PHYSICS
253

ADVANCED
QUANTUM
MECHANICS

<u>PHYSICS 253</u>

Fall, 1961

<u>Summary of Special Relativity</u>

Special relativity deals with the equivalence of <u>inertial systems</u> of reference - all rectangular coordinates, all in uniform translational motion relative to each other. General relativity deals with equivalence of more general systems of reference - curvi-linear coordinates, accelerated and rotational motions.

Newton's equations hold for inertial systems without the use of "fictitious forces". The "fictitious forces" are of the same type as gravitational forces - proportional to mass. General relativity has a main feature the theory of gravitation.

For two protons

$$\frac{\text{Gravitational force}}{\text{Electric force}} = \frac{Gmm}{e\ e} \quad \frac{7\times10^{-8}(2\times10^{-24})^2}{5\times10^{-10})^2} \quad 10^{-36}$$

This gives some grounds for regarding gravitation as unimportant in microscopic phenomena, and supposing that the difficulties of existing quantum theories should be solved in special relativity.

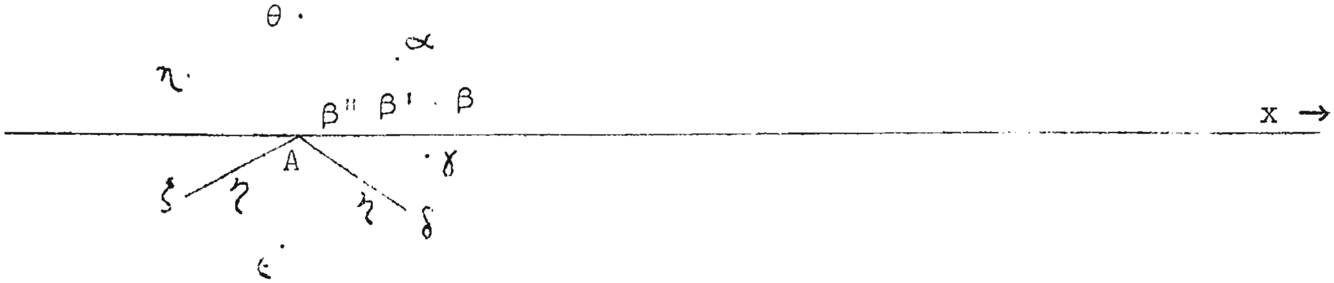
We shall not discuss experiments. From the Michelson-Morley experiment and for other reasons, Einstein in 1905 set up his two principles of special relativity:

I. All equations of physics must take the same form in all inertial systems.

II. The speed of light in free space is the same for all observers working in inertial systems, and is independent of the motion of the source.

I is the relativity principle. II serves to exclude "emission theories", in which the speed of light is affected by the motion of the source; such ideas are excluded by astronomical evidence.

Consider two observers in uniform relative motion. As they pass, let a spark be struck between them. At future times, <u>each</u> will find that <u>he</u> is always at the center of the spherical wave-front (whose passage across various points of space may be revealed by scattering from various small obstacles.) This result cannot be reconciled with our traditional ideas of space, time, and simultaneity. Indeed, the explanation



is found to be that the two observers do not agree about simultaneity. If A announces that he saw the wave pass simultaneously across $\alpha\beta\gamma\delta\epsilon\zeta\eta\theta$, all distant r from A, then B, who receives the message at B' and who (as observed by A) was at B'' at the time $\frac{\overline{AB''}+r}{c}$ earlier when the alleged simultaneous passage occurred, will agree that all the reflected rays reached A at once, but deny that the reflections from $\alpha\beta\gamma\delta\epsilon\zeta\eta\theta$ were simultaneous: he will say that the passages across $\alpha\beta\gamma\delta$ occurred earlier than those across $\epsilon\zeta\eta\theta$, but that A went to meet the reflected rays from the latter and ran from those from the former.

The changes in ideas of space, time, and simultaneity mean changes in the transformation equations relating A's variables (x,y,z,t) and B's variables (x', y', z', t'). The traditional Galilei transformation.

$$x' = x-vt, \quad y' = y, \quad z' = z, \quad t' = t \tag{1}$$

must be modified accordingly. The key fact in determining the new form is that the two equations

$$\begin{aligned} x^2+y^2+z^2-c^2t^2 &= 0 \\ x'^2+y'^2+z'^2-c^2t'^2 &= 0 \end{aligned} \tag{2}$$

which express the fact that each observer finds himself always at the center of the spherical wave-front, must each be a consequence of the other. Together with simple requirements — linearity of the equations (homogeneity of space and time), and obtainability of inverse transformation by replacing v by -v — (2) leads by an algebraic argument to the Lorentz transformation

$$x' = \frac{x-vt}{\sqrt{1-v^2/c^2}}, \quad y' = y, \quad z' = z, \quad t' = \frac{t-(v/c^2)x}{\sqrt{1-v^2/c^2}} \tag{3}$$

(1) and (3) are of course not the most general transformations possible in the two theories. They are specialized by choice of origin, of relative orientation of space axes, and of direction of relative velocity. To get the most general transformation we must use first a space rotation, to get the x-axis along v, then (1) or (3), then another space rotation, and also at some stage the addition of constants to shift the origin. The shift of origin is so trivial that it is almost always ignored. The combining of all these operations into a single set of equations is comparatively simple in non-relativity theory, using (1), but is a nasty job with (3).

We omit special arguments about Lorentz-contraction, time-dilation, etc; they can be made either from (3) or from arguments about the way A and B conduct their measurements, using light as a means of signaling to their assistants who ride the respective frames. We mention only that (3) implies $v < c$, and that it has to be assumed that no relative velocity as observed by one of the objects in question, and also no signal velocity, can exceed the value c. Indeed, the disagreements about simultaneity which are basic in the theory would be impossible if infinitely rapid signalling were possible.

If we write as variables

$$x_1, x_2, x_3, x_4 = ict, \tag{4}$$

then (3) takes the form

$$x_1' = x_1 \cos\theta + x_4 \sin\theta,\; x_2' = x_2,\; x_3' = x_3$$
$$x_4' = x_4 \cos\theta - x_1 \sin\theta, \text{ with} \tag{5}$$

$$\cos\theta = \frac{1}{\sqrt{1-v^2/c^2}}\,,\quad \sin\theta = i\,\frac{v/c}{\sqrt{1-v^2/c^2}} \tag{6}$$

This is just the formal expression for a rotation in the x_1-x_4 plane of a 4-dimensional Euclidean space, but the angle used is imaginary ($\theta = i \tanh^{-1} v/c$)

Such rotations leave the square of a 4-dimensional interval

$$s^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2 \tag{7}$$

invariant. Thus using the notation (4), the general Lorentz transformation is just the orthogonal transformation

$$x_\mu' = \sum_\nu a_{\mu\nu}\, x_\nu\,, \tag{8}$$

with the coefficients subject to the orthogonality conditions,

$$\sum_\nu a_{\mu\nu} a_{\rho\nu} = \delta_{\mu\rho}\;;\; \sum_\mu a_{\mu\nu}\, a_{\mu\sigma} = \delta_{\nu\sigma} \tag{9}$$

with the reality conditions:

$$a_{\mu\nu} = \text{pure real},\; \mu \neq 4 \neq \nu$$
$$a_{\mu\nu} = \text{pure imaginary},\; \mu = 4 \neq \nu \text{ or } \mu \neq 4 = \nu \tag{10}$$
$$a_{44} = \text{pure real positive}$$

The requirement $a_{44} > 0$ restricts us to transformations which do not reverse the time-coordinate. The matrix $\|a_{\mu\nu}\|$ is <u>orthogonal</u>, <u>not unitary</u>.

The second equation in (9) is not independent of the first, and the first allows arbitrary choice of just six of the $a_{\mu\nu}$. That the general Lorentz transformation has six parameters can be seen in three ways:

Algebraically, six $a_{\mu\nu}$ determine the rest

Geometrically, choose angle of rotation in each of six planes.

Physically: $\left.\begin{array}{l}\text{Direction and magnitude of } \bar{v}\text{: } 3\\ \text{Relative orientation of space-frames: } 3\end{array}\right\}\ 6$

Most of the relativistic technique used in practice, both in classical and quantum theories, is concerned with the recognition and use of <u>vector and tensor quantities</u>. In the general case it is necessary to distinguish two kinds of vectors:

A <u>contravariant vector</u> A^μ transforms like the coordinate-differentials dx^μ.

A <u>covariant vector</u> B transforms in such a way that the sum $\sum B_\mu dx^\mu$ is <u>invariant</u>. Any <u>physical vector quantity</u> can be represented by either kind of vector, or in other ways. It may be remarked that in many three-dimensional calculations in the literature physical vector quantities in curvilinear coordinates are presented by components which satisfy neither definition. — In consequence of the definition,

$$\sum_{\mu} A^{\mu} B_{\mu} = \text{invariant.} \tag{10'}$$

To form an invariant in this way, covariant components must be used for one vector, contravariant for the other. The summation-sign is often omitted, it being a commonly used convention that all repeated indices are summation-indices or "dummies" unless the contrary is stated explicitly.

Comparison of (10) with the invariant expression (7) illustrates the fact that in a Euclidean space, with rectangular coordinates, there is no distinction between covariant and contravariant vectors. This is the great usefulness of the choice of notation (4), and outweighs the strangeness of having one component take imaginary values. It makes it possible to dispense with superscript indices; otherwise we ought from the start to have written x^{μ}.

It may be mentioned that the notation x^1, x^2, x^3, $x^0 = ct$ is also in fairly common use — sometimes x^4 is written for x^0, alas! — Then $x_0 = -ct$, and in general indices can be raised and lowered freely, with a change of sign whenever a o is raised or lowered. We shall stick to (4). — In general relativity the distinction is non-trivial.

Tensor components transform like products of vector components. Tensors can have contravariant or covariant behavior with regard to each index, but by (4) and (7) we need not consider this.

New vector or tensor quantities can be formed in three ways:

(a) Contraction, by such generalizations of (10) as

$$\begin{aligned} A_{\mu} T_{\mu\nu} &= C_{\nu} \\ T_{\mu\mu} &= T \end{aligned} \tag{11}$$

and so on

Here we have not worried about upper indices, but have used the summation convention. A very common vector with which contraction occurs is $\frac{\partial}{\partial x_{\mu}}$; we get equations like

$$\frac{\partial}{\partial x_{\mu}} A_{\mu} = B \tag{12}$$

$$\frac{\partial}{\partial x_{\mu}} T_{\mu\nu} = S_{\nu}$$

This is called the four-dimensional divergence.

(b) Outer multiplication, as in

$$\begin{aligned} A_{\mu} C_{\nu} &= D_{\mu\nu} \\ \frac{\partial B}{\partial x_{\mu}} &= C_{\mu} \end{aligned} \tag{13}$$

The latter of these equations gives the four-dimensional gradient.

(c) Differentiation by a world-scalar (also multiplication, which is trivial). The important scalar here is the proper time of a moving particle, defined by

$$d\tau^2 = d\tau^2 - \frac{dx^2 + dy^2 + dz^2}{c^2} = - \frac{ds^2}{c^2} = dt^2\ (1-\beta^2) \tag{14}$$

(Here β means the speed of the particle divided by c). In the case of a particle moving uniformly or with negligible acceleration, τ is the time measured by a clock carried with the particle; in any case, $d\tau$ is the infinitesimal time-interval measured in a reference frame in which the particle is momentarily at rest. That $d\tau > dt$ is called the "time-dilatation". The observational case <u>par excellence</u> is the half-life of muon decay. — We have equations like

$$\frac{dA_\mu}{d\tau} = B_\mu$$
$$\frac{dF_{\mu\nu}}{d\tau} = G_{\mu\nu}, \text{ and so on.} \tag{15}$$

The <u>antisymmetric</u> tensors are important. An antisymmetric tensor of the second rank satisfies

$$F_{\mu\nu} = - F_{\nu\mu} \tag{16}$$

It has six independent non-vanishing components, and is sometimes called a "six-vector"; a vector A is called a "four-vector". The property of antisymmetry is preserved on transformation:

$$F'_{\nu\mu} = a_{\nu\rho} a_{\mu\sigma} F_{\rho\sigma} = a_{\mu\sigma} a_{\nu\rho} (-F_{\sigma\rho}) = -F'_{\mu\nu} \tag{17}$$

An antisymmetric tensor of the third rank,

$$A_{\mu\nu\rho} = -A_{\mu\rho\nu} = -A_{\nu\mu\rho} = -A_{\rho\nu\mu} = A_{\nu\rho\mu} = A_{\rho\mu\nu}, \tag{18}$$

has four independent components. A completely antisymmetric tensor of the fourth rank has only one. By arguments like (17) we can show that the antisymmetry of these two kinds of tensor is preserved on transformation. For the fourth rank,

$$A_{\mu\nu\rho\tau} = \pm A_{1234}\ ; \tag{19}$$

The sign is + if $\mu\nu\rho\sigma$ is an even permutation of 1234, - it it is odd. The <u>numerical value</u> of the single component changes at most its sign on transformation:

$$A'_{1234} = a_{1\mu} a_{2\nu} a_{3\rho} a_{4\tau}\, A_{\mu\nu\rho\sigma} = A_{1234} \sum_{\text{Perm.}} \pm\, a_{1\mu} a_{2\nu} a_{3\rho} a_{4\sigma} =$$
$$A_{1234}\ \text{Det}\ (a_{\mu\nu}) = \pm A_{1234} \tag{20}$$

The last step is based on the fact that by (8) the transposed matrix of $\|a_{\mu\nu}\|$ is its reciprocal, so that Det $(a_{\mu\nu})$ is a square root of unity. Since $a_{44} > 0$, any transformation with Det $(a_{\mu\nu}) = -1$ can be regarded as the product of one with Det $(a_{\mu\nu}) = 1$ and one with $a_{\mu\nu} = 2\delta_{4\mu}\delta_{4\nu} = 1$: This latter is the <u>inversion of the space</u>

coordinates — change from right-handed to left-handed system. The single independent element A_{1234} has the properties of a pseudoscalar — it is invariant except for change of sign on inversion.

Let $\epsilon_{1234} = 1$ for right-handed system, and $\epsilon_{\alpha\beta\gamma\delta}$ be anti-symmetric. Then if $A_{\lambda\mu\nu}$ is antisymmetric, multiplication and contraction gives a vector, $\frac{1}{6}\epsilon_{\lambda\mu\nu\rho}A_{\mu\nu\rho}$. Thus by setting

$$\overset{*}{A}_1 = A_{234},\ \overset{*}{A}_2 = A_{314},\ \overset{*}{A}_3 = A_{412},\ \overset{*}{A}_4 = A_{132} \tag{21}$$

we get a quantity which behaves like a vector except for an added change of sign on inversion. $\overset{*}{A}$ is the dual of $A_{\mu\nu\sigma}$ and is an axial vector.

Also from an antisymmetric tensor $F_{\mu\nu}$ we get another, $\frac{1}{2}\epsilon_{\lambda\mu\nu\rho}F_{\nu\rho}$, so that by setting

$$\overset{*}{F}_{12} = F_{34},\ \overset{*}{F}_{23} = F_{14},\ \overset{*}{F}_{31} = F_{24},\ \overset{*}{F}_{14} = F_{23},\ \overset{*}{F}_{24} = F_{31},\ \overset{*}{F}_{34} = F_{12} \tag{22}$$

we get as dual of $F_{\mu\nu}$ an $\overset{*}{F}_{\mu\nu}$ which has all the transformation properties of a six-vector except for wrong behavior on inversion. To write (21) and (22) we make the subscripts always even permutations of 1234.

To get equations of physics in relativistic form, use is made of the fact that equations which equate tensors of the same kind are automatically covariant. The equations of electromagnetism (in free space) are left essentially unchanged, the burden being thrown on transformation properties; thus our electrodynamics is good in relativity even if we learned it while innocent of relativity. Mechanics, however, is distinctly changed.

Electrodynamics:

Charge and current four-vector:

$$s_1 = j_x/c,\ s_2 = j_y/c,\ s_3 = j_z/c,\ s_4 = i\rho \tag{23}$$

Potentials:

$$\phi_1 = A_x,\ \phi_2 = A_y,\ \phi_3 = A_z,\ \phi_4 = i\phi \tag{24}$$

Fields:

$$F_{23} = H_x,\ F_{31} = H_y,\ F_{12} = H_z \tag{25}$$

$$F_{41} = iE_x,\ F_{42} = iE_y,\ F_{43} = iE_z \qquad (F_{\mu\nu} = -F_{\nu\mu})$$

Relations:

$$F_{\mu\nu} = \frac{\partial\phi_\nu}{\partial x_\mu} - \frac{\partial\phi_\mu}{\partial x_\nu} \qquad \left\{\begin{aligned} \vec{H} &= \operatorname{curl}\vec{A} \\ \vec{E} &= -\frac{1}{c}\frac{\partial\vec{A}}{\partial t} - \nabla\phi \end{aligned}\right\} \tag{26}$$

The field-tensor is the four-dimensional curl of the four-potential.

Maxwell's equations:

$$\frac{\partial F_{\mu\nu}}{\partial x_\nu} = 4\pi s_\mu \qquad \left\{ \begin{array}{c} \mathrm{curl}\ \vec{H} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \frac{4\pi \vec{j}}{c} \\ \mathrm{div}\ E = 4\pi\rho \end{array} \right\} \tag{27}$$

$$\frac{\partial F^*_{\mu\nu}}{\partial x_\nu} = 0 \qquad \left\{ \begin{array}{c} \mathrm{curl}\ \vec{E} + \frac{1}{c}\frac{\partial \vec{H}}{\partial t} = 0 \\ \mathrm{div}\ \vec{H} = 0 \end{array} \right\} \tag{28}$$

Mechanics:

Four-velocity:

$$U = \frac{dx_\mu}{d\tau}: (U_1, U_2, U_3) = \frac{\vec{v}}{\sqrt{1-\beta^2}},\ U_4 = \frac{ic}{\sqrt{1-\beta^2}} \tag{29}$$

This satisfies the identity

$$u_\mu u_\mu = -c^2 \tag{30}$$

Kinetic energy - momentum four-vector

$$Mu_\mu = \left(\frac{m\vec{v}}{\sqrt{1-\beta^2}}, \frac{imc}{\sqrt{1-\beta^2}}\right) = \left(p_{kin}, i\,\frac{E_{kin}}{c}\right) \tag{31}$$

Here m is the rest-mass, $\frac{m}{\sqrt{1-\beta^2}}$ the relativistic mass. The proof that $E_{kin} = mc^2/\sqrt{1-\beta^2}$ appears a bit later.

By analogy with non-relativity theory, we should expect to set

$$\frac{d}{d\tau} mu_\mu = K_\mu, \tag{32}$$

where K_μ is a four-vector whose first three components are the force on the particle. Here comes in a limitation due to (30), namely

$$K_\mu u_\mu = 0 \tag{33}$$

If we choose $F \;:\; \frac{e}{c} u$ for K, (33) is satisfied because of the antisymmetry of $F_{\mu\nu}$, and by (25) and (29) we have

$$(K_1, K_2, K_3) = \frac{e\left\{\vec{E} + \left[\frac{\vec{v}}{c} \times \vec{H}\right]\right\}}{\sqrt{1-\beta^2}},\ K_4 = \frac{i(e\vec{E}\cdot\frac{\vec{v}}{c})}{\sqrt{1-\beta^2}} \tag{34}$$

Apart from the factor $\sqrt{1-\beta^2}$, which would be removed in getting $\frac{d}{dt}$ rather than $\frac{d}{d\tau}$ on the left side of (32), these expressions just represent the rates of increase of momentum and $\frac{i}{c}$ times energy, caused by the Lorentz force, $e\vec{E} + \left[\frac{\vec{v}}{c} \times \vec{H}\right]$. Accordingly,

$$\frac{d}{d\tau} mu_\mu = \frac{e}{c} F_{\mu\nu} u_\nu \tag{35}$$

gives the Newtonian equations of motion for a charged particle in an electromagnetic field.

These equations can be derived from the variation principle

$$\delta \int_{\tau_1}^{\tau_2} \left(\tfrac{1}{2}\, m u_\mu u_\mu + \frac{e}{c}\, \phi_\mu u_\mu\right) d\tau = 0 \tag{36}$$

by varying the x_μ (u_μ means $\frac{dx}{d\tau}$). The Euler-Lagrange equations are

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial u_\mu} - \frac{\partial L}{\partial x_\mu}\right) = \frac{d}{d\tau}\left(m u_\mu + \frac{e}{c}\,\phi_\mu\right) - \frac{e}{c}\frac{\partial \phi_\nu}{\partial x_\mu} u_\nu =$$

$$= \frac{d}{d\tau}\, mu + \frac{e}{c}\left(\frac{\partial \phi_\mu}{\partial x_\nu} - \frac{\partial \phi_\nu}{\partial x_\mu}\right) u_\nu = 0, \text{ which is (35).}$$

The generalized moments are

$$p_\mu = \frac{\partial L}{\partial u_\mu} = m u_\mu + \frac{e}{c}\,\phi_\mu \tag{37}$$

The integral corresponding to the integral of energy is provided by (30), and can be written

$$\sum_\mu \left(p_\mu - \frac{e}{c}\,\phi_\mu\right)^2 = -\, m^2 c^2. \tag{38}$$

The equations of motion can also be derived from a non-relativistic appearing variation principle using t as variable of intepration (in a chosen coordinate system):

$$\delta \int_{t_1}^{\tau_2} L\, dt = \delta \int_{\tau_1}^{\tau_2} \left(- \sqrt{1 - \frac{v^2}{c^2}}\, mc^2 - e\phi + \frac{e}{c}\,(\vec{A}\cdot\vec{v})\right) d\tau = 0. \tag{39}$$

Proof omitted; x, y, z are varied, and $v_x = \dot{x}$, etc. The generalized momenta $\frac{\partial L}{\partial v_x}, \frac{\partial L}{\partial v_y}, \frac{\partial L}{\partial v_x}$ agree with p_1, p_2, p_3, and $H = \sum_1^3 p\,\dot{q} - L$

equals $-icp_4$. Thus p is the <u>total</u> energy-momentum vector, and mu_μ is the <u>kinetic</u> energy-momentum vector.

On the idea that not all forces are necessarily due to an electromagnetic field, we might want to try some other expression for K_μ, such as, say, $-\frac{\partial V}{\partial x_\mu}$, where V is a world-scalar. But this cannot agree with (33) for arbitrary initial u_μ unless V = const. Then we would like K_1, K_2, K_3 to agree approximately with $-\frac{\partial V}{\partial x_\mu}$, at least for $v \ll c$ and $V \ll mc^2$. It turns out that the only simple way is to set

$$K_\mu = -\frac{\partial V}{\partial x_\mu} - \frac{1}{c^2}\frac{d}{d\tau}(V u_\mu) \tag{40}$$

Then (32) becomes

$$\frac{d}{d\tau}\left(m + \frac{V}{c^2}\right) u_\mu = - \frac{\partial V}{\partial x_\mu} \tag{41}$$

We shall not bother to give a variation principle. The electric and non-electric K_μ's could both be present at once, of course.

We have had to admit much physics — discussions of transformation properties of fields, the invariance of charge, etc., etc.— which would appear in a <u>course</u> <u>on</u> <u>relativity</u>.

PHYSICS 253

ADVANCED QUANTUM MECHANICS

INSTRUCTOR: FURRY | ROOM J 356 : MWF 12

LECTURE 1 : 9-25-61

Course Outline:
Dirac Electron
Second Quantitization
Electrodynamics

References: Dirac
Pauli in H. d. Phys.
Rose (relativistic electron theory, new)
on Field Theory:
Heitler : Theory of Radiation
Akhiezer & Berestetskii, 1st ed.
Jauch & Rohrlich
Schwinger : Q.E.D.

The Dirac Equation:

We shall start with the simplest case, that is, the free particle, as this is the usual starting place in quantum theory. We begin by defining the usual relativistic four-vectors:

$$x_\mu = (\vec{r}, ict)$$
$$p_\mu = (\vec{p}, i\tfrac{E}{c})$$

For the free particle, we have:

$$\frac{E}{c^2} - p^2 = m^2c^2$$

or $$p_\mu p_\mu = -m^2c^2 \qquad (1)$$

There is a sum implied over any repeated index.

sometimes one sees the notation compressed for the manner of writing the product of the momentum four vector to the form $p\bar{u}$ or p^2, or for the scalar product of distance and momentum, px is sometimes seen.

In going to the quantum mechanical formulation, we assume the form of the usual de Broglie relation is upheld:

$$p_\mu \rightarrow -i\hbar \frac{\partial}{\partial x_\mu} \qquad (2)$$

In NRQM (non-relativistic quantum mechanics), time is considered a parameter while all other variables are operators. Relativistically, however, this is intolerable. One often speaks of two types of RQM (relativistic quantum mechanics):

"small" RQM: all variables are operators

"large" RQM: none of the variables are operators; the operators are the field components which results in or from second quantitization.

What we will consider right now is the "small" theory. We shall treat time as an operator. Hence (2) means:

$$\vec{p} \rightarrow -i\hbar \nabla \; ; \; E \rightarrow i\hbar \frac{\partial}{\partial t}$$

Thus (1) becomes, upon operating on some wave function ψ:

$$\left.\begin{aligned} &\underbrace{\nabla^2 \psi - \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2}}_{\Box^2 \psi} - \frac{m^2 c^2}{\hbar^2}\psi = 0 \\ \text{or:} \quad &\frac{\partial^2 \psi}{\partial x_\mu \partial x_\mu} - \frac{m^2 c^2}{\hbar^2}\psi = 0 \end{aligned}\right] \qquad (3)$$

This is called the Gordon-Klein equation.

However, the GK (Gordon - Klein) equation presents difficulties in the interpretation of the probability density, $|\psi|^2$, recalling its interpretation in NRQM. Recall the NRQM Schroedinger equation:

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi$$

and the equation of continuity:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0$$

$$\rho = \psi \psi^* \; ; \quad \vec{j} = \frac{\hbar}{2mi} \left\{ \psi^* \nabla \psi - \psi \nabla \psi^* \right\}$$

We can perform the same operation with the GK equation, noting that the same GK equation holds for ψ^*, viz:

$$\frac{\partial^2 \psi^*}{\partial x_\mu \partial x_\mu} - \frac{m^2 c^2}{\hbar^2} \psi^* = 0 \qquad (3)^*$$

There is no need to change x_μ, because it is real for $\mu = 1, 2, 3$ and even though $x_4 = ict$, the product with itself erases the difference between $-i$ and $+i$.

We now write the relativistic continuity equation:

$$\frac{\partial S_\mu}{\partial x_\mu} = 0 \; ; \quad S_\mu = (\vec{j}, ic\rho) \qquad (4)$$

We now form from the GK equation:

$$\left.\begin{aligned} S_\mu &= \frac{\hbar}{2mi} \left(\psi^* \frac{\partial \psi}{\partial x_\mu} - \psi \frac{\partial \psi^*}{\partial x_\mu} \right) \\ \rho &= -\frac{\hbar}{2mc^2 i} \left\{ \psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t} \right\} \end{aligned}\right] \qquad (5)$$

Note that the NRQM ρ is a positive definite number. Not so with the RQM ρ. Although it may be real, it is not necessarily positive. Negative probability densities have little meaning for electrons, but do have a meaning for some particles like π mesons.

However, if we go to the "large" RQM, we are saved from the interpretation of negative probability by the fact that the number of particles is not constant. However, we are still in trouble for electrons because of spin which π mesons do not have.

Originally, the GK equation was considered no good at all because of the interpretation that was put on ρ. Hence, the first thing Dirac wanted in his formulation was the old interpretation of ρ. He considered ψ to have components and ρ to be the sum of these.

$$\rho = \sum_i \psi_i^* \psi_i$$

He also had strong reasons for wanting to preserve the first derivative or linear nature in t. Thus he was motivated to write:

$$\left[i\hbar \frac{\partial}{\partial t} + \alpha_x \frac{\hbar c}{i} \frac{\partial}{\partial x} + \alpha_y \frac{\hbar c}{i} \frac{\partial}{\partial y} + \alpha_z \frac{\hbar c}{i} \frac{\partial}{\partial z} + \beta mc \right] \psi = 0$$

being suggested by the form of the GK equation and the form of the relativistic Hamiltonian (Dirac, p. 255). Now recall $H\psi = i\hbar \frac{\partial \psi}{\partial t}$, and we see $H = -(c\vec{\alpha}\cdot\vec{p} + \beta mc^2)$. However, now-a-days we change the sign and write:

$$\left.\begin{aligned} H &= c\vec{\alpha}\cdot\vec{p} + \beta mc^2 \\ H &= \frac{\hbar c}{i} \vec{\alpha}\cdot\nabla + \beta mc^2 \end{aligned}\right] \quad (6)$$

We have changed β to include an extra c. Hence, the equation becomes:

$$\left[i\hbar \frac{\partial}{\partial t} - \frac{\hbar c}{i} \vec{\alpha}\cdot\nabla - \beta mc^2 \right] \psi = 0 \qquad (7)$$

We rewrite in a more convenient form:

$$\left[\frac{\partial}{c\,\partial t} + \vec{\alpha}\cdot\nabla + \frac{imc}{\hbar}\beta\right]\psi = 0 \qquad (B)$$

However, Dirac insisted that a connection with the GK equation be made. We see that we can make a start toward this by applying $\frac{\partial}{c\partial t} - \vec{\alpha}\cdot\nabla - \frac{imc}{\hbar}\beta$ onto the above equation.

If $\alpha_x^2 = \alpha_y^2 = \alpha_z^2 = 1$; $\beta^2 = 1$, we will get the squared terms that give the GK equation. However, what about the 20 cross-product terms? 8 will cancel naturally. To get more to go, we must have:

$$\left.\begin{aligned} \alpha_x\alpha_y + \alpha_y\alpha_x &= 0 \\ \alpha_y\alpha_z + \alpha_z\alpha_y &= 0 \\ \alpha_z\alpha_x + \alpha_x\alpha_z &= 0 \end{aligned}\right] \quad \text{2 terms apiece}$$

$$\alpha_k\beta + \beta\alpha_k = 0$$

These can all be expressed at once by:

$$\alpha_k\alpha_l + \alpha_l\alpha_k = 2\,\delta_{kl} \quad ; \quad \beta^2 = 1$$

These commutation relations define the Dirac commutators.

LECTURE 2 : 9-27-61

According to the Bohr and de Broglie relations, one must characterize relativistically de Broglie waves by the GK equation just as one characterizes them non-relativistically by the Schroedinger equation.

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} - \frac{m^2c^2}{\hbar^2}\psi = 0 \qquad (3)$$

However, Dirac points out that the ρ derived from (3) lacks the property of being positive definite as it should be.

Dirac takes the total wave function to be composed of components such that, taking a clue from Pauli spin theory,

$$\rho = \psi^*\psi = \sum_\sigma \psi_\sigma^* \psi_\sigma$$

hence: $\psi^* = (\psi_1^* \; \psi_2^* \cdots)$; $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \end{pmatrix}$

Now the Schroedinger equation yields positive definite ρ, $\underline{H}\psi = i\hbar \frac{\partial \psi}{\partial t}$. However, ψ must also satisfy the KG equation. This necessity helps determine $\underline{H}$ and in particular certain commutation relations. Dirac takes as the Hamiltonian:

$$\underline{H} = c\vec{\alpha}\cdot\frac{\hbar}{i}\nabla + \beta mc^2 = \frac{\hbar c}{i}\left(\alpha_x \frac{\partial}{\partial x} + \alpha_y \frac{\partial}{\partial y} + \alpha_z \frac{\partial}{\partial z}\right) + \beta mc^2 \quad (6)$$

We fit this Hamiltonian to the KG equation by forming:

$$\left(i\hbar\frac{\partial}{\partial t} + \underline{H}\right)\left(i\hbar\frac{\partial}{\partial t} - \underline{H}\right) = 0$$

$$\text{or} \quad \left(-\hbar^2\frac{\partial^2}{\partial t^2} - \underline{H}^2\right)\psi = 0$$

To make the identity with the KG equation:

$$\underline{H}\,\underline{H} = -\hbar^2c^2\nabla^2 + m^2c^4$$

The squared terms are all right, but the other terms must cancel, which gives rise to anticommutation rules for α's and β:

$$\left.\begin{aligned} \alpha_k\alpha_l + \alpha_l\alpha_k &= 2\delta_{kl} \\ \alpha_k\beta + \beta\alpha_k &= 0 \\ \beta^2 &= 1 \end{aligned}\right] \quad (9)$$

Matrices already existed when Dirac wrote these relations; they are the Pauli spin matrices.

We will underline all 2×2 matrices and not underline 4×4 matrices.

$$\underline{\sigma_1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; ; \; \underline{\sigma_2} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \; ; \; \underline{\sigma_3} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\underline{\sigma_k}\,\underline{\sigma_l} + \underline{\sigma_l}\,\underline{\sigma_k} = 2\,\delta_{kl}$$

How do we go from 2×2 to 4×4 matrices which are needed to handle the four operators α_k, β? For three of them Dirac just repeated $\underline{\sigma_1}, \underline{\sigma_2}, \underline{\sigma_3}$ twice:

$$\sigma_1 = \begin{pmatrix} \underline{\sigma_1} & 0 \\ 0 & \underline{\sigma_1} \end{pmatrix} \; ; \; \sigma_2 = \begin{pmatrix} \underline{\sigma_2} & 0 \\ 0 & \underline{\sigma_2} \end{pmatrix} \; ; \; \sigma_3 = \begin{pmatrix} \underline{\sigma_3} & 0 \\ 0 & \underline{\sigma_3} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

He also defined the ρ matrices, noting that these also satisfy the required commutation rules.

$$\rho_1 = \begin{pmatrix} \underline{0} & \underline{1} \\ \underline{1} & \underline{0} \end{pmatrix} \; ; \; \rho_2 = \begin{pmatrix} \underline{0} & -i\underline{1} \\ i\underline{1} & \underline{0} \end{pmatrix} \; ; \; \rho_3 = \begin{pmatrix} \underline{1} & \underline{0} \\ \underline{0} & -\underline{1} \end{pmatrix}$$

We note that all the σ's commute with all the ρ's. Hence, we have the following rules:

$$\left.\begin{aligned} \sigma_k\sigma_l + \sigma_l\sigma_k &= 2\,\delta_{kl} \\ \rho_k\rho_l + \rho_l\rho_k &= 2\,\delta_{kl} \\ \rho_k\sigma_l &= \sigma_l\rho_k \end{aligned}\right] \quad (11)$$

Now Dirac proceed to pick four matrices from the above six. He choose to combine:

$$\left.\begin{aligned} \alpha_k &= \rho_1\sigma_k \\ \beta &= \rho_3 \end{aligned}\right] \quad (12)$$

This choice is not unique since new ones can be generated by similitude transformations, which will then have the same commutation properties.

An example is:

$$\bar{\alpha}_k = T^{-1}\alpha_k T \quad ; \quad \bar{\beta} = T^{-1}\beta T \qquad (13)$$

We will soon see that α_k, β must be Hermitian hence this requires that T be unitary or $T^\dagger T = 1$.

There is a theorem by which one can generate all sorts of matrices that obey the above commutation rules from a given set that do such as σ_k, ρ_k. Consider the generation of matrices with more rows and columns than the original set by the rule:

$$A_k = \begin{pmatrix} T^\dagger \alpha_k T & \underset{\sim}{0} & \cdots \\ \underset{\sim}{0} & T^\dagger \alpha_k T & \vdots \\ \vdots & \ddots & \vdots \\ \vdots & \cdots & \end{pmatrix} \qquad (13)_A$$

The mark $\sim$ means 4 rows.

There is also a theorem by which, in this case, matrices of more than 8×8 can be reduced to diagonal form. This again can be done by a similitude transformation:

$$A_k' = J^\dagger A_k J$$

Here A_k' will have the same 4×4 matrix repeated along the diagonal. How is this proved? One way is to show by constructing a Clifford algebra. An easier way is by group theory. It is almost done in one of the papers collected and edited by Schwinger.

Only 4×4 matrices or multiples will appear in this course. Any greater than 4×4 can be reduced and all must be some multiple of 4×4.

Some notes on notation:

Wave functions: ψ: 1 column, 4 rows such that the following operation takes place:

$$\alpha_1 \psi = \begin{pmatrix} \psi_4 \\ \psi_3 \\ \psi_2 \\ \psi_1 \end{pmatrix}, \text{ according to matrix multiplication.}$$

$*$ complex conjugate
$+$ Hermitian adjoint of 4×4 matrix.
To avoid confusion with the wave function, we stipulate that ψ or ψ^* can either a one column or one row matrix depending on how they stand in relation to the operator matrix.

Requirement that α_k, β be Hermitian:

Since a physical theory requires that the operators be Hermitean, especially the Hamiltonian, we show that α_k, β are also Hermitean. Recall:

$$\frac{\partial \psi}{\partial t} + c\,\nabla\cdot\vec{\alpha}\,\psi + \frac{imc^2}{\hbar}\beta\psi = 0 \qquad (8)$$

Take the complex conjugate:

$$\frac{\partial \psi^*}{\partial t} + c\,\nabla\cdot\vec{\alpha}^*\psi^* - \frac{imc^2}{\hbar}\beta^*\psi^* = 0 \qquad (8)^*$$

We will transpose. Consider first: $\sum_\delta (\alpha_1)^*_{\gamma\delta}\,\psi^*_\delta = \sum_\delta \psi^*_\delta (a_1^+)_{\delta\gamma}$
However, we want $\vec{\alpha}, \beta$ to be Hermitean, thus:
$\vec{\alpha}^+ = \vec{\alpha}$, $\beta^+ = \beta$ and:

$$\frac{\partial \psi^*}{\partial t} + c\,\nabla\cdot\psi^*\vec{\alpha} - \frac{i\,mc^2}{\hbar}\psi^*\beta = 0 \qquad (8)^*$$

Form: $\psi^*(8) + (8)^*\psi$ and get:

$$\frac{\partial}{\partial t}\psi^*\psi + \nabla\cdot c\,\psi^*\vec{\alpha}\,\psi = 0 \qquad (14)$$

Since $\frac{\partial \rho}{\partial t} + \nabla\cdot\vec{j} = 0$:

$$\rho = \psi^*\psi \quad ; \quad \vec{j} = c\,\psi^*\vec{\alpha}\,\psi \qquad (15)$$

LECTURE 3: 9-29-61

Recapitulation:

Recall:

$$\frac{\partial\psi}{c\,\partial t} + \vec{\alpha}\cdot\nabla\psi + \frac{imc}{\hbar}\beta\psi = 0 \qquad (8)$$

$$\frac{\partial\psi^*}{c\,\partial t} + \nabla\cdot\psi^*\vec{\alpha} - \frac{imc}{\hbar}\psi^*\beta = 0 \qquad (8)^*$$

$(8)^*$ is really the adjoint of (8). Now form $\psi^*(8) + (8)^*\psi$, and from $\frac{\partial\rho}{\partial t} + \nabla\cdot\vec{j} = 0$, we get:

$$\rho = \psi^*\psi \quad ; \quad \frac{\vec{j}}{c} = \psi^*\vec{\alpha}\psi \qquad (15)$$

Now $\vec{\alpha}$ is not really a vector but merely a convenient notation for the three α matrices. However, $\vec{j}$ is a vector. This can be seen from the fact that ψ, ψ^* are column-row matrices or 4-spinors and the "sandwich" of these spinors with the component α matrices are scalars with each one as a coefficient of a unit cartesian vector of $\vec{j}$. We will eventually see that ρ, $\frac{\vec{j}}{c}$ form a four vector.

