\pi^2\hbar^3 m}\int_0^\infty \frac{p^4 \, dp}{e^{(\frac{p^2}{2m}-J)/kT}+1}}$$

Having divided by V to get particle density.

(7) $$P = \frac{1}{3}\,\frac{(2mkT)^{5/2}}{\pi^2\hbar^3 m}\int_0^\infty \frac{z^4 \, dz}{e^{z^2-J/kT}+1}$$

$$\boxed{P = \frac{(2m)^{3/2}}{3\pi^2\hbar^3}(kT)^{5/2}\int_0^\infty \frac{z^{3/2} \, dz}{e^{z-J/kT}+1}}$$

(8) When $e^{-J/kT} = \frac{V}{4N\pi^{3/2}}\,\frac{(2\pi m kT)^{3/2}}{h^3} \gg 1$; $P = \frac{N}{V}kT$

as it should.

(9) Consider degenerate case : $T \approx 0$:

$$\int_0^\infty \frac{p^4\,dp}{e^{\frac{p^2}{2mkT}-\frac{\zeta}{kT}}+1} \longrightarrow \int_0^{\sqrt{2m\zeta_0}} p^4\,dp = \frac{1}{5}(2m\zeta_0)^{5/2}$$

(10) From Lecture : $\zeta_0 = \left\{3\pi^2\frac{N}{V}\right\}^{2/3}\frac{\hbar^2}{2m}$

$$\frac{1}{5}(2m\zeta_0)^{5/2} = \frac{\hbar^5}{5}\left\{3\pi^2\frac{N}{V}\right\}^{5/3}$$

(11) $$P = \frac{1}{3\pi^2\hbar^3 m}\cdot\frac{\hbar^5}{5}\left\{3\pi^2\frac{N}{V}\right\}^{5/3}$$

(13) $$\boxed{P = \frac{1}{5}\frac{\hbar^2}{m}(3\pi^2)^{2/3}\left(\frac{N}{V}\right)^{5/3}}$$

For Na : $\frac{N}{V} \approx 2.5\cdot10^{22}/cc$ assuming 1 free electron per atom. (Mott & Jones)

$$\frac{(3\pi^2)^{2/3}}{5} \cong 1.9 \; ; \; \frac{\hbar^2}{m} = \frac{(6.63\cdot10^{-34})^2}{(2\pi)^2\,9.11\cdot10^{-31}} = \frac{4.8\cdot10^{-37}}{(2\pi)^2} \; ; \; \left(\frac{N}{V}\right)^{5/3} = 46\cdot10^{35}\cdot10^{10}$$

$$\therefore P = \frac{(1.9)}{(2\pi)^2}\frac{(4.8)(.46)(10^5)}{10^{-5}} = \frac{4.2\,(10^5)}{10^{-5}}\frac{\text{joule}^2\ \sec^2}{kg\cdot m^5} = \frac{4.2\cdot10^5}{(2\pi)^2\,10^{-5}}\frac{nt}{m^2} \cong \frac{4.1}{4\pi^2}\cdot10^5\ \text{atm}$$

$$= 1.04\cdot10^4\ \text{atm.}$$

(14) Consider : $$P = \frac{(2m)^{3/2}}{3\pi^2\hbar^3}\int_0^\infty\frac{E^{3/2}\,dE}{e^{(E-\zeta)/kT}+1} \; ; \; T \neq 0$$

(15) From Lecture: $$I = \int_0^\infty f_0(E)\,g(E)\,dE = \int_0^\zeta g(E')\,dE' + \frac{\pi^2}{6}(kT)^2\left(\frac{\partial g}{\partial E}\right)_\zeta$$

$$+ \frac{7\pi^4}{360}(kT)^4\left(\frac{\partial^3 g}{\partial E^3}\right)_\zeta \; ; \text{ providing } \zeta \gg kT \text{ and } g(0)=0$$

Here $g(E) = E^{3/2}$; $\int_0^\zeta E^{3/2}\,dE = \frac{2}{5}\zeta^{5/2}$; $\left.\frac{\partial g}{\partial E}\right|_\zeta = \frac{3}{2}\zeta^{1/2}$

same correction for $\bar{E}$

(16) To quadratic terms in T :

$$\boxed{P = \frac{(2m)^{3/2}}{3\pi^2\hbar^3}\left\{\frac{2}{5}\zeta^{5/2} + \frac{\pi^2}{4}\zeta^{1/2}(kT)^2\right\}}$$

$$\frac{N}{V}\frac{2}{5}\zeta(0)\left(1+\frac{5\pi^2}{12}\left(\frac{kT}{\zeta(0)}\right)^2 + \right)$$

(17) Also, $$\boxed{P = \frac{2}{15\pi^2}\frac{\hbar^2}{2m}\left\{\frac{2m\zeta_0}{\hbar^2}\right\}^{5/2}}$$

$$= \frac{N}{V}\frac{2}{5}\zeta(0)$$

$\therefore$ For Na : $\zeta = 3.15\,ev$: $$P = \frac{1}{74}\frac{1}{1.63\cdot10^{27}}\cdot6.05\cdot10^{39} = 4.94\cdot10^4\ \text{atm}$$

Assignment #2
Continued

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6\. (1) From problem 1 and lecture, the criteria for the transition from FD to MB is:

$$e^{-\alpha} = \frac{V}{4N}\left(\frac{2mkT}{\pi\hbar^2}\right)^{3/2} \gg 1$$

(2) $$\frac{2mkT}{\pi\hbar^2} \gg 4^{2/3}\left(\frac{N}{V}\right)^{2/3} \; ; \quad T \gg \frac{2^{1/3}\pi\hbar^2}{mk}\left(\frac{N}{V}\right)^{2/3}$$