What about external fields? We make the usual replacements as we did in the Schroedinger equation but now in 4-vector notation.

$$\varphi_\mu = (\vec{A}, i\varphi)$$

$$p_\mu \rightarrow \pi_\mu = p_\mu - \frac{e}{c}\varphi_\mu$$

$$\left.\begin{aligned} &\vec{p} \rightarrow \vec{p} - \frac{e}{c}\vec{A} \;;\; \frac{\hbar}{i}\nabla \rightarrow \frac{\hbar}{i}\nabla - \frac{e}{c}\vec{A} \\ &i\hbar\frac{\partial}{\partial t} \rightarrow i\hbar\frac{\partial}{\partial t} - e\varphi \end{aligned}\right\} \quad (16)'$$

Now recall: $p_4 = \frac{\hbar}{i}\frac{\partial}{\partial x_4} \rightarrow -\hbar\frac{\partial}{c\,\partial t} - i\frac{e}{c}\varphi$

which gives finally:

$$\frac{\hbar}{i}\frac{\partial}{\partial x_\mu} \rightarrow \frac{\hbar}{i}\frac{\partial}{\partial x_\mu} - \frac{e}{c}\varphi_\mu \qquad (16)$$

We will now examine the effect of gauge transformations on the Dirac and GK equations. Recall the form of the gauge transformations:

$$\left.\begin{aligned} \vec{A} \to \vec{A}' &= \vec{A} + \nabla\lambda \\ \varphi \to \varphi' &= \varphi - \tfrac{1}{c}\,\partial\lambda/\partial t \end{aligned}\right\} \quad (18)$$

We recall that under gauge transformations, the electric and magnetic fields are left invariant:

$$\vec{E} = -\nabla\varphi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$
$$\vec{\mathcal{H}} = \nabla \times \vec{A}$$

Also recall the gauge transformation that left the Schroedinger equation invariant:

$$\psi \to \psi' = \psi e^{i\frac{e\lambda}{\hbar c}} \qquad (18)'$$

We now try these transformations on the Dirac equation. First write the Dirac equation in an EM field:

$$\frac{\partial\psi}{c\,\partial t} + \frac{ie}{\hbar c}\varphi\psi + \vec{\alpha}\cdot\left[\nabla - \frac{ie}{\hbar c}\vec{A}\right]\psi + \frac{imc}{\hbar}\beta\psi = 0$$

If we now substitute in the gauge transformations for the Schroedinger equation, we find that they leave the Dirac equation invariant also. Also, because the Dirac equation is linear in its derivatives, the potentials will <u>always</u> drop out when we gauge transform the continuity equation. This was <u>not</u> true in the case of the Schroedinger equation, where we get a term in A^2 because of the presence of ∇^2 in the wave equation. As a matter of fact, it is not necessary at all to perform the gauge transformation at all in order to see this.

If we employ the method of "factorization" to get the KG equation from the Dirac equation, we find that it will not work and we get mixed terms.

Hence the Dirac and GK equations part company when external forces are introduced. This fact essentially reflects the spin nature of the electron.

For non-electrical, not static electrical, fields we make the substitution of adding the new potential to the rest energy: $mc^2 \rightarrow mc^2 + V$

It will prove convenient from the point of Lorentz transformations and invariance to have matrix coefficients on all derivative terms and none on the mass term. The resulting equation was found by Pauli and it is called the γ form of Dirac's equation. We see that multiplication by β ($\beta^2 = 1$) will do the job but for future convenience form $-i\beta(8)$ instead. We define:

$$-i\beta\alpha_k = \gamma^k \ ; \ \beta = \gamma^4 \qquad (19)$$

The position of the index on γ is only conventional and has no other meaning. This operation allows us to write the first two terms in (8) in four-vector notation, thus giving for the γ equation:

$$\gamma^\mu \frac{\partial \psi}{\partial x^\mu} + \frac{mc}{\hbar}\psi = 0 \qquad (20)$$

or

$$\gamma^\mu \left[\frac{\partial}{\partial x^\mu} - \frac{ie}{\hbar c}\varphi_\mu\right]\psi + \frac{mc}{\hbar}\psi = 0 \qquad (20)'$$

We easily find from the commutation rules for $\vec{\alpha}, \beta$ that:

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\,\delta_{\mu\nu} \qquad (21)$$

These are just the relations needed to form the GK equation by operating with $\gamma^\mu \frac{\partial}{\partial x^\mu} - \frac{mc}{\hbar}$ on (20):

$$\left[\frac{\partial^2}{\partial x^\mu \partial x^\mu} - \frac{m^2c^2}{\hbar^2}\right]\psi = 0$$

However, we cannot do this with potentials present.

What about forming the complex conjugate of (20)'. We must consider γ^μ Hermitean, as can be seen from the Hermiticity of α^k, and β:

$$\gamma^{k\dagger} = i\alpha_k\beta = -i\beta\alpha_k \quad , \text{ since } \alpha_k\beta + \beta\alpha_k = 0 .$$

Then, taking the complex conjugate:

$$\frac{\partial \psi^*}{\partial x^*_\mu}\gamma^\mu + \frac{mc}{\hbar}\psi^* = 0$$

NB!! We must conjugate x_μ because $x_4 = ict$. How can we fix this? The trick is to multiply on the right by $-\gamma^4$ and then define as the partner in γ notation (due to Schwinger) of ψ:

$$\psi^*\gamma^4 = \bar{\psi} \qquad (22)$$

In the $\vec{\alpha}, \beta$ notation, the partner of ψ is ψ^*. Pauli used the notation $i\psi^*\gamma^4 = \psi^+$, called the Pauli adjoint. Doing the operation we get:

$$-\frac{\partial \psi^*}{\partial x^*_\mu}\gamma^\mu\gamma^4 - \frac{mc}{\hbar}\bar{\psi}$$

$$= \frac{\partial \psi^*}{\partial x^*_\mu}\left[\gamma^4\gamma^\mu - 2\delta_{\mu 4}\right] - \frac{mc}{\hbar}\bar{\psi}$$

$$= \frac{\partial \psi^*}{\partial x_1}\gamma^4\gamma^1 + \cdots + \underbrace{\frac{\partial \psi^*}{-ic\,\partial t}\Big[\underbrace{\gamma^4\gamma^4}_{1} - 2\Big]}_{\frac{\partial \psi^*}{ic\,\partial t}\gamma^4\gamma^4} - \frac{mc}{\hbar}\bar{\psi}$$

Therefore:

$$\frac{\partial \bar{\psi}}{\partial x_\mu}\gamma^\mu - \frac{mc}{\hbar}\bar{\psi} = 0$$

or for the case of an EM field:

$$\left(\frac{\partial}{\partial x_\mu} + \frac{ie}{\hbar c}\varphi_\mu\right)\bar{\psi}\gamma^\mu - \frac{mc}{\hbar}\bar{\psi} = 0 \qquad (23)$$

To derive the equation of continuity or conservation of current, form $\bar{\Psi}(20)' + (23)\psi$ and get:

$$\bar{\Psi}\left(\gamma^{\mu}\frac{\partial}{\partial x^{\mu}}\psi\right) - \bar{\Psi}\left(\gamma^{\mu}\frac{ie}{\hbar c}\varphi_{\mu}\psi\right) + \left(\frac{\partial}{\partial x^{\mu}}\bar{\Psi}\gamma^{\mu}\right)\psi + \left(\frac{ie}{\hbar c}\varphi_{\mu}\bar{\Psi}\gamma^{\mu}\right)\psi$$

$= 0.$ This leads to:

$$\frac{\partial}{\partial x^{\mu}}\bar{\Psi}\gamma^{\mu}\psi = 0$$

now from $\dfrac{\partial S_{\mu}}{\partial x_{\mu}} = 0$, $S_{\mu} = i\bar{\Psi}\gamma^{\mu}\psi$

we have:

$$S_4 = i\psi^{*}\psi = i\rho$$

$$S_k = i\psi^{*}\gamma^{4}(-i\gamma^{4}\alpha_k)\psi$$
$$= \psi^{*}\alpha_k\psi = s_k/c$$

Hence it is very simple to get continuity equation and show that no potentials arise in it when we formulate the problem in the γ notation. This is because:

$$\frac{\partial}{\partial x_{\mu}}(\bar{\Psi}\gamma^{\mu}\psi) = \bar{\Psi}\gamma^{\mu}\frac{\partial}{\partial x_{\mu}}\psi + \left(\frac{\partial}{\partial x_{\mu}}\bar{\Psi}\gamma^{\mu}\right)\psi$$

and; φ_{μ} is just a component of a four vector and a function of the coordinates hence it can be positioned at will among the ψ's and γ's since these are matrices and φ_{μ} commutes with each element of them.

We now bring up the subject of Lorentz transformations and Lorentz invariance. The definition of Lorentz invariance is that lengths of 4-vectors or distances are left unchanged when transformed to a new coordinate system. Otherwise expressed as:

$$x_{\mu}x_{\mu} = x'_{\mu}x'_{\mu} \quad (\text{sum implied})$$

In all following work, a repeated index is to be summed on.

We take as the matrix form of the Lorentz transform:

$$x_\mu = a_{\mu\nu} x'_\nu \qquad (24)$$

Since lengths are preserved, $a_{\mu\nu}$ describes an orthogonal transformation. A property of an orthogonal matrix is that the transpose equals the reciprocal, something like a unitary matrix when the elements are infinite in number. Form:

$$\left.\begin{array}{lll} a_{\mu\nu} a_{\mu\sigma} = \delta_{\nu\sigma} & : & a^T a = 1 \\ a_{\mu\nu} a_{\lambda\nu} = \delta_{\mu\lambda} & : & a\, a^T = 1 \end{array}\right] \quad (25)$$

Recall that the above is only true in infinite matrices for unitary matrices since one cannot invert an infinite matrix and hence find out something about whether or not the determinant exists as we do with finite matrices to determine if a given matrix possesses a reciprocal.

Another condition on the Lorentz transformation is that we must transpose real to real and imaginary to imaginary because of x_k, x_4. We find the following conditions:

$$\left.\begin{array}{l} a_{kl} \text{ real} \\ a_{k4}, a_{4k} \text{ imaginary} \\ a_{44} \text{ real} \end{array}\right] \quad (26)$$

We also say that they are proper transformations in that they do not cause time reversal nor an inversion of an axis (improper rotation). Lorentz transforms must preserve the sense of direction of time and cause only true space rotations (proper rotation). This requires:

$$a_{44} > 0 \;;\; \mathrm{Det}(a_{kl}) > 0 \qquad (27)$$

LECTURE 4: 10-2-61

Recall:

$$\gamma^\mu \left(\frac{\partial}{\partial x_\mu} - \frac{ie}{\hbar c}\varphi_\mu\right)\psi + \frac{mc}{\hbar}\psi = 0$$

$$\left(\frac{\partial}{\partial x_\mu} + \frac{ie}{\hbar c}\varphi_\mu\right)\bar{\psi}\gamma^\mu - \frac{mc}{\hbar}\bar{\psi} = 0$$

where $\bar{\psi} = \psi^*\gamma^4$

Make a Lorentz transformation:

$$x_\mu = a_{\mu\nu}\, x'_\nu \qquad (24)$$

This same law holds for any four-vector. A note on notation: If we take as the four-vector (x, y, z, ct) we must use the covariant - contravariant notation of raised and lowered subscripts because we must change sign on the time term to get proper form for invariant quantity. That is:

$$x^2 + y^2 + z^2 - c^2t^2 = \text{invariant}$$

or $x_\mu x^\mu = \text{invariant}$

with:

$$x^\mu : x, y, z, ct$$
$$x_\mu : x, y, z, -ct$$

Here we do not need to do this as we take for four vector (x, y, z, ict) so no distinguishing between covariant or contravariant vectors or raised or lowered indices is necessary. Continuing then, the operators in the γ equation transform as:

$$\frac{\partial}{\partial x_\mu} \pm \frac{ie}{\hbar c}\varphi_\mu = a_{\mu\nu}\left(\frac{\partial}{\partial x'_\nu} \pm \frac{ie}{\hbar c}\varphi'_\nu\right) \qquad (24)$$

On substitution:

$$\gamma^\mu a_{\mu\nu}\left(\frac{\partial}{\partial x'_\nu} - \frac{ie}{\hbar c}\varphi'_\nu\right)\psi + \frac{mc}{\hbar}\psi = 0 \qquad (28)$$

$$\left.\begin{array}{l} a_{\mu\nu}\left(\frac{\partial}{\partial x'_\nu} + \frac{ie}{\hbar c}\varphi'_\nu\right)\bar{\psi}\gamma^\mu - \frac{mc}{\hbar}\bar{\psi} = 0 \\ \text{or} \quad \left(\frac{\partial}{\partial x'_\nu} + \frac{ie}{\hbar c}\varphi'_\nu\right)\bar{\psi}\gamma^\mu a_{\mu\nu} - \frac{mc}{\hbar}\bar{\psi} = 0 \end{array}\right] \quad (29)$$

We now want to see if we can restore to original form and prove invariance.

We first discuss an <u>incorrect</u> way to do this. Define an new γ:

$$\Gamma^\nu = \gamma^\mu a_{\mu\nu} \qquad (30)$$

This would make equations (28) and (29) equivalent in form to (20)' and (23). See how these Γ's commute:

$$\Gamma^\mu\Gamma^\nu + \Gamma^\nu\Gamma^\mu = \underbrace{\left(\gamma^\sigma\gamma^\lambda + \gamma^\lambda\gamma^\sigma\right)}_{2\delta_{\lambda\sigma}} a_{\sigma\mu} a_{\lambda\nu}$$

$$= 2 a_{\lambda\mu} a_{\lambda\nu} = 2\delta_{\mu\nu} \qquad (31)$$

recalling: $a_{\lambda\mu} a_{\lambda\sigma} = \delta_{\mu\sigma}$
$a_{\lambda\mu} a_{\sigma\mu} = \delta_{\lambda\sigma}$

so we might say that the Γ's are new γ''s since their commutation properties are the same. That is, we could write $\gamma^{\nu\prime} = \gamma^\mu a_{\mu\nu}$ and $a_{\lambda\nu}\gamma^{\nu\prime} = \gamma^\lambda$.

However, this is completely <u>phony</u>. Recall that the original γ's were Hermitean. However, Γ is <u>not Hermitean</u> because all the a's are <u>not real</u>. Thus Γ does not satisfy the requirements for a good wave equation. That is:

$$\Gamma^{\nu\dagger} = a^*_{\nu\mu}\gamma^\mu \neq \Gamma^\nu \quad \text{since } a^*_{\nu\mu} \neq a_{\mu\nu}$$

from equation (26). Perhaps we could find a frame of reference in which Γ is Hermitean but this would create a preferred frame of reference which is contrary to the laws of relativity. Also, there is no reason to believe ψ is not transformed.

As a matter of fact, wave functions must be transformed or rotated in space because of spin. This would also lead to a priviledged reference system for ψ, contrary to relativity.

We do use the fact that Γ commutes like γ. Pauli's Theorem states that there is only one set of γ's or those related to it by a similitude transformation. Since Γ commutes like γ, it should be possible to relate them by a similitude transformation since a similitude transformation does not change the commutation rules of operators. Hence:

$$S^{-1} \Gamma^\mu S = \gamma^\mu \qquad (32)$$
$$S^{-1} S = 1$$

Because Γ is not Hermitean, S is not unitary hence the reason for S^{-1} and not S^+. When we find S, we can form:

$$\psi = S \psi' \qquad (33)$$

and get for (28):

$$\gamma^\mu \left(\frac{\partial}{\partial x_\mu'} - \frac{ie}{\hbar c} \varphi_\mu' \right) \psi' + \frac{mc}{\hbar} \psi' = 0$$

We get this by operating on (28) like: $S^{-1}(28) S$, or:

$$S^{-1} \gamma^\mu a_{\mu\nu} S S^{-1} \left(\frac{\partial}{\partial x_\nu'} - \frac{ie}{\hbar c} \varphi_\nu' \right) S S^{-1} \psi + \frac{mc}{\hbar} S^{-1} \psi = 0$$

which gives the above. Now take:

$$\bar{\psi} = B \bar{\psi}' S^{-1} \qquad (34)$$

where B is an ordinary number or a constant times the unit matrix. We get by operation on (29):

$$\left(\frac{\partial}{\partial x_\mu'} + \frac{ie}{\hbar c} \varphi_\mu' \right) \bar{\psi}' \gamma^\mu - \frac{mc}{\hbar} \bar{\psi}' = 0$$

We have not shown invariance yet. We do this by showing in the new form $\bar{\Psi} = \psi^* \gamma^4$ by the proper selection of B. Form:

$$\bar{\Psi}' = \underset{\text{one row}}{\psi'^* \gamma^4} = \underset{\text{one column}}{\gamma^{4T} \psi'^*} = \gamma^{4T} S^{-1*} \psi^*$$

Treat again as one row by transposing:

$$\bar{\Psi}' = \psi^* S^{-1\dagger} \gamma^4$$

On the other hand: $\bar{\Psi} = B \bar{\Psi}' S^{-1}$. Then:

$$\bar{\Psi} = B \psi^* S^{-1\dagger} \gamma^4 S^{-1} = \psi^* \gamma^4 \text{ (by definition)}$$

Thus we must have:

$$B S^{-1\dagger} \gamma^4 S^{-1} = \gamma^4$$

$$\text{or } B = S^\dagger \gamma^4 S \gamma^4$$

which is or must be either a number times a unit matrix or a numerical factor. We could prove this by refering to Shur's Lemma:

Shur's Lemma: Any matrix that commutes with all irreducible representations of a group is just a multiple of the unit matrix. Hence we can show this by commuting B with all γ's.

We form the irreducible representations of the group of γ by choosing suitable products of the γ^μ's.

$$1$$
$$\gamma^1, \gamma^2, \gamma^3, \gamma^4$$
$$\gamma^1\gamma^2, \gamma^2\gamma^3, \gamma^3\gamma^1, \gamma^2\gamma^4, \gamma^3\gamma^4$$
$$\gamma^1\gamma^2\gamma^3, \gamma^1\gamma^2\gamma^4, \gamma^2\gamma^3\gamma^4, \gamma^3\gamma^1\gamma^4$$
$$\gamma^1\gamma^2\gamma^3\gamma^4$$

<u>and</u> their negatives hence we have 32 elements in the group. We shall see that B commutes with all elements, and hence is a constant.

We assert that the trace of each element in the group, except 1, is zero.

Proof: $Tr(AB) = \sum_{\alpha\beta} A_{\alpha\beta} B_{\beta\alpha} = \sum_{\alpha\beta} B_{\beta\alpha} A_{\alpha\beta} = Tr(BA)$

Note, however, that this does not mean
$Tr(ABC) = Tr(BAC)$

The trace of the group vanishes whatever the representation. Take $Tr(S^{-1}\gamma^4 S) = 0$ whatever S is.
Also;

$$Tr(\gamma^\mu \gamma^\nu) = Tr(\gamma^\nu \gamma^\mu) = 0$$

since: $\gamma^\mu \gamma^\nu = -\gamma^\nu \gamma^\mu$.

Another way: the positive elements of the group are usually called γ^A, the negative elements γ^B, $A, B = 1, \ldots, 16$. Recall the definition of linear independence:

$$\sum c_A \gamma^A = 0 \quad , \text{ hence } c_A = 0$$

must be true. Take the trace and get $4c_0$. Multiply the group by γ^B. This scrambles the elements in the group, getting the same ones back however, but now:

$$\sum c_A' \gamma^B = 0$$

and taking trace gives $c_0' = \pm c_B$
We can show that B commutes with any linear combination that can be formed from the group. A matrix that commutes with any matrix must be a multiple of the unit matrix.

LECTURE 5: 10-4-61

Recapitulation:

Recall the Lorentz transformation and its properties:

$$x'_\mu = a_{\mu\nu} x_\nu$$

$$\left.\begin{aligned} a_{\mu\nu} a_{\mu\lambda} &= \delta_{\nu\lambda} \\ a_{\mu\sigma} a_{\nu\sigma} &= \delta_{\mu\nu} \end{aligned}\right\} \quad (25)$$

a_{kl}, a_{44} are real
a_{4k}, a_{k4} are imaginary

Under this transformation, the Dirac equation is invariant if we put:

$$\psi = S\psi' \qquad (33)$$

$$S^{-1}\gamma^\nu a_{\nu\mu} S = \gamma^\mu \qquad (32)$$
$$\gamma'^\mu = \gamma^\nu a_{\nu\mu}$$

Now we would like to have $\bar{\psi} = \bar{\psi}' S^{-1}$ because then we could operate on the Dirac γ equation in the $\bar{\psi}$ form and, applying S^{-1} from the right, get the invariant form. However, if we have to include a multiplicative constant this will be all right. Also, and more important, we want the definition of $\bar{\psi} = \psi^* \gamma^4$ to hold in both systems, i.e., $\bar{\psi}' = \psi'^* \gamma^4$.
Using these definitions:

$$\bar{\psi} = \psi^* \gamma^4 \quad ; \quad \psi = S\psi' \quad ; \quad \psi^* = \psi'^* S^+ = \bar{\psi}' \gamma^4 S^+$$

or $$\bar{\psi} = \bar{\psi}' \gamma^4 S^+ \gamma^4$$

where we use the fact that $\bar{\psi} = \psi^* \gamma^4$ in both systems. We see that S^{-1} must be equivalent to $\gamma^4 S^+ \gamma^4$. We said before that we would settle for a multiplicative factor in the transformation, that is:

$$\bar{\psi} = \bar{\psi}' S^{-1} B \qquad (34)$$

Note that we retain the matrix character of B.

We can then identify:

$$B = S\gamma^4 S^\dagger \gamma^4 \qquad (35)$$

This time we are not assuming B a scalar matrix. Now we can form the group of the γ's as before which we can use to expand any matrix. Hence if B commutes with all the elements of γ it commutes with any 4×4 matrix and hence must be scalar. We make the following remarks:

$$S^{-1}\gamma^\nu a_{\nu\mu} S = \gamma^\mu$$

Remember $a_{\nu\mu}$ is a matrix element and hence acts as a number. Operate with S:

$$S\gamma^\mu = \gamma^\nu a_{\nu\mu} S$$

$$a_{\lambda\mu}\Big\} \quad \gamma^\lambda S = a_{\lambda\mu} S\gamma^\mu$$

Take Hermitean adjoint:

$$\gamma^\mu S^\dagger = S^\dagger \gamma^\nu a^*_{\nu\mu}$$

$$S^\dagger\gamma^\lambda = a^*_{\lambda\mu}\gamma^\mu S^\dagger$$

Use this last relation with $\lambda = 4$ to form:

$$B = a^*_{4\mu} S\gamma^4\gamma^\mu S^\dagger = a_{4\mu} S\gamma^\mu\gamma^4 S^\dagger$$

OK for $\mu = 4$, since a_{44} is real and $\gamma^4\gamma^4 = 1$. For $\mu = k$, a_{4k} is imaginary, $\gamma^k\gamma^4 = -\gamma^4\gamma^k$ so OK again.

Now write $\gamma^4 B$:

$$\gamma^4 B = a_{4\mu}\gamma^4 S\gamma^\mu\gamma^4 S^\dagger = a_{4\mu} a_{4\nu} S\gamma^\nu\gamma^\mu\gamma^4 S^\dagger$$

Now, since μ, ν are dummies, we change them into each other with no contradiction. Hence:

$$\gamma^4 B = a_{4\mu} a_{4\nu} S\gamma^\mu\gamma^\nu\gamma^4 S^\dagger$$

now take half of each and add:

$$\gamma^4 B = \tfrac{1}{2} a_{4\mu} a_{4\nu} S \underbrace{(\gamma^\nu\gamma^\mu + \gamma^\mu\gamma^\nu)}_{2\delta_{\mu\nu}} \gamma^4 S^\dagger$$

$$= \delta_{\mu\nu} a_{4\nu} a_{4\mu} \; S\gamma^4 S^\dagger = a_{4\mu} a_{4\mu} \; S\gamma^4 S^\dagger$$

$$= S\gamma^4 S^\dagger = B\gamma^4$$

Thus B and γ^4 commute. How does B commute with γ^k?
Use:

$$B = a_{4\mu} S\gamma^\mu\gamma^4 S^\dagger$$

Operate from right with: $\gamma^k = S^{-1\dagger}\gamma^\nu a^*_{k\nu} S^\dagger$
Get:

$$B\gamma^k = a_{4\mu} a^*_{k\nu} S\gamma^\mu\gamma^4\gamma^\nu S^\dagger$$

To interchange $\gamma^4\gamma^\nu$ and remove cc on $a^*_{k\nu}$ will cost a change of sign. Can be seen with same argument given above.

$$B\gamma^k = - a_{4\mu} a_{k\nu} S\gamma^\mu\gamma^\nu\gamma^4 S^\dagger$$

Now operate on B from left with $\gamma^k = a_{k\nu} S\gamma^\nu S^{-1}$.

$$\gamma^k B = a_{4\mu} a_{k\nu} S\gamma^\nu\gamma^\mu\gamma^4 S^\dagger$$

and: $$\gamma^k B - B\gamma^k = a_{4\mu} a_{k\nu} S \underbrace{\{\gamma^\nu\gamma^\mu + \gamma^\mu\gamma^\nu\}}_{2\delta_{\nu\mu}} \gamma^4 S^\dagger$$

$$= 2 a_{4\mu} a_{k\mu} \; S\gamma^4 S^\dagger = 0 \quad \text{since } \delta_{4k} = 0$$

Thus B commutes with all γ's and thus with all elements of the γ group and is hence a scalar matrix. It is also Hermitean:

$$B^\dagger = \gamma^4 S\gamma^4 S^\dagger = a_{4\mu} a^*_{\nu 4} S\gamma^\mu S^\dagger \gamma^\nu$$

$$\gamma^4 B^\dagger = S\gamma^4 S^\dagger = B^\dagger\gamma^4 \;; \quad B^\dagger = S\gamma^4 S^\dagger \gamma^4 = B$$

Thus B is real:

$$\begin{pmatrix} B & & & 0 \\ & B & & \\ & & B & \\ 0 & & & B \end{pmatrix} = B\,\mathbb{1}$$

We can fix the magnitude of B by choosing correctly the arbitrary constants involved in the similitude transformation S. We choose this so $B = 1$, however, since $SS^{-1} = \mathbb{1}$, there is still some arbitrariness about the sign, so we really must choose $B = \pm 1$.
Then:

$$\bar{\psi} = \bar{\psi}' S^{-1}$$
$$\text{or } \bar{\psi} = -\bar{\psi}' S^{-1}$$

However, we can show that the +1 must hold for a proper Lorentz transformation. We can see this by taking an infinitesimal proper rotation and translation. To the first degree, $\bar{\psi}$ hardly changes let alone change sign so + sign must hold.

We have now established the Lorentz transformation properties of the Dirac equation.

However, we have not found S and we proceed to do so now. Now the Lorentz transforms form a group in the usual sense in that any one operation can be generated by "products" of the group elements. These can be applied all at once or in sequence as it is the final state that is of importance. Now recall:

$$\text{Det } a_{\mu\nu} > 0 \ : \ \text{proper rotation}$$
$$\text{Det } a_{\mu\nu} < 0 \ : \ \text{space inversion}$$

However, we can generate any space inversion from the proper rotations by:

$$\begin{pmatrix} -1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & 1 \end{pmatrix} \underbrace{|a_{\mu\nu}|}_{\text{proper}} \quad \text{gives space inversion operation.}$$

On the other hand, for time reversal:

$$\begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & -1 \end{pmatrix} |a_{44}|$$ gives time reversal from a proper time transformation.

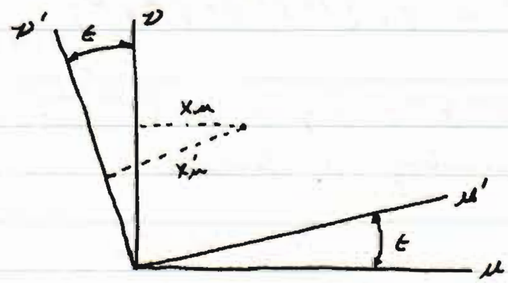
All this is because we can regard the result of successive transformations as being lumped into one taking place at one time.

$x_\mu = a_{\mu\nu} x'_\nu$; $x'_\nu = b_{\nu\sigma} x''_\sigma$; $x_\mu = a_{\mu\nu} b_{\nu\sigma} x''_\sigma = c_{\mu\sigma} x''_\sigma$

and $\psi' = \Lambda \psi''$; $\psi = S\Lambda \psi''$

This establishes the group properties of the Lorentz Transformation. We will consider proper transformations with space and time inversion.

Consider an infinitesimal rotation in the μ-ν plane, μ, ν being no longer dummy indices



If ϵ is infinitesimal, we can work to first order in it and we can say x_μ is almost x'_μ.

Then: $$x_\mu = x'_\mu - \epsilon x'_\nu$$
$$x_\nu = x'_\nu + \epsilon x'_\mu$$

Now for an infinitesimal rotation, S is almost a unit matrix and we want to preserve the property $SS^{-1} = 1$ to first order, hence we take to first order:

$$S = 1 + \epsilon T$$
$$S^{-1} = 1 - \epsilon T$$

Now: $S^{-1} a_{\lambda\sigma} \gamma^\sigma S = \gamma^\lambda$

$\nu \neq \lambda \neq \mu$: $S^{-1} \gamma^\lambda S = \gamma^\lambda$ to first order.

LECTURE 6: 10-6-61

Recapitulation of the proof that the group of γ is linearly independent. This proof depends on $Tr(\gamma^A) = 0$, $\gamma^A \neq 1, -1$; A going from 1 to 16.

Double γ:

$\mu \neq \nu$: $Tr(\gamma^\mu \gamma^\nu) = Tr(\gamma^\nu \gamma^\mu) = -Tr(\gamma^\mu \gamma^\nu) = 0$
because of anti-commutation.

Single γ:

$\mu \neq \nu$: $Tr(\gamma^\mu) = Tr(\underbrace{\gamma^\nu \gamma^\nu}_{1} \gamma^\mu)$

ν not summed

$= Tr(\gamma^\nu \gamma^\mu \gamma^\nu) = -Tr(\gamma^\mu) = 0$

Triple γ:

μ, ν, λ all different and not summed. Introduce σ, not summed : $Tr(\gamma^\lambda \gamma^\mu \gamma^\nu) = Tr(\gamma^\sigma \gamma^\sigma \gamma^\lambda \gamma^\mu \gamma^\nu)$

$= Tr(\gamma^\sigma \gamma^\lambda \gamma^\mu \gamma^\nu \gamma^\sigma) = -Tr(\gamma^\lambda \gamma^\mu \gamma^\nu) = 0$
by anti-commuting 3 times.

Four γ: $Tr(\gamma^1 \gamma^2 \gamma^3 \gamma^4) = Tr(\gamma^2 \gamma^3 \gamma^4 \gamma^1) = -Tr(\gamma^1 \gamma^2 \gamma^3 \gamma^4) = 0$
by anti-commuting 3 times.

Hence we have shown all traces to be zero and all the elements of the γ group to be linearly independent.

Now $\gamma^1 \gamma^2 \gamma^3 \gamma^4 = \gamma^5$ is often written which is a quantity which anti-commutes with all the γ's. Then:

$$\gamma^5 \gamma^\mu + \gamma^\mu \gamma^5 = 0$$

Furthermore:

$$\gamma^5 \gamma^5 = \gamma^1 \gamma^2 \gamma^3 \gamma^4 \gamma^1 \gamma^2 \gamma^3 \gamma^4 = -\gamma^2 \gamma^3 \gamma^4 \gamma^2 \gamma^3 \gamma^4$$

$$= -\gamma^3 \gamma^4 \gamma^3 \gamma^4 = 1$$

We could use Greek letters to index 1-5 and write:

$$\{\gamma^\Delta, \gamma^\Lambda\} = 2\delta_{\Delta\Lambda}$$

Also, γ^5 is Hermitean:

$$\gamma^{5+} = \gamma^4\gamma^3\gamma^2\gamma^1 = \gamma^1\gamma^2\gamma^3\gamma^4 = \gamma^5$$

Thus we could write the group as:

1	:	1
γ^μ	:	4
$\gamma^\mu\gamma^\nu$	:	6
$\gamma^5\gamma^\mu$	:	4
γ^5	;	1
		16

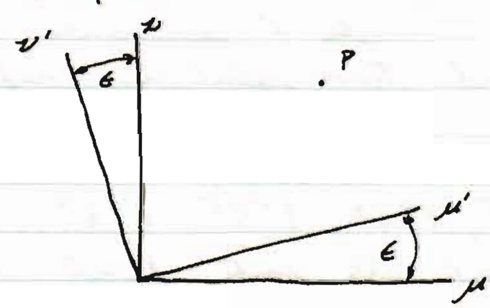
Rotations in the μ,ν plane:

Let us say that $\mu, \nu, \lambda, \sigma$ is some permutation of 1234, that is:

$$\mu, \nu, \lambda, \sigma = P(1234)$$

These are not dummy indices in as for example:

$X_\alpha = a_{\alpha\beta} X'_\beta$



$$X_\mu = X'_\mu - \epsilon X'_\nu$$
$$X_\nu = X'_\nu + \epsilon X'_\mu$$
$$X_\lambda = X'_\lambda$$
$$X_\sigma = X'_\sigma$$

Working to first order in ϵ, we take S to be:

$$S = 1 + \epsilon T$$
$$S^{-1} = 1 - \epsilon T$$

We want: $\quad S^{-1}\gamma^\beta \underbrace{a_{\beta\alpha}}_{\text{transposed } a_{\alpha\beta}} S = \gamma^\alpha$ $\quad \therefore \; X'_\mu = X_\mu + \epsilon X_\nu$

or, taking $\alpha \to \mu$; and operating on a first order change in γ:

$$(1-\epsilon T)(\gamma^\mu + \epsilon\gamma^\nu)(1+\epsilon T) = \gamma^\mu$$
$$(1-\epsilon T)(\gamma^\nu - \epsilon\gamma^\mu)(1+\epsilon T) = \gamma^\nu$$
$$(1-\epsilon T)\gamma^\lambda(1+\epsilon T) = \gamma^\lambda$$
$$(1-\epsilon T)\gamma^\sigma(1+\epsilon T) = \gamma^\sigma$$

Work out to first order in ϵ:

ϵ^1 : $\quad -T\gamma^\mu + \gamma^\mu T + \gamma^\nu = 0$

or: $\quad [T, \gamma^\mu] = \gamma^\nu$
$[T, \gamma^\nu] = -\gamma^\mu$
$[T, \gamma^\lambda] = 0$
$[T, \gamma^\sigma] = 0$

We could guess T but we will work it out anyway. The γ's are 4×4 matrices so T must be too. We can hence expand T in the complete set of γ's:

$$T = \sum c_A \gamma^A = 1 + \cdots$$

There is no reason why one term should not be the unit matrix as it commutes and will not disrupt anything.

We cannot use the single γ elements as they will not satisfy the last two relations (refer to anti-commutating rules).

Double γ: $\quad \gamma^\mu\gamma^\nu,\ \gamma^\mu\gamma^\lambda,\ \gamma^\mu\gamma^\sigma,\ \gamma^\nu\gamma^\lambda,\ \gamma^\nu\gamma^\sigma,\ \gamma^\lambda\gamma^\sigma$

We can rule out all but first: $\gamma^\mu\gamma^\nu$

Because:

$$[\gamma^\mu\gamma^\nu, \gamma^\lambda] = \gamma^\mu \underbrace{[\gamma^\nu, \gamma^\lambda]}_{\substack{\gamma^\nu\gamma^\lambda - \gamma^\lambda\gamma^\nu \\ = 2\gamma^\nu\gamma^\lambda}} + \underbrace{[\gamma^\mu, \gamma^\lambda]}_{\substack{\gamma^\mu\gamma^\lambda - \gamma^\lambda\gamma^\mu \\ = 2\gamma^\mu\gamma^\lambda}}\gamma^\nu$$

$$= 2\gamma^\mu\gamma^\nu\gamma^\lambda + 2\gamma^\mu\gamma^\lambda\gamma^\nu = 0$$

Triple γ: $\quad \gamma^\mu\gamma^\nu\gamma^\lambda,\ \gamma^\mu\gamma^\nu\gamma^\sigma,\ \gamma^\mu\gamma^\lambda\gamma^\sigma,\ \gamma^\nu\gamma^\lambda\gamma^\sigma$

Only $\gamma^\mu\gamma^\lambda\gamma^\sigma$, $\gamma^\nu\gamma^\lambda\gamma^\sigma$ satisfy the last two commutation relations above.

γ^5 : Cannot use, does not satisfy two lower relations.