(3) We define the threshold temperature as:

$$\boxed{T = \frac{2^{1/3}\pi\hbar^2}{mk}\left(\frac{N}{V}\right)^{2/3}}$$

(4) $$\left(\frac{N}{V}\right)^{2/3} = 8.5\cdot 10^{14} ; \quad \frac{h^2}{mk} = \frac{(6.63)^2\cdot 10^{-54}}{(9.11\cdot 10^{-28})(1.38\cdot 10^{-16})} = 3.5\cdot 10^{-10}$$

$$\frac{2^{1/3}}{4\pi} = \frac{1}{2^{5/3}\pi} \cong .1 = 1\cdot 10^{-1}$$

(5) $$T = (8.5)(3.5)\cdot 10^{3} \cong 30{,}000°K$$

$\frac{N}{V}$ from Mott & Jones

(6) Also, since $J_0 = \left\{3\pi^2 \frac{N}{V}\right\}^{2/3} \frac{\hbar^2}{2m}$

$$\boxed{T \gg \left(\frac{16}{9\pi}\right)^{1/3} \frac{J_0}{k}} \; ; \quad \text{For Na: } J_0 = 3.15\,\text{ev}$$

(7) $$T \gg \left(\frac{16}{9\pi}\right)^{1/3} \frac{3.15}{.86\cdot 10^{-4}} = 3.66\cdot 10^{4}\cdot(.83)$$

$$= 30{,}500°K$$

(8) Actually we should choose a threshold temperature about 10 times that above:

$\therefore$ T (threshold) $\cong$ 300,000 °K

7. (1) De Broglie Relation: $\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mE}}$

(2) $\lambda_F = \frac{h}{\sqrt{2m J_0}}$; $J_0 = \left\{ 3\pi^2 \frac{N}{V} \right\}^{2/3} \frac{\hbar^2}{2m}$

(3) $\lambda_F = \frac{h}{\hbar} \frac{1}{\left\{ 3\pi^2 \frac{N}{V} \right\}^{1/3}}$

or $\boxed{\lambda_F = \left\{ \frac{8\pi}{3\left(\frac{N}{V}\right)} \right\}^{1/3}}$

(4) For Lithium: $\frac{N}{V} = 4.7 \cdot 10^{22}$ atoms/cc $= 4.7 \cdot 10^{22}$ electrons/cc

assuming each atom donates an electron (Mott & Jones).

$a = 3.46$ Å (CRC)

(5) $\frac{8(3.14)}{3(4.7)} \cdot 10^{-22} \cong 1.8 \cdot 10^{-22} = 18 \cdot 10^{-21}$

$\lambda_F = 2.6 \cdot 10^{-7}$ cm $= 26$ Å

$\therefore \frac{\lambda_F}{a} = 7.5$

(6) Another Way: $\lambda_F = \frac{h}{\sqrt{2m J_0}}$

From lecture; $J_0 = 4.76$ ev $= 7.6 \cdot 10^{-12}$ erg

$\frac{h^2}{2m} = \frac{(6.61)^2 \cdot 10^{-54}}{2 \cdot 9.1 \cdot 10^{-28}} = 2.4 \cdot 10^{-26}$

$\lambda_F = \sqrt{\frac{2.4 \cdot 10^{-26}}{7.6 \cdot 10^{-12}}} = \sqrt{.316 \cdot 10^{-14}} = \sqrt{31.6 \cdot 10^{-16}} = \underline{5.62 \text{ Å}}$

$\therefore \frac{\lambda_F}{a} = 1.62$

There must be a considerable error in assuming one electron per atom in Lithium.

Assignment #2
Continued

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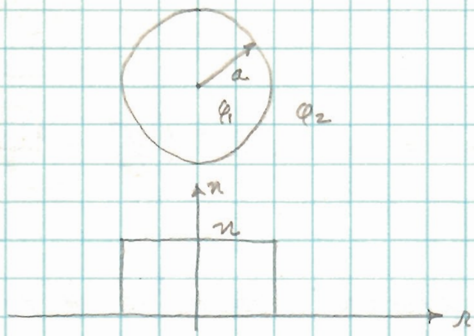
8.



(1) Boundary Conditions:

$\varphi_2(\infty) = 0$
$\varphi_1(a) = \varphi_2(a)$
φ_1 must be finite at $r = 0$

Gauss' Law:

$$\int_S \vec{E}\cdot d\vec{s} = 4\pi Q = -4\pi \tfrac{4}{3}\pi a^3 ne$$
$$= E\, 4\pi r^2$$

(2) $\therefore \quad E(a) = -\left(\frac{d\varphi_2}{dr}\right)_a = -\frac{4}{3}\pi a ne$

(3) Laplace's Equation: $\frac{d}{dr}\left(r^2 \frac{d\varphi_2}{dr}\right) = 0 \quad ; \; r > a$

Poisson's Equation: $\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\varphi_1}{dr}\right) = 4\pi ne \quad ; \; r < a$

(4) L: $\frac{d\varphi_2}{dr} = \frac{A}{r^2} \quad ; \quad \frac{4}{3}\pi a ne = \frac{A}{a^2} \quad ; \quad A = \frac{4}{3}\pi a^3 ne$

$$\varphi_2 = -\frac{4/3\, \pi a^3 ne}{r} \quad , \text{ since } \varphi_2(\infty) = 0$$

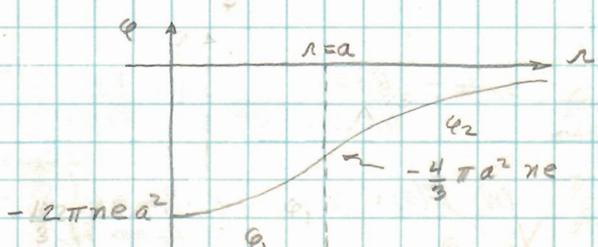
(5) P; $\frac{d\varphi_1}{dr} = \frac{4}{3}\pi ne r + \frac{C}{r^2}$

$$\varphi_1 = \frac{2}{3}\pi ne r^2 - \frac{C}{r} + D$$

$C = 0$ since φ_1 must be finite.