So far, we have only tried to satisfy the two lower relations. What about the two upper?

$$\underbrace{\gamma^\mu\gamma^\lambda\gamma^\sigma\cdot\gamma^\mu}_{\gamma^\lambda\gamma^\sigma} - \underbrace{\gamma^\mu\cdot\gamma^\mu\gamma^\lambda\gamma^\sigma}_{\gamma^\lambda\gamma^\sigma} = 0$$

$$\gamma^\mu\gamma^\lambda\gamma^\sigma\cdot\gamma^\nu - \gamma^\nu\cdot\gamma^\mu\gamma^\lambda\gamma^\sigma = 2\gamma^\nu\gamma^\mu\gamma^\lambda\gamma^\sigma = \pm 2\gamma^5, \text{ etc.}$$

Hence we cannot use the two that were left from the Triple γ set. Therefore:

$$T = a\,1 + b\gamma^\mu\gamma^\nu$$

We cannot find a since the unit matrix commutes with with all γ. Can find b, however, from:

$$[T, \gamma^\mu] = \gamma^\nu$$

$$b\,[\gamma^\mu\gamma^\nu, \gamma^\mu] = \gamma^\nu = b\Big[\underbrace{\gamma^\mu\gamma^\nu\gamma^\mu}_{-\gamma^\nu} - \underbrace{\gamma^\mu\gamma^\mu\gamma^\nu}_{-\gamma^\nu}\Big] = -2b\gamma^\nu$$

Hence $b = -\frac{1}{2}$ and:

$$T = a\,1 - \tfrac{1}{2}\gamma^\mu\gamma^\nu = a\,1 + \tfrac{1}{2}\gamma^\nu\gamma^\mu$$

We see that we could very well put $a=0$ since 1 commutes with all γ and everything else so it will never change anything, and we can still satisfy the first order commutation relations above. However, we can follow Pauli in requiring that S be unimodular, that is, $\det(S) = 1$, as is usually the case for a similitude transformation. This will make $a = 0$. To show this, it is necessary to introduce a representation for S. Now note that $i\gamma^\nu\gamma^\mu$ is Hermitean, so it is possible to find a representation that diagonalizes it and we can take the product of the diagonal elements for det(S). We know the trace of $\gamma^\nu\gamma^\mu = 0$. Now form:

$$i\gamma^\nu\gamma^\mu \cdot i\gamma^\nu\gamma^\mu = 1$$

Then in *any* representation with $\gamma^\nu\gamma^\mu$ diagonal:

$$\begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \lambda_3 & \\ 0 & & & \lambda_4 \end{pmatrix}^2 = 1 \quad , \quad \text{since} \quad A^{-1} 1 A = 1$$

or λ's must be all the roots of $\sqrt{1}$ or ± 1.
From the relation:

$$S = \left[1 a + \tfrac{1}{2}\gamma^\nu\gamma^\mu \right]\epsilon + 1$$

we can construct in a representation in which $\gamma^\nu\gamma^\mu$ is diagonal:

$$S = \begin{pmatrix} (1+\epsilon a) + \frac{i\epsilon}{2} & & & 0 \\ & (1+\epsilon a) + \frac{i\epsilon}{2} & & \\ & & (1+\epsilon a) - \frac{i\epsilon}{2} & \\ 0 & & & (1+\epsilon a) - \frac{i\epsilon}{2} \end{pmatrix}$$

Now the determinant is the diagonal product and is to first order in ϵ: $\det(S) = 1 + 4\epsilon a$. Thus, for S to be unimodular, we must choose $a = 0$. The best intuitive reason is that it simplifies the T. Hence:

$$S = 1 + \frac{\epsilon}{2}\gamma^\nu\gamma^\mu \quad ; \quad \measuredangle\, \epsilon << 1$$

$$S^{-1} = 1 - \frac{\epsilon}{2}\gamma^\nu\gamma^\mu \quad ; \quad \measuredangle\, \epsilon << 1$$

Suppose we now want to find a finite rotation through an angle θ. Then we apply the infintesimal rotation $\frac{\theta}{\epsilon}$ times while letting $\epsilon \to 0$.

$$S = \lim_{\epsilon \to 0} \left(1 + \frac{\epsilon}{2}\gamma^\nu\gamma^\mu\right)^{\frac{\theta}{\epsilon}}$$

Use the definition $\quad e = \lim_{x \to 0} (1+x)^{\frac{1}{x}}$

Then $S = e^{+\frac{\theta}{2}\gamma^\nu\gamma^\mu}$ or an exponential series of matrices.

Expand the series:

$$S = 1 + \frac{\theta}{2}\gamma^\nu\gamma^\mu + \frac{1}{2}\left(\frac{\theta}{2}\gamma^\nu\gamma^\mu\right)^2 + \frac{1}{6}\left(\frac{\theta}{2}\gamma^\nu\gamma^\mu\right)^3 + \cdots$$

now, $\gamma^\nu\gamma^\mu \cdot \gamma^\nu\gamma^\mu = -1$, hence we can write,

$$S = \left\{1 - \frac{1}{2}\left(\frac{\theta}{2}\right)^2 + \frac{1}{4!}\left(\frac{\theta}{2}\right)^4 - \cdots\right\}$$
$$+ \gamma^\nu\gamma^\mu\left\{\frac{\theta}{2} - \frac{1}{3!}\left(\frac{\theta}{2}\right)^3 + \cdots\right\}$$

or:

$$S = \cos\frac{\theta}{2} + \gamma^\nu\gamma^\mu \sin\frac{\theta}{2}$$

$$S^{-1} = \cos\frac{\theta}{2} - \gamma^\nu\gamma^\mu \sin\frac{\theta}{2}$$

now sandwich a γ^μ in between S, S^{-1} and get:

$$\underbrace{e^{-\frac{\theta}{2}\gamma^\nu\gamma^\mu}\gamma^\mu}_{\substack{\text{reversing changes}\\ \text{sign in exponent}}} e^{\frac{\theta}{2}\gamma^\nu\gamma^\mu} = \gamma^\mu e^{\theta\gamma^\nu\gamma^\mu}$$

$$= \gamma^\mu\left(\cos\theta + \gamma^\nu\gamma^\mu \sin\theta\right) = \gamma^\mu\cos\theta - \gamma^\nu\sin\theta$$

which is the usual form of the finite rotation.

LECTURE 7: 10-9-61

Recapitulation on the determination of B:
Recall:

$$B = \gamma^4 S^{+} \gamma^4 S = \pm 1$$

If B is unity, certainly its determinant is also, regardless of whether $B = \pm 1$.

$$1 = 1 (\mathrm{Det}\, S)^{*}\, 1\, (\mathrm{Det}\, S)$$

Now to the first order,

$$\mathrm{Det}\,(S) = 1 + 4\epsilon a$$

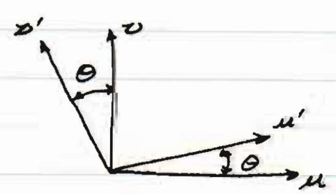
and $$|\mathrm{Det}\,(S)|^2 = (1+4\epsilon a)(1+4\epsilon a)^{*} = 1 + 4\epsilon\,(a + a^{*})$$

Thus the unit condition on B does __not__ require that a be zero, because a being pure imaginary would also be satisfactory. We choose $a = 0$ on the requirement of unimodularity of S.

Recall for a finite rotation:

$$S = e^{\frac{\theta}{2}\gamma^\ell \gamma^k} \quad \text{or more generally:}$$

$$S = e^{\frac{\theta}{2}\gamma^\mu \gamma^\nu}$$



Now this is just what one would expect in form from the rotation expressed in Pauli spin matrices:

$$S = e^{i\frac{\theta}{2}\sigma_m}$$

Now suppose a rotation in the k-4 plane of the Lorentz variety. However, our space is not Euclidean as can be seen from the fact that angles in our 4-space are imaginary (see relativity notes).

$$\theta = i \tanh^{-1}\frac{v}{c}$$

$$x_k = \cos\theta \; x_k' - \sin\theta \; x_4'$$
$$x_4 = \sin\theta \; x_k' + \cos\theta \; x_4'$$

Then: $$\cos\theta = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

and we see that θ is imaginary. Notice now that in a kl rotation S is unitary:

$$S = e^{\frac{\theta}{2}\gamma^l\gamma^k}$$
$$S^+ = e^{\frac{\theta}{2}\gamma^k\gamma^l} = S^{-1}$$

In case of the full Lorentz transformation including the x^4 coordinate:

$$S = e^{\frac{\theta}{2}\gamma^4\gamma^k}$$
$$S^+ = e^{-\frac{\theta}{2}\gamma^k\gamma^4} = e^{\frac{\theta}{2}\gamma^4\gamma^k} = S$$

thus here S is not unitary but Hermitean. This should not be unitary since $\psi^*\psi$ should not be constant under a Lorentz transformation. We can see the same thing from a consideration of the definition of B. If $B = +1$:

$$\gamma^4 S^+ \gamma^4 = S^{-1}$$

Take a proper Lorentz rotation: $S^+ = e^{\frac{\theta}{2}\gamma^4\gamma^k}$

$$= \cos\frac{\theta}{2} + \gamma^4\gamma^k \sin\frac{\theta}{2}$$

$$\gamma^4 S^+ \gamma^4 = \cos\frac{\theta}{2} + \gamma^k\gamma^4 \sin\frac{\theta}{2} = \cos\frac{\theta}{2} - \gamma^4\gamma^k \sin\frac{\theta}{2}$$
$$= e^{-\frac{\theta}{2}\gamma^4\gamma^k} = S^{-1}$$

For a kl rotation, $S^+ = e^{\frac{\theta}{2}\gamma^k\gamma^l} = S^{-1}$, and $\gamma^4 S^+ \gamma^4 = S^{-1}$.
Therefore, for all proper Lorentz transformations, B is $+1$.

What about improper rotations?

Consider the space inversion:

$$x_k = -x_k'$$
$$x_4 = x_4'$$

This could be two observers, one in a left-handed coordinate system and the other in a right-handed system. Then what we want for transforming the γ's are:

$$S\gamma^k S^{-1} = -\gamma^k$$
$$S\gamma^4 S^{-1} = \gamma^4$$

taken from:

$$S^{-1}\gamma^\nu a_{\nu\mu} S = \gamma^\mu$$

or $S\gamma^\mu S^{-1} = \gamma^\nu a_{\nu\mu}$

where $a_{\nu\mu}$ has been chosen to give the desired transformation.

The obvious choice for S, considering the anti-commutation rules is:

$$S = f\gamma^4$$

where f is some arbitrary constant. Now we want S unimodular, that is, $\det S = 1$. Since $\det \gamma^4 = 1$ we must have:

$$f^4 = 1 \ ; \quad f = 1, i, -1, -i$$

At this point Pauli chose $S = \gamma^4$. However, Racah chose $S = i\gamma^4$. We discuss reasons some other time. Carrying thru the same process as before, we again see that $B = +1$, Hence B is $+1$ for all Lorentz transformations and space inversions. This is not so for time reversal.

At this time, we will only consider that kind of time reversal called Geometric Time Reflection:

$$x_k = x_k'$$
$$x_4 = -x_4'$$

This cannot be realized in reality, however, it is possible to formulate it mathematically.

We then want:

$$S\gamma^k S^{-1} = \gamma^k$$
$$S\gamma^4 S^{-1} = -\gamma^4$$

Choose: $S = \gamma^1\gamma^2\gamma^3$ by consideration of the commutation rules.

Now:

$$B = S\gamma^4 S^+ \gamma^4 = \gamma^1\gamma^2\gamma^3\gamma^4\gamma^3\gamma^2\gamma^1\gamma^4 = -1$$

and then:

$$\bar{\psi} = -\bar{\psi}' S^{-1}$$

This is all on time reversal for now.

Groups:

There are four requirements for the formation of a group.

1. The application of consecutive operators should give another single operation.
2. The operations must associate.
3. There must exist an identity operation.
4. The reciprocal of each operation should exist.

Now the Lorentz Transformations satisfy all of these requirements and hence they form a group. Look at successive operations:

$$\psi = S_1\psi', \quad \psi' = S_2\psi'', \quad \psi = S\psi'' = S_1S_2\psi''$$

and $S = S_1S_2$ so successive operations act like multiplication, and give another operation of the group. They also satisfy the requirements 2 thru 4. Thus the S's form an explicit representation of the Lorentz group, or, is isomorphic with the Lorentz group.

What about rotation thru 360°?

$$S = e^{\frac{\theta}{2}\gamma^k\gamma^l} = e^{\pi\gamma^k\gamma^l} = \cos\pi + \gamma^k\gamma^l \sin\pi = -1 \quad !$$

This means the representation is <u>double-valued</u>. For every operation there is a negative. These are the <u>spinor</u> representations because they transform vectors. Tensors can be regarded as even spinors.

From group theoretical considerations, the simplest representation of the proper Lorentz group is 2×2. This can be seen from the fact that the Pauli matrices are 2×2. This is similar to a problem considered by Cayley. This representation is called the Weyl representation of γ^μ. We show how the two rows can be formed, 2×2 representation that is.

Recall:

Dirac Representation: $\alpha_k = \rho_1 \sigma_k \;;\; \beta = \rho_3$

now:

Weyl Representation: $\alpha_k = \rho_3 \sigma_k = \begin{pmatrix} \underline{\sigma_k} & \underline{0} \\ \underline{0} & -\underline{\sigma_k} \end{pmatrix}$

$$\beta = \rho_1 = \begin{pmatrix} \underline{0} & \underline{1} \\ \underline{1} & \underline{0} \end{pmatrix}$$

$$\gamma^k = -i\beta\alpha_k$$

$$\gamma^k\gamma^l = (-i\beta\alpha_k)(-i\beta\alpha_l) = \alpha_k\alpha_l = \sigma_k\sigma_l = i\,\epsilon_{klm}\,\sigma_m$$

Assume $\epsilon = 1$ for the moment: Then:

$$S = e^{\frac{\theta}{2}\gamma^k\gamma^l} = \begin{pmatrix} e^{i\frac{\theta}{2}\sigma_m} & 0 \\ 0 & e^{i\frac{\theta}{2}\sigma_m} \end{pmatrix}$$

Thus we see because of the diagonal arrangement, one can transform two components of the wave function at a time. What about time?

$$\gamma^4\gamma^k = \beta(-i\beta\alpha_k) = -i\,\alpha_k = -i\,\sigma_k\rho_3$$

Therefore:

$$S = \begin{pmatrix} e^{-i\frac{\theta}{2}\sigma_k} & 0 \\ 0 & e^{i\frac{\theta}{2}\sigma_k} \end{pmatrix}$$ However, here θ is imaginary.

We see this process gives two by two matrices on the diagonal. Each matrix applies to a pair of components of the Dirac wave function.

LECTURE 8 : 10-11-61

Recall from last time:

$$S = e^{\frac{\theta}{2}\gamma^\nu\gamma^\mu} \quad (\mu, \nu; \theta) \quad (\text{rotation})$$

$$S = i\gamma^4 \quad (\text{space inversion})$$

Dirac Representation:

$$\alpha_k = \begin{pmatrix} \underline{0} & \underline{\sigma_k} \\ \underline{\sigma_k} & \underline{0} \end{pmatrix} \quad ; \quad \beta = \begin{pmatrix} \underline{1} & \underline{0} \\ \underline{0} & -\underline{1} \end{pmatrix}$$

Weyl Representation:

$$\alpha_k = \begin{pmatrix} \underline{\sigma_k} & \underline{0} \\ \underline{0} & -\underline{\sigma_k} \end{pmatrix} \quad ; \quad \beta = \begin{pmatrix} \underline{0} & \underline{1} \\ \underline{1} & \underline{0} \end{pmatrix}$$

In this representation, from the fact that $\gamma^\ell\gamma^k = (-i\beta\alpha_\ell)(-i\beta\alpha_k)$ and:

$$\sigma_\ell\sigma_k = \delta_{k\ell} + i\,\epsilon_{\ell k m}\,\sigma_m$$

we obtain:

$$S(k,\ell;\theta) = \begin{pmatrix} e^{\mp\frac{i\theta}{2}\sigma_m} & \underline{0} \\ \underline{0} & e^{\mp\frac{i\theta}{2}\sigma_m} \end{pmatrix} \quad \begin{array}{l} - : k\ell m = 123, 231, 312 \\ + : k\ell m = 132, 321, 213 \\ \theta \text{ real} \end{array}$$

and:

$$S(k,4;\theta) = \begin{pmatrix} e^{-\frac{i\theta}{2}\sigma_k} & \underline{0} \\ \underline{0} & e^{\frac{i\theta}{2}\sigma_k} \end{pmatrix} \quad ; \ \theta \text{ imaginary.}$$

Thus we can say in the Weyl representation that ψ forms two invariant subsets since S does not mix them. Of course ψ is different in both the Dirac and Weyl representations.

Thus the <u>simplest</u> representation of the proper (no inversion) Lorentz group are the 2×2 matrices of the Weyl representation. The four components of ψ represent spin (2) and energies (2, + and −). The Weyl representation does not mix the two subsets of ψ.

An example of the explicit representation of S would be, using $k, l = 3, 2$ and:

$$e^{\frac{i\theta}{2}\sigma_m} = \cos\frac{\theta}{2} + i\sigma_m \sin\frac{\theta}{2}$$

Here $\sigma_m = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

$$S(3,2;\theta) = \begin{pmatrix} \cos\frac{\theta}{2} & i\sin\frac{\theta}{2} & 0 & 0 \\ i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} & 0 & 0 \\ 0 & 0 & \cos\frac{\theta}{2} & i\sin\theta/2 \\ 0 & 0 & i\sin\theta/2 & \cos\theta/2 \end{pmatrix}$$

What is the effect of the Weyl representation on the Dirac equation?

$$\frac{\partial\psi}{c\,\partial t} + \vec{\alpha}\cdot\nabla\psi = -\frac{i\beta mc}{\hbar}\psi$$

We write $\psi = \begin{pmatrix} \underline{\varphi} \\ \underline{\chi} \end{pmatrix}$, where $\underline{\varphi}$, $\underline{\chi}$ are the two invariant subsets in the Weyl representation.

Then:

$$\left.\begin{aligned} \frac{\partial\varphi}{c\,\partial t} + \vec{\sigma}\cdot\nabla\varphi &= -\frac{imc}{\hbar}\chi \\ \frac{\partial\chi}{c\,\partial t} - \vec{\sigma}\cdot\nabla\chi &= -\frac{imc}{\hbar}\varphi \end{aligned}\right\}$$ note that these equations are coupled by the <u>mass</u> term.

For a particle with no mass, there is nothing that connects the two equations. Thus we can say that for massless particles, we have only two component wave functions. If we invert the space axes, we get:

$$\frac{\partial\varphi}{c\,\partial t} - \vec{\sigma}\cdot\nabla\varphi = 0$$

so that this is not invariant under an inversion.

suppose we write $-\vec{\sigma} = \vec{\sigma}'$. We then get the commutation rules in the ' system:

$$\sigma_k' \sigma_l' = \delta_{kl} - i\,\epsilon_{klm}\,\sigma_m'$$

Before: $\sigma_k \sigma_l = \delta_{kl} + i\,\epsilon_{klm}\,\sigma_m$

This says the massless Dirac equation is not mirror invariant. In 1931 Pauli said that this could not be so because nature should not distinguish between left and right handed worlds. This is no longer correct in view of the non-conservation of parity in the β decay of Co^{60}. Hence the two component wave function does represent right and left handedness.

If we are using de Broglie waves, viz:

$$\varphi \sim e^{-iEt/\hbar + i\frac{\vec{p}}{\hbar}\cdot\vec{r}}$$

Then we get for the massless φ equation:

$$-\frac{E}{c}\varphi + \vec{\sigma}\cdot\vec{p}\,\varphi = 0$$

$$\text{or} \quad -|\vec{p}| + \vec{\sigma}\cdot\vec{p} = 0$$

which gives: $$\frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|} = +1$$

or the spin of this particle is in the direction of motion or a right-handed motion. This presented some difficulty at first with the neutrino which was first thought to right-handed but later was found to left-handed. The anti-neutrino is right-handed.

We now consider the Lorentz transformation properties of some other quantities, not including time reversal.

Consider the probability current: Recall:

$$\bar{\psi} = \psi^* \gamma^4 \quad ; \quad \bar{\psi} = \bar{\psi}' S^{-1} \quad ; \quad \psi = S\psi'$$

Then $I = \bar{\psi}\psi = \bar{\psi}'\psi'$ is invariant and is called a world scalar. This is example of the transformation of scalar quantities. Now look at the probability current vector:

$$\frac{\partial j_\mu}{\partial x_\mu} = 0 \; ; \quad j_\mu = i\bar{\psi}\gamma^\mu\psi = (\underbrace{\psi^* \vec{\alpha}\psi}_{\frac{\vec{J}}{c}}, \underbrace{i\psi^*\psi}_{i\rho})$$

using $\gamma^k = -i\beta\alpha_k$; $\gamma^4 = \beta$. We show that this should transform like a 4-vector. Recall:

$$x_\nu = a_{\nu\mu} x'_\mu$$
$$S^{-1}\gamma^\nu a_{\nu\mu} S = \gamma^\mu \quad ; \quad S^{-1}\gamma^\sigma S = a_{\sigma\mu}\gamma^\mu \; ; \quad S^{-1}\gamma^\mu S = a_{\mu\lambda}\gamma^\lambda$$

Now make the Lorentz Transformation: $\bar{\psi}\gamma^\mu\psi = \bar{\psi}' S^{-1}\gamma^\mu S\psi'$

$= a_{\mu\lambda}\bar{\psi}'\gamma^\lambda\psi'$. Then, $j_\mu = a_{\mu\lambda} j'_\lambda$ or the probability current transforms like a four vector.

Now consider a tensor, say: $M_{\mu\nu} = \bar{\psi}\,\frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\psi$

or: $$M_{\mu\nu} = \left\{ \begin{array}{cc} \bar{\psi}\gamma^\mu\gamma^\nu\psi & \mu \neq \nu \\ 0 & \mu = \nu \end{array} \right\} \therefore \text{antisymmetric}$$

$$M_{\mu\nu} = \bar{\psi}'\,\tfrac{1}{2}\left(S^{-1}\gamma^\mu S S^{-1}\gamma^\nu S - S^{-1}\gamma^\nu S S^{-1}\gamma^\mu S\right)\psi'$$

$$= a_{\mu\lambda}a_{\nu\sigma}\,\bar{\psi}'\,\tfrac{1}{2}(\gamma^\lambda\gamma^\sigma - \gamma^\sigma\gamma^\lambda)\psi' = a_{\mu\lambda}a_{\nu\sigma}M'_{\lambda\sigma}$$

which is the rule for the transformation of a second rank antisymmetric tensor.

LECTURE 9 : 10-13-61

We define the following notation:

Ⓢ Scalar
Ⓥ Vector
Ⓣ Tensor
Ⓐ axial vector
Ⓟ Pseudo-scalar

$I = \bar{\psi}\psi$ is an invariant $= \psi^*\beta\psi$ Ⓢ

$j_\mu = i\bar{\psi}\gamma^\mu\psi = (\psi^*\vec{\alpha}\psi, i\psi^*\psi)$ Transforms as a 4-vector Ⓥ

$M_{\mu\nu} = i\bar{\psi}\frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\psi$ Transforms as an antisymmetric tensor. Ⓣ

We now digress for a moment. What does $M_{\mu\nu}$ look like in the old notation

$\mu\nu$

$12: \quad \gamma^1\gamma^2 = (-i\beta\alpha_1)(-i\beta\alpha_2) = \alpha_1\alpha_2$

$M_{12} = i\psi^*\beta\,\alpha_1\alpha_2\psi$

$41: \quad \gamma^4\gamma^1 = -i\alpha_1 \quad ; \quad M_{41} = \psi^*\beta\alpha_1\psi$

Then:
$$\left.\begin{matrix}41\\42\\43\end{matrix}\right\} (\psi^*\beta\vec{\alpha}\psi)$$

Now we have introduced the quantities $\rho_1, \rho_2, \rho_3, \sigma_1, \sigma_2, \sigma_3$, which can now be given two meanings or interpretations.

The first meaning is that given by Dirac in that these are defined as definite numerical matrices:

$$\sigma_n = \begin{pmatrix}\underline{\sigma_n} & \underline{0}\\ \underline{0} & \underline{\sigma_n}\end{pmatrix} ; \quad \rho_1 = \begin{pmatrix}\underline{0} & \underline{1}\\ \underline{1} & \underline{0}\end{pmatrix} ; \quad \rho_2 = \begin{pmatrix}\underline{0} & -i\underline{1}\\ i\underline{1} & \underline{0}\end{pmatrix} ; \quad \rho_3 = \begin{pmatrix}\underline{1} & \underline{0}\\ \underline{0} & -\underline{1}\end{pmatrix}$$

Now recall that Dirac built α, β arbitrarily from σ, ρ.

The second meaning is an algebraic one, that is, the matrices ρ, σ are governed by the choice of α_k, β rather than vice-versa. We say that, in Dirac sense, ρ_k, σ_k are such that $\alpha_k = \rho_1 \sigma_k$; $\beta = \rho_3$. If we choose α_k, β as did Dirac, we obviously get the ρ_k, σ_k he used. One of the absolute or algebraic things that can always be said about these matrices is that their commutation rules are:

$$\left.\begin{aligned} \rho_k \rho_l &= \delta_{kl} + i\, \epsilon_{klm} \rho_m \\ \sigma_k \sigma_l &= \delta_{kl} + i\, \epsilon_{klm} \sigma_m \end{aligned}\right\} \quad [\rho_k, \sigma_l] = 0$$

These are not really commutation rules but rather properties these matrices have to obey.

From these rules we get for the α's:

$$\alpha_k \alpha_l = \rho_1 \sigma_k \rho_1 \sigma_l = \delta_{kl} + i\, \epsilon_{klm} \sigma_m$$

or: $k \neq l$:
$$\alpha_k \alpha_l = i\, \epsilon_{klm} \sigma_m$$
$$\sigma_k \sigma_l = i\, \epsilon_{klm} \sigma_m$$

This gives the σ's in terms of the α's. For the ρ's, look at:

$$\alpha_1 \alpha_2 \alpha_3 = \rho_1 \sigma_1 \rho_1 \sigma_2 \rho_1 \sigma_3 = i \rho_1$$

so:
$$\rho_1 = -i\, \alpha_1 \alpha_2 \alpha_3$$
$$\rho_2 = -\beta\, \alpha_1 \alpha_2 \alpha_3$$
$$\rho_3 = \beta$$

Therefore, if we define ρ_k, σ_k this way, we can write $M_{\mu\nu}$ as a six vector:

$$M_{\mu\nu} = -\underbrace{(\psi^* \beta \vec{\sigma} \psi)}_{\text{space} \times \text{space}},\ \underbrace{(\psi^* \beta \vec{\alpha} \psi)}_{\text{time} \times \text{time}}$$

The relativistic analog of this quantity is the EM field tensor:

$$F_{\mu\nu} = \vec{H}, i\vec{E}$$

Then: $-(\psi^* \beta \vec{\sigma} \psi)$ acts like $\vec{H}$
and: $(\psi^* \beta \vec{\alpha} \psi)$ acts like $\vec{E}$

So up to now we have the following operations or quantities in terms of the γ's:

S : 1 part
V : 4 parts
T : 6 parts

We have used up 11 combinations of γ and we have 5 more to go.

Now define the quantity:

$$K_{\mu\nu\sigma} = \begin{cases} -i\,\bar{\psi}\gamma^\mu\gamma^\nu\gamma^\sigma\psi & ; \mu \neq \nu \neq \sigma \\ 0 & ; \text{otherwise} \end{cases}$$

This is known as an antisymmetric tensor of the third rank and we wish to show that it transforms into another one under a Lorentz transformation.

$$K_{\mu\nu\sigma} = -i\, a_{\mu\alpha}\, a_{\nu\beta}\, a_{\sigma\epsilon}\;\bar{\psi}\gamma^\alpha\gamma^\beta\gamma^\epsilon\psi'$$

$$= a_{\mu\alpha}\, a_{\nu\beta}\, a_{\sigma\epsilon}\; K'_{\alpha\beta\epsilon}$$

We must prove this by showing that terms with equal $\alpha\beta$, $\beta\epsilon$, or $\alpha\epsilon$ cancel each other.

There are 64 terms in the sum, 24 different, 40 not and they break down this way:

4 : all equal $\alpha = \beta = \epsilon$
12 : $\alpha\alpha\beta$ - like or $\beta\beta\alpha$ - like
12 : $\alpha\beta\beta$ - like
12 : $\alpha\beta\alpha$ - like

Consider $\beta\beta\alpha$ - like including $\alpha = \beta$:

$$\sum_{\beta\alpha} \underbrace{a_{\mu\beta}\, a_{\nu\beta}}_{\delta_{\mu\nu}}\, a_{\sigma\alpha}\ \bar{\psi}'\, \underbrace{\gamma^\beta \gamma^\beta}_{1} \gamma^\alpha \psi'$$

However, by the statement of the problem, $\mu \neq \nu$ so this sum vanishes.

Consider $\alpha\beta\beta$ - like including $\alpha = \beta$:

$$\sum_{\beta\alpha} a_{\mu\alpha}\, \underbrace{a_{\nu\beta}\, a_{\sigma\beta}}_{\delta_{\mu\sigma}}\ \bar{\psi}'\, \gamma^\alpha \underbrace{\gamma^\beta \gamma^\beta}_{1} \psi' \qquad = 0 \quad \text{since } \mu \neq \sigma$$

Consider $\alpha\beta\alpha$, $\alpha \neq \beta$:

$$\sum_{\beta} \left\{ \sum_{\alpha \neq \beta} a_{\mu\alpha}\, a_{\nu\beta}\, a_{\sigma\alpha}\ \bar{\psi}'\, \underbrace{\gamma^\alpha \gamma^\beta \gamma^\alpha}_{-1} \psi' - a_{\mu\beta}\, a_{\nu\beta}\, a_{\sigma\beta}\, \bar{\psi}' \gamma^\beta \underbrace{\gamma^\beta \gamma^\beta}_{1} \psi' \right\}$$

Since in $\beta\beta\alpha$ and $\alpha\beta\beta$ we counted $\alpha = \beta$, we counted the equal $(\alpha = \beta = \epsilon)$ terms twice so now we must subtract once. This is reason for second term above. We combine to give:

$$- \sum_{\alpha\beta} \underbrace{a_{\mu\alpha}\, a_{\sigma\alpha}}_{\delta_{\mu\sigma}}\, a_{\nu\beta}\ \bar{\psi}' \gamma^\beta \psi' \qquad = 0 \quad \text{since } \mu \neq \sigma$$

Hence we have shown the Lorentz transformation properties of $K_{\mu\nu\sigma} = -i\, \bar{\psi}\, \gamma^\mu \gamma^\nu \gamma^\sigma \psi$. We examine this quantity explicitly:

Possible $\mu\nu\sigma$: 423, 431, 412 and 123

412: $\beta(-i\beta\alpha_1)(-i\beta\alpha_2) = -\beta\alpha_1\alpha_2 = -i\beta\sigma_3$

123: $(-i\beta\alpha_1)(-i\beta\alpha_2)(-i\beta\alpha_3) = -\beta\rho_1$

$\therefore \quad K_{\mu\nu\sigma} = (\psi^* \vec{\sigma}\, \psi),\ i(\psi^* \rho_1 \psi)$

Thus we see that $K_{\mu\nu\sigma}$ has four components or is vector-like. This quantity is known as an axial vector, denoted by (A)

The last quantity to be discussed is one containing all γ's and is defined as:

$$N = \bar{\psi}\gamma^5\psi = \bar{\psi}\gamma^1\gamma^2\gamma^3\gamma^4\psi$$

Transforming:

$$N = a_{1\lambda}\, a_{2\mu}\, a_{3\nu}\, a_{4\sigma}\; \bar{\psi}'\gamma^\lambda\gamma^\mu\gamma^\nu\gamma^\sigma\psi'$$

$$= \left(\sum_P \pm a_{1\lambda}\, a_{2\mu}\, a_{3\nu}\, a_{4\sigma}\right) N'$$

Recall: $\mathrm{Det}\,(a_{\mu\nu}) = +1$ for proper Lorentz transformation
$= -1$ for space inversion.

Note that N acts like a scalar for proper transformations yet goes negative on inversion which ordinary scalars do not do. For this reason, Pauli called it a pseudoscalar (P).

$$N = i\bar{\psi}\gamma^5\psi = \psi^*\rho_2\psi$$

Note that, from previous developments, we could have written (A) as:

$$K_\mu = i\bar{\psi}\gamma^5\gamma^\mu\psi$$

Thus displaying its vector character more clearly.

The quantities we have found here are the five famous covariant quantities of Pauli.

LECTURE 10 : 10-16-61

Recall we had finished with five covariant quantities, the last being the pseudoscalar N.

$$N = i\bar{\Psi}\gamma^1\gamma^2\gamma^3\gamma^4\Psi = i\bar{\Psi}\gamma^5\Psi$$

It is hard to show how this transforms since there are 256 terms in the sum, 232 with two or more of the terms equal. However, by using S we can represent any Lorentz transformation because we can rotate twice in 3-space to get a major axis in the direction of the velocity, then rotate in the plane containing 4 and this major axis to set in motion, and then rotate three times in 3-space to get back proper position. Thus we can get any Lorentz transformation by using 6 rotations. Form:

$$N = \bar{\Psi}\gamma^5\Psi = \bar{\Psi}SS^{-1}\gamma^5SS^{-1}\Psi = \bar{\Psi}'S^{-1}\gamma^5S\Psi' \quad (i \text{ understood})$$

Choose the proper Lorentz transformation $S = e^{\frac{\theta}{2}\gamma^\mu\gamma^\nu}$. γ^5 passes thru $\gamma^\mu\gamma^\nu$ with two canceling sign changes so. $S^{-1}\gamma^5S = \gamma^5$. For space inversion, recall we had the choice of $S = \sqrt[4]{1}\,\gamma^4$. Choose:

$$S = -i\gamma^4 \quad , \text{ then } S^{-1} = i\gamma^4$$

so: $$S^{-1}\gamma^5S = \gamma^4\gamma^1\gamma^2\gamma^3 = -\gamma^5$$

and hence for the space inversion operation:

$$N = -N'$$

while for the proper Lorentz transformation: $N = N'$.

Table of Five Covariant Quantities

Quantity	Form	Analog
(S) scalar	$I = \bar{\psi}\psi = \psi^* \beta \psi$	proper time, rest mass, charge; not many in relativity theory.
(V) vector	$S_\mu = i\bar{\psi}\gamma^\mu \psi = (\psi^* \vec{\alpha}\psi, \; i\psi^*\psi)$	Current: $\frac{\vec{j}}{c}$, $i\rho$
(T) tensor	$M_{\mu\nu} = i\bar{\psi}\frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\psi = (-\underset{23\ 31\ 12}{\psi^*\beta\vec{\sigma}\psi}, \; \underset{41\ 42\ 43}{\psi^*\beta\vec{\alpha}\psi})$	EM field tensor: $F_{\mu\nu} = (\vec{H}, i\vec{E})$
(A) axial vector	$K_\mu = i\bar{\psi}\gamma^5\gamma^\mu\psi = (\psi^*\vec{\sigma}\psi, \; i\psi^*\rho_1\psi)$	no analog in relativity. somewhat like angular momentum in 3-space.
(P) pseudoscalar	$N = i\bar{\psi}\gamma^5\psi = \psi^*\rho_2\psi$	no known analog

Example of use: Beta decay, or interaction of neutron-proton with electron-neutrino: Vector interaction:

$$(\bar{\psi}_N \gamma^\mu \psi_P)(\bar{\psi}_e \gamma^\mu \psi_\nu)$$

In general, we can speak of any of above 5 acting as the interaction operator denoted by O:

$$(\bar{\psi}_n O \psi_P)(\psi_e O \psi_\nu)$$

We return to a consideration of the Dirac particle. Furry says that only the first 3 operations above are possible on the Dirac particle. Consider the Dirac equation:

$$\gamma^\mu \left(\frac{\partial}{\partial x_\mu} - \frac{ie}{\hbar c}\varphi_\mu\right)\psi + \frac{mc}{\hbar}\psi = 0$$

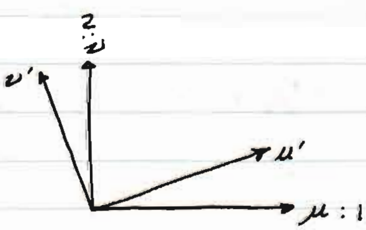
Note that we could add the term $\frac{i\kappa}{2} F_{\mu\nu}\gamma^\mu\gamma^\nu\psi$ to the equation without destroying its invariance (noticed by Pauli). When we put the equation in Hamiltonian form, this term becomes:

$$\underbrace{-\kappa\beta\vec{\sigma}\cdot\vec{H}}_{\substack{\text{like}\\ \text{magnetic}\\ \text{moment } \vec{\mu}}} + \underbrace{i\kappa\beta\vec{\alpha}\cdot\vec{E}}_{\substack{\text{like}\\ \text{dipole}\\ \text{moment}}}$$

We could also add a gravitational term without destroying the Lorentz invariance

NB: the added term gives an anomalous or extra moment besides that arising naturally from the spin of the Dirac particle.

We now discuss the inherent spin of the Dirac particle. Consider an infinitesimal rotation in the 12 plane:



$$\psi = S\psi'$$

$$S = 1 + \frac{\epsilon}{2}\gamma^\nu\gamma^\mu$$

Then: $\psi' = S^{-1}\psi = \left(1 + \frac{\epsilon}{2}\gamma^1\gamma^2\right)\psi = \left(1 + \frac{\epsilon}{2} i\sigma_3\right)\psi$

since $(-i\beta\alpha_1)(-i\beta\alpha_2) = i\sigma_3$ using the second meaning of σ, ρ.

Recall that we usually represent a rotation by:

$$1 + \epsilon\frac{iM_z}{\hbar} \longrightarrow 1 + \epsilon\frac{\partial}{\partial\varphi}$$

general angular momentum — for scalar ψ or orbital angular momentum

Now we take the product of the rotation on a scalar ψ with the rotation on the particle (Dirac vector ψ) and identify first order terms with the general angular momentum term:

$$i\frac{M_z}{\hbar} = \frac{\partial}{\partial\varphi} + \frac{1}{2}\sigma_3$$

or: $\quad M_z = \frac{\hbar}{i}\frac{\partial}{\partial\varphi} + \frac{\hbar}{2}\sigma_3 \quad = L_z + S_z$

However, Dirac demonstrated the spin by considering what were the constants of motion in a central field or what commuted with the Hamiltonian. Recall:

$$i\hbar \frac{\partial \psi}{\partial t} = \underline{H}\,\psi$$

$$\underline{H} = c\vec{\alpha}\cdot\vec{p} + \beta m c^2 + V(r) \quad ; \; V(r) \text{ for a central field.}$$

Now: $\quad \frac{dA}{dt} = \frac{i}{\hbar}[H, A] + \frac{\partial A}{\partial t}$

Consider the orbital angular momentum which we will find no longer has a vanishing time derivative or is a constant of the motion.