$\varphi_1(a) = \varphi_2(a) \quad ; \quad D + \frac{2}{3}\pi ne a^2 = -\frac{4}{3}\pi a^2 ne$

$\therefore \; D = -2\pi a^2 ne \quad ; \quad \varphi_1 = \frac{2}{3}\pi ne r^2 - 2\pi a^2 ne$



$$\boxed{\begin{aligned} \varphi_1 &= 2\pi ne\left\{\frac{r^2}{3} - a^2\right\} \\ \varphi_2 &= -\frac{4/3\,\pi a^3 ne}{r} \end{aligned}}$$

(6) Now the total energy is given by the well-known expression:

$$W = \frac{1}{2}\int_\tau \rho\,\varphi\,d\tau + \frac{1}{2}\int_s \sigma\,\varphi\,dS$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho\,\varphi_1\,r^2 dr\,\sin\vartheta\,d\vartheta\,d\phi$$

$$+ \frac{1}{2}\int_s \sigma\,\varphi\,dS$$

(7) $\frac{1}{2}\int_\tau \rho\varphi\,d\tau = -(2\pi n e)^2 \int_0^a \left(\frac{r^2}{3} - a^2\right) r^2 dr = -(2\pi n e)^2 \left(\frac{r^5}{15} - \frac{a^2 r^3}{3}\right)\Big|_0^a$

$$= \frac{4}{15}(2\pi n e)^2 a^5$$

(8) Since there are no conductors present, there is no surface integral to be taken.

(9) $\therefore$ $\boxed{W = \frac{4}{15}(2\pi n e)^2 a^5}$, which is the total energy of the system and hence is the work done assembling the system.

(10) Upon exploding the cloud, we invoke conservation of energy:
$\Delta KE = (KE)_f - KE)_i = (PE)_i - (PE)_f$ or $E_f = (PE)_i$

(11) For some arbitrary electron $(PE)_i = -e\,\varphi_1$, then:

P.E $= E_F = \frac{4}{3}\pi n e^2 r^2$

$\boxed{E_f = 2\pi n e^2\left\{a^2 - \frac{r^2}{3}\right\}}$ $\boxed{E_{f\,max} = 2\pi n e^2 a^2;\ E_{f\,min} = \frac{4}{3}\pi n e^2 a^2}$

(12) The number of electrons in a differential volume is:

$$dN = n\,dx\,dy\,dz = 4\pi n r^2 \cancel{dr} = N(r)\,dr = 4\pi n r\,(r\,dr)$$

$$dE_f = -\frac{4}{3}\pi n e^2\,r\,dr\ ;\quad r = \sqrt{3}\left\{2\pi n e^2 a^2 - E_f\right\}^{1/2}\cdot\frac{1}{\sqrt{2\pi n}\,e}$$

(13) $\therefore$ $\boxed{N(E_f) = \frac{3\sqrt{3}\,a}{e^2}\left\{1 - \frac{E_f}{2\pi n e^2 a^2}\right\}^{1/2} = \frac{3\sqrt{3}\,a}{e^2}\left\{1 - \frac{E_f}{E_{f\,max}}\right\}^{1/2}}$

Since $dN = -N(E)\,dE$

(14) Now:

$$F(E_f) = \frac{N(E_f)}{\int_{E_{f\,min}}^{E_{f\,max}} N(E_f)\,dE_f} = \frac{9\sqrt{3}}{4\pi n a^2 e^2}\left\{1 - \frac{E_f}{E_{f\,max}}\right\}^{1/2}$$

(15) or $\boxed{F(E_f) = \frac{9/2\sqrt{3}}{E_{f\,max}}\left\{1 - \frac{E_f}{E_{f\,max}}\right\}^{1/2}}$

(16) $F(E_f < 1/2\,E_{f\,max}) = \frac{9/2\sqrt{3}}{E_{f\,max}}\int_{E_{f\,min} = \frac{2}{3}E_{max}}^{\frac{1}{2}E_{f\,max}} \left\{1 - \frac{E_f}{E_{f\,max}}\right\}^{1/2} dE_f$

(17) $F(E_f < 1/2\,E_{f\,max}) = 0$, since $1/2\,E_{f\,max}$ is less than $E_{f\,min}$.

Assignment #2
Continued

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Problem 8
Continued

(18) If we define ½ the maximum final energy to be ½ of the amount between E_{fmax} and E_{fmin} ($= \frac{2}{3} E_{fmax}$) we have:

$$\tfrac{1}{2} E = \frac{1 + \frac{2}{3}}{2} E_{fmax} = \frac{5}{6} E_{fmax}$$

(19) $$\therefore \; F\left(E_f < \tfrac{5}{6} E_{fmax}\right) = \frac{9/2\sqrt{3}}{E_{fmax}} \int_{\frac{2}{3}E_{max}}^{\frac{5}{6}E_{fmax}} \left\{1 - \frac{E_f}{E_{fmax}}\right\}^{1/2} dE_f$$

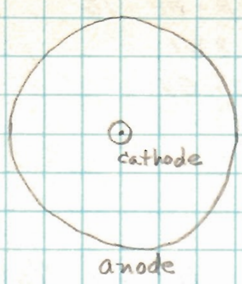
$$= 9/2\sqrt{3} \int_{1/6}^{1/3} \sqrt{x}\, dx = \frac{9/2\sqrt{3}}{3/2} \left.\frac{1}{x\sqrt{x}}\right]_{1/6}^{1/3} = 3\sqrt{3}\left[\frac{1}{3\sqrt{3}} - \frac{1}{6\sqrt{6}}\right]$$

$$= 1 - \frac{1}{2\sqrt{2}} = 1 - \frac{\sqrt{2}}{4} = 1 - .35 = .65$$

~~see posted solution~~

In first integral you assumed all charge on sphere. Thus first answer correct but N(E) wrong. The effect of false assumptions is to add this

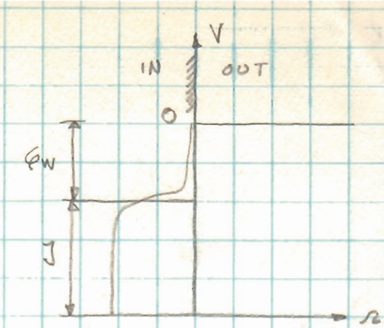
9.



ID = .4 mm = a
OD = 1 cm = b

T = 2000°K
A = 60 amp/cm² K°²

$$\frac{e\varphi_W}{k} = 52500\ °K$$



(1) The number of particles escaping is:

$$dn = \int_{p_r' = \sqrt{2m(\zeta+\varphi_W)}}^{\infty} \frac{n(p_r)}{V}\,\frac{p_r}{m}\,dp_r\,dt\,dA$$

with charge $d(ne)$.

(2) $$n(p) = \frac{V}{4\pi^3\hbar^3}\,\frac{1}{e^{\frac{p^2}{2mkT}-\zeta/kT}+1}$$

since $\frac{p_r^2}{2m} > \zeta + \varphi_W$; $n(p) = \frac{V}{4\pi^2\hbar^3}\,e^{\zeta/kT}\,e^{-\frac{p^2}{2mkT}}$

(3) In Cylindrical coordinates: $p_x^2 + p_y^2 = p_r^2$

$$\therefore\quad n(p) = \frac{V}{4\pi^2\hbar^3}\,e^{\zeta/kT}\,e^{-\frac{p_r^2}{2mkT}}\,e^{-\frac{p_z^2}{2mkT}}\,p_r$$

$$\int_0^{2\pi}\int_{-\infty}^{\infty} e^{-\frac{p_z^2}{2mkT}}\,dp_z\,d\theta = 2\pi\sqrt{2mkT}\int_{-\infty}^{\infty} e^{-z^2}\,dz$$

$$= 2\pi\sqrt{2m\pi kT}$$

$$\therefore\quad n(p_r) = \frac{V\sqrt{2m\pi kT}}{2\pi\hbar^3}\,e^{\zeta/kT}\,e^{-\frac{p_r^2}{2mkT}}$$

(4) $$J = \frac{d(ne)}{dt\,dA} = \frac{e}{\hbar^3}\sqrt{\frac{kT}{2m\pi}}\,e^{\zeta/kT}\int_{p_r'}^{\infty} p_r^2\,e^{-\frac{p_r^2}{2mkT}}\,dp_r$$

$$= \frac{e}{\hbar^3}\sqrt{\frac{kT}{2m\pi}}\,\{2mkT\}^{3/2}\,e^{\zeta/kT}\int_{\frac{p_r'}{\sqrt{2mkT}} = a}^{\infty} z^2\,e^{-z^2}\,dz$$

(5) Apparently, this integral is intractable or leads to an error function, hence, as a first approximation, we take from lecture the equation for the parallel plates, multiplied by the circumference of the cathode to form a linear current density:

$$\boxed{I_0 = \pi a A T^2 e^{-\varphi_W/kT}} \qquad \text{(No field)}$$

Reflection effects are taken to be contained in A.

Assignment # 2
Continued

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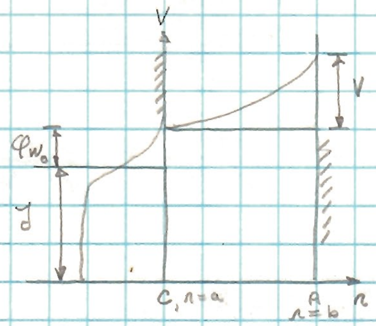
Problem 9
Continued:

(6) $\therefore$ at $T = 2000°K$: $I_0 = (3.14)(4\cdot10^{-2})(60)(2\cdot10^3)^2 e^{-26.3}$

$= \frac{151}{e^{26.3}} = \frac{151\cdot 2\cdot 10^3 \cdot 10^2}{(2.2\cdot10^4)(2.2\cdot10^4)(5.05\cdot10^2)} = 12.4\cdot10^{-5}$ amp/cm

$= \underline{124 \ \mu amp/cm}$

a.



Retarding Potential Case

(1) $\varphi_W = \varphi_{W_0} + \Delta\varphi_W$; $\Delta\varphi_W = eV$

(2) $\therefore$ $\boxed{I = I_0 e^{-eV/kT}}$

(3) $\frac{eV}{kT} = \frac{1}{17.2\cdot10^{-5}\cdot 2\cdot10^3} = \frac{100}{34.4}$

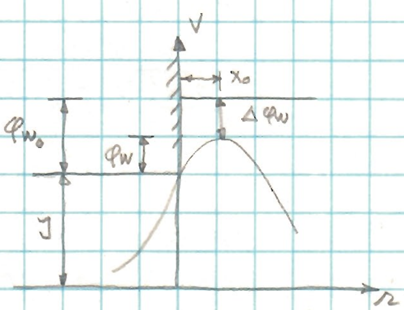
$= 2.9$

(4) $e^{-2.9} = 5.5\cdot10^{-2}$

(5) $\therefore$ $I = (124)(.55\cdot10^{-1}) = \underline{6.8 \ \mu amp/cm}$

parallel plate

b. c.