Note:
$\vec{p}\,V = \frac{\hbar}{i} V' \frac{\vec{r}}{r}$ and $\vec{r}\times\vec{r} = 0$ from $\vec{L} = \vec{p}\times\vec{r}$ when placed in the commutator. so only term left is $c\vec{\alpha}\cdot\vec{p}$. α's commute with $\vec{r}, \vec{p}$ as α's are numbers. Now, using:

$$L_i = \epsilon_{ijk}\, x_j\, p_k$$

we have: $\frac{d}{dt}L_x = \frac{i}{\hbar}[H, L_x] = \frac{ic}{\hbar}\epsilon_{1jk}\,\alpha_l\,[p_l, x_j p_k]$

$$= \frac{ic}{\hbar}\alpha_l \frac{\hbar}{i}\delta_{lj}\epsilon_{1jk}\,p_k = c\,\alpha_l\,\epsilon_{1lk}\,p_k = c\left[\vec{\alpha}\times\vec{p}\right]_x$$

Thus L_x is not conserved. Now compute:

$$\frac{d}{dt}\frac{\hbar}{2}\sigma_x = \frac{i}{2}c\,p_l\left[\alpha_l, \sigma_1\right] = \frac{i}{2}c\,p_l\,\rho_1\left[\sigma_l, \sigma_1\right]$$

$$= -c\,p_l\,\rho_1\,\epsilon_{l1j}\,\sigma_j = -c\,\epsilon_{l1j}\,p_l\,\alpha_j = -c\,\epsilon_{1jl}\,\alpha_j\,p_l = -c\left[\vec{\alpha}\times\vec{p}\right]_x$$

since $\quad [\sigma_k, \sigma_l] = 2i\,\epsilon_{klj}\,\sigma_j$, using the second meaning of ρ, σ.

Thus the sum $L_x + \frac{\hbar}{2}\sigma_x$ is conserved and this is how the presence of spin is usually demonstrated in that it must be added to conserve angular momentum.

LECTURE 11 : 10-18-61

Recall that Pauli found it was possible to add an extra term to the Dirac equation without destroying its Lorentz invariance.

$$\gamma^\mu \left(\frac{\partial}{\partial x_\mu} - \frac{ie}{\hbar c} \varphi_\mu \right) \psi + \frac{i\kappa}{2} F_{\mu\nu} \gamma^\mu \gamma^\nu \psi + \frac{mc}{\hbar} \psi = 0$$

The Pauli term causes an anomalous electric and magnetic moment and is sometimes said to give rise to the neutron moment. If we write the Dirac equation in Hamiltonian form, $i\hbar \frac{\partial \psi}{\partial t} = H \psi$, and use $F_{\mu\nu} = (\vec{\mathcal{H}}, i\vec{E})$, and the meaning of Ⓣ for $\gamma^\mu \gamma^\nu$, we get:

Magnetic Term: $-\kappa \hbar c \, \beta \vec{\sigma} \cdot \vec{\mathcal{H}}$; $\vec{\mu} = \kappa \hbar c \, \beta \vec{\sigma}$
Electric Term: $\kappa \hbar c \, i \beta \vec{\alpha} \cdot \vec{E}$; $\vec{p} = i \kappa \hbar c \, \beta \vec{\alpha}$

However, any effects from the Pauli term are in addition to the usual electron spin effects. It is not surprising that the Pauli term gives an electric as well as a magnetic term as the two fields should appear together in a relativistic framework. Note that, because of the anticommutation properties of $\vec{\alpha}, \beta$, the i is needed to make the electric term Hermitean. To get an idea of the size of the Pauli terms, we examine in the NR limit:

$$\left(\gamma^\mu \frac{\partial}{\partial x^\mu} + \frac{mc}{\hbar} \right) \psi = 0 \quad ; \quad \psi = u \, e^{-iEt/\hbar}$$

Let: $\frac{v}{c} \ll 1$; $\frac{\hbar k}{mc} \ll 1$; $\therefore E \approx mc^2 \left[1 + O\left(\frac{v^2}{c^2}\right) \right]$

now $-ict = -x_4$, therefore: $\psi \approx u \, e^{-\frac{mc}{\hbar} x_4}$
which leads to:

$$\gamma^4 \left(-\frac{mc}{\hbar} \right) \psi + \frac{mc}{\hbar} \psi = 0$$

thus $\gamma^4 \approx 1$. Hence we see that the anamolous magnetic moment is of first order. This is not so with the electric moment as $\vec{\alpha}$ is of the order $\frac{v}{c}$ as will be seen later in the course.

There are other instances where terms like the Pauli terms arise. Recall that one cannot get the KG equation from the Dirac equation when there are potentials present. If all cross-terms did cancel we would have for the KG form:

$$\left(\frac{\partial}{\partial x_\mu} - \frac{ie}{\hbar c} \varphi_\mu \right)^2 \psi + \left(\frac{mc}{\hbar} \right)^2 \psi = 0$$

Instead we really have:

$$\left(\frac{\partial}{\partial x_\mu} - \frac{ie}{\hbar c} \varphi_\mu \right)^2 \psi + \left(\frac{mc}{\hbar} \right)^2 \psi - \frac{ie}{\hbar c} \sum_{\mu \neq \nu} \frac{\partial \varphi_\mu}{\partial x_\nu} \gamma^\nu \gamma^\mu \psi = 0$$

Since: $\gamma^\nu \gamma^\mu = \frac{1}{2} \left(\gamma^\nu \gamma^\mu - \gamma^\mu \gamma^\nu \right)$, the "extra" term becomes:

$$- \frac{ie}{2\hbar c} \underbrace{\left(\frac{\partial \varphi_\mu}{\partial x_\nu} - \frac{\partial \varphi_\nu}{\partial x_\mu} \right)}_{F_{\mu\nu}} \gamma^\nu \gamma^\mu \psi$$

For the kl part: $- \frac{ie}{2\hbar c} \underbrace{F_{kl}}_{H_m} \underbrace{(-i\beta\alpha_k)(-i\beta\alpha_l)}_{i\sigma_m}$

$= \frac{+e}{2\hbar c} \vec{\sigma} \cdot \vec{H}$; could define: $\vec{\mu} = \frac{\hbar}{2} \frac{e}{mc} \vec{\sigma}$

For the $4k$ part: $- \frac{ie}{2\hbar c} \underbrace{F_{4k}}_{i\mathcal{E}_k} \underbrace{(\beta)(-i\beta\alpha_k)}_{-i\alpha_k}$

$= \frac{-ie}{2\hbar c} \vec{\alpha} \cdot \vec{\mathcal{E}}$

Thus, the "extra" term in the KG equation with a potential are of the Pauli form.

Recall that the probability current is of the form: $S_\mu = i \bar{\psi} \gamma^\mu \psi$, not at all like the expression for the KG current. However, let's see what happens when we try to force into the KG form.

First define:

$$\partial_\mu \equiv \frac{\partial}{\partial x_\mu} \ ; \ q \equiv \frac{e}{\hbar c} \ ; \ \kappa = \frac{mc}{\hbar},$$

not the same as before.

The Dirac equation becomes:

$$(\partial_\mu - i q \varphi_\mu)\gamma^\mu \psi + \kappa \psi = 0$$

$$(\partial_\mu + i q \varphi_\mu)\bar{\psi}\gamma^\mu - \kappa \bar{\psi} = 0$$

Solve for the $\kappa \psi$'s and plug in S_μ written as:

$$S_\mu = i \bar{\psi}\gamma^\mu \psi = i\left(\tfrac{1}{2}\bar{\psi}\gamma^\mu\psi + \tfrac{1}{2}\bar{\psi}\gamma^\mu\psi\right)$$

$$S_\mu = \frac{\hbar}{2mci}\Big\{\bar{\psi}\gamma^\mu(\partial_\nu - i q \varphi_\nu)\gamma^\nu\psi$$

$$- \Big[(\partial_\nu + i q \varphi_\nu)\bar{\psi}\Big]\gamma^\nu\gamma^\mu\psi\Big\}$$

To get in the KG form, write $S_\mu = S_\mu^{(1)} + S_\mu^{(2)}$, where for $\mu = \nu$, we have $S_\mu^{(1)}$; $\mu \neq \nu$, $S_\mu^{(2)}$:

$$S_\mu^{(1)} = \frac{\hbar}{2mci}\Big\{\underbrace{\bar{\psi}}_{\psi^* \text{ in NR limit}}(\partial_\mu - i q \varphi_\mu)\psi - \Big[(\partial_\mu + i q\varphi_\mu)\underbrace{\bar{\psi}}_{\psi^* \text{ in NR limit}}\Big]\psi\Big\}$$

$$= \frac{\hbar}{2mci}\Big\{\psi^*\partial_\mu\psi - \psi\,\partial_\mu\psi^*\Big\} - \frac{e}{2mc^2}\varphi_\mu\psi^*\psi$$

which is practically the KG expression for the probability current and for $\mu = k$, we have the Schroedinger probability current since $S_k = j_k/c$.
For $\mu \neq \nu$:

$$S_\mu^{(2)} = \frac{\hbar}{4mci}\bar{\psi}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\partial_\nu\psi + \frac{\hbar}{4mci}\partial_\nu\bar{\psi}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\psi$$

For $\mu = 4$; $\nu = k$:

$$S_4^{(2)} = \frac{\hbar}{2mci}\bar{\psi}\gamma^4\gamma^k\partial_k\psi + \frac{\hbar}{2mci}\partial_k\bar{\psi}\gamma^4\gamma^k\psi$$

$$= \frac{\hbar}{2mci}\partial_k\left(\bar{\psi}\gamma^4\gamma^k\psi\right) = \frac{\hbar}{2mci}\partial_k\left(\psi^*\gamma^k\psi\right)$$

$$= -\frac{\hbar}{2mc}\nabla\cdot\psi^*\beta\vec{\alpha}\psi = i\rho^{(2)}$$

Recall in a polarized medium, we have the following expression for a fictitious charge density:

$$\rho' = -\nabla \cdot \vec{P}$$

where $\vec{P}$ is the polarization. From our conclusions on Ⓣ and the Pauli term and the above equation, we can immediately write:

$$\rho^{(2)} = -\nabla \cdot \vec{P} \; ; \text{ where: } \quad \vec{P} = \frac{-i\hbar}{2mc} \psi^* \beta \vec{\alpha} \psi$$

now, for $\mu = k$, $\nu = 4, k, l$; $k, l \neq 4$:

$$S_k^{(2)} = \underbrace{\frac{-\hbar}{2mi} \partial_4 \; \psi^* (-i\beta\alpha_k) \psi}_{\frac{\partial}{\partial t} P_k} + \frac{\hbar}{2mi} \partial_l \underbrace{\psi^* \gamma^4 \gamma^k \gamma^l \psi}_{\underbrace{\psi^* \beta (-i\beta\alpha_k)(-i\beta\alpha_l)\psi}_{i\epsilon_{klm}\sigma_m}}$$

or:

$$S_k^{(2)} = \frac{\partial}{\partial t} P_k + c \left[\nabla \times \vec{m} \right]_k$$

where $\vec{m} = \dfrac{\hbar}{2mc} \psi^* \beta \vec{\sigma} \psi$

We see that the total probability current computed from the Dirac equation consists of two parts: a KG part in the limit of $\beta = 1$ and an electromagnetic current involving polarization and magnetization terms of the type developed from the properties of the tensor Ⓣ. The total current is positive definite only when <u>both</u> $S_\mu^{(1)}$ and $S_\mu^{(2)}$ are taken together.

LECTURE 12 : 10-20-61

We will now demonstrate that the Pauli electron results in the non-relativistic limit from the Dirac electron. We have for the energy:

$$E = mc^2 \left[1 + O\left(\frac{v^2}{c^2}\right)\right]$$

Now $mc^2 \sim .5$ Mev while the usual kinetic energy of an everyday electron is about 10 ev, so that the non-relativistic limit is a very good one. In this limit, the time dependent part of the wave function goes as:

$$e^{-i\frac{mc^2}{\hbar}t} = e^{-\frac{mc}{\hbar}x_4}$$

We now introduce some convenient notation and an operator formalism to help construct the non-relativistic limit of the Dirac equation.

$$D_\mu \equiv \frac{\partial}{\partial x_\mu} - \frac{ie}{\hbar c}\phi_\mu \quad ; \quad \kappa = \frac{mc}{\hbar}$$

The Dirac equation is then:

$$\gamma^\mu D_\mu \psi + \kappa\psi = 0$$

or :

$$\boxed{\gamma^4 D_4 \psi + \gamma^k D_k \psi + \kappa\psi = 0}$$

We choose as projection operators, the forms:

$$\boxed{P_+ = \tfrac{1}{2}(1+\gamma^4) \quad ; \quad P_- = \tfrac{1}{2}(1-\gamma^4)}$$

If these are projection operators, they should have the property of idempotence:

$$P_+P_+ = \tfrac{1}{4}(1+\gamma^4)^2 = P_+ \quad ; \quad P_-P_- = \tfrac{1}{4}(1-\gamma^4)^2 = P_-$$

$$P_+P_- = 0 \quad ; \quad P_+ + P_- = 1$$

These operators also have the following relations with respect to the γ^μ's:

$$\boxed{\begin{array}{ll} [\gamma^4, P_\pm] = 0 \quad ; & \{P_\pm, \gamma^k\} = \gamma^k \\ P_\pm \gamma^k = \gamma^k P_\mp \quad ; & \gamma^4 P_\pm = \pm P_\pm \end{array}}$$

All these relations are easily verified from the definitions above. The purpose of projection operators is to separate wave functions into their components. Let, then:

$$P_+ \psi = \varphi \quad ; \quad \gamma^4 \varphi = \gamma^4 P_+ \psi = \varphi$$
$$P_- \psi = \chi \quad ; \quad \gamma^4 \chi = \gamma^4 P_- \psi = -\chi$$

Also: $\quad (P_+ + P_-)\psi = \psi = \varphi + \chi$

Now, in the non-relativistic limit, we have shown that $\gamma^4 \psi \simeq \psi$. Hence:

$$\gamma^4 \psi = \varphi - \chi \simeq \varphi + \chi$$

so evidently $\chi \ll \varphi$ must be the case or χ is the small part of the wave function.

Now apply P_+ on the Dirac equation, keeping in mind all of the above relations:

$$(D_4 + \kappa)\varphi + \gamma^k D_k \chi = 0$$

Operate with P_-:

$$(-D_4 + \kappa)\chi + \gamma^k D_k \varphi = 0$$

Now: $\varphi, \chi \sim e^{-\frac{mc}{\hbar} x_4}$, so that:

$$D_4 + \kappa \sim \mathcal{O}\left(\frac{v^2}{c^2}\kappa\right) \quad ; \text{ indicates } \varphi \text{ should be large,}$$

$$-D_4 + \kappa \sim 2\kappa \quad ; \text{ indicates } \chi \text{ should be small,}$$

because the terms in χ, φ, must balance each other in each equation.

The separation of the wave function into large and small parts justifies the introduction of the projection operators. Since in the NR limit, $\gamma^4 \approx 1$, it is seen that P_- is small and $P_- \psi = \chi$ leads to a small quantity. We have: $\gamma^4 \varphi = \varphi$ and $\gamma^4 \chi = -\chi$ so the eigenvalues of γ^4 are ± 1. ψ is a four component wave function which we have now written in "vector" form consisting of the two components $\varphi + \chi$. However, φ and χ are still two component functions. We know that a vector can be written in row or column form, that is, we can now write:

$$\psi = \begin{pmatrix} \varphi' \\ \chi' \end{pmatrix}$$

We write the primes to indicate we are including the time factors. Later we will split these off and write φ, χ instead. We are now interested in finding a representation for γ^4 which is diagonal with respect to φ', χ' or φ, χ. Such a representation is obviously:

$$\gamma^4 = \rho_3 = \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&-1&0 \\ 0&0&0&-1 \end{pmatrix}$$

With this representation for γ^4, we have, in the first meaning of the α's and β's,

$$\beta = \rho_3 \;; \quad \alpha_n = \rho_1 \sigma_n \;; \qquad \rho_3 = \begin{pmatrix} \underline{1} & \underline{0} \\ \underline{0} & -\underline{1} \end{pmatrix} ; \; \sigma_n = \begin{pmatrix} \underline{\sigma_n} & \underline{0} \\ \underline{0} & \underline{\sigma_n} \end{pmatrix} ; \; \rho_1 = \begin{pmatrix} \underline{0} & \underline{1} \\ \underline{1} & \underline{0} \end{pmatrix}$$

$$\text{and,} \quad \alpha_n = \begin{pmatrix} 0 & \underline{\sigma_n} \\ \underline{\sigma_n} & 0 \end{pmatrix}.$$

Furthermore:

$$P_+ = \tfrac{1}{2}(1+\beta) = \begin{pmatrix} \underline{1} & \underline{0} \\ \underline{0} & \underline{0} \end{pmatrix} ; \; P_- = \tfrac{1}{2}(1-\beta) = \begin{pmatrix} \underline{0} & \underline{0} \\ \underline{0} & \underline{1} \end{pmatrix}$$

Now, in the canonical form, $\vec{\pi} = \vec{p} - \frac{e}{c}\vec{A}$, where e is the charge of the particle, and the Dirac equation becomes:

$$i\hbar \frac{\partial \psi}{\partial t} = c\,(\vec{\alpha}\cdot\vec{\pi})\,\psi + \beta m c^2 \psi + e\phi\,\psi$$

substituting in the matrix forms of the operators β, α and the wave function ψ:

$$\boxed{\begin{aligned} i\hbar \frac{\partial \varphi'}{\partial t} - mc^2 \varphi' - e\phi\varphi' &= c(\vec{\sigma}\cdot\vec{\pi})\chi' \\ i\hbar \frac{\partial \chi'}{\partial t} + mc^2 \chi' - e\phi\chi' &= c(\vec{\sigma}\cdot\vec{\pi})\varphi' \end{aligned}}$$

Everything is still exact, however, the two equations are coupled by a kinetic energy term which will now be made small compared to mc^2 so we can proceed to the NR limit. The main part of the energy is mc^2 and we now split off the time dependent part in mc^2 from the wave function:

$$\psi = \begin{pmatrix} \varphi' \\ \chi' \end{pmatrix} = e^{-\frac{imc^2t}{\hbar}} \begin{pmatrix} \varphi \\ \chi \end{pmatrix}$$

and get:

$$\boxed{\begin{aligned} i\hbar \frac{\partial \varphi}{\partial t} - e\phi\varphi &= c(\vec{\sigma}\cdot\vec{\pi})\chi \\ i\hbar \frac{\partial \chi}{\partial t} + 2mc^2\chi - e\phi\chi &= c(\vec{\sigma}\cdot\vec{\pi})\varphi \end{aligned}}$$

Everything is still exact, but now notice that the χ equation contains $2mc^2\chi$ which in the NR limit will be enormous compared to the rest of the terms hence demanding that χ be small. We start the approximation then by saying that this term is so large that to a good degree of approximation we have:

$$\boxed{\chi \simeq \frac{1}{2mc}(\vec{\sigma}\cdot\vec{\pi})\varphi}$$

We obtain a better one by resubstituting in the χ equation and solving again for the $2mc^2\chi$ term:

$$\chi \simeq \frac{1}{2mc}(\vec{\sigma}\cdot\vec{\pi})\varphi - \frac{i\hbar}{4m^2c^3}\frac{\partial}{\partial t}(\vec{\sigma}\cdot\vec{\pi})\varphi + \frac{e\phi}{4m^2c^3}(\vec{\sigma}\cdot\vec{\pi})\varphi$$

Now we are assuming that all the NR energies are of order $\frac{v^2}{c^2}$, so that $e\phi \sim v^2$ and $i\hbar \frac{\partial}{\partial t} \sim v^2$. Hence, the first term above is of order 1 in its coefficient of $(\vec{\sigma}\cdot\vec{\pi})\varphi$ and the last two are of order v^2/c^2 in their coefficients. Another resubstitution would yield another order in v/c.

Now plug our value for χ in the RHS of the φ equation:

$$i\hbar \frac{\partial \varphi}{\partial t} - e\phi\varphi = \frac{1}{2m}(\vec{\sigma}\cdot\vec{\pi})(\vec{\sigma}\cdot\vec{\pi})\varphi - \frac{i\hbar}{4m^2c^2}(\vec{\sigma}\cdot\vec{\pi})\frac{\partial}{\partial t}(\vec{\sigma}\cdot\vec{\pi})\varphi$$

$$+ \frac{e}{4m^2c^2}(\vec{\sigma}\cdot\vec{\pi})\,\phi\,(\vec{\sigma}\cdot\vec{\pi})\varphi$$

We will do some convenient rearrangement. When we switch $\frac{\partial}{\partial t}(\vec{\sigma}\cdot\vec{\pi})$ to $(\vec{\sigma}\cdot\vec{\pi})\frac{\partial}{\partial t}$, we omit the the term:

$$-\frac{i\hbar e}{4m^2c^2}(\vec{\sigma}\cdot\vec{\pi})\left(-\frac{1}{c}\frac{\partial \vec{A}}{\partial t}\cdot\vec{\sigma}\right)\varphi$$ which then must be

added by putting it in the equation. When we switch $\phi(\vec{\sigma}\cdot\vec{\pi})$ to $(\vec{\sigma}\cdot\vec{\pi})\phi$, we must put in the term:

$$\frac{-e}{4m^2c^2}(\vec{\sigma}\cdot\vec{\pi})\frac{\hbar}{i}(\nabla\phi\cdot\vec{\sigma})\varphi = \frac{-i\hbar e}{4m^2c^2}(\vec{\sigma}\cdot\vec{\pi})(-\nabla\phi\cdot\vec{\sigma})$$

However, we note that $\vec{\mathcal{E}} = -\nabla\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$ so that when we make the changes and add back the "extra" terms we have:

$$i\hbar\frac{\partial\varphi}{\partial t} - e\phi\varphi = \frac{1}{2m}(\vec{\sigma}\cdot\vec{\pi})(\vec{\sigma}\cdot\vec{\pi})\varphi - \frac{1}{4m^2c^2}(\vec{\sigma}\cdot\vec{\pi})(\vec{\sigma}\cdot\vec{\pi})\left(i\hbar\frac{\partial}{\partial t} - e\phi\right)\varphi$$

$$- \frac{ie\hbar}{4m^2c^2}(\vec{\sigma}\cdot\vec{\pi})(\vec{\sigma}\cdot\vec{\mathcal{E}})\varphi$$

Consider now the vector relation:

$$(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = a_k b_l \sigma_k \sigma_l = a_k b_l(\delta_{kl} + i\epsilon_{klm}\sigma_m)$$

$$= (\vec{a}\cdot\vec{b}) + i\,\vec{\sigma}\cdot[\vec{a}\times\vec{b}]$$

We make use of this as follows:

$$(\vec{\sigma}\cdot\vec{\pi})(\vec{\sigma}\cdot\vec{\pi}) = \pi^2 + i\vec{\sigma}\cdot[\vec{\pi}\times\vec{\pi}]$$

Recall: $\vec{\pi}\times\vec{\pi} = -\frac{e\hbar}{ci}\vec{\mathcal{H}}$

We get for the first term: $\frac{\pi^2}{2m}\varphi - \frac{e\hbar}{2mc}(\vec{\sigma}\cdot\vec{\mathcal{H}})\varphi$

The coupling of the spin with the field comes out of the first order term.

The second term becomes:

$$-\left(\frac{\pi^2}{2m} - \frac{e\hbar}{2mc}(\vec{\sigma}\cdot\vec{\mathcal{H}})\right)\left(\frac{1}{2mc^2}\left[i\hbar\frac{\partial}{\partial t} - e\phi\right]\right)\varphi$$

Now we know $\frac{\pi^2}{2m} \sim (\frac{v}{c})^2$ and $\frac{e\hbar}{2mc}(\vec{\sigma}\cdot\vec{\mathcal{H}})$ is somewhat less than this, say $\sim (\frac{v}{c})^3$. Thus, if we wish to proceed no further than $(\frac{v}{c})^4$, say, in our approximation, we can write $i\hbar\frac{\partial}{\partial t} - e\phi$ as $\frac{\pi^2}{2m}$ immediately from the NR Schroedinger equation as to go any further would just bring in higher order terms than $(\frac{v}{c})^4$. As a matter of fact, $\frac{e\hbar}{2mc}(\vec{\sigma}\cdot\vec{\mathcal{H}})\cdot\frac{\pi^2}{2m} \sim (\frac{v}{c})^5$ and we can hence drop this.

The second term then simply becomes:

$$-\frac{\pi^2\pi^2}{8m^3c^2}$$

If we consider the third term, and, $(\vec{\sigma}\cdot\vec{\pi})(\vec{\sigma}\cdot\vec{\mathcal{E}}) = \vec{\pi}\cdot\vec{\mathcal{E}} + i\vec{\sigma}\cdot[\vec{\pi}\times\vec{\mathcal{E}}]$, we have:

$$-\frac{ie\hbar}{4m^2c^2}(\vec{\pi}\cdot\vec{\mathcal{E}})\varphi + \frac{e\hbar}{4m^2c^2}\left(\vec{\sigma}\cdot[\vec{\pi}\times\vec{\mathcal{E}}]\right)\varphi$$

We consider this term as being at least of order $(\frac{v}{c})^4$ as it is the spin-orbit coupling. Finally the RHS becomes:

$$\Big\{\underbrace{\frac{\pi^2}{2m}}_{(\frac{v}{c})^2} - \underbrace{\frac{1}{8m^3c^2}\pi^2\pi^2}_{(\frac{v}{c})^4} - \underbrace{\frac{e\hbar}{2mc}(\vec{\sigma}\cdot\vec{\mathcal{H}})}_{(\frac{v}{c})^3} - \underbrace{\frac{ie\hbar}{4m^2c^2}(\vec{\pi}\cdot\vec{\mathcal{E}}) + \frac{e\hbar}{4m^2c^2}\left(\vec{\sigma}\cdot[\vec{\pi}\times\vec{\mathcal{E}}]\right)}_{>(\frac{v}{c})^4}\Big\}\varphi$$

$\frac{1}{8m^3c^2}\pi^2\pi^2$ is called the mass change term from the NR limit of the classical value: $E = \sqrt{m^2c^4 + c^2\pi^2} \simeq mc^2 + \frac{\pi^2}{2m} - \frac{\pi^2\pi^2}{8m^3c^2} + \cdots$

LECTURE 13 : 10-23-61

Recall that we were working to get the NR Pauli electron wave equation from the Dirac wave equation. Recall we had:

$$\psi = e^{-imc^2 t/\hbar} \begin{pmatrix} \varphi \\ \chi \end{pmatrix}$$

with $\chi = \mathcal{O}\left(\frac{v}{c}\varphi\right)$, since $\chi \approx \frac{1}{2mc}(\vec{\sigma}\cdot\vec{\pi})\varphi$.

We finally get for the equation of the Pauli electron:

$$\boxed{\begin{aligned} i\hbar\frac{\partial\varphi}{\partial t} - e\phi\varphi &= \frac{\pi^2}{2m}\varphi - \underbrace{\frac{\pi^2\pi^2}{8m^3c^2}}_{\text{mass correction}}\varphi \\ &\underbrace{-\frac{e\hbar}{2mc}(\vec{\sigma}\cdot\vec{\mathcal{H}})\varphi}_{\text{Zeeman effect}} + \underbrace{\frac{e\hbar}{4m^2c^2}\left[\vec{\sigma}\cdot(\vec{\pi}\times\vec{\mathcal{E}})\right]}_{\text{spin-orbit coupling}}\varphi - \underbrace{\frac{ie\hbar}{4m^2c^2}(\vec{\pi}\cdot\vec{\mathcal{E}})}_{\text{Darwin Term}}\varphi \end{aligned}}$$

Recall that when getting the mass correction term we dropped $\frac{\pi^2}{2m}$ times the Zeeman term. We stipulated the order of the spin-orbit term as being $> \left(\frac{v}{c}\right)^4$ to keep it in the calculation. Actually, it is bigger than the Zeeman term.

The mass and Darwin terms are important in the hydrogen atom in that they shift the usual energy levels; however, they do not split any degeneracies and are hence rather unimportant in the heavier atoms.

Note that the Darwin term is not Hermitean. Since $i\hbar\frac{\partial\psi}{\partial t} = \underline{H}\psi$, $\frac{d}{dt}\int|\psi|^2 d\vec{r} = 0$ as $\underline{H}$ is Hermitean. However, this is true then for the total wave function ψ and does <u>not</u> have to be so for φ. In fact, it is just the Darwin term that is needed to make $\frac{d}{dt}\int|\varphi|^2 d\vec{r} = 0$ (see homework problem #2).

The non-Hermiticity of the Darwin term caused some trouble historically in trying to understand it.

Also, the Darwin term is needed for agreement with the results of the Sommerfeld fine structure formula. As we will see, it only has an effect on the s state and no others. We can use first order perturbation theory to find the effect of the Darwin term. Use as a model the hydrogen atom, where $\vec{\mathcal{E}} = -\nabla\phi$ and $\vec{\pi} = \vec{p} = -i\hbar\nabla$:

$$E^{(1)} = -\frac{ie\hbar}{4m^2c^2}\int \varphi^*(\vec{\pi}\cdot\vec{\mathcal{E}})\varphi\, d\vec{r} = \frac{e\hbar^2}{4m^2c^2}\int \varphi^*\,\nabla\cdot\left[(\nabla\phi)\varphi\right] d\vec{r}$$

$$= \frac{e\hbar^2}{4m^2c^2}\, I$$

now: $$\varphi^*\,\nabla\cdot\left[(\nabla\phi)\varphi\right] = \nabla\cdot\left\{\varphi^*\left[(\nabla\phi)\varphi\right]\right\} - \left[(\nabla\phi)\varphi\right]\cdot\nabla\varphi^*$$

Use the divergence theorem for the first term on the RHS:

$$\int_V \nabla\cdot\vec{Q}\, dV = \int_S \vec{Q}\cdot\hat{n}\, dS$$

Choose as a surface a sphere of radius a and let $a \to 0$:

$$\lim_{a\to 0}\int_\Gamma |\varphi|^2 \left(\frac{\partial\phi}{\partial r}\right)_{r=a} a^2\, d\Omega = \int \nabla\cdot\left\{\varphi^*\left[(\nabla\phi)\varphi\right]\right\} d\vec{r}$$

$$\phi = -\frac{Ze}{r} \; ; \quad \frac{\partial\phi}{\partial r} = \frac{Ze}{r^2} \; ; \quad \varphi = R_{en}(r)\, Y_{\ell m}$$

where the $Y_{\ell m}$ are normalized. Thus:

$$\lim_{a\to 0}\int Y_{\ell m}^2\, d\Omega \cdot Ze\,|R_{en}(a)|^2 = Ze\,|R_{en}(0)|^2$$

now: $$(\nabla\phi)\cdot(\nabla\varphi) = \left(\hat{e}_r\frac{\partial\phi}{\partial r}\right)\cdot(\nabla\varphi) = \frac{\partial\phi}{\partial r}\frac{\partial\varphi}{\partial r}$$

$$= \frac{Ze}{r^2}\frac{\partial}{\partial r}R_{en}(r)\, Y_{\ell m}$$

$$\int\left[(\nabla\phi)\varphi\right]\cdot\nabla\varphi^*\, d\vec{r} = \int_0^\infty R_{en}(r)\,\frac{Ze}{r^2}\frac{\partial}{\partial r}R_{en}(r)\, r^2\, dr \cdot \int_\Omega Y_{\ell m}^2\, d\Omega$$

$$= -\frac{1}{2}\, Ze\,|R_{en}(0)|^2$$

Hence $I = \frac{3}{2} Ze \, |R_{\ell n}(0)|^2$, and, writing $E^{(1)} = E_{Darwin}$:

$$\boxed{E_{Darwin} = \frac{3 Ze^2 \hbar^2}{8 m^2 c^2} \, |R_{\ell n}(0)|^2}$$

now: $R_{\ell n}(r) = \left(\frac{2Z}{n a_0}\right)^{3/2} \left\{ \frac{(n-\ell-1)!}{2n[(n+\ell)!]^3} \right\}^{1/2} e^{-\rho/2} \rho^{\ell} L_{n+\ell}^{2\ell+1}(\rho)$

where $\rho = \frac{2Zr}{n a_0}$

We see that for $\ell \neq 0$, $R_{\ell n}(0) = 0$, hence the Darwin term only effects the s states (as far as first order perturbation theory is concerned).

The Dirac Electron in a Coulomb Potential

We will take the wave function ψ for the stationary state case:

$$\psi = e^{-iEt/\hbar} \begin{pmatrix} \varphi \\ \chi \end{pmatrix}$$

and use the Dirac representation for $\vec{\alpha}$ and β:

$$\vec{\alpha} = \begin{pmatrix} \underline{0} & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \quad ; \quad \beta = \begin{pmatrix} \underline{1} & \underline{0} \\ \underline{0} & -\underline{1} \end{pmatrix}$$

and substitute these in the Dirac equation:

$$i\hbar \frac{\partial \psi}{\partial t} - \beta mc^2 \psi - e\phi\psi = c\,(\vec{\alpha} \cdot \vec{\pi})\,\psi \, .$$

We obtain:

$$\boxed{\begin{aligned} \left(E - mc^2 + \frac{Ze^2}{r}\right)\varphi &= c\,(\vec{\sigma} \cdot \vec{p})\,\chi \\ \left(E + mc^2 + \frac{Ze^2}{r}\right)\chi &= c\,(\vec{\sigma} \cdot \vec{p})\,\varphi \end{aligned}}$$

where $\phi = \phi(r) = -\frac{Ze}{r}$

The most useful approach to most quantum mechanical problems is to begin by classifying the states according to the constants of the motion. In the central field problem, the most convenient constants of the motion to use are the energy, the square of the total angular momentum, and the z component of the total angular momentum. Recall that we had found for the total angular momentum:

$$\vec{r} \times \vec{p} + \frac{\hbar}{2}\vec{\sigma} = \hbar\left(\vec{L} + \frac{\vec{\sigma}}{2}\right)$$

now, we know that $\left|\vec{r}\times\vec{p} + \frac{\hbar}{2}\vec{\sigma}\right|^2$ has as an eigenvalue $j(j+1)\hbar^2$ and $(\vec{r}\times\vec{p} + \frac{\hbar}{2}\vec{\sigma})_z$ has $m\hbar$, so j and m are quantum numbers of our problem and we will use them to help classify our solution.

Now, we will also find that l can be used as a quantum number for the states φ or χ alone. We now explore the nature of the two component wave functions with eigenvalues j, m and will see that we will have two possible choices for l when j, m are fixed. For convenience, we work with φ alone at this time, knowing that it may be χ as we will see.

$$\left|\vec{L} + \tfrac{1}{2}\vec{\sigma}\right|^2 \varphi = j(j+1)\varphi \quad ; \quad \left(L_z + \tfrac{1}{2}\sigma_z\right)\varphi = m\varphi$$

We would also like: $|\vec{L}|^2\varphi = l(l+1)\varphi$. Now expand $|\vec{L} + \frac{1}{2}\vec{\sigma}|^2\varphi$ and carry through explicitly for $l = j - 1/2$:

$$\left|\vec{L} + \tfrac{1}{2}\vec{\sigma}\right|^2\varphi = |\vec{L}|^2\varphi + \tfrac{1}{4}|\vec{\sigma}|^2\varphi + \vec{L}\cdot\vec{\sigma}\,\varphi = j(j+1)\varphi$$

now: $\vec{L}\cdot\vec{\sigma} = L_x\sigma_x + L_y\sigma_y + L_z\sigma_z = L_z\sigma_z + \frac{1}{2}(L_x + iL_y)(\sigma_x - i\sigma_y)$

$$+ \tfrac{1}{2}(L_x - iL_y)(\sigma_x + i\sigma_y) = L_z\sigma_z + \tfrac{1}{2}L_+\sigma_- + \tfrac{1}{2}L_-\sigma_+$$

Recall:

$$\boxed{\begin{array}{l} \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ; \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} ; \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\[2ex] \sigma_+ = 2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} ; \quad \sigma_- = 2\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} ; \quad \sigma^2 = 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}}$$

Then we have:

$$\underbrace{\underbrace{\ell}_{(j-1/2)}\underbrace{(\ell+1)}_{(j+1/2)}}_{j^2-1/4}\varphi + \frac{3}{4}\varphi + \left\{\begin{pmatrix} L_z & 0 \\ 0 & -L_z \end{pmatrix} + \begin{pmatrix} 0 & L_- \\ L_+ & 0 \end{pmatrix}\right\}\varphi = \underbrace{j(j+1)}_{j^2+j}\varphi$$

Hence, for $\ell = j - 1/2$:

$$\left\{\begin{pmatrix} L_z & 0 \\ 0 & -L_z \end{pmatrix} + \begin{pmatrix} 0 & L_- \\ L_+ & 0 \end{pmatrix}\right\}\varphi = (j-1/2)\,\varphi \quad ; \quad \begin{pmatrix} 0 & L_- \\ L_+ & 0 \end{pmatrix} \text{ switches components of } \varphi$$

Recall:

$$\boxed{\begin{aligned} L_+ Y_j^m &= \sqrt{(j-m)(j+m+1)}\; Y_j^{m+1} \\ L_- Y_j^m &= \sqrt{(j+m)(j-m+1)}\; Y_j^{m-1} \\ L^2 Y_j^m &= j(j+1)\, Y_j^m \\ L_z Y_j^m &= m\, Y_j^m \end{aligned}}$$

Hence, we immediately see that we can choose for φ:

$$\varphi = \begin{pmatrix} a\, Y_{j-1/2}^{\mu} \\ b\, Y_{j-1/2}^{\mu+1} \end{pmatrix}$$

where a, b include the radial function and appropriate constants. Now:

$$\left(L_z + \tfrac{1}{2}\sigma_z\right)\varphi = \begin{pmatrix} L_z + 1/2 & 0 \\ 0 & L_z - 1/2 \end{pmatrix}\varphi = \begin{pmatrix} (\mu+\frac{1}{2})\,a\, Y_{j-1/2}^{\mu} \\ (\mu+1/2)\,b\, Y_{j-1/2}^{\mu+1} \end{pmatrix}$$

We then see that a more appropriate choice for φ would be to let $\mu \to m - 1/2$, for then $(L_z + \frac{1}{2}\sigma_z)\varphi = m\varphi$ and:

$$\varphi = \begin{pmatrix} a\, Y_{j-1/2}^{m-1/2} \\ b\, Y_{j-1/2}^{m+1/2} \end{pmatrix}$$

LECTURE 14 : 10-25-61

Recall:
$$\left\{ \begin{pmatrix} L_z & 0 \\ 0 & -L_z \end{pmatrix} + \begin{pmatrix} 0 & L_- \\ L_+ & 0 \end{pmatrix} \right\} \varphi = (j - 1/2)\, \varphi$$

$$\varphi = \begin{pmatrix} a \; Y_{j-1/2}^{m-1/2} \\ b \; Y_{j-1/2}^{m+1/2} \end{pmatrix}$$

Now operate with the { } operator: We obtain the two equations:

$$(m - 1/2)\, a + \sqrt{(j+m)(j-m)}\; b = (j - 1/2)\, a$$

$$\sqrt{(j+m)(j-m)}\; a - (m + 1/2)\, b = (j - 1/2)\, b$$

or:
$$-(j-m)\, a + \sqrt{(j+m)(j-m)}\; b = 0$$

$$\sqrt{(j+m)(j-m)}\; a - (j+m)\, b = 0$$

Then:
$$\frac{a}{b} = \sqrt{\frac{j+m}{j-m}}$$

If we demand normalization, this fixes a and b, that is, $a^2 + b^2 = 1$, or:

$$b^2 + b^2\, \frac{j+m}{j-m} = 1 \quad ; \quad b^2 = \frac{j-m}{2j} \quad ; \quad a^2 = \frac{j+m}{2j}$$

Hence we finally get for φ, including the radial function:

$$l = j - 1/2 \; ; \quad \varphi_{j,\, j-1/2,\, m} = \begin{pmatrix} \sqrt{\dfrac{j+m}{2j}} \; Y_{j-1/2}^{m-1/2} \; f(r) \\ \sqrt{\dfrac{j-m}{2j}} \; Y_{j-1/2}^{m+1/2} \; f(r) \end{pmatrix}$$

$$= \varphi_{l+1/2,\, l,\, \mu+1/2} = \begin{pmatrix} \sqrt{\dfrac{l+\mu+1}{2l+1}} \; Y_l^{\mu} \; f(r) \\ \sqrt{\dfrac{l-\mu}{2l+1}} \; Y_l^{\mu+1} \; f(r) \end{pmatrix}$$

We now redo the calculation explicitly for $l = j + 1/2$:

$$\underbrace{\underbrace{l}_{(j+1/2)}\underbrace{(l+1)}_{(j+3/2)}}_{j^2+2j+\frac{3}{4}}\varphi + \frac{3}{4}\varphi + \vec{\sigma}\cdot\vec{L}\,\varphi = \underbrace{j(j+1)}_{j^2+j}\varphi$$

$$\therefore \quad \vec{\sigma}\cdot\vec{L}\,\varphi = (-j-\tfrac{3}{2})\varphi = -(j+\tfrac{3}{2})\varphi$$

$$\varphi = \begin{pmatrix} a\; Y_{j+1/2}^{\mu} \\ b\; Y_{j+1/2}^{\mu+1} \end{pmatrix} = \begin{pmatrix} a\; Y_{j+1/2}^{m-1/2} \\ b\; Y_{j+1/2}^{m+1/2} \end{pmatrix}$$

Operating with $\vec{\sigma}\cdot\vec{L}$, we obtain:

$$(m-1/2)a + \sqrt{(j-m+1)(j+m+1)}\,b = -(j+3/2)a$$

$$\sqrt{(j+m+1)(j-m+1)}\,a - (m+1/2)b = -(j+3/2)b$$

or:

$$(j+m+1)a + \sqrt{(j+m+1)(j-m+1)}\,b = 0$$

$$\sqrt{(j+m+1)(j-m+1)}\,a + (j-m+1)b = 0$$

$$\therefore \quad \frac{a}{b} = -\sqrt{\frac{(j-m+1)}{(j+m+1)}}$$

$$a^2+b^2 = b^2 + b^2\frac{(j-m+1)}{(j+m+1)} = b^2\frac{2(j+1)}{(j+m+1)}$$

Then we can write: $a = \sqrt{\dfrac{(j-m+1)}{2(j+1)}}$; $b = -\sqrt{\dfrac{(j+m+1)}{2(j+1)}}$

Then:

$$\varphi_{j,j+1/2,m} = \begin{pmatrix} \sqrt{\dfrac{j-m+1}{2(j+1)}}\; Y_{j+1/2}^{m-1/2}\, f(r) \\ -\sqrt{\dfrac{j+m+1}{2(j+1)}}\; Y_{j+1/2}^{m+1/2}\, f(r) \end{pmatrix}$$

$$= \varphi_{l-1/2,l,\mu+1/2} = \begin{pmatrix} \sqrt{\dfrac{l-\mu}{2l+1}}\; Y_l^{\mu}\, f(r) \\ -\sqrt{\dfrac{l+\mu+1}{2l+1}}\; Y_l^{\mu+1}\, f(r) \end{pmatrix}$$

We see that $\varphi_{l+1/2,\,l,\,\mu+1/2}$ and $\varphi_{l-1/2,\,l,\,\mu+1/2}$ are orthogonal. Now we assume that φ will be in terms of one of the φ's above and χ will be in terms of the other, with a different $f(r)$, of course. We will now see that the choice of one for φ leads to the other for χ.