Accelerating Potential Case

(1) We argue that x_0 is very small compared to the radius of curvature of the cathode, hence the Schottky effect for parallel plates can be taken from lecture:

$\boxed{I = I_0 e^{\frac{\sqrt{e^3 E}}{kT}}}$; E is at $r = a$

(2) $\frac{1}{r}\frac{d}{dr}\left(r\frac{dV}{dr}\right) = 0$; $V(a) = V_a$; $V(b) = V_b$

(3) $\frac{dV}{dr} = \frac{C}{r}$; $V = C\ln r - C\ln D$

$= C\ln\frac{r}{D}$; $V_a = C\ln\frac{a}{D}$; $V_b = C\ln\frac{b}{D}$; $C = \frac{V_a - V_b}{\ln a/b}$

(4) $E = -\frac{dV}{dr} = -\frac{\Delta V}{r\ln a/b}$; $E(a) = -\frac{2\Delta V/a}{\ln a/b} = \frac{2\Delta V/a}{\ln b/a}$

b. (5) $\Delta V = \frac{300}{300} = 1$ statvolt : $\ln\frac{b}{a} = \ln 25 = 3.22$

$\therefore E(a) = \frac{2\cdot25}{3.22} = 15.5$ stV/cm ; $kT = 1.38\cdot10^{-16}\cdot2000 = 2.76\cdot10^{-13}$ ergs

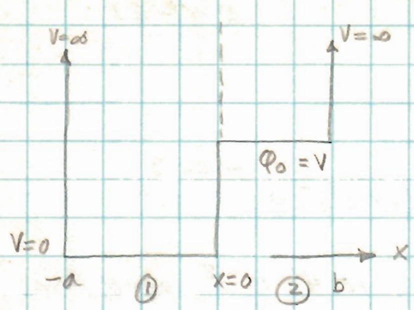
$\sqrt{e^3 E_a} = \left\{(4.8)^3\cdot10^{-30}\cdot15.5\right\}^{1/2} = 41.3\cdot10^{-15}$ ergs $= 4.13\cdot10^{-14}$ ergs

(6) $e^{.150} = 1.16$; $\therefore$ $I = (124)(1.16) = \underline{144 \ \mu amp/cm}$

c. (7) $\Delta V = \frac{5000}{300} = 16.6$ stv. ; $e^{.61} = 1.85$; $\therefore$ $I = \underline{230 \ \mu amp/cm}$

Tunneling current is of the order e^{-10^4}, so is $\underline{\text{negligible}}$.

10. (1) We shall choose an illustrative model similar to the one used in lecture to demonstrate equilibrium exchange:



(2) The flux density of particles across $x=0$ from ① to ② is:

$$\frac{dN_1}{dt\,dA} = \int n(v_1)\, v_x \, d v_1 = \frac{1}{4\pi^3\hbar^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{\sqrt{\frac{2e\varphi_0}{m}}}^{\infty} \frac{v_{1x}\, dv_{1x}\, dv_y\, dv_z}{e^{+\frac{\frac{1}{2}mv_1^2 - J_1}{kT}} + 1}$$

(3) The flux density of particles moving from ② to ① is:

$$\frac{dN_2}{dt\,dA} = \int n(v_2)\, v_{2x}\, dv_2 = \frac{1}{4\pi^3\hbar^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{-\infty} \frac{v_{2x}\, dv_{2x}\, dv_y\, dv_z}{e^{+\frac{\frac{1}{2}mv_2^2 - J_2 + e\varphi_0}{kT}} + 1}$$

(4) Under conditions of thermal equilibrium, $\frac{dN_1}{dt\,dA} = \frac{dN_2}{dt\,dA}$

$$\therefore \int_{\sqrt{\frac{2e\varphi_0}{m}}}^{\infty} \frac{v_{1x}\, dv_{1x}}{e^{+\frac{\frac{1}{2}mv_1^2 - J_1}{kT}} + 1} = \int_{0}^{-\infty} \frac{v_{2x}\, dv_{2x}}{e^{+\frac{\frac{1}{2}mv_2^2 - J_2 + e\varphi_0}{kT}} + 1}$$

(5) From conservation of energy: $\frac{m}{2} v_{1x}^2 = \frac{m}{2} v_{2x}^2 + e\varphi_0$

and $v_{1x}\, dv_{1x} = v_{2x}\, dv_{2x}$

(6) $$\therefore \int_{\sqrt{\frac{2e\varphi_0}{m}}}^{\infty} \frac{v_{1x}\, dv_{1x}}{e^{\frac{\frac{1}{2}m(v_{1x}^2 + v_y^2 + v_z^2) - J_1}{kT}} + 1} = \int_{0}^{\infty} \frac{v_{2x}\, dv_{2x}}{e^{\frac{\frac{1}{2}m(v_{2x}^2 + v_y^2 + v_z^2) + e\varphi_0 - J_1}{kT}} + 1}$$

(7) When the integrand is <u>odd</u> one can change the signs of the limits without changing the sign of the integral; Therefore, from (4), it is obvious:

$$\frac{1}{2} m (v_{2x}^2 + v_y^2 + v_z^2) + e\varphi_0 - J_1 = \frac{1}{2} m (v_{2x}^2 + v_y^2 + v_z^2) + e\varphi_0 - J_2$$

$$\therefore \boxed{J_1 = J_2} \quad \text{QED}$$

(8) The generalization of the proof by approximating arbitrary potential with steps is immediate.

Midterm Examination Applied Physics 231

March 23, 1961

1. In a synchrotron, the radius ρ of the particle's equilibrium orbit is kept constant by changing the magnetic field H at ρ.

   a) Calculate relativistically the angular velocity ω for a particle in a synchrotron as a function of its relativistic kinetic energy T.

   b) Show that your formula gives the correct result in the limit of both high and low velocities.

   c) Obtain an expression for H as a function of the relativistic kinetic energy of the particle.

2. A charged particle enters a magnetic mirror moving with relativistic velocity v.

   a) Show that the equation of motion for the component of momentum ($p_{||}$) along the axis of the mirror (z axis) is given by

$$\frac{dp_{||}}{dt} = -\frac{M}{\left(\frac{1}{\sqrt{1 - (\frac{v}{c})^2}}\right)} \frac{\partial H}{\partial z}$$

   where

$$M = \frac{p_{\perp}^2}{2mH} = \text{magnetic moment.}$$

   b) Using the equation of motion, show that M is an adiabatic invariant.

   c) If p is the total momentum of the particle show that reflection occurs when $H = H_r$ where

$$H_r = \frac{p^2}{2mM}$$

   m = mass of particle.

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HARVARD UNIVERSITY

FACULTY OF ARTS and SCIENCES

Examination Book

Name: Paul Grant

Date: 3-23-61

Subject:

Section:

B+

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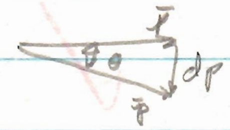
16.5

J. L. Hammett Co., Cambridge, Mass.

① (a) (1) $\frac{d\bar{p}}{dt} = \frac{e}{c} v \times H$

(2)  $\frac{d\bar{p}}{p} = \theta$, $\frac{dp}{dt} = p\omega$

$= p v \rho$

(3) $p\rho = \frac{e}{c} H$, $p = \frac{eH}{c\rho}$

(4) $T^2 = p^2 c^2 + m^2 c^4$

$T^2 = \left(\frac{eH}{\rho}\right)^2 + m^2 c^4$

(5) $\frac{m}{\sqrt{1-\beta^2}} \frac{dv}{dt} = \frac{evH}{c}$

(6) since v constant,

$\omega = \omega_0 \sqrt{1-\beta^2}$, $\omega_0 = \frac{eH}{mc}$ 2

$T = \frac{mc^2}{\sqrt{1-\beta^2}}$, $\sqrt{1-\beta^2} = \left(\frac{T}{mc^2}\right)^{-1}$

$\omega = \frac{\omega_0 mc^2}{T}$ ← elimate H

(b) (1) $\omega = \omega_0 \sqrt{1-\beta^2}$, for $v \ll c$ $\beta^2 \to 0$ 1.5

and $\omega = \omega_0$

(2) For $\sqrt{1-\beta^2} \ll 1$, $T \sim pc$

$\sim \frac{eH}{\rho}$, $\therefore$ $\omega \simeq \frac{\omega_0 \, mc^2}{\frac{eH}{\rho}}$ as $v \to c$

$\boxed{\cancel{\omega = \omega_0 \rho \frac{mc^2}{eH} = \rho c}}$ But $\omega = \frac{c}{\rho}$

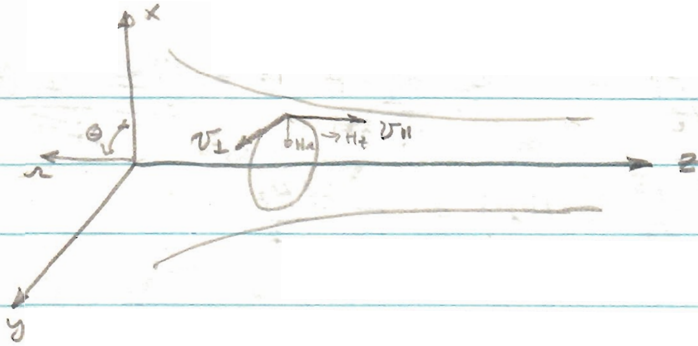
① (c) Please see ① (a)

$$\frac{eH}{\rho} = \sqrt{T^2 - m^2c^4}$$

or $$H = \frac{\rho}{e}\sqrt{T^2 - m^2c^4}$$

3

(2)



(1) Take $H = H_r(r,z)\,\hat{1}_r + H_z(r,z)\,\hat{1}_z$

<u>Assume</u> $H_r(r,z) = H_r(r)$, $H_z(r,z) = H_z(z)$

(2) div $H = 0$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\,H_r(r)\right) + \frac{\partial H_z}{\partial z} = 0$$

or $$\frac{1}{r}H_r(r) + \frac{\partial H_r}{\partial r} + \frac{\partial H_z}{\partial z} = 0$$

(3) Curl $H = 0$

$$\begin{vmatrix} \frac{\hat{1}_r}{r} & 1_\theta & \frac{1_z}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ H_r & 0 & H_z \end{vmatrix}$$

$$= \hat{1}_\theta\left(\frac{\partial H_z}{\partial r} - \frac{\partial H_r}{\partial z}\right) = 0$$

(3) $\dfrac{d\vec{p}_{\parallel}}{dt} = \dfrac{e}{c}\dfrac{\vec{v}_{\perp}\times\vec{H}}{\sqrt{1-\beta^2}} = \dfrac{e}{c}\dfrac{v_{\perp} H_r}{\sqrt{1-\beta^2}}$

$$\frac{1}{r}\frac{d}{dr}(r H_r) = -\frac{\partial H_z}{\partial z}$$

$$\frac{d}{dr}(r H_r) = -\frac{\partial H_z}{\partial z} r$$

$$\cancel{r} H_r = -\frac{\partial H_z}{\partial z}\frac{r^{\cancel{2}}}{2}$$

$$H_r = -\frac{1}{2}\frac{\partial H_z}{\partial z} r$$

(4) $\therefore \dfrac{dp_{\parallel}}{dt} = -\dfrac{1}{2}\dfrac{\partial H_z}{\partial z} r \cdot \dfrac{e}{c} \cdot \dfrac{v_{\perp}}{\sqrt{1-\beta^2}}$