If we consider the equations for χ and φ for a moment;

$$\left(E - mc^2 + \frac{Ze^2}{r}\right)\varphi = c(\vec{\sigma}\cdot\vec{p})\chi$$

$$\left(E + mc^2 + \frac{Ze^2}{r}\right)\chi = c(\vec{\sigma}\cdot\vec{p})\varphi$$

and now suppose that we multiply the top equation by φ and the bottom by χ and integrate, we then will get matrix elements on the RHS in the form: $(\varphi, \vec{p}\chi)$ and $(\chi, \vec{p}\varphi)$. Clearly if the equations are to be coupled, these matrix elements cannot vanish.

For now we will assume:

χ transforms like $Y_{l'}$
φ transforms like Y_l
and we know $\vec{p}$ transforms like Y_1

By elementary group theory, the direct product of the representations of $\vec{p}$ and φ must contain that of χ in order for the interaction not to vanish. Then,

$$D^{(1)} \times D^{(l)} = D^{(l+1)} + D^{(l)} + D^{(l-1)}$$

Thus, $l' = l+1, l, l-1$; however, in the case of a single electron, $\Delta l = 0$ is forbidden, thus $\Delta l = \pm 1$.

Now, if we choose the subscripts on ψ to be representative of those on φ, then we have for $\psi_{\dots,l,\dots}$:

$$\varphi \sim Y_l\ ,\quad \chi \sim Y_{l+1} \text{ or } Y_{l-1}$$

so there are in fact two choices.

Now for $\psi_{j,\, l-1/2,\, m} = \psi_{l+1/2,\, l,\, \mu+1/2}$ we choose $\varphi_{l+1/2,\, l,\, \mu+1/2}$ to represent φ and $\varphi_{l-1/2,\, l,\, \mu+1/2}$ with $l \to l+1$ to represent χ. We then have:

$$\psi_{l+1/2,\, l,\, \mu+1/2} = \begin{pmatrix} \sqrt{\dfrac{l+\mu+1}{2l+1}} \; Y_l^{\mu} \; f(r) \\ \sqrt{\dfrac{l-\mu}{2l+1}} \; Y_l^{\mu+1} \; f(r) \\ \sqrt{\dfrac{l-\mu+1}{2l+3}} \; Y_{l+1}^{\mu} \; g(r) \\ \sqrt{\dfrac{l+\mu+2}{2l+3}} \; Y_{l+1}^{\mu+1} \; g(r) \end{pmatrix}$$

For $\psi_{j,\, j+1/2,\, m} = \psi_{l-1/2,\, l,\, \mu+1/2}$ we choose $\varphi_{l-1/2,\, l,\, \mu+1/2}$ to represent φ and $\varphi_{l+1/2,\, l,\, \mu+1/2}$ with $l \to l-1$ to represent χ. We then have:

$$\psi_{l-1/2,\, l,\, \mu+1/2} = \begin{pmatrix} \sqrt{\dfrac{l-\mu}{2l+1}} \; Y_l^{\mu} \; f(r) \\ -\sqrt{\dfrac{l+\mu+1}{2l+1}} \; Y_l^{\mu+1} \; f(r) \\ \sqrt{\dfrac{l+\mu}{2l-1}} \; Y_{l-1}^{\mu} \; g(r) \\ \sqrt{\dfrac{l-\mu-1}{2l-1}} \; Y_{l-1}^{\mu+1} \; g(r) \end{pmatrix}$$

Now in order to circumvent becoming involved in spherical harmonics and recurrence relations, we take a page from Dirac and consider building some convenient operators. Consider what happens when we form:

$$(\vec{\sigma}\cdot\vec{r})(\vec{\sigma}\cdot\vec{p}) = (\vec{r}\cdot\vec{p}) + i\hbar\,(\vec{\sigma}\cdot\vec{L})$$

We see in this operator, $\vec{r}\cdot\vec{p}$ operates only on the radial part of the wave function while $\vec{\sigma}\cdot\vec{L}$ operates only on the spherical harmonics. What we are trying to do is develop a form of $\vec{\sigma}\cdot\vec{p}$ which is convenient to operate on ψ with.

now operate again, but this time with $\frac{\vec{\sigma}\cdot\vec{r}}{r^2}$.

First, note that $\vec{r}\cdot\vec{p} = -i\hbar r \frac{\partial}{\partial r}$ and $(\vec{\sigma}\cdot\vec{r})(\vec{\sigma}\cdot\vec{r}) = r^2$

Then:

$$\frac{\vec{\sigma}\cdot\vec{r}}{r^2}\ (\vec{\sigma}\cdot\vec{r})(\vec{\sigma}\cdot\vec{p}) = \vec{\sigma}\cdot\vec{p}$$

$$= -i\hbar\ \frac{\vec{\sigma}\cdot\vec{r}}{r}\frac{\partial}{\partial r} + \frac{i\hbar}{r}\ \frac{\vec{\sigma}\cdot\vec{r}}{r}\ (\vec{\sigma}\cdot\vec{L})$$

note that $\frac{(\vec{\sigma}\cdot\vec{r})^2}{r^2}$ is a unit operator

now: $\vec{r} = \hat{i}\, r\cos\varphi\sin\theta + \hat{j}\, r\sin\varphi\sin\theta + \hat{k}\, r\cos\theta$

Then:

$$\frac{\vec{\sigma}\cdot\vec{r}}{r} = \sigma_x \cos\varphi\sin\theta + \sigma_y \sin\varphi\sin\theta + \sigma_z\cos\theta$$

$$= \begin{pmatrix} 0 & \cos\varphi\sin\theta \\ \cos\varphi\sin\theta & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i\sin\varphi\sin\theta \\ i\sin\varphi\sin\theta & 0 \end{pmatrix} + \begin{pmatrix} \cos\theta & 0 \\ 0 & -\cos\theta \end{pmatrix}$$

Hence:

$$\frac{\vec{\sigma}\cdot\vec{r}}{r} = \begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\varphi} \\ \sin\theta\, e^{i\varphi} & -\cos\theta \end{pmatrix}$$

We see that $\frac{\vec{\sigma}\cdot\vec{r}}{r}$ commutes with $\frac{\partial}{\partial r}$. We will next apply this on the wave function.

LECTURE 15 : 10-27-61

We reformulate some of the results of the previous lecture in a more compact form. So far in our solution of our coulomb potential problem, we have:

$$\left(E - mc^2 + \frac{Ze^2}{r}\right)\varphi = c\,(\vec{\sigma}\cdot\vec{p})\,\chi$$

$$\left(E + mc^2 + \frac{Ze^2}{r}\right)\chi = c\,(\vec{\sigma}\cdot\vec{p})\,\varphi$$

with the φ's and χ's of the form: $\varphi_{j,l,m} = Y_{j,l,m}\, f(r)$

We now compact the notation for the spherical harmonics and write:

$$\mathcal{Y}^{(1)}_{l+1/2,\,l,\,\mu+1/2} = \begin{pmatrix} \sqrt{\dfrac{l+\mu+1}{2l+1}}\; Y_l^{\mu} \\ \sqrt{\dfrac{l-\mu}{2l+1}}\; Y_l^{\mu+1} \end{pmatrix}$$; $\vec{\sigma}\cdot\vec{L} = j - 1/2$ operating on this $\mathcal{Y}$

$$\mathcal{Y}^{(2)}_{l-1/2,\,l,\,\mu+1/2} = \begin{pmatrix} \sqrt{\dfrac{l-\mu}{2l+1}}\; Y_l^{\mu} \\ -\sqrt{\dfrac{l+\mu+1}{2l+1}}\; Y_l^{\mu+1} \end{pmatrix}$$; $\vec{\sigma}\cdot\vec{L} = -(j+3/2)$ operating on this $\mathcal{Y}$

Now, there are two possibilities for the total wave function, viz.,

$$\left.\begin{matrix} \psi_{j,\,j-1/2,\,m} \\ \psi_{l+1/2,\,l,\,\mu+1/2} \end{matrix}\right\} = \begin{pmatrix} \mathcal{Y}^{(1)}_{l+1/2,\,l,\,\mu+1/2} & f(r) \\ \mathcal{Y}^{(2)}_{l+1/2,\,l+1,\,\mu+1/2} & g(r) \end{pmatrix} = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}_{j-1/2}$$

$$\left.\begin{matrix} \psi_{j,\,j+1/2,\,m} \\ \psi_{l-1/2,\,l,\,\mu+1/2} \end{matrix}\right\} = \begin{pmatrix} \mathcal{Y}^{(2)}_{l-1/2,\,l,\,\mu+1/2} & f(r) \\ \mathcal{Y}^{(1)}_{l-1/2,\,l-1,\,\mu+1/2} & g(r) \end{pmatrix} = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}_{j+1/2}$$

We return to consideration of the operator $(\vec{\sigma}\cdot\vec{p})$.

since $\frac{\vec{\sigma}\cdot\vec{r}}{r}$ commutes with $\frac{\partial}{\partial r}$, we have:

$$\vec{\sigma}\cdot\vec{p} = \frac{\hbar}{i}\frac{\partial}{\partial r}\left(\frac{\vec{\sigma}\cdot\vec{r}}{r}\right) + \frac{i\hbar}{r}\frac{(\vec{\sigma}\cdot\vec{r})}{r}(\vec{\sigma}\cdot\vec{L})$$

now notice that $\left(\frac{\vec{\sigma}\cdot\vec{r}}{r}\right)^2$ is unity, and $\frac{\vec{\sigma}\cdot\vec{r}}{r}$ is Hermitean, then:

$\frac{\vec{\sigma}\cdot\vec{r}}{r}$ is both unitary and Hermitean.

We will now see that $\frac{\vec{\sigma}\cdot\vec{r}}{r}$ commutes with all the components of the angular momentum.

$$\left[L_k + \tfrac{1}{2}\sigma_k, \frac{\vec{\sigma}\cdot\vec{r}}{r}\right] = \left[L_k + \frac{\sigma_k}{2}, \frac{\sigma_n x_n}{r}\right]$$

$$= \sigma_n\left[L_k, \frac{x_n}{r}\right] + \left[\frac{\sigma_k}{2}, \sigma_n\right]\frac{x_n}{r}$$

$$= i\sigma_n \epsilon_{kns}\frac{x_s}{r} + i\sigma_s \epsilon_{kns}\frac{x_n}{r} = 0$$

recalling some of the properties of the commutators above, namely:

$$L_k = \frac{1}{\hbar}\epsilon_{kjl}\, x_j p_l \; ; \quad [L_k, x_n] = \frac{1}{\hbar}[x_j p_l, x_n]\,\epsilon_{kjl}$$

$$= \epsilon_{kjl}\, x_j \cdot \frac{1}{\hbar}[p_l, x_n] = -i\,\epsilon_{kjn}\, x_j = i\,\epsilon_{knj}\, x_j$$

and $[\sigma_k, \sigma_n] = \sigma_k\sigma_n - \sigma_n\sigma_k$

$$= (\delta_{kn} + i\,\epsilon_{kns}\sigma_s) - (\delta_{nk} + i\,\epsilon_{nks}\sigma_s) = 2i\,\epsilon_{kns}\sigma_s$$

now, from the fact that $\frac{\vec{\sigma}\cdot\vec{r}}{r}$ is independent of r and commutes with all the components of the total angular momentum leads us to wonder and suspect that its operation on the spherical harmonics has a meaning. It certainly would for it commutes with $L_z + \frac{\sigma_z}{2}$ and also the ladder operators and hence would have the same eigenfunctions as these. Note that it doesn't have any single eigenfunction, which leads us to think of a more general relation.

Recalling that "adjacent" spherical harmonics are connected by recurrence relations, let us consider if:

$$\left(\frac{\vec{\sigma}\cdot\vec{r}}{r}\right) \mathcal{Y}^{(1)}_{l+1/2,\, l,\, \mu+1/2} = c \;\; \mathcal{Y}^{(2)}_{l+1/2,\, l+1,\, \mu+1/2}$$

has any meaning. We will try to determine c, considering for simplicity the case for $\mu = 0$.
From Condon and Shortley:

$$Y_l^\mu = \left(\frac{-\mu}{|\mu|}\right)^\mu \sqrt{\frac{2l+1}{4\pi}} \sqrt{\frac{(l-|\mu|)!}{(l+|\mu|)!}} \; P_l^{|\mu|}(\cos\theta)\, e^{i\mu\varphi}$$

This is in the form necessary for the ladder operators to work. Also:

$$P_l^{|\mu|}(\cos\theta) = (\sin\theta)^{|\mu|} \frac{d^{|\mu|}}{d(\cos\theta)^{|\mu|}} P_l(\cos\theta)$$

Then:

$$Y_l^0 = \sqrt{\frac{2l+1}{4\pi}} \; P_l(\cos\theta)$$

$$Y_l^1 = -\sqrt{\frac{2l+1}{4\pi}} \sqrt{\frac{(l-1)!}{(l+1)!}} \; P_l^1(\cos\theta)\, e^{i\varphi}$$

$$= -\sin\theta \sqrt{\frac{2l+1}{4\pi}} \sqrt{\frac{1}{l(l+1)}} \; P_l'(\cos\theta)\, e^{i\varphi}$$

$$\mathcal{Y}^{(1)}_{l+1/2,\, l,\, +1/2} = \sqrt{\frac{1}{4\pi}} \cdot \sqrt{\frac{1}{l+1}} \begin{pmatrix} (l+1)\, P_l(\cos\theta) \\ -\sin\theta\, P_l'(\cos\theta)\, e^{i\varphi} \end{pmatrix}$$

$$\mathcal{Y}^{(2)}_{l+1/2,\, l+1,\, +1/2} = \sqrt{\frac{1}{4\pi}} \cdot \sqrt{\frac{1}{l+1}} \begin{pmatrix} (l+1)\, P_{l+1}(\cos\theta) \\ +\sin\theta\, P_{l+1}'(\cos\theta)\, e^{i\varphi} \end{pmatrix}$$

Recall:

$$\left(\frac{\vec{\sigma}\cdot\vec{r}}{r}\right) = \begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\varphi} \\ \sin\theta\, e^{i\varphi} & -\cos\theta \end{pmatrix} = \begin{pmatrix} \omega & (1-\omega^2)^{1/2} e^{-i\varphi} \\ (1-\omega^2)^{1/2} e^{i\varphi} & -\omega \end{pmatrix}$$

letting $\omega = \cos\theta$.

Then operation gives:

$$(l+1)\, w\, P_l(w) - (1-w^2)\, P_l'(w) = c\,(l+1)\, P_{l+1}(w)$$

$$(l+1)\, P_l(w) + w\, P_l'(w) = c\, P_{l+1}'(w)$$

Multiply the last equation by w and subtract and get:

$$c\,(l+1)\, P_{l+1}(w) - c\, w\, P_{l+1}'(w) + P_l'(w) = 0$$

Change $l \to l-1$ and multiply by -1:

$$c\, w\, P_l'(w) - P_{l-1}'(w) - c\, l\, P_l(w) = 0$$

This corresponds to a well known recursion relation from Pauling and Wilson, eq. 19-4:

$$z\, P_l'(z) - P_{l-1}'(z) - l\, P_l(z) = 0$$

hence $c = +1$. Thus:

$$\left(\frac{\vec{\sigma}\cdot\vec{r}}{r}\right) \mathcal{Y}^{(1)}_{l+1/2,\, l,\, \mu+1/2} = \mathcal{Y}^{(2)}_{l+1/2,\, l+1,\, \mu+1/2}$$

and, since $\left(\frac{\vec{\sigma}\cdot\vec{r}}{r}\right)^2 = 1$

$$\left(\frac{\vec{\sigma}\cdot\vec{r}}{r}\right) \mathcal{Y}^{(2)}_{l+1/2,\, l+1,\, \mu+1/2} = \mathcal{Y}^{(1)}_{l+1/2,\, l,\, \mu+1/2}$$

now, for $l = j - 1/2$; $\varphi = \mathcal{Y}^{(1)}_{l+1/2,\, l,\, \mu+1/2}\, f(r)$

$$\chi = \mathcal{Y}^{(2)}_{l+1/2,\, l+1,\, \mu+1/2}\, g(r)$$

We must operate with:

$$\boxed{\vec{\sigma}\cdot\vec{p} = \frac{\hbar}{i}\frac{d}{dr}\left(\frac{\vec{\sigma}\cdot\vec{r}}{r}\right) + \frac{i\hbar}{r}\left(\frac{\vec{\sigma}\cdot\vec{r}}{r}\right)(\vec{\sigma}\cdot\vec{L})}$$

and recalling:

$$(\vec{\sigma}\cdot\vec{L})\begin{pmatrix} \mathcal{Y}^{(1)}_{l+1/2,\, l,\, \mu+1/2} \\ \mathcal{Y}^{(2)}_{l-1/2,\, l,\, \mu+1/2} \end{pmatrix} = \begin{pmatrix} j-1/2 & \\ & -(j+3/2) \end{pmatrix}\begin{pmatrix} \mathcal{Y}^{(1)}_{l+1/2,\, l,\, \mu+1/2} \\ \mathcal{Y}^{(2)}_{l-1/2,\, l,\, \mu+1/2} \end{pmatrix}$$

we then have for the Dirac equations for $l = j - 1/2$:

$$\boxed{\left.\begin{aligned} \left(E - mc^2 + \frac{Ze^2}{r}\right) f(r) &= \frac{\hbar c}{i}\frac{d\,g(r)}{dr} - \frac{i\hbar c}{r}(j+3/2)\,g(r) \\ \left(E + mc^2 + \frac{Ze^2}{r}\right) g(r) &= \frac{\hbar c}{i}\frac{d\,f(r)}{dr} + \frac{i\hbar c}{r}(j-1/2)\,f(r) \end{aligned}\right\} l = j - 1/2}$$

For $l = j + 1/2$:

$$\varphi = \mathcal{Y}^{(2)}_{l-1/2,\, l,\, \mu+1/2}\; f(r)$$
$$\chi = \mathcal{Y}^{(1)}_{l-1/2,\, l-1,\, \mu+1/2}\; g(r)$$

Then:

$$\boxed{\left.\begin{aligned} \left(E - mc^2 + \frac{Ze^2}{r}\right) f(r) &= \frac{\hbar c}{i}\frac{d\,g(r)}{dr} + \frac{i\hbar c}{r}(j-1/2)\,g(r) \\ \left(E + mc^2 + \frac{Ze^2}{r}\right) g(r) &= \frac{\hbar c}{i}\frac{d\,f(r)}{dr} - \frac{i\hbar c}{r}(j+3/2)\,f(r) \end{aligned}\right\} l = j + 1/2}$$

We now introduce a new quantum number k in order to compress the notation:

$$\boxed{|k| = j + 1/2}$$

$k = 1, 2, 3, \cdots$ for $j = l - 1/2$
$k = -1, -2, -3, \ldots$ for $j = l + 1/2$, Then:

$$\boxed{\begin{aligned} \left(E - mc^2 + \frac{Ze^2}{r}\right) f(r) &= \frac{\hbar c}{i}\frac{d\,g(r)}{dr} + (k-1)\frac{i\hbar c}{r}\,g(r) \\ \left(E + mc^2 + \frac{Ze^2}{r}\right) g(r) &= \frac{\hbar c}{r}\frac{d\,f(r)}{dr} - (k+1)\frac{i\hbar c}{r}\,f(r) \end{aligned}}$$

Other notation for k: Dirac: k Kramers: $-k$
Rose: κ Furry: k

LECTURE 16 : 10-30-61

Recall:
$$\psi_{j,l,m} = \begin{pmatrix} \varphi \\ \chi \end{pmatrix} = \begin{pmatrix} \mathcal{Y}^{(1,2)}_{j,l,m} \quad f(r) \\ \mathcal{Y}^{(1,2)}_{j,\,l\pm 1,\,m} \quad g(r) \end{pmatrix}$$

Dirac's Method of Obtaining the Coupled Wave Equations for the Central Field Model:

We have found:

$$(\vec{\sigma}\cdot\vec{L})\,\mathcal{Y}_{j,j-1/2,m} = (j-1/2)\,\mathcal{Y}_{j,j-1/2,m}$$

$$(\vec{\sigma}\cdot\vec{L})\,\mathcal{Y}_{j,j+1/2,m} = -(j+3/2)\,\mathcal{Y}_{j,j+1/2,m}$$

We proceeded on the basis of finding rules for the operation of $\vec{\sigma}\cdot\vec{p}$ on these spherical harmonics. Dirac's approach was to look for a new way to classify states by finding another constant of the motion. Consider $\vec{\sigma}\cdot\vec{L}$. However this does not commute with the Hamiltonian. On the other hand, Dirac noticed that:

$$\mathcal{K} = \beta\left\{(\vec{\sigma}\cdot\vec{L})+1\right]$$

does and we consider its eigenvalue to be k and we classify the states by j, m, k and whatever the radial quantum is. The wave equation is:

$$\left(E - \beta mc^2 + \frac{Ze^2}{r} - c(\vec{\alpha}\cdot\vec{p})\right)\psi = 0$$

where now:

$$(\vec{\alpha}\cdot\vec{p}) = \left(\frac{\vec{\alpha}\cdot\vec{r}}{r}\right)\frac{\hbar}{i}\frac{\partial}{\partial r} + \frac{i\hbar}{r}\left(\frac{\vec{\alpha}\cdot\vec{r}}{r}\right)(\vec{\sigma}\cdot\vec{L})$$

found by the same method as $\vec{\sigma}\cdot\vec{p}$. We can define the new operator:

$$\epsilon = \frac{\vec{\alpha}\cdot\vec{r}}{r}$$

and write $\vec{\sigma}\cdot\vec{L} = \beta\mathcal{K} - 1$.

We must find a representation for ϵ. We know from the anticommutation rules for $\vec{\alpha}$ and β that:

$$\{\epsilon, \beta\} = 0$$

and also that $\epsilon = \frac{\vec{\alpha}\cdot\vec{r}}{r}$ is Hermitean. Also, since only 2 matrices appear in the equation, we need only a 2×2 representation for ϵ, β and a two element form for ψ. We choose β as usual and take ϵ so that it satisfies the requirements and gives our former answer:

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad ; \quad \epsilon = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

We take this for ϵ instead of the equally good $\epsilon = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ as we want our former answer. We have:

$$(\vec{\alpha}\cdot\vec{p})\psi = \frac{\hbar}{i}\frac{d}{dr}\,\epsilon\begin{pmatrix} \varphi \\ \chi \end{pmatrix} + \frac{i\hbar}{r}\,\epsilon\,(\beta k - 1)\begin{pmatrix} \varphi \\ \chi \end{pmatrix}$$

$$= \frac{\hbar}{i}\frac{d}{dr}\begin{pmatrix} \chi \\ \varphi \end{pmatrix} + \frac{i\hbar}{r}\left[k\begin{pmatrix} -\chi \\ \varphi \end{pmatrix} - \begin{pmatrix} \chi \\ \varphi \end{pmatrix}\right]$$

To put into the Furry form, we would have to go back and redefine k as its negative. Here we will just let $k \to -k$ and obtain as before; splitting off the angular parts:

$$\left(E - mc^2 + \frac{Ze^2}{r}\right) f = \frac{\hbar c}{i}\frac{dg}{dr} + (k-1)\frac{i\hbar}{r} g$$

$$\left(E + mc^2 + \frac{Ze^2}{r}\right) g = \frac{\hbar c}{i}\frac{df}{dr} - (k+1)\frac{i\hbar}{r} f$$

We now introduce the Dirac scales of length:

$$a_1 = \frac{\hbar c}{E + mc^2} \qquad \left(\text{in NR limit, very near Compton wavelength, } \hbar/mc \sim .004\,\text{Å, very small.}\right)$$

$$a_2 = \frac{\hbar c}{-E + mc^2} \qquad (\text{in NR limit and otherwise, } a_2 \gg a_1)$$

$$a = \sqrt{a_1 a_2} = \frac{\hbar c}{\sqrt{(mc^2)^2 - E^2}} \approx \frac{\hbar}{\sqrt{2m(mc^2 - E)}} \approx \frac{\hbar}{mv} \quad \text{or about The Bohr radius}$$

Note that since a_2 is negative, a is pure imaginary.
Also define:

$$\gamma = \frac{Ze^2}{\hbar c} = Z\alpha = \frac{Z}{137}$$

where α is the fine structure constant.
Recapitulate:

$$\boxed{\begin{array}{ll} a_1 = \dfrac{\hbar c}{E + mc^2} \quad ; & a_2 = \dfrac{\hbar c}{-E + mc^2} \\ a = \sqrt{a_1 a_2} \quad ; & \gamma = \dfrac{Ze^2}{\hbar c} \end{array}}$$

Substitution yields:

$$\boxed{\begin{array}{l} \left(-\dfrac{1}{a_2} + \dfrac{\gamma}{r}\right) f(r) = -i \dfrac{dg(r)}{dr} + i\,(k-1)\,\dfrac{g(r)}{r} \\ \left(\dfrac{1}{a_1} + \dfrac{\gamma}{r}\right) g(r) = -i \dfrac{df(r)}{dr} - i\,(k+1)\,\dfrac{f(r)}{r} \end{array}}$$

Examine the asymptotic solution for large r, that is, $r \gg a_2$:

$$\left.\begin{array}{l} \dfrac{dg}{dr} = -i \dfrac{f}{a_2} \\ \dfrac{df}{dr} = i \dfrac{g}{a_1} \end{array}\right\} \quad \begin{array}{ll} f'' = \dfrac{f}{a_1 a_2} = \dfrac{f}{a^2} \quad ; & f \sim e^{-r/a} \\ g'' = \dfrac{g}{a_1 a_2} = \dfrac{g}{a^2} \quad ; & g \sim e^{-r/a} \end{array}$$

We take the minus to give convergence at infinity. If the argument is imaginary, sign does not make any difference so choose $-$ sign anyhow. Recall that it is customary to take $E < 0$ (or here mc^2) for the hydrogen atom, so a will turn out real for the bound case. Recall from the NR case that the solution went as r^{-1} also, so we suspect it will help to take the solutions as:

$$\boxed{\begin{array}{l} f = r^{-1} e^{-r/a} F \\ g = -i\, r^{-1} e^{-r/a} G \end{array}}$$

We use the $-i$ in g to eliminate the i's in the equations.

Then we have:

$$\boxed{\begin{aligned} &\left(-\frac{1}{a_2} + \frac{\gamma}{r}\right) F + \left(\frac{d}{dr} - \frac{k}{r} - \frac{1}{a}\right) G = 0 \\ &\left(\frac{1}{a_1} + \frac{\gamma}{r}\right) G + \left(\frac{d}{dr} + \frac{k}{r} - \frac{1}{a}\right) F = 0 \end{aligned}}$$

Note on notation: Dirac: f, g, α, j
Furry: G, F, γ, k

The above equations could be solved by contour integration exactly. Here we will use the usual series expansion method which is exact also.
Write:

$$\boxed{F = \sum_{s=s_0}^{\infty} c_s' r^s \quad ; \quad G = \sum_{s=s_0}^{\infty} c_s r^s}$$

Substituting and equating coefficients of r^{s-1}:

$$(s-k)\, c_s + \gamma c_s' = \frac{1}{a} c_{s-1} + \frac{1}{a_2} c_{s-1}'$$

$$-\gamma c_s + (s+k)\, c_s' = \frac{1}{a_1} c_{s-1} + \frac{1}{a} c_{s-1}' = \frac{a}{a_1}\left(\frac{1}{a} c_{s-1} + \frac{1}{a_2} c_{s-1}'\right)$$

Now for $s = s_0$, the right-hand side of each equation vanishes and for a non-trivial solution, the coefficient determinant of the LHS must vanish. This gives for the indicial equation:

$$s_0^2 - k^2 + \gamma^2 = 0 \quad ; \qquad s_0 = \sqrt{k^2 - \gamma^2} \quad ; \text{ real, since } \gamma = \frac{Z}{137} < 1$$

We take the + root because we want no singularities stronger than r^{-1}.
Multiply the upper equation by a and the lower by a_1 and subtract:

$$\left[a(s-k) + a_1 \gamma\right] c_s = \left[a_1 (s+k) - a\gamma\right] c_s'$$

Note that in the limit of high s: $\frac{c_s'}{c_s} \approx \sqrt{\frac{a_2}{a_1}} > 1$; $c_s' > c_s$
as befits the large wave part

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Note the strange fact that the reality of $s_0 = \sqrt{k^2 - \gamma^2}$ is governed by $\alpha Z < 1$, so far; that is, Z would have to be about 137 for trouble to occur. It is strange that this type of dependence is present.

We now see if we can develop a single series out of the results of the last lecture. Define:

$$\boxed{q_s = c_s + \frac{a}{a_2} c_s' = c_s + \frac{a_1}{a} c_s'}$$

We will try to develop a recurrence relation in terms of the q_s's. We have the relations:

$$\left[a(s-k) + a_1\gamma\right] c_s - \left[a_1(s+k) - a\gamma\right] c_s' = 0$$

$$c_s' = \frac{a_2}{a} q_s - \frac{a_2}{a} c_s = \frac{a}{a_1} q_s - \frac{a}{a_1} c_s$$

$$c_s = q_s - \frac{a}{a_2} c_s' = q_s - \frac{a_1}{a} c_s'$$

$$c_s = \frac{(s+k) - \frac{a}{a_1}\gamma}{2s + \left(\frac{a_1}{a} - \frac{a}{a_1}\right)\gamma} q_s$$

$$c_s' = \frac{(s-k) + \frac{a_1}{a}\gamma}{2s + \left(\frac{a_1}{a} - \frac{a}{a_1}\right)\gamma} \frac{a}{a_1} q_s$$

Note that our original equations can now be written:

$$(s-k) c_s + \gamma c_s' = \frac{1}{a} q_{s-1}$$

$$-\gamma c_s + (s+k) c_s' = \frac{1}{a_1} q_{s-1}$$

We now substitute in one of these equations the c_s, c_s' in terms of q_s and arrive at a recurrence relation for q_s.

We have:

$$q_s = \frac{2s + \left(\frac{a_1}{a} - \frac{a}{a_1}\right)\gamma}{s^2 - k^2 + \gamma^2} \; \frac{1}{a} \, q_{s-1}$$

where $s^2 - k^2 + \gamma^2 = (s - s_0)(s + s_0) = (s - s_0)\left(2s_0 + [s - s_0]\right)$

Now recall some facts about the confluent hypergeometric function:

$${}_1F_1(a; b; x) = 1 + \frac{a}{1!\, b} x + \frac{a(a+1)}{2!\, b(b+1)} x^2 + \cdots = \sum_{k=0}^{\infty} c_k x^k$$

where:

$$c_k = \frac{k - 1 + a}{k\,(k + b - 1)} \; c_{k-1}$$

Form: $$\sum_{s=s_0}^{\infty} q_s r^s = r^{s_0} \sum_{s-s_0=0}^{\infty} q_{s-s_0} \, r^{s-s_0}$$

We readily identify: $k \to s - s_0$; $b \to 2s_0 + 1$; $x \to \frac{2r}{a}$

$a \to s_0 + 1 + \frac{1}{2}\left(\frac{a_1}{a} - \frac{a}{a_1}\right)\gamma$

hence we can immediately write:

$$\sum_{s=s_0}^{\infty} q_s r^s = q_0 \, r^{s_0} \, {}_1F_1\left\{ s_0 + 1 + \tfrac{1}{2}\left(\tfrac{a_1}{a} - \tfrac{a}{a_1}\right)\gamma \, ; \; 2s_0 + 1 \, ; \; \tfrac{2r}{a} \right\}$$

What we really want is the sum over c_s's. We put

$$\frac{q_0}{2s_0 + \left(\frac{a_1}{a} - \frac{a}{a_1}\right)\gamma} = c \, \frac{a_1}{a}$$

and use the recurrence relations in terms of the c's and q and find upon algebraic reduction:

$$\boxed{\begin{aligned} \sum_{s=s_0}^{\infty} c_s r^s &= C\, r^{-k+1+\frac{a}{a_1}\gamma} \frac{d}{dr} \, r^{s_0 + k - \frac{a}{a_1}\gamma} \, \frac{a_1}{a} \, {}_1F_1\left\{ s_0 - \tfrac{1}{2}\left(\tfrac{a}{a_1} - \tfrac{a_1}{a}\right)\gamma \, ; \; 2s_0 + 1 \, ; \; \tfrac{2r}{a} \right\} \\ \sum_{s=s_0}^{\infty} c_s' r^s &= C\, r^{k+1-\frac{a_1}{a}\gamma} \frac{d}{dr} \, r^{s_0 - k + \frac{a_1}{a}\gamma} \, {}_1F_1\left\{ s_0 - \tfrac{1}{2}\left(\tfrac{a}{a_1} - \tfrac{a_1}{a}\right)\gamma \, ; \; 2s_0 + 1 \, ; \; \tfrac{2r}{a} \right\} \end{aligned}}$$

now recall: $a_1 = \dfrac{\hbar c}{E + mc^2}$; $a_2 = \dfrac{\hbar c}{-E + mc^2}$; $a = \sqrt{a_1 a_2}$

Either $E < mc^2$, for which a is real and we have a bound particle, or, $E > mc^2$, for which a is imaginary and we have a free particle problem. $E = mc^2$ corresponds to $E = 0$ in the NR case, in which case the electron is not bound but it is not moving either. Here we consider exclusively $E < mc^2$.

We recall the asymptotic behaviour of ${}_1F_1(a;b;x)$:

${}_1F_1(a;b;x) \to e^x$ for large x.

Now we know that $f(r)$ and $g(r)$ both go as $r^{-1} e^{-r/a}\, {}_1F_1$ but ${}_1F_1 \sim e^{2r/a}$ for large r so we cannot have an admissible solution if we keep all the terms in ${}_1F_1$. Thus we must break the series off just as we did in the NR case. This is done by setting a in ${}_1F_1(a;b;x)$ equal to a negative integer. Hence, for $E < mc^2$:

$$\boxed{S_0 - \frac{1}{2}\left(\frac{a}{a_1} - \frac{a_1}{a}\right)\gamma = -n'}$$

; $n' = 1, 2, 3, \dots$ and is called the radial quantum number.

The question of whether $n' = 0$ is open and will answered later on.