(5) Now: $\dfrac{v_{\perp}}{\sqrt{1-\beta^2}} = r\omega = r\omega_0\sqrt{1-\beta^2}$

$$= r\frac{eH}{mc}\sqrt{1-\beta^2}$$

(6) $\dfrac{dp_{\parallel}}{dt} = -\dfrac{1}{2}\dfrac{e}{c}\underbrace{\dfrac{eH}{mc}}_{\omega_0} r^2\sqrt{1-\beta^2}\,\dfrac{\partial H}{\partial z}$

(7) $p_{\perp} = \dfrac{m v_{\perp}}{\sqrt{1-\beta^2}} = m r\omega$

$$p_{\perp}^2 = m^2 r^2\omega^2 = m^2 r^2 \omega_0^2(1-\beta^2)$$

(8) Put in (6): $r^2\omega_0\sqrt{1-\beta^2}\ \omega_0\sqrt{1-\beta}$

$$= \frac{p_{\perp}^2}{m^2}, \quad r^2\omega_0\sqrt{1-\beta^2} = \frac{p_{\perp}^2}{m^2\omega_0\sqrt{1-\beta^2}}$$

4

(9) $\therefore \quad \dfrac{dp_{\parallel}}{dt} = -\dfrac{p_{\perp}^2}{2m H_z}\sqrt{1-\beta^2}\,\dfrac{\partial H}{\partial z}$

we define $M = \dfrac{p_{\perp}^2}{2m H}$

(b) (1) $\dfrac{dT}{dt} = 0$ since no energy is gained in mag. field

(2) $T = \dfrac{mc^2}{\sqrt{1-\beta}}$

(3) $T^2 = p^2c^2 + m^2c^2$

$T\dfrac{dT}{dt} = \vec{p}\cdot\dfrac{d\vec{p}}{dt} = 0$

or $\vec{p}\cdot\dfrac{d\vec{p}}{dt} = 0$

or $p_{\parallel}\dfrac{dp_{\parallel}}{dt} + p_{\perp}\dfrac{dp_{\perp}}{dt} = 0$

(4) $p_{\parallel}\left(-M\sqrt{1-\beta^2}\,\dfrac{\partial H}{\partial z}\right) + \dfrac{1}{2}\dfrac{d}{dt}\left(p_{\perp}^2\right) = 0$

(5) $\dfrac{d}{dt}\left(p_{\perp}^2\right) = \dfrac{d}{dt}\left(2mHM\right)$

$= 2m\dfrac{dH}{dt} + 2m\dfrac{dM}{dt}$

(6) By definition of derivative;

$\dfrac{dH}{dt} = \underbrace{\dfrac{\partial H}{\partial t}}_{0 \text{ no explicit time dependence}} + v_{\parallel}\dfrac{\partial H}{\partial z} = \quad p_{\parallel} = \dfrac{mv_{\parallel}}{\sqrt{1-\beta^2}}$

(7) $\therefore \dfrac{dH}{dt} = \dfrac{p_u}{m}\sqrt{1-\beta^2}\,\dfrac{\partial H}{\partial z}$

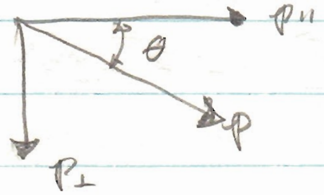
(8) substituting in (4)

$$2m\,\frac{dM}{dt} = 0$$

$\therefore \dfrac{dM}{dt} = 0$ and does not change with time.

4

(c)



$$M = \frac{p_\perp^2}{2mH}$$

since $\dfrac{dM}{dt} = 0$

$$\frac{p_{\perp 0}^2}{2mH_0} = \frac{p_\perp^2}{2mH} \quad \text{at any point}$$

implies

$$\frac{\sin^2\theta_0}{H_0} = \frac{\sin^2\theta}{H} = \frac{p^2}{2mH}$$

$$\therefore \sin^2\theta_c = \frac{p^2 H_0}{2mH} = \frac{H_0}{H_r}$$

reflects when

$$H_r = \frac{p^2}{2mM}$$

✓

Final Examination Applied Physics 231 June 5, 1961

ANSWER 5 OF THE FOLLOWING QUESTIONS

1. Define and discuss briefly each of the following terms:

a) Mott shielding length

b) Magnetic mirror

c) Thomas precession

d) Diamagnetic (Lamb) shielding

e) Betatron oscillations in particle accelerators

f) The Fermi function

g) " Cutoff" in a magnetron

2. Derive the dimensionless Fermi-Thomas equation for an atom making clear the physical ideas on which the equation is based. (10)

Remember: $\zeta(r) = \left(\frac{3}{8\pi}\right)^{2/3} \frac{h^2}{2m} \left(n(r)\right)^{2/3}$

3. Derive expressions for the following two parameters of a fully ionized plasma:

a) The Debye length

b) The plasma oscillation frequency (6)

Explain physically the significance of each of these parameters. (4)

4. a) Show that the temperature (T) dependence of the Fermi energy (ζ) for an assembly of electrons with a density of states n(ϵ) is given by:

$$\zeta(T) - \zeta(0) = -\frac{\pi^2}{6}(kT)^2\left[\frac{1}{n(\epsilon)}\frac{\partial n(\epsilon)}{\partial \epsilon}\right]_{\zeta(0)}$$

b) Show that the internal energy (E) of the assembly is given at temperature (T) by:

$$E(T) - E(0) = +\frac{\pi^2}{6}(kT)^2\left[n(\epsilon)\right]_{\zeta(0)}$$

5. Consider an electron moving in a circular orbit of radius (r), with velocity (v), around a nucleus.

a) Obtain an expression for the orbital magnetic moment in terms of r and v. (2)

b) Show that the action of a magnetic field (H) on the magnetic moment (M) produces a precession of the plane of the orbit around the field with the Larmor frequency:

$$\omega_L = eH/2mc$$

c) Let the electron orbit be in a plane normal to the applied field. Calculate the change in velocity produced by <u>changing</u> the magnetic field from zero to (H). Show that this result is identical with that calculated from the Larmor precession.