For $E > mc^2$, there are no restrictions to be placed on ${}_1F_1$ because the exponents involved are imaginary.

We now examine the NR limit for $E < mc^2$. This means that:

$$\gamma = \frac{Z}{137} \ll 1 \; ; \qquad S_0^2 = -\gamma^2 + k^2 \simeq k^2$$

Also:

$$\frac{1}{2}\frac{a}{a_1} = \frac{1}{2}\sqrt{\frac{a_2}{a_1}} = \frac{1}{2}\sqrt{\frac{E + mc^2}{-E + mc^2}} \simeq \frac{1}{2}\sqrt{\frac{2mc^2}{-E}} = c\sqrt{\frac{m}{-2E}}$$

where in the last two terms, E is the usual NR form going as $\frac{1}{2}mv^2$.

Then $\frac{1}{2}\frac{a}{a_1} \sim \frac{c}{v}$ while $\frac{1}{2}\frac{a_1}{a} \sim \frac{v}{c}$, hence:

$$s_0 - \frac{1}{2}\left(\frac{a}{a_1} - \frac{a_1}{a}\right)\gamma \rightarrow k - \frac{Ze^2}{\hbar}\sqrt{\frac{m}{-2E}} = -n'$$

One should show that $k = l+1$ here, however, Professor Furry said he was not able to do it at this time. However, one can see how one is lead to the Rydberg formula from the above by letting the principle quantum $n = n' + k$ so that:

$$\boxed{E = -\frac{Z^2 e^4 m}{2 n^2 \hbar^2}}$$

Proceeding now to the full relativistic solution:

$$\left(\frac{a}{a_1} - \frac{a_1}{a}\right)\gamma = 2(s_0 + n')$$

$$\frac{a}{a_1} - \frac{a_1}{a} = \sqrt{\frac{a_2}{a_1}} - \sqrt{\frac{a_1}{a_2}} = \sqrt{\frac{E+mc^2}{-E+mc^2}} - \sqrt{\frac{mc^2-E}{mc^2+E}}$$

$$= \frac{2E}{\sqrt{m^2c^4 - E^2}} = \frac{2}{\sqrt{\frac{m^2c^4}{E^2} - 1}}$$

or: $$\frac{m^2c^4}{E^2} - 1 = \frac{\gamma^2}{(s_0+n')^2} = \frac{Z^2\alpha^2}{(\sqrt{k^2-\gamma^2} + n')^2}$$

$$= \frac{Z^2\alpha^2}{(\sqrt{k^2 - Z^2\alpha^2} + n')^2}$$, using $\alpha = Z\alpha$ and $s_0 = \sqrt{k^2 - Z^2\alpha^2}$

Upon rearrangement, we finally obtain the Sommerfeld Fine Structure Formula:

$$\boxed{E = \frac{mc^2}{\left[1 + \frac{Z^2\alpha^2}{(\sqrt{k^2 - Z^2\alpha^2} + n')^2}\right]^{1/2}}}$$

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Recall:

$$E = \frac{mc^2}{\left[1 + \frac{Z^2\alpha^2}{(\sqrt{k^2 - Z^2\alpha^2} + n')^2}\right]^{1/2}} \approx mc^2 - \underbrace{\frac{Z^2\alpha^2}{2n^2}}_{\text{Rydberg term.}}$$

We will now expand this and see how some of the lower order terms enter. First define some useful notation and new "quantum numbers".

$$n \equiv n' + |k|$$

$$n_1 \equiv \sqrt{k^2 - Z^2\alpha^2} + n' = s_0 + n'$$

$$= n' + |k| \sqrt{1 - \frac{Z^2\alpha^2}{k^2}} = n' + |k| \left\{ 1 - \frac{1}{2} \frac{Z^2\alpha^2}{k^2} + \cdots \right\}$$

$$\approx n - \frac{Z^2\alpha^2}{2|k|}$$

providing $|k| >> Z\alpha$ which is true for the hydrogen atom since $Z\alpha \sim \frac{1}{137}$ and $|k|$ is an integer, apparently $\neq 0$ as can be seen. Also we can write:

$$n_2 \equiv \sqrt{n_1^2 + Z^2\alpha^2} = n_1 \sqrt{1 + \frac{Z^2\alpha^2}{n_1^2}}$$

$$\approx n_1 + \frac{Z^2\alpha^2}{2n_1}$$

for the H atom case for the same reasons as above. However, we will not use this right away.

Substituting the definition for n_1 in the Sommerfeld fine structure formula, we have:

$$E = \frac{mc^2}{\sqrt{1 + \frac{Z^2\alpha^2}{n_1^2}}}$$

note that in the NR limit, $E \sim mc^2$ and what is left over is the usual Rydberg expression for the binding energy.

Now, since $n_1 \gg Z\alpha$, we can immediately use the expansion:

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{x}{2} + \frac{3x^2}{8} - \cdots$$

to write to order $(Z\alpha)^4$:

$$E - mc^2 \cong -\frac{mc^2 Z^2\alpha^2}{2n_1^2} + \frac{3mc^2 Z^4\alpha^4}{8n_1^4}$$

But recall that $n_1 \cong n - \frac{Z^2\alpha^2}{2|k|}$ and hence use the expansion:

$$\frac{1}{(1-x)^2} = 1 + 2x + \cdots$$

to get to order $(Z\alpha)^4$:

$$E - mc^2 \cong \frac{-mc^2 Z^2\alpha^2}{2n^2} - \frac{mc^2 Z^4\alpha^4}{2n^3|k|} + \frac{3mc^2 Z^4\alpha^4}{8n^4}$$

or; since $Rhc = \frac{mc^4e^4}{2\hbar^2c^4} = \frac{mc^2}{2}\left(\frac{e^2}{\hbar c}\right)^2 = \frac{mc^2\alpha^2}{2}$

we can write:

$$\boxed{E - mc^2 = -Z^2Rhc\left[\frac{1}{n^2} + \frac{Z^2\alpha^2}{n^3|k|} - \frac{3Z^2\alpha^2}{4n^4}\right]}$$

The first term is the usual Rydberg formula, while the second term, because of $|k|$, gives a splitting effect. The third term is merely a level shift and is very small ($n^2 \gg Z\alpha$).

We are now ready to use our definition of n_2 in writing the the large and small components of the wave function.

$$\boxed{\begin{aligned} f(r) &= C\, e^{\frac{Zr}{n_2 a_0}}\, r^{k-n_2+n_1} \frac{d}{dr}\, r^{s_0-k+n_2-n_1}\, {}_1F_1\left(-n'; 2s_0+1; \frac{2Zr}{n_2 a_0}\right) \\ g(r) &= -i\, C\, \frac{Z\alpha}{n_2+n_1}\, e^{\frac{Zr}{n_2 a_0}}\, r^{-k+n_2+n_1} \frac{d}{dr}\, r^{s_0+k-n_2-n_1}\, {}_1F_1\left(-n'; 2s_0+1; \frac{2Zr}{n_2 a_0}\right) \end{aligned}}$$

We will now consider some of the simpler term levels of the relativistic hydrogenic atom. Recall the definition of k:

$$|k| = j + \tfrac{1}{2} \quad ; \quad \begin{aligned} k &= 1, 2, 3, \cdots \text{ for } j = l - \tfrac{1}{2} \\ k &= -1, -2, -3, \cdots \text{ for } j = l + \tfrac{1}{2} \end{aligned}$$

<u>Case of $k > 0$:</u>

$$j = l - \tfrac{1}{2} \quad \text{or} \quad l = j + \tfrac{1}{2}$$

now $n = n' + |k|$; $|k| = j + \tfrac{1}{2} = l$, therefore:

$$\boxed{n = n' + l}$$

Suppose we know that l takes on $0, 1, 2, 3, \ldots$ as it always can, but we do not know if n' can be zero or not. However, if $n' = 0$, we see that $n_1 = s_0$ and $n_2 = |k|$. This means that, for $k > 0$, $-k + |k| = 0$, so that in both $g(r)$ and $f(r)$ above we end up taking the derivative of a constant and hence $g(r)$ and $f(r)$ vanish. So for $k > 0$, $n' \neq 0$. (note that this is not the case for $k < 0$). Also $k \neq 0$ as was seen from the expansion to order $(Z\alpha)^4$ of $E - mc^2$. Hence:

$$\boxed{k > 0: \quad l = k \quad ; \quad n = l+1, l+2, \cdots \quad ; \quad j = l - \tfrac{1}{2}}$$

and we can immediately write as term states:

$2p_{1/2}$, $3p_{1/2}$, $4p_{1/2}$, . . .
$3d_{3/2}$, $4d_{3/2}$, . . .
$4f_{5/2}$, . . .
⋮

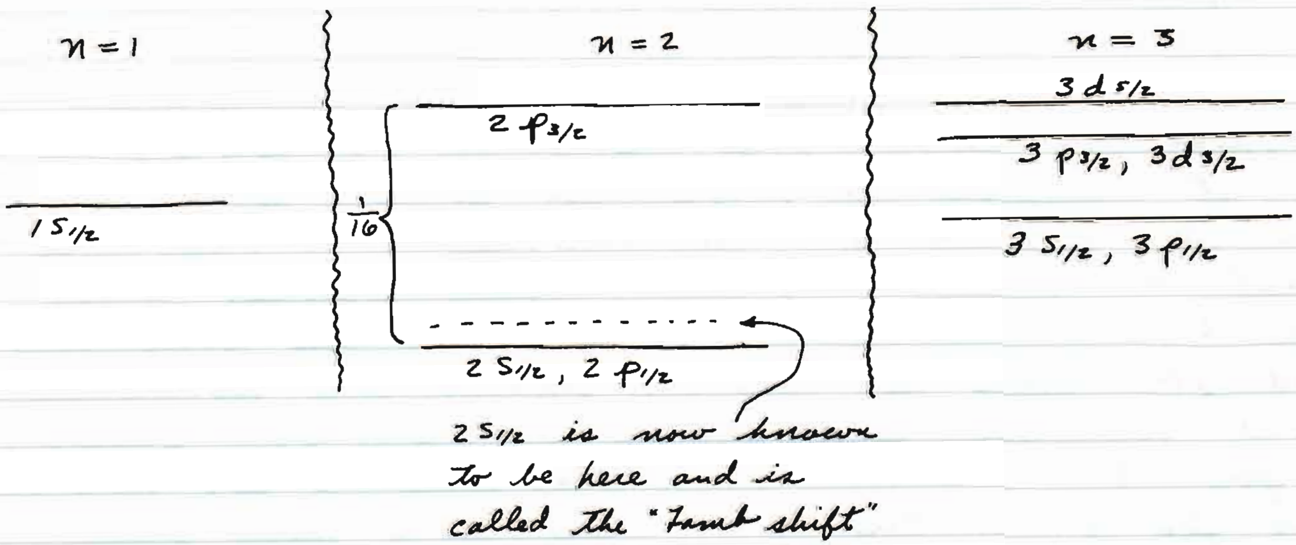
Case of $k < 0$: $\quad l = j - 1/2 = |k| - 1 \; ; \; n = n' + |k|$
$$= n' + l + 1$$

Now here, $n' = 0$ is possible, so that we have for the rules determining the term states for $k < 0$:

$$\boxed{k < 0 : \quad l = |k| - 1 \; ; \; n = l+1, l+2, \cdots \; ; \quad j = l + 1/2}$$

giving:
$$\begin{array}{l} 1\,s_{1/2},\ 2\,s_{1/2}, \cdots \\ 2\,p_{3/2},\ 3\,p_{3/2}, \cdots \\ 3\,d_{5/2},\ 4\,d_{5/2}, \cdots \\ \vdots \end{array}$$

What do the states of each principle quantum number n look like with respect to each other? Use a scale of units: $Z^2 Rhc \cdot Z^2 \alpha^2$



Note that $2\,s_{1/2}, 2\,p_{1/2}$ have the same value of n and $|k|$ and hence the same energy. Of course, we have not here any scale on $n = 1, 2, 3$; they are drawn side by side for convenience.

Prior to 1946, the Dirac theory accounted exactly for the observed hydrogenic spectra but then the Lamb shift was discovered which was not accounted for by the theory and is only accounted for by field theory.

The Free Dirac Electron

We now treat the Dirac equation for the free electron. We will introduce "natural units" in which $m = \hbar = c = 1$. In these units the Dirac equation becomes:

$$\boxed{i\frac{\partial \psi}{\partial t} = \frac{1}{i}\vec{\alpha}\cdot\nabla\psi + \beta\psi = \underline{H}\psi \qquad (1)}$$

For stationary states where we may write

$$\boxed{\psi = u\, e^{-iEt} \qquad (2)}$$

we have:

$$\boxed{E u = \frac{1}{i}\vec{\alpha}\cdot\nabla u + \beta u = \underline{H} u \quad (3)}$$

We may take for a plane wave solution:

$$\psi = a\, e^{-iEt + i\vec{p}\cdot\vec{r}} \qquad (4)$$

where a is a 4 element column matrix and substitution into (1) gives:

$$\left\{E - \vec{\alpha}\cdot\vec{p} - \beta\right\} a = 0 \quad (5)$$

Now what we want is the determinant of (5) to vanish. We could introduce a representation for $\vec{\alpha}$ and β but we do not need to as the determinant of a matrix is invariant under a unitary transformation and hence the determinant is independent of the representation:

Let: $\alpha_n' = T^{-1}\alpha_n T \;;\; \beta' = T^{-1}\beta T \;;\; TT^{+} = 1$

But: $\det \alpha_n' = |T^{-1}|\,|\alpha_n|\,|T| = \det \alpha_n$

and so forth.

Instead of using a representation, we form in the spirit of the factorization of the Dirac equation the "square" of the determinant of (5). That is, since we know that the determinants of $\vec{\alpha}$ and β are invariant under a unitary transformation, we could pick one that just changes the sign of $\vec{\alpha}$ and β. Hence we can write:

$$\Delta^2 = \left\{ \det\left(E - \vec{\alpha}\cdot\vec{p} - \beta \right) \right\}^2$$

$$= \det\left[(E + \vec{\alpha}\cdot\vec{p} + \beta)(E - \vec{\alpha}\cdot\vec{p} - \beta) \right] = 0$$

or:

$$\Delta^2 = \det\left(E^2 - p^2 - 1\right) = (E^2 - p^2 - 1)^4 = 0$$

now: $\Delta = (E^2 - p^2 - 1)^2 = (E - \sqrt{p^2+1})^2 (E + \sqrt{p^2+1})^2 = 0$

We thus have four roots, but only two are non-identical.

$E = +\sqrt{p^2+1}$; 2 identical roots

$E = -\sqrt{p^2+1}$; 2 identical roots

If we indroduce the notation $p_0 = \sqrt{p^2+1}$, we have:

$$\boxed{E = +p_0 \,, -p_0}$$

This demonstrates the existence of the negative energy states.

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The requirement that solutions to the Dirac equation must satisfy the KG equation leads us to take plane wave solutions of the form:

$$\psi = a\, e^{-iEt + i\vec{p}\cdot\vec{r}} \qquad (4)$$

which gives $\quad (E - \vec{\alpha}\cdot\vec{p} - \beta)\, a = 0 \quad (5)$

and whose determinantial solution gives:

$$E = +p_0,\ -p_0 \qquad (8)$$

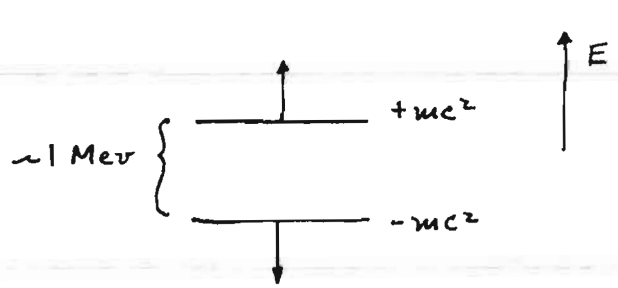
where: $\quad p_0 = \sqrt{p^2 + 1} \quad (7)$

Returning to CGS units in (8) gives:

$$E = \sqrt{m^2 c^4 + c^2 p^2}$$

or the usual relativistic result. However, we also have a negative energy root which was originally disregarded by many people including Pauli. It was known that these negative energies also arose from the KG equation, but Dirac took his equation very seriously and was hence concerned about the negative roots. Dirac interpreted correctly that the negative energy really represented that of a positively charged particle, a theory which Pauli originally disagreed with.

We see that the energy diagram for a free particle must look like:



now the presence of the negative root also occurs classically, ($E^2 = m^2c^4 + c^2p^2$) but a classical particle cannot or could not make the transition across the gap. However, a quantum particle can make the transition under favorable circumstances.

Dirac postulated that all the negative energy states are filled as a normal condition of nature. However, we can only detect these negative energy states when one or more of them are empty, or when the electron may be thought of as having positive energy and positive charge. This was verified by the discovery of the positron in 1932-33.

Today the negative energy states are accepted and the whole problem is treated formally without so much queerness. Since we have four eigenvalues we may associate a wave function with each of these. In this case, there should be four different (perhaps) values of a. We will denote each by a superscript, viz, $a^{(r)}$. However, $a^{(r)}$ is also a 4 element column or row matrix, so we denote an element of a particular eigenvector by $a_i^{(r)}$. It is now our job to find the a's that satisfy:

$$\left[E - H(\vec{p})\right] a = 0 \qquad \text{or} \qquad (E - \vec{\alpha}\cdot\vec{p} - \beta) a = 0$$

We now try to find a transformation which when applied to the $H(\vec{p})$ brings it into the form where all the eigenvalues are represented as diagonal elements. That is, we want S, a matrix such that:

$$S^{+} H(\vec{p}) S = \begin{pmatrix} p_0 \underline{1} & \underline{0} \\ \underline{0} & -p_0 \underline{1} \end{pmatrix} = \rho_3 \, p_0 \qquad (9)$$

where we are taking ρ_3 in the "first" sense so that $\rho_3 = \beta$. We take $H(p) = \vec{\alpha}\cdot\vec{p} + \beta$ and S to be Hermitean.

If we wanted to use some other representation in which $\beta \neq \rho_3$ in the "first" sense, we could find a new S matrix as follows:

Let: $\alpha_k = T^\dagger \alpha_k' T \quad ; \quad \beta = T^\dagger \beta' T \qquad (10)$

where T is unitary (T could, for example, connect the Dirac and Weyl representations). Then we would have:

$$S^\dagger T^\dagger H'(p) T S = S'^\dagger H'(p) S'$$

so that: $\qquad S = T^\dagger S' \qquad (11)$

and gives a relation for finding S in any representation as long as we know S' and T.

However, we will work in a representation that has:

$$\beta = \begin{pmatrix} \underline{1} & \underline{0} \\ \underline{0} & -\underline{1} \end{pmatrix} \qquad (12)$$

and find S in this representation. We must now guess at a form for S. We will hope that:

$$S(\vec{p}) = C\left[H(\vec{p}) + p_0 \beta \right]$$

does the job, choosing this because $H(\vec{p})$ brings in $\vec{\alpha}\cdot\vec{p}$ and it looks as though this might be needed. We find C by the condition that $S(\vec{p})$ be unitary.

$$S^\dagger S = 1 = |C|^2 \left\{ H^2 + p_0 \{\beta, H\} + p_0^2 \right\}$$

$$= |C|^2 \left[p^2 + 1 + 2p_0 + p_0^2 \right] = |C|^2 \left(2p_0^2 + 2p_0 \right)$$

Then:

$$C = \frac{1}{\sqrt{2p_0(1+p_0)}}$$

We now must see if this choice of $S(\vec{p})$ gives us (9).

$$S^{+}(\vec{p})\, H(\vec{p})\, S(\vec{p}) = |c|^2 (H + p_0\beta) H (H + p_0\beta)$$

$$= |c|^2 \Big[\underbrace{H^3}_{p_0^2 H} + p_0 \underbrace{\{\beta, H^2\}}_{2p_0^3\beta} + \underbrace{p_0^2 \beta H \beta}_{\underbrace{p_0^2(\underbrace{\beta\vec{\alpha}\cdot\vec{p}\beta}_{-\vec{\alpha}\cdot\vec{p}} + \beta)}_{p_0^2(-H + 2\beta)}} \Big]$$

$$= 2|c|^2 (p_0^3 + p_0^2)\beta = \;= |c|^2\, 2p_0(1+p_0)\, p_0\beta = p_0\beta$$

It checks. Now, to make a connection between $S(p)$ and the a's, we can write:

$$S^{+}(\vec{p}) \left[E - H(\vec{p})\right] S(\vec{p}) = E - p_0 \rho_3$$

or:

$$\left[E - H(\vec{p})\right] S(\vec{p}) = S(\vec{p})(E - p_0\rho_3) \qquad (14)$$

writing in the form such that if $S(p)$ were a wave function the RHS of (14) would be zero. Now we have, using ρ_3 in the first sense:

$$E - p_0\rho_3 = \begin{pmatrix} E - p_0 & & & \\ & E - p_0 & & 0 \\ 0 & & E + p_0 & \\ & & & E + p_0 \end{pmatrix}$$

For $E = +p_0$, we see that the first two columns on the RHS matrix of (14) are zero. This means that the first two columns of $S(\vec{p})$ are eigenvectors of $H(\vec{p})$ with eigenvalues $+p_0$. It is easy to see that the last two columns are the eigenvectors of the states of $-p_0$ or the negative energy states.

We are now able to identify $S(\vec{p})$ with a in the following way:

$$\boxed{a_i^{(r)} = S_{ir} \qquad (15)}$$

Here r is the eigenvector index on a and the column index on S while i is the row index on both a and S. Of course, $r = 1,2$ for $E = p_0$ and $r = 3,4$ for $E = -p_0$. We have of course the usual orthogonality relation between the eigenvectors.

$$\sum_i a_i^{(r)*} a_i^{(r')} = \delta_{rr'} \qquad (16)$$

and since $S(\vec{p})$ is unitary, we have a sort of "closure" relation:

$$\sum_r a_i^{(r)} a_j^{(r)*} = \delta_{ij} \qquad (17)$$

We can now write out the full representation of the free Dirac electron. We do not include time dependence, although this counts when talking about pair production because a time varying EM field is involved.

We write, combining all notations:

$$\langle \vec{r}, i \,|\, \vec{p}, r \rangle = u_i^{(r)}(\vec{r}\,;\vec{p}) = \frac{1}{(2\pi)^{3/2}}\, a_i^{(r)}(\vec{p})\, e^{i\vec{p}\cdot\vec{r}} \qquad (18)$$

Of course, we have:

$$\sum_i \int u_i^{(r)*}(\vec{r}\,;\vec{p})\, u_i^{(r')}(\vec{r}\,;\vec{p}\,')\, d\vec{r} = \delta_{rr'}\, \delta(\vec{p}-\vec{p}\,') \qquad (19)$$

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Because our $u_i^{(\tau)}(\vec{r};\vec{p})$ form a complete orthonormal set, we can expand any wave function in terms of them and write:

$$\psi_j(\vec{r}) = \sum_\tau \int d\vec{p}\; c^{(\tau)}(\vec{p})\; u_j^{(\tau)}(\vec{r};\vec{p}) \qquad (20)$$

and: $$c^{(\tau)}(\vec{p}) = \sum_i \int d\vec{r}\; u_i^{(\tau)*}(\vec{p};\vec{r})\; \psi_i(\vec{r}) \qquad (21)$$

Or, in the Dirac notation:

$$\langle \vec{r};j|\;\rangle = \sum_\tau \int d\vec{p}\; \langle \vec{r},j|\vec{p},\tau\rangle\langle\vec{p},\tau|\;\rangle \qquad (20),\{(21)\}$$

Let us now examine the explicit representation of $S(\vec{p})$. Recall:

$$\boxed{S(p) = \frac{1}{\sqrt{2p_0(p_0+1)}}\left\{H(\vec{p}) + p_0\beta\right\}}$$

which holds in any representation which has:

$$\beta = \begin{pmatrix} \underline{1} & \underline{0} \\ \underline{0} & -\underline{1} \end{pmatrix}$$

such as the original Dirac representation. We may then write:

$$S(\vec{p}) = \frac{1}{\sqrt{2p_0(p_0+1)}}\left\{\rho_1(\vec{\sigma}\cdot\vec{p}) + \rho_3(p_0+1)\right\}$$

and immediately have for $S_{i\tau}$; not including the prefactor $\frac{1}{\sqrt{2p_0(p_0+1)}}$:

$i \backslash \tau$	1	2	3	4
1	p_0+1	0	p_z	p_x-ip_y
2	0	p_0+1	p_x+ip_y	$-p_z$
3	p_z	p_x-ip_y	$-(p_0+1)$	0
4	p_x+ip_y	$-p_z$	0	$-(p_0+1)$

(22)

The columns of course give $a^{(\tau)}$.

We now consider wave packets whose motion obeys the Schroedinger equation written with the Dirac Hamiltonian.

$$i \frac{\partial \psi}{\partial t} = \underline{H}\, \psi \qquad (23)$$

On the basis of this equation we can define the time derivative of an operator $\underline{A}$ as:

$$\frac{d\underline{A}}{dt} = i \left[\underline{H}, \underline{A} \right] + \frac{\partial \underline{A}}{\partial t} \qquad (24)$$

We take the most general Dirac Hamiltonian:

$$\underline{H} = \vec{\alpha} \cdot (\vec{p} - e\vec{A}) + \beta(1 + V) + e\varphi \qquad (25)$$

V could be some gravitational potential and we could have also included the Pauli term. Let us use this $\underline{H}$ to determine $\dot{x}$ or what we will call the velocity operator $\underline{v}$:

$$\underline{v} = \frac{dx}{dt} = i \left[\underline{H}, \underline{x} \right] = \alpha_x \qquad (26)$$

as is readily seen, so the velocity operator is α_x. In CGS units:

$$\boxed{\vec{v} = c\,\vec{\alpha} \qquad (27)}$$

From now on we will restrict our discussion mostly to the free particle. Recall:

$$\underline{H} = \vec{\alpha} \cdot \vec{p} + \beta$$

Compare this with the classical expression for the Hamiltonian in terms of the Lagrangian.

$$H = \sum p\dot{q} - L$$

and we can draw the analogy: $p \to p\,;\ \dot{q} \to \alpha\,;\ L \to -\beta$

now in classical relativity theory we have:

$$L = -mc^2\sqrt{1-\frac{v^2}{c^2}}$$

so that the analogy can be made that

$$\sqrt{1-\frac{v^2}{c^2}} \to \beta \qquad (28) \qquad \text{NR limit}, \ \beta \to 1$$

Recall that the various components of $\vec{\alpha}$ do not commute which means that we have a finite probability of having velocity c in the y and z directions when we know it is c in the x direction (very strange). However, the average value of the velocity operator in the y, z directions is zero as will now be shown. We know:

$$\alpha_x \psi = b\psi \quad ; \quad b = \begin{cases} +1 \\ -1 \end{cases}$$ in a representation which diagonalizes α_x.

now form : $\psi^* \alpha_y \psi = \frac{1}{b}\psi^* \alpha_y \alpha_x \psi = \frac{1}{b}\psi^* \alpha_x \alpha_y \psi$

$$= \frac{1}{2b}\psi^*(\alpha_y\alpha_x + \alpha_x\alpha_y)\psi = 0$$

and the same happens for α_z. Now ψ above cannot be stationary because $[\alpha_x, \underline{H}] \neq 0$, however, we can think of it as stationary for one instant although this will no longer be true 10^{-22} seconds later as the crossterms of ψ^*, ψ involving the factors e^{ip_0t} and e^{-ip_0t} will no longer cancel and hence will lead to a rapidly varying matrix element as will be seen later. These crossterms arise from the mixing of the negative energy component with the positive energy component of the wave function ψ.

Schroedinger suggested that we could get rid of this "difficulty" if when considering the matrix representation of an operator we considered only the elements connecting purely plus components and those connecting purely minus components as having any existence.

Those elements connecting plus with minus and vice versa were to be considered as zero. Schroedinger introduced the notion of the Schroedinger even and odd operators:

Even: <u>Schroedinger</u> Odd:

r' \ r	1	2	3	4
1	x	x	o	o
2	x	x	o	o
3	o	o	x	x
4	o	o	x	x

r' \ r	1	2	3	4
1	o	o	x	x
2	o	o	x	x
3	x	x	o	o
4	x	x	o	o

Schroedinger chose as Evens, $\vec{p}$, $\underline{H}$, I, and in general observables while he took for odds operators like $\vec{\alpha}$, β, and others. Schroedinger claimed that one could transform to a "semi-even" form and then throw out the off-diagonal as not physically meaningful. However, this conjecture was not successful as it contradicts now known physical facts.

The Schroedinger even and odd scheme is not to be confused with the Dirac even and odd scheme which is true only for operators in the Dirac representations.

Even: <u>Dirac</u> Odd:

j \ i	1	2	3	4
1	x	x	o	o
2	x	x	o	o
3	o	o	x	x
4	o	o	x	x

j \ i	1	2	3	4
1	o	o	x	x
2	o	o	x	x
3	x	x	o	o
4	x	x	o	o

The Schroedinger and Dirac even-odd schemes are only equivalent for particles at rest.

The Schroedinger effort was to get rid of $+E$ to $-E$ transitions but today we know that these transitions are some of the simplest phenomena in nature.

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In this lecture, we explore the Schroedinger even-odd scheme further although it is not correct. However, we can gain some connection with the classical limit under these incorrect representations. In particular, we consider the matrix elements of α_x. We commence by transforming from the Dirac representation of α_x to the Schroedinger representation of α_x for free electrons. Consider:

$$\langle \vec{r}, i | \alpha_x | \vec{r}', j \rangle = \delta(\vec{r} - \vec{r}')\,(\alpha_x)_{ij} \qquad (29)$$

We use the free electron transformation function

$$\langle \vec{r}, i | \vec{p}, r \rangle = \frac{1}{(2\pi)^{3/2}}\, e^{i\vec{p}\cdot\vec{r}}\, S_{ir}(\vec{p}) \qquad (30)$$

to get:

$$\langle \vec{p}, r | \alpha_x | \vec{p}', r' \rangle = \sum_{ij} \int \langle \vec{p}, r | \vec{r}, i \rangle \underbrace{\langle \vec{r}, i | \alpha_x | \vec{r}', j \rangle}_{\delta(\vec{r}-\vec{r}')(\alpha_x)_{ij}} \langle \vec{r}', j | \vec{p}', r' \rangle\, d\vec{r}\, d\vec{r}'$$

$$= \delta(\vec{p} - \vec{p}') \left\{ S^{+} \alpha_x S \right\}_{rr'}$$

Hence we are in the Schroedinger representation. We use a representation where:

$$\beta = \begin{pmatrix} \underline{1} & \underline{0} \\ \underline{0} & -\underline{1} \end{pmatrix} \quad ; \quad S(p) = \frac{1}{\sqrt{2p_0(p_0+1)}} \left[H(\vec{p}) + p_0 \beta \right]$$

and we write $\vec{\alpha}\cdot\vec{p} + \beta = H(\vec{p}) = \rho_1 \vec{\sigma}\cdot\vec{p} + \rho_3$.
Then:

$$\left\{ S^{+} \alpha_x S \right\}_{rr'} = \frac{1}{2p_0(p_0+1)} \left[\rho_1 \vec{\sigma}\cdot\vec{p} + \rho_3 (1+p_0) \right] \rho_1 \sigma_1 \left[\rho_1 \vec{\sigma}\cdot\vec{p} + \rho_3 (1+p_0) \right]$$

We will now examine for Schroedinger evenness and oddness. The Schroedinger even part will contain no ρ_1's as they mix + and − components.

Then:

$$\langle \vec{p},\tau | \alpha_x | \vec{p}',\tau'\rangle_{even} = \left[\frac{(\vec{\sigma}\cdot\vec{p})\,\sigma_1\rho_3 + \rho_3\sigma_1\,(\vec{\sigma}\cdot\vec{p})}{2p_0}\right]_{\tau\tau'} \delta(\vec{p}-\vec{p}')$$

$$= \left[\frac{p_x}{p_0}\,\rho_3\right]_{\tau\tau'} \delta(\vec{p}-\vec{p}') \qquad (31)$$

Hence we can write:

"α_x"

$\tau \backslash \tau'$	1	2	3	4
1	p_x/p_0	0	unwanted	
2	0	p_x/p_0		
3	unwanted		$-p_x/p_0$	0
4			0	$-p_x/p_0$

We see that the identification of α_x with velocity is complete in the Schroedinger representation.

We can do the same for β and get:

$$\{S^+\beta S\}_{\tau\tau'} = \frac{1}{2p_0(p_0+1)}\left[\rho_1\vec{\sigma}\cdot\vec{p}+\rho_3(1+p_0)\right]\rho_3\left[\rho_1\vec{\sigma}\cdot\vec{p}+\rho_3(1+p_0)\right]$$

now the Schroedinger even part is:

$$\langle\vec{p},\tau|\beta|\vec{p}',\tau'\rangle_{even} = \underbrace{\left[\frac{-\rho_3\,p^2+\rho_3\,(1+p_0)^2}{2p_0\,(1+p_0)}\right]_{\tau\tau'}}_{\frac{\rho_3(-p^2+1+2p_0+p_0^2)}{2p_0(p_0+1)} = \frac{2\rho_3(1+p_0)}{2p_0(1+p_0)}} \delta(\vec{p}-\vec{p}')$$

$$= \left\{\frac{1}{p_0}\,\rho_3\right\}_{\tau\tau'} \delta(\vec{p}-\vec{p}') \quad , \text{ since } p_0^2-p^2=1.$$

"β"

$\tau \backslash \tau'$	1	2	3	4
1	$\sqrt{1-\frac{v^2}{c^2}}$	0	unwanted	
2	0	$\sqrt{1-\frac{v^2}{c^2}}$		
3	unwanted		$-\sqrt{1-\frac{v^2}{c^2}}$	0
4			0	$-\sqrt{1-\frac{v^2}{c^2}}$

Again the classical correspondence is clear in the Schroedinger even representation.

We will now consider the motion of a wave packet using only that part of the wave function that comes from the negative energy states alone or the positive energy components alone. We will use positive energy components. Take:

$$\psi = \sum_{r=1}^{2} \int d\vec{p}\; c^{(r)}(\vec{p})\; a^{(r)}(\vec{p})\; \frac{e^{i\vec{p}\cdot\vec{r} - i p_0 t}}{(2\pi)^{3/2}}$$

Strictly speaking there should be:

$$\sum_{r=3}^{4} \int d\vec{p}\; c^{(r)}(\vec{p})\; a^{(r)}(\vec{p})\; \frac{e^{i\vec{p}\cdot\vec{r} + i p_0 t}}{(2\pi)^{3/2}}$$

however, as we said, we are avoiding negative energy terms and here they would give crossterms in the calculation of expectation values of e^{i2p_0t}, e^{-i2p_0t} or terms that are very rapidly oscillating.

We now calculate $\bar{x}$:

$$\bar{x} = \int \psi^* x\, \psi\, d\vec{r} = \int d\vec{r}\; \psi^* x \sum_{r=1}^{2} \int d\vec{p}\; c^{(r)}(\vec{p})\, a^{(r)}(\vec{p})\; \frac{e^{i\vec{p}\cdot\vec{r} - i p_0 t}}{(2\pi)^{3/2}}$$

$$= \int d\vec{r}\; \psi^* \sum_{r=1}^{2} \int d\vec{p} \left\{ \frac{1}{i} \frac{\partial}{\partial p_x} e^{i\vec{p}\cdot\vec{r}} \right\} c^{(r)}(\vec{p})\, a^{(r)}(\vec{p})\; \frac{e^{-ip_0t}}{(2\pi)^{3/2}}$$

Integrating by parts:

$$\bar{x} = \int d\vec{r}\; \psi^* \sum_{r=1}^{2} \int d\vec{p}\; e^{i\vec{p}\cdot\vec{r}} \left\{ i \frac{\partial}{\partial p_x} \right\} c^{(r)}(\vec{p})\, a^{(r)}(\vec{p})\; \frac{e^{-ip_0t}}{(2\pi)^{3/2}}$$

$$= \int d\vec{r} \sum_{r,r'=1}^{2} \int d\vec{p}\,'\; c^{(r')*}(\vec{p}\,')\; a^{(r')*}(\vec{p}\,')\; e^{ip_0't}\; \frac{e^{i(\vec{p}-\vec{p}\,')\cdot r}}{(2\pi)^3}\; i\frac{\partial}{\partial p_x} c^{(r)}(\vec{p})\, a^{(r)}(\vec{p})\, e^{-ip_0t}\, d\vec{p}$$

$$= \sum_{r,r'=1}^{2} \int d\vec{p}\; c^{(r')*}(\vec{p})\; a^{(r')*}(\vec{p})\; e^{ip_0t}\; i\frac{\partial}{\partial p_x}\; c^{(r)}(\vec{p})\; a^{(r)}(\vec{p})\; e^{-ip_0t}$$

Then:

$$\bar{x} = \sum_{r,r'=1}^{2} \int d\vec{p}\; C^{(r')*}(\vec{p})\, a^{(r')*}(\vec{p})\; i\frac{\partial}{\partial p_x}\; C^{(r)}(\vec{p})\, a^{(r)}(\vec{p})$$

$$+ \sum_{r,r'=1}^{2} \int d\vec{p}\; \left[C^{(r)}(\vec{p})\, C^{(r')*}(\vec{p}) \right] \underbrace{\left[a^{(r')*}(\vec{p})\, a^{(r)}(\vec{p}) \right]}_{\delta_{rr'}} \frac{\partial p_0}{\partial p_x}\, t$$

$$= \overline{x(0)} + \sum_{r=1}^{2} \int dp\, |C^{(r)}(p)|^2 \frac{\partial p_0}{\partial p_x}\, t$$

$\overline{x(0)}$ is just a constant (first term above) and can be thought of as an initial condition. The second term is the average value of $\frac{\partial p_0}{\partial p_x}$ times time.

$$\frac{\partial p_0}{\partial p_x} = \frac{\partial}{\partial p_x}\sqrt{1+p^2} = \frac{p_x}{p_0}$$

or:

$$\frac{\partial p_0}{\partial p_x} \rightarrow \frac{\partial E}{\partial p_x} = (v_x)_{\text{classical}}$$

So we see that the second term gives the classical relativistic velocity. Hence we finally have:

$$\bar{x} = \overline{x(0)} + \overline{\frac{\partial p_0}{\partial p_x}}\, t = \overline{x(0)} + \overline{(v_x)}_{\text{classical}}\, t$$

If the negative energy terms had been included in the wave function, we would have obtained a term with $e^{\pm i 2 p_0 t}$ giving oscillations in $\bar{x}$. See Dirac for discussion of this case (Zitterbewegung).

We see that the wave packet spreads in time and the result for $\bar{x}$ is generally like the NRQM result.

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We will now calculate the operator $\dot{p}_x$ and its expectation value $\overline{\dot{p}_x}$ with respect to the positive energy wave functions using for the Dirac Hamiltonian:

$$\underline{H} = \vec{\alpha}\cdot(\vec{p} - e\vec{A}) + \beta(1+V) + e\varphi$$

$$\dot{p}_x = i\,[\underline{H}, p_x] = -ie\,\alpha_x\,[A_x, p_x] + i\beta\,[V, p_x] + ie\,[\varphi, p_x]$$

$$-ie\,\alpha_y\,[A_y, p_x] - ie\,\alpha_z\,[A_z, p_x]$$

Using $p_x = -i\frac{\partial}{\partial x}$ and assuming implicit operation on a wave function, we have:

$$\dot{p}_x = -\beta\frac{\partial V}{\partial x} - e\frac{\partial\varphi}{\partial x} + e\left[\vec{\alpha}\times[\nabla\times\vec{A}]\right]_x + e(\vec{\alpha}\cdot\nabla)A_x$$

What is the last term? Consider:

$$\dot{A}_x = i\,[\underline{H}, A_x] + \frac{\partial A_x}{\partial t} = (\vec{\alpha}\cdot\nabla)A_x + \frac{\partial A_x}{\partial t}$$

We have also used the following identities:

$$[\vec{\alpha}\times\nabla\times\vec{A}]_x = \alpha_y\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - \alpha_z\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)$$

$$(\vec{\alpha}\cdot\nabla)A_x = \alpha_x\frac{\partial A_x}{\partial x} + \alpha_y\frac{\partial A_x}{\partial y} + \alpha_z\frac{\partial A_x}{\partial z}$$

Then:

$$\dot{p}_x = -\beta\frac{\partial V}{\partial x} + e\dot{A}_x + e\,[\vec{\alpha}\times\vec{\mathcal{H}}]_x + e\,\mathcal{E}_x$$

We now take the average with respect to the wave packet of a positive energy free particle, which means replacing $\vec{\alpha}$ by $\vec{v}$ and β by $\sqrt{1-v^2}$. Hence:

$$\boxed{\frac{d}{dt}\overline{(p_x - eA_x)} = -\overline{\sqrt{1-v^2}\frac{\partial V}{\partial x}} + e\,\overline{[\vec{v}\times\vec{\mathcal{H}}]_x} + \overline{e\,\mathcal{E}_x}}$$

= gravitational force + Lorentz Force.

Here we have Ehrenfest's Theorem. However, we actually have $-E$ terms that give rise to further time dependence.

Miscellaneous Transformation Theory

We now return to a discussion of transformation theory beginning with some ways of handling β-decay. Recall the Pauli covariant quantities:

$$I = \bar{\Psi}\psi = \psi^* \beta \psi \qquad \text{(S)}$$
$$S_\mu = i\bar{\Psi}\gamma^\mu\psi = (\psi^*\vec{\alpha}\psi,\ i\psi^*\psi) \qquad \text{(V)}$$
$$i\bar{\Psi}\gamma^\mu\gamma^\nu\psi \qquad \text{(T)}$$
$$i\bar{\Psi}\gamma^5\gamma^\mu\psi \qquad \text{(A)}$$
$$i\bar{\Psi}\gamma^5\psi \qquad \text{(P)}$$

The beta decay process is the decay of a neutron into a proton and an electron and an anti neutrino. Recall that conjugated wave functions represent the final state while un-conjugated wave functions represent the initial state. This is important as in second quantitization the wave functions take on annihilation - creation meanings.

Now the usual form of the β-decay interactions is written:

$$\int (\underbrace{\bar{\Psi}_p\gamma^\mu\psi_N)(\bar{\Psi}_e}_{\text{creation}}\underbrace{\gamma^\mu\psi_\nu)}_{\text{annihilation}}\, d\vec{r}$$

Today we know that there are only V-A interactions so we must put in $1-\gamma^5$ in the proper places.

However, instead of the above form of the interaction, Fermi first evolved the expression

$$(\psi_p^* \quad \psi_N)(\psi_e^*\,\delta \quad \psi_\nu^*) - (\psi_p^*\vec{\alpha}\psi_N)(\psi_e^*\,\delta\,\vec{\alpha}\,\psi_\nu^*)$$

while working out the β-decay reaction. The blank spaces above are for the insertion of the proper interaction operator. The quantity δ is chosen to make $\psi_1\delta\gamma^A\psi_2$ transform under a Lorentz transformation like $\psi_1^*\gamma^A\psi_2$. γ^A is any one of the covariant quantities.

However, since we are more used to working with $\bar{\psi}$ rather than ψ^*, we will restate the problem to see if we can find ξ such that $\psi_1 \xi \gamma^A \psi_2$ transforms like $\bar{\psi}_1 \gamma^A \psi_2$. We have from before:

$$\psi = S\psi' \quad ; \quad \bar{\psi} = \bar{\psi}' S^{-1}$$

Then: $\quad \bar{\psi}_1 \gamma^A \psi_2 \quad \longrightarrow \quad \bar{\psi}_1' S^{-1} \gamma^A S \psi_2'$

$$\psi_1 \xi \gamma^A \psi_2 \quad \longrightarrow \quad \psi_1' \tilde{S} \xi \gamma^A S \psi_2'$$

where the "snake ~" denotes transposition. Now, we see that we must have:

$$\psi_1' \tilde{S} \xi \gamma^A S \psi_2' \;=\; \psi_1' \xi S^{-1} \gamma^A S \psi_2'$$

so evidently the equation to be solved for ξ is:

$$\boxed{\tilde{S} \xi = \xi S^{-1}}$$

Recall now the form of the S's:

$$S = e^{\frac{\theta}{2}\gamma^\nu \gamma^\mu} = \cos\frac{\theta}{2} + \gamma^\nu \gamma^\mu \sin\frac{\theta}{2}$$

$$S^{-1} = e^{-\frac{\theta}{2}\gamma^\nu \gamma^\mu} = \cos\frac{\theta}{2} - \gamma^\nu \gamma^\mu \sin\frac{\theta}{2}$$

$$\tilde{S} = \cos\frac{\theta}{2} + \tilde{\gamma}^\mu \tilde{\gamma}^\nu \sin\frac{\theta}{2}$$

Hence the equation to handle becomes:

$$\boxed{\tilde{\gamma}^\mu \tilde{\gamma}^\nu \xi = \xi \gamma^\mu \gamma^\nu}$$

Recall that for space inversion we had a choice of phase factor to make. Pauli chose $S = \gamma^4$ while it was indicated that Racah's choice of $S = i\gamma^4$ might be more convenient. We will work both choices out, denoting ξ_1 for the choice of 1 for the phase factor and ξ_i for Racah's choice of i.

$\underline{S = \gamma^4}$: $\quad \tilde{\gamma}^4 \xi_1 = \xi_1 \gamma^4$

Now, since: $\tilde{\gamma}^\mu \tilde{\gamma}^4 \xi_1 = \xi_1 \gamma^\mu \gamma^4 = \tilde{\gamma}^\mu \xi_1 \gamma^4$
we also have:

$$\tilde{\gamma}^\mu \xi_1 = \xi_1 \gamma^\mu$$

Now Fermi used the correspondence $\psi^* \to \psi \delta$ whereas we used $\bar{\psi} \to \psi \xi$ or $\psi^* \beta \to \psi \xi$, hence we conclude:

$$\delta = \xi \beta \qquad \text{or} \qquad \delta \beta = \xi$$

Substituting above for ξ_1, we have:

$$\tilde{\beta} \delta = \delta \beta$$

Also, since $\gamma^k = -i \beta \alpha_k$; $\tilde{\gamma}^k = -i \tilde{\alpha}_k \tilde{\beta}$, we have:

$$\tilde{\alpha}_k \tilde{\beta} \delta \beta = \delta \beta \beta \alpha_k$$

or:
$$\tilde{\alpha}_k \delta = \delta \alpha_k$$

Thus we have determined δ essentially because we see that its action when moving from right to left across $\tilde{\alpha}_k$ or $\tilde{\beta}$ is to remove the "snake". Now, if we use the Dirac representation we know that $\beta, \alpha_1, \alpha_3$ are real and symmetric while α_2 is imaginary and antisymmetric.

$$\alpha_1 = \begin{pmatrix} 0&0&0&1 \\ 0&0&1&0 \\ 0&1&0&0 \\ 1&0&0&0 \end{pmatrix}; \quad \alpha_2 = \begin{pmatrix} 0&0&0&-i \\ 0&0&i&0 \\ 0&-i&0&0 \\ i&0&0&0 \end{pmatrix}; \quad \alpha_3 = \begin{pmatrix} 0&0&1&0 \\ 0&0&0&-1 \\ 1&0&0&0 \\ 0&-1&0&0 \end{pmatrix}; \quad \beta = \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&-1&0 \\ 0&0&0&-1 \end{pmatrix}$$

Thus, the sign of α_2 must change as we pass δ thru it while the signs of $\alpha_1, \alpha_3, \beta$ must not. The obvious choice for δ in the Dirac representation is:

$$\delta = \alpha_1 \alpha_3 \beta$$

$\underline{S = i\gamma^4}$: From $\tilde{S}\xi = \xi S^{-1}$, we get:

$$i\tilde{\gamma}_4 \xi_i = -i\, \xi_i \gamma^4 \quad \text{or:} \quad \tilde{\gamma}_4 \xi_i = -\xi_i \gamma^4$$

From $\tilde{\gamma}^\mu \tilde{\gamma}^\nu \xi = \xi \gamma^\mu \gamma^\nu$, we conclude:

$$\tilde{\gamma}^\mu \xi_i = -\xi_i \gamma^\mu \quad (\text{NB: sign changes unlike } \xi_1)$$

Now we know $\xi = \delta\beta$, so: $\tilde{\beta}\delta\beta = -\delta\beta\beta$, or

$$\tilde{\beta}\delta = -\delta\beta$$

Also: $\tilde{\alpha}_k \tilde{\beta}\delta\beta = -\delta\beta\beta\alpha_k = -\delta\alpha_k$

or: $\tilde{\alpha}_k \delta = \delta\alpha_k$

We could proceed on to work out a representation for δ or ξ_i.

The purpose of all the above discussion was to lead into the topic of charge conjugation which we touch on now. The process of charge conjugation is defined by the operation:

$$\boxed{\psi_c = C\bar{\psi}}$$

This, of course, alone does not define anything but gives an indication, from $\bar{\psi} \to \psi\xi$ or $\psi \to \tilde{\xi}^{-1}\bar{\psi}$, that C could be related to $\tilde{\xi}^{-1}$ or at least have very similar properties. ψ_c is called the charge conjugate wave function. For the moment, let us assume that $C = \tilde{\xi}_i^{-1}$, Taking ξ_i because of the sign change above above suggesting the possible future change in sign of charge when the Dirac equation is charge conjugated. We have from:

$$\tilde{\gamma}^\mu \xi_i = -\xi_i \gamma^\mu \; ; \; \xi_i = \tilde{C}^{-1} \text{ giving } \tilde{\gamma}^\mu \tilde{C}^{-1} = -\tilde{C}^{-1}\gamma^\mu$$

or $\boxed{\gamma^\mu C = -C\tilde{\gamma}^\mu}$

LECTURE 23 : 11-15-61

The Foldy - Wouthuysen Transformation

This transformation has as its object the transformation of the Hamiltonian into the Dirac even form, that is, of the form:

$$\underline{H}' = \begin{pmatrix} \times\times & \circ\circ \\ \times\times & \circ\circ \\ \circ\circ & \times\times \\ \circ\circ & \times\times \end{pmatrix}$$

This transformation can be effected with weak EM fields applied. Strong fields create particles so that single particle theory breaks down, or, there are transitions from - energy to + energy so that the transformation cannot be done. The transformation is written:

$$\psi \rightarrow \psi' = e^{-iS}\psi$$

$$\underline{H}' = e^{-iS}\,\underline{H}\,e^{iS}$$

Here S is none of those considered before. For the case of free particles we recall:

$$\underline{H}' = e^{-iS}\,\underline{H}\,e^{iS} = \beta S^{\dagger}(\vec{p})\;H\;S(\vec{p})\beta = \beta\beta p_0\beta = \beta p_0$$

We see that we can add β to the old transformation $S(\vec{p})$ without any change. S and $S(\vec{p})$ are not the same and only related through $e^{iS} = S(\vec{p})\beta$.

The full FW transformation can be worked out to successive approximations in the field applied to the single electron.

The FW transformation applied to other operators does not necessarily lead to an even representation and recall that the throwing away of the odd parts is not correct. The operators x and α are not pure even and cause transitions between negative and positive energies.

However, there is some interest in the even terms in x. Call the operator that we transform to the even form X. That is, we want:

$$\beta\ S^{+}(\vec{p})\ X\ S(\vec{p})\,\beta \Rightarrow x \longrightarrow \begin{pmatrix} x & x & 0 & \\ & 0 & x & x \end{pmatrix}$$

Hence:

$$X = S(\vec{p})\ x\ S^{+}(\vec{p})$$

$$= S(\vec{p})\ i\frac{\partial}{\partial p_x}\ S^{+}(\vec{p})$$

Now, implicit operation of X on a wave function is assumed, so that we can write:

$$X = S(\vec{p})\ S^{+}(\vec{p})\ x + S(\vec{p})\left\{ i\frac{\partial}{\partial p_x} S^{+}(\vec{p})\right\}$$

or: $$X = x + S(\vec{p})\left\{ i\frac{\partial}{\partial p_x} S^{+}(\vec{p})\right\}$$

Now, the second term above is connected with the ordinary indefiniteness in the position of the electron of the order of $\frac{\hbar}{mc}$ or the Compton wavelength.

Charge Conjugation

We define charge conjugation by the operation:

$$\boxed{\psi_c = C\bar{\Psi}}$$

and the requirement that C be such that if $\bar{\Psi}$ Lorentz transforms then ψ_c must also Lorentz transform. It is this requirement that enabled us to identify C with $\tilde{\xi}^{-1}$ previously and hence find the relation for C from the already known relation for ξ. We chose ξ_1 hoping that the sign change involved might come in handy.

However, we now proceed directly to find the relation determining C. Now:

$$\bar{\Psi} = \bar{\Psi}' S^{-1} \qquad (\bar{\Psi} \to \text{row matrix})$$
$$\bar{\Psi} = \tilde{S}^{-1} \bar{\Psi}' \qquad (\bar{\Psi} \to \text{column matrix})$$

One must be careful with the row or column representation of the wave functions. Now what we want is:

$$\psi_c = S\psi_c' = C\bar{\Psi} = C\tilde{S}^{-1}\bar{\Psi}'$$

or: $$\psi_c' = S^{-1}C\tilde{S}^{-1}\bar{\Psi}'$$

But for invariance of the charge conjugation operation in the new frame of reference, we must have:

$$\psi_c' = C\bar{\Psi}'$$

Thus: $$\boxed{SC = C\tilde{S}^{-1}}$$

For the space inversion, let us choose Racah's $S = i\gamma^4$ for the reasons expressed above. First of all, using

$$S = e^{\frac{\theta}{2}\gamma^\mu\gamma^\nu} = \cos\frac{\theta}{2} + \gamma^\mu\gamma^\nu \sin\frac{\theta}{2}$$

We have:

$$\boxed{\gamma^\mu \gamma^\nu C = C\, \tilde{\gamma}^\mu \tilde{\gamma}^\nu}$$

Now for $S = i\gamma^4$ we have directly: $\gamma^4 C = -C\tilde{\gamma}^4$
and by substitution into the above we have generally:

$$\boxed{C\,\tilde{\gamma}^\mu = -\gamma^\mu C}$$

Always note that the sign depends on our choice of phase for the space inversion.

We now pose the question of whether or not charge conjugation is self-reciprocal, that is, can we write?

$$\psi = c\,\bar{\psi}_c$$

We begin by writing $\bar{\psi} = C^{-1}\psi_c$; $\bar{\psi} = \psi^* \gamma^4$; $\bar{\psi} = \tilde{\gamma}^4 \psi^*$.

Then: $\psi^* = \tilde{\gamma}^4 C^{-1}\psi_c = -C^{-1}\gamma^4\psi_c$

Now note that: $\tilde{\gamma}^\mu C^{-1} = -C^{-1}\gamma^\mu$
and: $\tilde{\gamma}^\mu C^+ = -C^+\gamma^\mu$

so that when C^{-1} or C^+ is determined, so is the other and hence we can take C to be unitary, $C^{-1} = C^+$, as $\tilde{\gamma}^\mu$ is just as good a representation as γ^μ. Being unitary fixes any multiplicative constants in C.
Now we can write:

$$\psi^* = -C^+\gamma^4\psi_c \quad ; \quad \psi = -\tilde{C}\,\tilde{\gamma}^4\psi_c^* = -\tilde{C}\,\bar{\psi}_c$$

Now to be self-reciprocal, it is evident that we require:

$$\boxed{-\tilde{C} = C}$$

or that C be antisymmetric.

Let us try showing that $\tilde{C} = bC$ where b will have to be either $+1$ or -1. We begin by showing that $\tilde{C}C^{-1}$ commutes with the γ group.

$$C\tilde{\gamma}^\mu = -\gamma^\mu C$$
$$\gamma^\mu \tilde{C} = -\tilde{C}\tilde{\gamma}^\mu$$
$$\tilde{\gamma}^\mu = -C^{-1}\gamma^\mu C$$

Then: $\gamma^\mu \tilde{C} = +\tilde{C}C^{-1}\gamma^\mu C$

or:

$$\gamma^\mu \tilde{C} C^{-1} = \tilde{C}C^{-1}\gamma^\mu$$

and hence $\tilde{C}C^{-1}$ must be a multiple of the unit matrix, $\tilde{C}C^{-1} = b\mathbb{1}$, or:

$$\tilde{C} = bC \quad \text{and} \quad C = b\tilde{C}$$
$$\text{or} \quad C = b^2 C \;, \text{ so that } b = \pm 1$$

We must now choose the sign of b. We do this by reflecting that a matrix group comprised of 4×4 matrices cannot have more than 6 linearly independent antisymmetric matrices in it. This is because there are 6 off diagonal elements that can be reflected by transposition. What we do now is form the product of C with each element of the γ group and determine the sign of b necessary to satisfy the antisymmetric requirement. We have:

$\tilde{C} = bC$	1 matrix
$\widetilde{(\gamma^\mu C)} = bC\tilde{\gamma}^\mu = -b\gamma^\mu C$	4 matrices
$\widetilde{(\gamma^\mu\gamma^\nu C)} = bC\tilde{\gamma}^\nu\tilde{\gamma}^\mu = -b\gamma^\mu\gamma^\nu C \;;\; \nu \neq \mu$	6 matrices
$\widetilde{(\gamma^5\gamma^\mu C)} = b\gamma^5\gamma^\mu C$	4 matrices
$\widetilde{(\gamma^5 C)} = b\gamma^5 C$	1 matrix
	16 total

The last two expressions were obtained with the help of:

$$C\tilde{\gamma}^5 = C\tilde{\gamma}^4\tilde{\gamma}^3\tilde{\gamma}^2\tilde{\gamma}^1 = \gamma^4\gamma^3 C\tilde{\gamma}^2\tilde{\gamma}^1 = \gamma^4\gamma^3\gamma^2\gamma^1 C$$
$$= -\gamma^1\gamma^4\gamma^3\gamma^2 C = -\gamma^1\gamma^2\gamma^4\gamma^3 C = \gamma^5 C$$

We see that $b=+1$ gives us 10 antisymmetric matrices while $b=-1$ gives 6. Hence we have shown $b=-1$ and $-\tilde{C} = C$ so charge conjugation is self-reciprocal.

LECTURE 24: 11-17-61

Recapitulation: so far, we have shown the following relations from the basic definition:

$$\psi_c = C\bar{\psi}$$

plus the requirement that ψ_c behave properly under a Lorentz transformation and that we take Racah's choice for the phase factor of the space inversion $S = i\gamma^4$:

$$\boxed{\begin{array}{l} \psi_c = C\bar{\psi} \quad ; \quad C\tilde{\gamma}^\mu = -\gamma^\mu C \\ \tilde{C} = -C \quad ; \quad C^+ = C^{-1} \quad ; \quad \psi = C\bar{\psi}_c = (\psi_c)_c \end{array}}$$

We now consider the operation of charge conjugation on the Dirac equation. Recall:

$$\boxed{\begin{array}{l} \gamma^\mu \left\{ \frac{\partial}{\partial x_\mu} - \frac{ie}{\hbar c} \varphi_\mu \right\} \psi + \frac{mc}{\hbar} \psi = 0 \\ \left\{ \frac{\partial}{\partial x_\mu} + \frac{ie}{\hbar c} \varphi_\mu \right\} \bar{\psi} \gamma^\mu - \frac{mc}{\hbar} \bar{\psi} = 0 \end{array}}$$

now substitute: $\bar{\psi} = C^{-1}\psi_c$ $\qquad \psi = C\bar{\psi}_c$

$\bar{\psi} = \psi_c \tilde{C}^{-1}$ $\qquad \psi = \bar{\psi}_c \tilde{C}$

and get:

multiply by $-C^{-1}$, use $\tilde{\gamma}^\mu = -C^{-1}\gamma^\mu C$ and then transpose.

$$\left(\frac{\partial}{\partial x_\mu} - \frac{ie}{\hbar c} \varphi_\mu \right) \gamma^\mu C \bar{\psi}_c + \frac{mc}{\hbar} C \bar{\psi}_c = 0$$

$$\left(\frac{\partial}{\partial x_\mu} + \frac{ie}{\hbar c} \varphi_\mu \right) \psi_c \tilde{C}^{-1} \gamma^\mu - \frac{mc}{\hbar} \psi_c \tilde{C}^{-1} = 0$$

multiply by $-\tilde{C}$, use $-\tilde{C}^{-1}\gamma^\mu\tilde{C} = -C^{-1}\gamma^\mu C = \tilde{\gamma}^\mu$ and then transpose.

This gives us:

$$\boxed{\begin{aligned} &\left\{\frac{\partial}{\partial x_\mu} - \frac{ie}{\hbar c}\varphi_\mu\right\}\bar{\psi}_c\gamma^\mu - \frac{mc}{\hbar}\bar{\psi}_c = 0 \\ &\left\{\frac{\partial}{\partial x_\mu} + \frac{ie}{\hbar c}\varphi_\mu\right\}\gamma^\mu\psi_c + \frac{mc}{\hbar}\psi_c = 0 \end{aligned}}$$

This is the same set of equations as before but now the sign of the charge is switched, so evidently ψ represents the state of the electron and ψ_c that of the positron. We see the importance now of using Racah's $S = i\gamma^4$ as if we had taken $S = \gamma^4$ for space reflection, we would have ended up by changing the sign of the mass also.

We see that because charge conjugation is self-reciprocal, we have complete symmetry between electrons and positrons. We also see that the time dependent part of the wave function behaves properly since if $\psi \sim e^{-iEt/\hbar}$, then $\psi_c \sim \bar{\psi} \sim e^{iEt/\hbar} = e^{-i(-E)t/\hbar}$. Now if there is a change in the sign of the charge, $-E$ becomes positive and the time direction remains the same.

Time Reversal

The meaning of a "time reversal" operation is that it is possible to run many processes "backward in time" and cover the same path in phase space that the process has just passed over while running forward. In QM we would like to do this by letting $\psi(t) \to \psi(-t)$ but this will reverse the energy also. We get something like reversing the time when we take the conjugate of the Schroedinger equation:

$$\underline{H}\,\psi^*(t) = -i\hbar\frac{\partial}{\partial t}\psi^*(t)$$

However, this operation is incomplete if the Hamiltonian contains an imaginary factor. Otherwise we could put $\psi'(-t) = \psi^*(t)$. We then form a more general time reversal operation defined as:

$$\boxed{\psi'(-t) = T\psi^*(t)}$$

We might have included the conjugation operation in T but this would make it non-linear. A linear operator is one which obeys the rule:

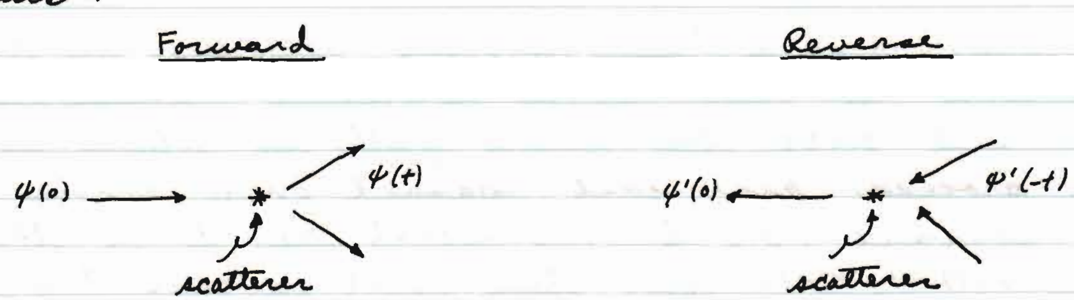
$$L(c_1\psi_1 + c_2\psi_2) = c_1 L\psi_1 + c_2 L\psi_2$$

If we had defined $\mathcal{T}\psi = T\psi^*$, then we would have:

$$\mathcal{T}(c_1\psi_1 + c_2\psi_2) = c_1^* T\psi_1^* + c_2^* T\psi_2^*$$

which is not linear. Thus we prefer to operate with a linear operator and then take the complex conjugate separately.

We will talk about the Schroedinger election for a while. The physical picture of time reversal applied to a scattering problem and considered at the same moment of time in both its forward and backward motions is something like this:



In mathematical language for the physics, we say that at corresponding times we want:

$$\langle \vec{r} \rangle_{\text{reverse}} = \langle \vec{r} \rangle \quad ; \quad \langle \vec{p} \rangle_{\text{reverse}} = -\langle \vec{p} \rangle$$

In other words:

$$\int \psi^{*\prime}(-t)\, \vec{r}\, \psi'(-t)\, d\vec{r} = \int \psi^*(t)\, \vec{r}\, \psi(t)\, d\vec{r}$$

$$\int \psi^{*\prime}(-t)\, \vec{p}\, \psi'(-t)\, d\vec{r} = -\int \psi^*(t)\, \vec{p}\, \psi(t)\, d\vec{r}$$

$$= \int \psi(t)\, \vec{p}\, \psi^*(t)\, d\vec{r}$$

by integrating by parts. From these two equations it is easy to conclude that $\psi'(\vec{r},-t) = \psi^*(\vec{r},t)$ so that in the $\vec{r}$ representation, $T = 1$ or $T_{\vec{r}} = 1$.

What is T in the momentum representation? We know that generally we have:

$$\varphi'(\vec{p},-t) = T_{\vec{p}}\, \varphi^*(\vec{p},t)$$

Recall the Fourier transform relationship between the $\vec{r}$ and $\vec{p}$ representations which we now express in operator form, $\mathcal{F}$ being the Fourier operator:

$$\psi = \mathcal{F}\varphi$$
$$\psi' = \mathcal{F}\varphi'$$

We then can write:

$$\psi'(-t) = \mathcal{F}\varphi'(-t) = 1\cdot\{\mathcal{F}\varphi(t)\}^* = 1\cdot\psi^*(t) = \mathcal{F}^*\varphi^*(t)$$

$$\varphi'(-t) = \mathcal{F}^{-1}\psi'(-t) = \mathcal{F}^{-1}\mathcal{F}^*\varphi^*(t) = T_{\vec{p}}\,\varphi^*(t)$$

Now $\psi = \mathcal{F}\varphi$ means explicitly:

$$\psi(\vec{r}) = \int \langle\vec{r}|\vec{p}\rangle\, d\vec{p}\, \varphi(\vec{p}) = \langle\vec{r}|\rangle = \int \langle\vec{r}|\vec{p}\rangle\, d\vec{p}\, \langle\vec{p}|\rangle$$

$$\varphi(\vec{p}) = \langle\vec{p}|\rangle = \int \langle\vec{p}|\vec{r}\rangle\, d\vec{r}\, \langle\vec{r}|\rangle$$

so we see that $\mathcal{F}^{-1} = \mathcal{F}^*$ when we write:

$$\langle\vec{r}|\vec{p}\rangle = \frac{e^{i\vec{p}\cdot\vec{r}}}{(2\pi)^{3/2}}$$

Proceeding then:

$$\varphi'(-t) = \mathcal{F}^{-1}\mathcal{F}^{*}\ \varphi^{*}(t) = \frac{1}{(2\pi)^3}\int e^{-i(\vec{p}\,'+\vec{p})\cdot\vec{r}}\, d\vec{p}\ \ \varphi^{*}(\vec{p})\, d\vec{r}$$

$$= \int \delta(\vec{p}\,'+\vec{p})\, d\vec{p}\ \ \varphi^{*}(\vec{p},t)$$

Thus we see that:

$$\varphi'(\vec{p},-t) = \varphi^{*}(-\vec{p},t)$$

a result which we could have surmised from the beginning. Thus the action of $T_{\vec{p}}$ is to change the sign on $\vec{p}$ or reverse the momentum.

To check these results, we calculate $\langle\vec{r}\rangle$ reverse in the $\vec{p}$ representation:

$$\langle\vec{r}\rangle_{\text{reverse}} = \int \varphi'^{*}(\vec{p},-t)\ \vec{r}\ \varphi'(\vec{p},-t)\, d\vec{p}$$

$$= \int \varphi(-\vec{p},t)\ \vec{r}\ \varphi^{*}(-\vec{p},t)\, d\vec{p}$$

$$= -\int \varphi^{*}(-\vec{p},t)\ \vec{r}\ \varphi(-\vec{p},t)\, d\vec{p}$$

writing $\vec{r} = i\hbar\left(\hat{i}\frac{\partial}{\partial p_x} + \hat{j}\frac{\partial}{\partial p_y} + \hat{k}\frac{\partial}{\partial p_z}\right)$ and doing by parts.

We now change the variables of integration to $-p_x, -p_y, -p_z$ and get:

$$\langle\vec{r}\rangle_{\text{reverse}} = \int \varphi^{*}(\vec{p},t)\ \vec{r}\ \varphi(\vec{p},t)\, d\vec{p}$$

$$= \langle\vec{r}\rangle$$

Hence we check out all right.

LECTURE 25: 11-20-61

Now that we have found the action of T in both the $\vec{r}$ and the $\vec{p}$ representations, we can immediately write down their matrix elements:

$$\langle \vec{r} | T | \vec{r}' \rangle = \langle \vec{r} | \vec{r}' \rangle = \delta(\vec{r} - \vec{r}')$$

$$\langle \vec{p} | T | \vec{p}' \rangle = \langle \vec{p} | -\vec{p}' \rangle = \delta(\vec{p} + \vec{p}')$$

since $\psi'(\vec{r}, -t) = T_{\vec{r}}\, \psi^*(\vec{r}, t) = \psi^*(\vec{r}, t)$

$$\varphi'(\vec{p}, -t) = T_{\vec{p}}\, \varphi^*(\vec{p}, t) = \varphi^*(-\vec{p}, t)$$

Up to now we have been working in the Schroedinger picture. Let us now look at the Dirac picture or the Hiesenberg picture. Recall that in the Schroedinger picture the time dependence is included in the wave function in the following way:

$\psi(t) = U(t)\, \psi(0)$, where $U(t)$ is a unitary operator.

<u>Schroedinger Picture</u>: $\int \psi^* O \psi \, d\tau$

$$= \int \underbrace{\psi^*(0)\, U^+(t)}\; O\; \underbrace{U(t)\, \psi(0)}\, d\tau$$

time dependence in wave function

<u>Hiesenberg Picture</u>:

$$\int \psi^*(0)\, \underbrace{U^+(t)\, O\, U(t)}\, \psi(0)\, d\tau$$

time dependence in operator

Thus: $O(t) = U^+(t)\, O\, U(t)$

Now what is $U(t)$? Evidently it must satisfy:

$$\psi'(-t) = T\, U^*(t)\, \psi^*(0)$$

We could define another operator $U_{rev}(-t)$ such that:

$$\psi'(-t) = U_{rev}(-t)\,\psi'(0) = U_{rev}(-t)\,T\,\psi^*(0)$$

We can get $U_{rev}(-t)$ in terms of $U(t)$ by comparing the two previous expressions together:

$$U_{rev}(-t) = T\,U^*(t)\,T^{-1}$$

If H depends on time, the most general form of U is:

$$U(t) = e^{-\frac{i}{\hbar}\int_0^t H(t')\,dt'}$$

To proceed further, we need not assume $H(t)$ independent of time as this would defeat the purpose of the procedure, but we can invoke the adiabatic approximation and assume $H(t)$ small compared to the natural periods of the system, that is, we assume the time dependence of H to be much slower than any of the stationary periods of the system. We then have for $U(t)$:

$$U(t) = 1 - i\,\frac{H(t)\,t}{\hbar}$$

Applying the formula for $U_{rev}(-t)$ we immediately have:

$$\boxed{H_{rev}(-t) = T\,H^*(t)\,T^{-1}}$$

The subscript "rev" denotes the operations to be done on $H(-t)$ to make it equal to $T\,H^*(t)\,T^{-1}$. For example, if the Hamiltonian is of the form:

$$H(t) = -\frac{\hbar^2}{2m}\nabla^2 + V(t)$$

and we are using the coordinate representation where $T\vec{r} = 1$, we have:

$$H_{rev}(t) = H^*(t) = H(t)$$

Hence, here the "rev" operation means just changing the sign of t as usual for time reversal (we assume always that $V(t)$ is real). However, suppose we have:

$$H(t) = -\frac{\hbar^2}{2m}\left\{\nabla - \frac{ie}{\hbar c}A(t)\right\}^2 + V(t)$$

$$H(-t) = -\frac{\hbar^2}{2m}\left\{\nabla - \frac{ie}{\hbar c}A(-t)\right\}^2 + V(-t)$$

$$H^*(t) = -\frac{\hbar^2}{2m}\left\{\nabla + \frac{ie}{\hbar c}A(t)\right\}^2 + V(t)$$

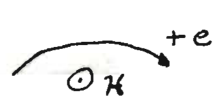
So we see that this Hamiltonian is not time reversable in the sense that "rev" means just changing the sign of the time. To have $H_{rev}(-t) = H^*(t)$ we must also change the sign of the <u>vector</u> potential. Note that changing the sign of the charge is ineffective as the charge is also involved in $V(t)$. In other words, to satisfy $H_{rev}(-t) = H^*(t)$, we must write:

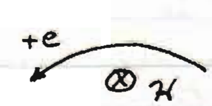
$$H_{rev}\left(m, e, \vec{\mathcal{E}}(-t), \vec{\mathcal{H}}(-t)\right) = H\left(m, e, \vec{\mathcal{E}}(t), -\mathcal{H}(t)\right)$$

Hence we see that the scalar potential is invariant under time reversal, while the vector potential changes sign. The electric field is left invariant because:

$$\vec{\mathcal{E}} = -\nabla\varphi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}$$

and we see that changing time and the sign of $\vec{A}$ are cancelling operations, while $\vec{\mathcal{H}}$ changes because $\vec{\mathcal{H}} = \nabla \times \vec{A}$. This is obvious from physical reasoning about a positive charge in a magnetic field. To have the same path of motion in both the direct and reverse system we must change the direction of $\vec{\mathcal{H}}$ in the reverse system:

Direct: 

Reverse: 

The fact that under time reversal we must also reverse the magnetic field implies that when spin is considered, these must be reversed also. This can be seen from a consideration of the Stern - Gerlach experiment.

Consider the Pauli electron in a magnetic field. In the Hamiltonian we have the Zeeman and spin - orbit terms:

Zeeman: $-\frac{e\hbar}{2mc}(\vec{\mathcal{H}}\cdot\vec{\sigma})$ spin-orbit: $\vec{\sigma}\cdot\left(\frac{\vec{p}}{mc}\times\vec{\mathcal{E}}\right)$

We see that $H_{rev}(-t) = H^*(t)$ if we include in the "rev" operation, $t\to -t$, $\vec{\mathcal{H}} \to -\vec{\mathcal{H}}$, $\vec{\sigma}\to -\vec{\sigma}$.

Even if we have spin - spin coupling in the absence of external fields we can still reverse the spins under time reversal because the coupling terms in the Hamiltonian are of the form:

$$\underbrace{\frac{\vec{\sigma}_1\cdot\vec{\sigma}_2}{r^3}}_{\text{in-line spins}} + \underbrace{\frac{(\vec{\sigma}_1\cdot\vec{r})(\vec{\sigma}_2\cdot\vec{r})}{r^5}}_{\text{out of line spins}}$$

Because now we are including spin we have two component wave functions and the time reversal operator must have a part that operates on the spin part of the wave function. Thus in the $\vec{r}$ representation we may write:

$$T = 1_{\vec{r}}\; \tau_{spin}$$

From our deduced results concerning $H_{rev}(-t) = T H^*(t) T^{-1}$ we see that:

$$T\,\sigma_j^*\,T^{-1} = -\sigma_j$$

must be true. That the above arises can be seen from a consideration of the spin-orbit term of the Hamiltonian.

$$H_{so\,rev}(-t) = T H_{so}^{*}(t) T^{-1} = T\, \vec{\sigma}^{*} \cdot \left(\frac{\vec{p}^{*}}{mc} \times \vec{\mathcal{E}} \right) T^{-1}$$

$$= T \vec{\sigma}^{*} T^{-1}\, T \cdot \left(\frac{-\vec{p}}{mc} \times \vec{\mathcal{E}} \right) T^{-1} = -T \vec{\sigma}^{*} T^{-1} \cdot \left(\frac{\vec{p}}{mc} \times \vec{\mathcal{E}} \right)$$

$$= \vec{\sigma} \cdot \left(\frac{\vec{p}}{mc} \times \vec{\mathcal{E}} \right)$$

We now ask what the form of the T must be. Recall the properties of the σ_j's:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ; \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} ; \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_j = \sigma_j^{+} = \sigma_j^{-1}$$

$$\sigma_x^{*} = \sigma_x \quad ; \quad \sigma_y^{*} = -\sigma_y \quad ; \quad \sigma_z^{*} = \sigma_z$$

$$\sigma_k \sigma_l = \delta_{kl} + i\, \epsilon_{klm}\, \sigma_m$$

We see that for a single electron $T = \sigma_y$ satisfies the requirements. For n particles, recalling $\psi'(-t) = T\psi^{*}(t)$ and that ψ is a product wave function, we must necessarily have:

$$T = \prod_{j=1}^{n} (\sigma_j)_y$$

The many electron picture is applicable to crystals and gives rise to Kramers degeneracy.