Hint: $$\nabla \times E = -\frac{1}{c}\frac{\partial H}{\partial t}$$

6. In connection with the use of nuclear fusion as a source of controlled power:

a) Discuss briefly:

   1) The process of fusion

   2) Advantages of fusion reactors (5)

   3) Basic requirements for controlled fusion

b) Discuss briefly the central ideas involved in the three major proposals for plasma confinement:

   1) The mirror machine

   2) The pinch (5)

   3) The stellerator

AP 231 – FINAL EXAMS

Spring 1960 : Answer 5 of the following:

(1) Define and discuss briefly the following terms:
a) Debye shielding length
b) Plasma oscillation frequency
c) Mirror machine
d) Pinch effect
e) Stellarator
f) Magnetic Pressure

(2) An electron is accelerated through a potential difference ΔV. Obtain a relativistically correct expression for the de Broglie wavelength in terms of ΔV.

(3) Show that or how the magnetic field in a cyclotron must vary with r so that the particles have radial stability, and stability about the median plane. (r is the radial distance from the center of the magnet gap.).

(4) Obtain the distribution function in v_x for those electrons which have been thermionically emitted from a metal surface. (The x direction is normal to the metal surface.).

(5) a) Obtain an expression or equation for the pressure exerted by the "free" electrons in a metal at $T=0$.

b) Obtain an equation for the Bulk modulus:
$B = V(dP/dV)_T$ at $T=0$

Recall from thermodynamics that $P = -\left(\frac{\partial F}{\partial V}\right)_T$ and $F = E - TS$.

(6) Using the Fermi-Thomas method, obtain an expression for the potential φ as a function of the distance r from the center of an impurity atom in a metal. Discuss briefly the physical significance of the result.

AP 231 FINAL EXAM 1959

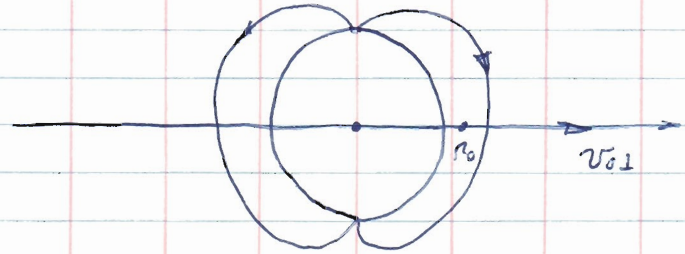
① Derive an expression for the electrostatic potential around a positive ion in an ionized gas. From the form of the potential distribution obtain the deBye shielding distance d. Explain physically why d is temperature dependent.

② Let metal A and metal B be two plates of a parallel plate capacitor. Also let these two plates be connected to one another thru a large resistance R. Show that by changing the distance d between the plates as $d = d_0(1 + r \cos \omega t)$ $(r \ll 1)$ that a sinusoidal voltage $V_R(t)$ will be developed across the resistor, which is proportional to the contact difference of potential V_{AB}: i.e, show that

$$V_R = R \omega C V_{AB} r \sin \omega t$$

where C is the capacity of the unperturbed condenser.

③ a) Consider a plane bisecting the North-South magnetic axis of the earth as is shown below:

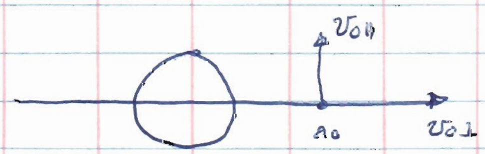


Assume that the strength of the earth's

(3) magnetic field falls off linearly with radius r in this plane. Let a charged particle be projected outward along a radius vector thru the center of the earth in this plane with velocity $v_{0\perp}$. Obtain an expression which gives approximately the time required for the particle to travel around the planet and return to its starting position. Assume:

$v_{0\perp} / (eH/mc) \ll r_0$, and that the particle starts at r_0.

b) Consider next a particle moving in the same field with a velocity component along the field lines as well, as shown below



Neglect the azimuthal drift in the equatorial plane, and assume that in its motion <u>along</u> the field lines the particle sees a field whose gradient along the field direction is a constant. Determine how long it takes to travel from the equatorial plane to the north pole and back. (Assume the total distance travelled in this trip is πr_0).

c) Discuss qualitatively the trajectory of ionized particles liberated by an explosion at r_0.

4. a) Use the Fermi-Thomas model to calculate the kinetic energy of the electrons in a free atom. In your calculations, include the shielding of the nucleus by the electrons, ie, use the solution $\chi(r)$ of the Fermi-Thomas equation to determine the form of the electronic charge distribution.

b) From your result in (a) determine the total ionization energy of an atom.

Hints: 1. $n(r) = \frac{8\pi}{3}\left(\frac{2me}{h^2}\right)^{3/2} \zeta^{3/2}(r)$

2. It can be shown that

$$\int_0^\infty \frac{\chi^{5/2}}{x^{1/2}}\,dx = \frac{5}{7}\left(\frac{\partial \chi}{\partial x}\right)_0$$

5. a) Derive the Child-Langmuir law relating the potential V to the distance x in a plane parallel diode under conditions of space charge limited current. Assume zero initial velocity for electrons emitted from the cathode.

b) Obtain the expression for the time required for an electron to travel from cathode to plate.

6. a) Starting from the Fermi function f_0 and the density of states $N(E) = (V/2\pi^2)(2m/\hbar^2)^{3/2}\sqrt{E}$, Obtain expressions for:

1. The energy distribution function $n(E)$
2. The momentum distribution function $n(p)$
3. The momentum component distribution function $n(p_x)$

b) Using the result you obtained in a (3) derive the Dushman-Richardson equation for thermionic emission.