Kramer's Degeneracy:

If there is no magnetic field and the Hamiltonian is independent of time, we have:

$$H = H_{rev} = T H^{*} T^{-1}$$

At this time, we should note that the change in the sign of the spin is a part of the operation of T, the time reversal operator, and not an "ad hoc" operation such as is necessary to reverse the magnetic field. However, note that it is the magnetic field reversal that was the originally responsible field that caused us to change the direction of the spin.

However, when the magnetic field was absent, an examination of the spin-spin coupling terms convinced us that reversal of spin left the Hamiltonian invariant anyway. This does not necessarily mean that the wave function giving the states of direct and reversed spins are necessarily the same and hence the reason for suspecting some sort of degeneracy to occur.

Now notice that since $H = H^*$, $HT = TH$ or H commutes with T. This means that H and T are diagonalized by the same unitary transformation or wave function. Proceeding:

$$H\psi = E\psi$$
$$H^*\psi^* = E\psi^* \quad ; \quad H\psi^* = E\psi^*$$

and:

$$TH^*T^{-1}T\psi^* = ET\psi^*, \text{ or } HT\psi^* = ET\psi^*$$

$$\text{or } TH\psi^* = ET\psi^* \quad \text{or} \quad H\psi^* = E\psi^* \text{ as above.}$$

Now, $T\psi^*$ is another eigenfunction of H. It may be just a multiple of ψ, that is.

$$T\psi^* = \lambda\psi$$

or it is entirely different, in which case we have Kramers degeneracy. We will now show that it takes an odd number of spin particles to give a Kramers degeneracy. Assume $T\psi^* = \lambda\psi$ and form:

$$(\psi')' = T(\psi')^* = TT^*\psi = T\lambda^*\psi^* = |\lambda|^2\psi$$

But: $TT^* = \prod_{j=1}^{n}\sigma_{jy}\prod_{j=1}^{n}\sigma_{jy}^* = \prod_{j=1}^{n}\sigma_{jy}\sigma_{jy}^* = \prod_{j=1}^{n}(-1) = (-1)^n$

So if $T\psi^* = \lambda\psi$, we have for $(\psi')'$:

$(\psi')' = (-1)^n\psi = |\lambda|^2\psi$ which is impossible if n is odd in which case we have a two-fold degeneracy which due to the general nature of H cannot be removed by any possible type of electric field.

LECTURE 26 : 11-22-61

Wigner Time Reversal

We now take up the consideration of time reversal and the Dirac electron. Recall that for the Schroedinger and Pauli elections we had:

$$\psi' = T\psi^* \quad \text{and} \quad \psi' = \tau\psi^*$$

For the Dirac election, we will have geometric time reflection ($x_k = x_k'$, $x_4 = -x_4'$) involved in the time reversal plus the usual reverse in the sign of the vector potential. Instead of writing the time reversal operation in terms of the Lorentz time reflection ($\psi' = S\psi$), let us take our cue from the NR case and write for the reversed wave function:

$$\boxed{\begin{array}{cc} \psi' = S\bar{\Psi} & \psi' = \bar{\Psi}\tilde{S} \\ \bar{\Psi} = \tilde{S}^{-1}\psi' & \bar{\Psi} = \psi' S^{-1} \\ \text{column} & \text{row} \end{array}}$$

S is <u>not</u> a Lorentz operator but a time reversal operator just like T in the NR case. Let us write down some useful relations:

$$\bar{\Psi} = \psi' S^{-1} \; ; \quad \bar{\Psi} = \psi^* \gamma^4 \; ; \quad \psi^* = \psi' S^{-1} \gamma^4$$

$$\psi = \psi'^* S^{-1*} \gamma^{4*} = \psi'^* \gamma^4 \gamma^4 S^{-1*} \gamma^{4*} = \bar{\Psi}' \gamma^4 S^{-1*} \gamma^{4*} = \bar{\Psi}' \gamma^4 S^{-1*} \gamma^{4T*}$$

$$= \bar{\Psi}' \gamma^4 S^{-1*} \tilde{\gamma}^4$$

We also have under time reversal, the relations:

$$x_4 = -x_4' = x_4'^* \quad ; \quad \varphi_\mu = -\varphi_\mu'^* \quad [\phi = \phi' \; ; \; A = -A']$$
$$x_k = x_k^*$$

recalling that: $x_\mu = (\vec{r}\,; ict)$; $\varphi_\mu = (\vec{A}, i\phi)$

Recall Dirac's equations:

$$\boxed{\begin{aligned}&\left(\frac{\partial}{\partial x_\mu} - \frac{ie}{\hbar c}\varphi_\mu\right)\gamma^\mu \psi + \frac{mc}{\hbar}\psi = 0\\ &\left(\frac{\partial}{\partial x_\mu} + \frac{ie}{\hbar c}\varphi_\mu\right)\bar{\psi}\gamma^\mu - \frac{mc}{\hbar}\bar{\psi} = 0\end{aligned}}$$

We now time reverse these equations. We get for the first equation:

$$\left(\frac{\partial}{\partial x'^*_\mu} + \frac{ie}{\hbar c}\varphi'^*_\mu\right)\bar{\psi}'\gamma^4 S^{-1*}\tilde{\gamma}^4\tilde{\gamma}^\mu + \frac{mc}{\hbar}\bar{\psi}'\gamma^4 S^{-1*}\gamma^4 = 0$$

Note that we have transformed the equation before operating on it or rather transposed it. We do the same to the second equation:

$$\left(\frac{\partial}{\partial x'^*_\mu} - \frac{ie}{\hbar c}\varphi'^*_\mu\right)\tilde{\gamma}^\mu\tilde{S}^{-1}\psi' - \frac{mc}{\hbar}\tilde{S}^{-1}\psi' = 0$$

We would like this last one to have the form:

$$\left(\frac{\partial}{\partial x'_\mu} - \frac{ie}{\hbar c}\varphi'_\mu\right)\gamma^\mu\psi' + \frac{mc}{\hbar}\psi' = 0$$

so that the Dirac equation will be invariant under time reversal. We start by multiplying on the left by $-\tilde{S}$ and get:

$$\left(\frac{\partial}{\partial x'^*_\mu} - \frac{ie}{\hbar c}\varphi'^*_\mu\right)\left(-\tilde{S}\tilde{\gamma}^\mu\tilde{S}^{-1}\right)\psi' + \frac{mc}{\hbar}\psi' = 0$$

so we see we have to have $-\tilde{S}\tilde{\gamma}^\mu\tilde{S}^{-1} = \gamma^\mu$ and we must be able to take off the $*$. This means that we must change the sign of γ^μ if $\mu = 4$ but not if $\mu = k$. Thus we must write:

$$-\gamma^4\gamma^\mu\gamma^4 = \begin{cases}-\gamma^4 & \mu = 4\\ \gamma^k & \mu = k\end{cases}$$

Then:

$$\boxed{\tilde{\gamma}^4\tilde{\gamma}^\mu\tilde{\gamma}^4 = S^{-1}\gamma^\mu S}$$

now S must be unitary, $S^{-1} = S^{+}$, as we may suspect from taking the Hermitean conjugate of the above, getting $\tilde{\gamma}^4 \tilde{\gamma}^\mu \tilde{\gamma}^4 = S^{+} \gamma^\mu S^{-1+}$. S cannot be determined explicitly in terms of the γ's unless we specify a representation to which the γ's belong (Dirac's or Weyl's).

We now repeat somewhat the above operations on the first Dirac equation by multiplying on the right by $-\tilde{\gamma}^4 S^{*} \gamma^4$ and getting:

$$\left(\frac{\partial}{\partial x_\mu'^{*}} + \frac{ie}{\hbar c} \varphi_\mu'^{*}\right) \bar{\psi}' \gamma^4 S^{-1*} \tilde{\gamma}^4 \tilde{\gamma}^\mu (-\tilde{\gamma}^4 S^{*} \gamma^4) - \frac{mc}{\hbar} \bar{\psi}' = 0$$

now we would like this equation in the form:

$$\left(\frac{\partial}{\partial x_\mu'} + \frac{ie}{\hbar c} \varphi_\mu'\right) \bar{\psi}' \gamma^\mu - \frac{mc}{\hbar} \bar{\psi}' = 0$$

so we see we must have:

$$-\gamma^4 S^{-1*} \tilde{\gamma}^4 \tilde{\gamma}^\mu \tilde{\gamma}^4 S^{*} \gamma^4 = \gamma^\mu$$

in addition to being able to remove the $*$. As above, this can be done by writing:

$$-\gamma^4 S^{-1*} \tilde{\gamma}^4 \tilde{\gamma}^\mu \tilde{\gamma}^4 S^{*} \gamma^4 = -\gamma^4 \gamma^\mu \gamma^4$$

or: $S^{-1*} \tilde{\gamma}^4 \tilde{\gamma}^\mu \tilde{\gamma}^4 S^{*} = \gamma^\mu$; $\tilde{\gamma}^4 \tilde{\gamma}^\mu \tilde{\gamma}^4 = S^{*} \gamma^\mu S^{-1*} = S^{*} \tilde{\gamma}^{\mu *} S^{-1*}$

$$\boxed{\begin{aligned} \gamma^4 \gamma^\mu \gamma^4 &= S \tilde{\gamma}^\mu S^{-1} \\ \tilde{\gamma}^4 \tilde{\gamma}^\mu \tilde{\gamma}^4 &= \tilde{S}^{-1} \gamma^\mu \tilde{S} \end{aligned}}$$

now this is not the same as the relation obtained from the first equation and suggests the relation:

$$\tilde{S} = \pm S$$

Indeed, $\tilde{S} = -S$, and this can be shown in exactly the same manner as it was shown for charge conjugation, by commuting $S^{-1}\tilde{S}$ with the γ's and showing that -1 gives the right number of antisymmetric matrices.

In the following, we will use the relation $\tilde{\gamma}_4 \tilde{\gamma}^\mu \tilde{\gamma}^4 = S^{-1}\gamma^\mu S$. What is S in the Dirac representation? Recall:

Dirac Representation: $\gamma^k = -i\beta\alpha_k$; $\tilde{\gamma}^k = -i\tilde{\alpha}_k\tilde{\beta}$
$= i\tilde{\beta}\tilde{\alpha}_k = i\beta\tilde{\alpha}_k$; $\gamma^4 = \beta$, $\tilde{\gamma}^4 = \gamma^4$.

From the properties of the α's, we have further:

$$\tilde{\gamma}^1 = -\gamma^1 \quad ; \quad \tilde{\gamma}^2 = \gamma^2 \quad ; \quad \tilde{\gamma}^3 = -\gamma^3 \quad ; \quad \tilde{\gamma}^4 = \gamma^4$$

Now, using $\gamma^4\gamma^\mu\gamma^4 = S\tilde{\gamma}^\mu S^{-1}$ as obtained from the first Dirac equation, we substitute for various values of μ:

$$\begin{aligned}
\mu = 1: &\quad \gamma^4\gamma^1\gamma^4 = S\tilde{\gamma}^1 S^{-1} = -\gamma^1 = -S\gamma^1 S^{-1} \\
\mu = 2: &\quad \gamma^4\gamma^2\gamma^4 = S\tilde{\gamma}^2 S^{-1} = -\gamma^2 = S\gamma^2 S^{-1} \\
\mu = 3: &\quad \gamma^4\gamma^3\gamma^4 = S\tilde{\gamma}^3 S^{-1} = -\gamma^3 = -S\gamma^3 S^{-1} \\
\mu = 4: &\quad \gamma^4\gamma^4\gamma^4 = S\tilde{\gamma}^4 S^{-1} = \gamma^4 = S\gamma^4 S^{-1}.
\end{aligned}$$

Thus we see that S commutes with γ^1, γ^3 and γ^4 but anti-commutes with γ^2. This result would also have been obtained if we had used the relation $\gamma^4\tilde{\gamma}^\mu\gamma^4 = S^{-1}\gamma^\mu S$. Now a matrix S that commutes with $\gamma^1, \gamma^3, \gamma^4$ but anticommutes with γ^2 must have the form $\gamma^1\gamma^3\gamma^4$. We see that this is unitary from:

$$(\gamma^1\gamma^3\gamma^4)^\dagger(\gamma^1\gamma^3\gamma^4) = \gamma^4\gamma^3\gamma^1\gamma^1\gamma^3\gamma^4 = 1$$

However, it is not Hermitean since:

$$(\gamma^1\gamma^3\gamma^4)^\dagger = \gamma^4\gamma^3\gamma^1 = -(\gamma^1\gamma^3\gamma^4)$$

To have S unitary along with the convenience of being Hermitean, we finally take:

$$\boxed{S = i\gamma^1\gamma^3\gamma^4 \ : \text{Dirac Representation}}$$

What is $\tilde{S}$? $\tilde{S} = (\widetilde{i\gamma^1\gamma^3\gamma^4}) = i\tilde{\gamma}^4\tilde{\gamma}^3\tilde{\gamma}^1 = i\gamma^4\gamma^3\gamma^1 = -S$.

Hence, in the the Dirac representation, we have:

$$S = S^\dagger = S^{-1} = -\tilde{S}$$

or S is unitary, Hermitean, and antisymmetric.

For the Dirac matrix form of S, we have:

$$S = i\gamma^1\gamma^3\gamma^4 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}$$

Let us try and find some of the properties of S in some other representation. We go to this new representation by applying a unitary transformation on the γ^μ's represented by a unitary matrix R:

$$\gamma^\mu = R\gamma'^\mu R^{-1} \quad ; \quad \tilde{\gamma}^\mu = \tilde{R}^{-1}\tilde{\gamma}'^\mu\tilde{R}$$

Substitute these in: $\tilde{\gamma}^4\tilde{\gamma}^\mu\tilde{\gamma}^4 = S^{-1}\gamma^\mu S$:

$$(\tilde{R}^{-1}\tilde{\gamma}'^4\tilde{R})(\tilde{R}^{-1}\tilde{\gamma}'^\mu\tilde{R})(\tilde{R}^{-1}\tilde{\gamma}'^4\tilde{R}) = S^{-1}(R\gamma'^\mu R^{-1})S = \tilde{R}^{-1}\tilde{\gamma}'^4\tilde{\gamma}'^\mu\tilde{\gamma}'^4\tilde{R}$$

or: $\tilde{\gamma}'^4\tilde{\gamma}'^\mu\tilde{\gamma}'^4 = (\tilde{R}S^{-1}R)\gamma'^\mu(R^{-1}S\tilde{R}^{-1})$, that is, in the new representation:

$$\tilde{R}S^{-1}R = S'^{-1} \quad \text{and} \quad R^{-1}S\tilde{R}^{-1} = S'$$

Are these unitary transformations? Look at $R^{-1}S\tilde{R}^{-1} = S'$. We know that $\tilde{R}^{-1} = \tilde{R}^{+} = R^*$ so that:

$$R^{-1}SR^* = S'; \quad S = RS'R^{-1*} \quad ; \quad S = RS'\tilde{R}$$

Hence the transformation from S' to S is not unitary because if it were, $S = RS'R^{-1}$ instead of the above. Now notice that since:

$$S' = R^{-1}S\tilde{R}^{-1}, \text{ then: } \tilde{S}' = R^{-1}\tilde{S}\tilde{R}^{-1} = R^{-1}(-S)\tilde{R}^{-1} = -S'$$

so that if S is antisymmetric in the original representation it remains so in the new as we already know from a more general treatment.

Let us examine further S in the Dirac representation:

$$S = i\gamma^1\gamma^3\gamma^4 = i(-i\beta\alpha_1)(-i\beta\alpha_3)\beta$$

$$S = -i(\beta\alpha_1)(\beta\alpha_3)\beta = i\beta\alpha_1\alpha_3 = i\beta\rho_1\sigma_1\rho_1\sigma_3$$

$$= i\beta\rho_1\rho_1\sigma_1\sigma_3 = i\beta\sigma_1\sigma_3 = i\beta(-i\sigma_2) = \beta\sigma_2$$

Then:
$$S = \beta\sigma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} +\sigma_2 & 0 \\ 0 & +\sigma_2 \end{pmatrix} = \begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix} = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}$$

Thus we see that for the large components of the wave function, S is essentially σ_2 thus corroborating the result obtained for the Pauli electron.

Let us now examine the behaviour of some of the Pauli covariant quantities under the operation of Wigner Time reversal. Let the covariant quantity be represented by $\bar{\Psi}\eta\psi$ and we now will make the usual substitutions:

$$\bar{\Psi}\eta\psi = \psi' S^{-1}\eta\gamma^4 S\tilde{\gamma}^4\bar{\Psi}' = \bar{\Psi}'\gamma^4\tilde{S}\tilde{\gamma}^4\tilde{\eta}\tilde{S}^{-1}\psi'$$

where we have transposed going to the RHS which is all right as $\bar{\Psi}\eta\psi$ is a scalar number and transposition of each matrix and the whole product cannot change the result.

However, at this point we may note that there is another very good reason for choosing $\bar{\Psi} = \psi' S^{-1}$ as the form of the time reversal operation. At corresponding times, as we are considering above, we would expect the <u>initial</u> state of the reversed system (ψ') to be related to the <u>final</u> state of the direct system ($\bar{\Psi}$). Hence the form $\bar{\Psi} = \psi' S^{-1}$ is justified physically. Consider some forms of η:

<u>$\eta = I$</u> : $\bar{\Psi}\psi = \bar{\Psi}'\gamma^4\tilde{S}\tilde{\gamma}^4\tilde{S}^{-1}\psi' = \bar{\Psi}'\gamma^4\gamma^4\gamma^4\gamma^4\psi' = \bar{\Psi}'\psi'$

Hence the scalar quantity is invariant under Wigner Time Reversal.

In what follows, there should strictly be included in η an "i" but this will be omitted for simplicity.

$\underline{\eta = \gamma^\mu}$: $\bar{\Psi}\gamma^\mu\psi = \bar{\Psi}'\gamma^4\tilde{S}\tilde{\gamma}^4\tilde{\gamma}^\mu\tilde{S}^{-1}\psi' = \bar{\Psi}'\gamma^4\tilde{S}\tilde{\gamma}^4\tilde{S}^{-1}\tilde{S}\tilde{\gamma}^\mu\tilde{S}^{-1}\psi'$

$= \bar{\Psi}'\gamma^4\gamma^\mu\gamma^4\psi'$

Hence: $\bar{\Psi}\gamma^4\psi = \bar{\Psi}'\gamma^4\psi'$
$\bar{\Psi}\gamma^k\psi = -\bar{\Psi}'\gamma^k\psi'$

$\underline{\eta = \gamma^\mu\gamma^\nu;\ \nu \neq \mu}$: $\bar{\Psi}\gamma^\mu\gamma^\nu\psi = \bar{\Psi}'\gamma^4\tilde{S}\tilde{\gamma}^4\tilde{\gamma}^\nu\tilde{\gamma}^\mu\tilde{S}^{-1}\psi'$

$= \bar{\Psi}'\gamma^4\tilde{S}\tilde{\gamma}^4\tilde{S}^{-1}\tilde{S}\tilde{\gamma}^\nu\tilde{S}^{-1}\tilde{S}\tilde{\gamma}^\mu\tilde{S}^{-1}\psi' = \bar{\Psi}'\gamma^4\gamma^\nu\gamma^\mu\gamma^4\psi' = -\bar{\Psi}'\gamma^4\gamma^\mu\gamma^\nu\gamma^4\psi'$

Hence: $\bar{\Psi}\gamma^k\gamma^l\psi = -\bar{\Psi}'\gamma^4\gamma^k\gamma^l\gamma^4\psi' = -\bar{\Psi}'\gamma^k\gamma^l\psi'$; $k \neq l$
$\bar{\Psi}\gamma^k\gamma^4\psi = -\bar{\Psi}'\gamma^4\gamma^k\gamma^4\gamma^4\psi' = \bar{\Psi}'\gamma^k\gamma^4\psi'$

This result agrees with the physical arguments that can be made from the Pauli interaction term $F_{\mu\nu}\gamma^\mu\gamma^\nu$. When $\gamma^\mu\gamma^\nu$ is $\gamma^k\gamma^4$, the Pauli term corresponds to the electric vector in the maxwell field tensor and we do not want this to change sign under time reversal. Also, when $\gamma^\mu\gamma^\nu$ is $\gamma^k\gamma^l$, we have the magnetic vector part of the field tensor. Hence F_{kl} changes sign under time reversal but so does $\gamma^k\gamma^l$, thus the interaction term remains the same. Thus, if we add the Pauli term to the Dirac equations, it still remains invariant under time reversal.

$\underline{\eta = \gamma^5}$: $\bar{\Psi}\gamma^5\psi = \bar{\Psi}'\gamma^4\tilde{S}\tilde{\gamma}^4\tilde{\gamma}^5\tilde{S}^{-1}\psi' = \bar{\Psi}'\gamma^4\tilde{S}\tilde{\gamma}^4\tilde{S}^{-1}\tilde{S}\tilde{\gamma}^5\tilde{S}^{-1}\psi'$

$= \bar{\Psi}'\tilde{S}\tilde{\gamma}^5\tilde{S}^{-1}\psi'$

Now: $\tilde{S}\tilde{\gamma}^5\tilde{S}^{-1} = \tilde{S}\tilde{\gamma}^4\tilde{S}^{-1}\tilde{S}\tilde{\gamma}^3\tilde{S}^{-1}\tilde{S}\tilde{\gamma}^2\tilde{S}^{-1}\tilde{S}\tilde{\gamma}^1\tilde{S}^{-1} = -\gamma^4\gamma^3\gamma^2\gamma^1 = -\gamma^5$

or: $\boxed{S^{-1}\gamma^5 S = -\tilde{\gamma}^5}$

Hence: $\bar{\Psi}\gamma^5\psi = -\bar{\Psi}'\gamma^5\psi'$

$\underline{\eta = \gamma^5\gamma^\mu}$: $\bar{\Psi}\gamma^5\gamma^\mu\psi = \bar{\Psi}'\gamma^4\tilde{S}\tilde{\gamma}^4\tilde{\gamma}^\mu\tilde{\gamma}^5\tilde{S}^{-1}\psi' = \bar{\Psi}'\tilde{S}\tilde{\gamma}^\mu\tilde{S}^{-1}\tilde{S}\tilde{\gamma}^5\tilde{S}^{-1}\psi'$
$= -\bar{\Psi}'\gamma^4\gamma^\mu\gamma^4\gamma^5\psi' = -\bar{\Psi}'\gamma^4\gamma^5\gamma^\mu\gamma^4\psi'$

Hence:

$\bar{\Psi}\gamma^5\gamma^4\psi = \bar{\Psi}'\gamma^5\gamma^4\psi'$
$\bar{\Psi}\gamma^5\gamma^k\psi = -\bar{\Psi}'\gamma^5\gamma^k\psi'$

Summary of Time Reversal and Charge Conjugation

Wigner Time Reversal: The operation on the wave function is given by $\bar{\psi} S = \psi'$. What this means physically is that we reverse the time $(t \to -t)$ and the magnetic field $(\vec{H} \to -\vec{H})$ in order to completely reverse the motion.

Charge Conjugation: The operation on the wave function is given by $\psi_c = C\bar{\psi}$. Physically this means we reverse the sign of the charge. Charge conjugation is not an invariant operation. To make it invariant, we must also change the sign of the electric and magnetic fields $(\vec{E} \to -\vec{E},\ \vec{H} \to -\vec{H})$ so that an oppositely signed charge will have the same motion.

Wigner Time Reversal and "Complete" Charge Conjugation: If we combine the operations of Wigner time reversal and invariant charge conjugation, we may write this operation on the wave function as $\psi = S_g \psi''$ and have the physical result that $e \to -e$, $\vec{E} \to -\vec{E}$, $t \to -t$. Now we then have for the charge density $\rho'' = -\rho$ but for the current vector $\vec{j}'' = \vec{j}$ because we have changed the sign of the charge, velocity, and electric field giving cancelling effects. This suggests that we might consider the Pauli covariant quantity:

$$s_\mu = i\bar{\psi}\gamma^\mu\psi = (\psi^*\vec{\alpha}\psi,\ i\psi^*\psi) = \left(\frac{\vec{j}}{c},\ i\rho\right)$$

subject to the transformations: $\psi = S_g\psi''$, $\bar{\psi} = \bar{\psi}'' S_g^{-1}$

Then:

$$\bar{\psi}\gamma^\mu\psi = \bar{\psi}'' S_g^{-1}\gamma^\mu S_g\psi'' = \begin{cases} -\bar{\psi}''\gamma^4\psi'' & ;\ \mu = 4 \\ \bar{\psi}''\gamma^k\psi'' & ;\ \mu = k \end{cases}$$

We see that:

$$\boxed{S_g^{-1}\gamma^\mu S_g = -\gamma^4\gamma^\mu\gamma^4}$$

satisfies the requirement.

However, we note that this requirement is precisely equal to the requirement for the geometric time inversion operation considered earlier.

We see that the requirement on S_g satisfies the requirement on ρ, that is:

$$\rho = \psi^* \psi = \bar{\psi} \gamma^4 \psi = \bar{\psi}'' S_g^{-1} \gamma^4 S_g \psi'' = -\bar{\psi}'' \gamma^4 \psi'' = -\psi^{*''} \psi''$$
$$= -\rho''$$

We do not get quite the same thing if we actually use the Lorentz transformation for geometrical time reflection because here $\psi = S \psi'$ and $\bar{\psi} = \bar{\psi}' S^{-1} B$ and B was shown to be -1 for geometrical time reflection, hence this would preserve the sign of ρ.

Today we do not think of ρ and $\vec{j}$ as probability densities and probability flows, but now think of them as charge densities and currents in accordance with the Dirac theory which has shown the existence of two oppositely charged, equal mass, particles.

LECTURE 27 : 11-24-61

Field Theory and Radiation Theory

The underlying feature of the physics of light and matter is the wave-particle duality. For the case of matter, this duality is very nearly resolved by Schroedinger equation or the Dirac equation. However, for radiation the question of wave-particle duality is still rather open.

The initial attempts at radiation theory were by Planck with his radiation formula and by Einstein who developed phenomenological constants of spontaneous and stimulated emission and absorption. The coefficient of spontaneous emission was called A while that for absorption and stimulated emission was called B and the relation between them is:

$$A_{nm} = \frac{8\pi h \nu^3}{c^3} B_{nm}$$

The stimulated transition rates are written as B_{nm} times a density of states factor and A_{nm} can be thought of as B_{nm} times a "standard" density ρ_0 or $A_{nm} = B_{nm}\,\rho_0(\nu)$.

The development of the Dirac time dependent perturbation theory gave an explicit form for B_{nm} but gave no indication whatever that there should be a spontaneous emission process characterized by A_{nm}.

One way that might occur to one to handle this problem would be to consider a stationary system without radiation incident upon it characterized by the wave function:

$$\psi = \sum_m C_m u_m e^{-i E_m t/\hbar}$$

Now, we can write the current as $\vec{j} = \psi^* \vec{j}_{op}\, \psi$ assuming non-degeneracy.

That is:

$$\vec{J} = \sum_{mn} (u_n^* \vec{j}_{op} u_m) \, C_n^* C_m \; e^{-i(E_m - E_n)t/\hbar}$$

This appears promising because we have an oscillating current which we should expect to radiate. Recall the retarded potential from classical electromagnetic theory:

$$\vec{A}(r,t) = \int \frac{\vec{J}\left(\vec{r}\,', t - \frac{|\vec{r}-\vec{r}\,'|}{c}\right)}{c\,|\vec{r}-\vec{r}\,'|} \, d\vec{r}\,'$$

Thus it looks good. Using the usual formulae that give the field quantities from the retarded potential and taking expectation values (since quantum mechanics is involved) we would get for some field quantity F (component of $\vec{E}$ or $\vec{H}$) the form of the expectation value:

$$\langle F \rangle = \sum_{nm} C_n^* F_{nm} C_m \; e^{-i(E_m - E_n)t/\hbar}$$

However, this is not the whole story. We must still consider the time average in order to get the net field flux of spontaneous emission. When we do this we obtain a δ function that says $\langle F \rangle$ only exists when when $E_m = E_n$ or when there are no transitions at all! Not a very satisfactory theory of an emission process.

However, our observable is really the Poynting vector and not one of the field quantities at all. Therefore we should ask for the mean value of F^2:

$$\langle F^2 \rangle = \sum_{mn} C_n^* \sum_{l} F_{nl} F_{ml} \; e^{-i(E_m - E_n)t/\hbar} C_m \; ; \quad \overline{\langle F^2 \rangle} = \sum_{n} |C_n|^2 \sum_{l} F_{nl} F_{ln}$$

Suppose we know the system is in the m state definitely, then:

$\overline{\langle F^2 \rangle} = \sum_{l} |F_{ml}|^2$. This approach gives use transition matrix elements. However, the $\sum_{l}$ goes below m as well as above so that as well as spontaneous emission we get spontaneous absorption!

LECTURE 28 : 11-27-61

Recall: $\overline{\langle F^2 \rangle} = \sum_{\ell} |F_{m\ell}|^2$: we can fix the problem of "spontaneous absorption" by writing from common sense:

$$\overline{\langle F^2 \rangle} = 2 \sum_{E_\ell < E_m} |F_{m\ell}|^2$$

This means: $F_{m\ell} F_{\ell m} \rightarrow 2 \underbrace{F_{m\ell}^{(-)}}_{e^{i\omega t},\ \omega<0} \underbrace{F_{\ell m}^{(+)}}_{e^{-i\omega t},\ \omega>0}$ } each term gives opposite directions.

This makes for radiation instead of absorption.

Also recall: $A_{nm} = \frac{8\pi h \nu^3}{c^3} B_{nm}$ as derived from phenomenological arguments by Einstein. The presence of spontaneous emission is tantamount to assigning an extra degree of freedom to the photon.

Look at it this way: Absorption is proportional to the amount of photons present before they are absorbed.

Emission is proportional to the amount of photons present <u>after</u> emission if we consider the emission process to be the time reversed absorption process.

That is, if time reversal is a valid physical concept, we can say that the spontaneous emission is stimulated by the photon to be emitted. We could go further and say that if only absorption and stimulated emission were possible, time reversal symmetry would not exist.

Second Quantization

References: Dirac's time-dependent perturbation theory in his book and his papers in Schwinger's Reprints.

We will follow Dirac's original argument. Recall that he had invented a time perturbation theory in which the principle relations are as follows:

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi \qquad H = H_0 + V$$

$$-i\hbar \frac{\partial \psi^*}{\partial t} = (H\psi)^* \qquad H_0 u_n = E_n u_n$$

We take $\psi = \sum_n b_n(t)\, u_n$

Upon substitution we have:

$$i\hbar \dot{b}_n = \sum_m H_{nm} b_m \quad ; \quad H_{nm} = \int u_n^* H u_m \, d\vec{r}$$

Now Dirac pointed out that if we introduce a quantity that we would formerly regard as an expectation value, we could write the equation for b_n in "Hamiltonian" form. Introduce:

$$\boxed{\bar{H} = \int \psi^* \underline{H} \psi \, d\vec{r} = \sum_{nm} b_n^* H_{nm} b_m = \sum_{nm} b_m^* H_{mn} b_n}$$

Then we could write:

$$\boxed{\begin{aligned} i\hbar \dot{b}_n &= \sum_m H_{nm} b_m \rightarrow \frac{d}{dt}(i\hbar b_n) = \frac{\partial \bar{H}}{\partial b_n^*} \\ -i\hbar \dot{b}_n^* &= \sum_m b_m^* H_{mn} \rightarrow \frac{d}{dt} b_n^* = -\frac{\partial \bar{H}}{\partial (i\hbar b_n)} \end{aligned}}$$

Recall Hamilton's equations of motion: $\dot{p}_n = -\frac{\partial H}{\partial q_n}$

and: $\dot{q}_n = \frac{\partial H}{\partial p_n}$, We see the analogy:

$$\boxed{\begin{aligned} p &\rightarrow b_n^* \\ q &\rightarrow i\hbar b_n \end{aligned}}$$

Now we know in quantum mechanics, the canonical variables p and q obey commutation rules, hence Dirac was led to write:

$$[i\hbar b_n, b_m^*] = i\hbar\,\delta_{nm}$$

or:

$$\boxed{\begin{array}{l} [b_n, b_m^*] = \delta_{nm} \\ [b_n^*, b_m^*] = [b_n, b_m] = 0 \end{array}}$$

Thus second Quantization was born. Now if we take n fixed, we see that the b_n, b_n^* satisfy exactly the same commutation rules as the harmonic oscillator ladder operators. That is, if we take, in natural units ($\hbar = m = \omega = 1$):

$$b = \frac{1}{\sqrt{2}}(Q + iP) \quad ; \quad b^* = \frac{1}{\sqrt{2}}(Q - iP) \quad ; \quad [Q,P] = i$$

we have $[b, b^*] = 1$. Since the particles in question will turn out to be bosons, we will retain the mneumonic b, b^*. Now recall some of the properties of the ladder operators:

$$b b^* = \tfrac{1}{2}(Q^2 + P^2) + \tfrac{1}{2}$$

$$b^* b = \tfrac{1}{2}(Q^2 + P^2) - \tfrac{1}{2} = \mathcal{H} - 1/2$$

where $\mathcal{H}$ is the harmonic oscillator Hamiltonian which has the eigenvalues $\mathcal{H}' = N' + 1/2$, or, we can write the operator $N = b^* b$ with eigenvalues $N' = 0, 1, 2, \ldots$

Matrix elements of b with respect to the eigenvectors of N can be formed:

$$\boxed{\begin{array}{l} \langle N' | b | N'' \rangle = \sqrt{N''}\,\delta_{N', N''-1} \\ \langle N' | b^* | N'' \rangle = \sqrt{N'}\,\delta_{N', N''+1} \end{array}}$$

Thus we have a matrix representation of b, b^*

We now drop the prime notation and write the above equations in more general form:

$$\boxed{\begin{aligned} \langle N_1 \cdots N_{n-1}, N_n - 1, N_{n+1} \cdots | b_n | N_1 \cdots N_n \cdots \rangle &= \sqrt{N_n} \\ \langle N_1 \cdots N_{n-1}, N_n + 1, N_{n+1} \cdots | b_n^* | N_1 \cdots N_n \cdots \rangle &= \sqrt{N_n + 1} \end{aligned}}$$

We see that b_n is an anihilation operator while b_n^* is a creation operator.

Since the b's are operators, then $\bar{H}$ is also, and from the way the operator nature of the b's was found we ought to be able to write:

$$\dot{b}_n = \frac{i}{\hbar}\left[\bar{H}, b_n\right] = \frac{i}{\hbar}\sum_{\ell m} H_{\ell m}\left[b_\ell^* b_m, b_n\right]$$

$$= \frac{i}{\hbar}\sum_{\ell m} H_{\ell m} \underbrace{\left[b_\ell^*, b_n\right]}_{-\delta_{\ell n}} b_m$$

$$\text{or} \quad \dot{b}_n = \frac{1}{i\hbar}\sum_m H_{nm} b_m$$

$$\text{Similarly,} \quad \dot{b}_n^* = \frac{i}{\hbar}\left[\bar{H}, b_n^*\right] = -\frac{1}{i\hbar}\sum_m b_m^* H_{mn}$$

Thus we have developed one more step in strengthening the position of the b's as operators.

Since we have seen that $\bar{H}$ behaves as a Hamiltonian, it is natural to consider the construction of a Schroedinger equation which is obeyed by a state function whose independent variables are the level occupation numbers. We write:

$$i\hbar \frac{\partial \Psi}{\partial t} = \bar{H}\,\Psi$$

$$\Psi = \Psi(N_1 \cdots) = \langle N_1 \cdots | \rangle$$

$$i\hbar \frac{\partial}{\partial t}\Psi(N_1 \cdots) = \sum_{nm} H_{nm}\, b_n^* b_m\, \Psi(N_1 \cdots)$$

We now make use of the properties of the b's to obtain:

$$\boxed{\begin{aligned} i\hbar \frac{\partial}{\partial t} \Psi(N_1 \cdots) = & \sum_{m \neq n} H_{nm} \left[N_n (N_m + 1) \right]^{1/2} \Psi(\cdots N_n - 1 \cdots N_m + 1 \cdots) \\ & + \sum_n H_{nn} N_n \Psi(N_1 \cdots) \end{aligned}}$$

In view of the nature of the creation-destruction operators, the first term on the RHS above may appear backwards. Consider, however, just what the operation is:

$$\langle N'| \rangle b = \langle N'|b| \rangle = \sum_{N''} \underbrace{\langle N'|b|N'' \rangle}_{\substack{\sqrt{N''}\,\delta_{N', N''-1} \\ \sqrt{N''}\,\delta_{N'+1, N''}}} \langle N''| \rangle = \sqrt{N'+1}\, \langle N'+1| \rangle$$

$$\langle N'|b^*| \rangle = \langle N'| \rangle b^* = \sum_{N''} \underbrace{\langle N'|b^*|N'' \rangle}_{\substack{\sqrt{N'}\,\delta_{N', N''+1} \\ \sqrt{N'}\,\delta_{N'-1, N''}}} \langle N''| \rangle = \sqrt{N'}\, \langle N'-1| \rangle$$

Hence we see that when the ladder operators operate on wave functions or representative functions, their usual action appears to be opposite.

Now the first term in the equation represents transitions among states labeled by n, m, $n \neq m$ while the second term on the RHS is a total energy term. Note that n labels the final state and m the initial state. We see that N_n in the first term on the RHS represents the number of particles in the final state after transition and knowing that this term squared will appear in the transition probability current it appears that the rate goes up as the number of particles to be in the final state. N_m+1 of course is the population of the initial state before transition. This is classically expected but not the dependence on the population of the final state after transition.

LECTURE 29: 11-29-61

We have seen how the transition rate depends on the population of the final state. This is not classical as if the transitions were governed by Boltzmann statistics, the population of the final state would have been one (1). Evidently the particles under description here (bosons, obeying Bose-Einstein statistics) like to congregate or bunch (gregarious).

Another distinction can be made between BE and Boltzmann statistics on the basis of their statistical weights. Speaking non-relativistically, the number of bosons in the system is conserved: $\sum_n N_n = \mathcal{N}$. The statistical weight of a system of $\mathcal{N}$ particles distributed among n levels, N_n per level is:

Boltzmann: $\dfrac{\mathcal{N}!}{N_1!\,N_2!\cdots}$

BE: 1

The statistical weight is the number of ways (different) that particles can be arranged. Since bosons are indistinguishable, they only have 1 independent arrangement.

The reason that the statistical weight is 1 for the BE statistics is that we are dealing with indistinguishable particles.

Dirac shows in his book that the wave functions for Bose-Einstein particles are symmetric, that is:

$$\psi(r_1, r_2, r_3, \cdots) = \sum_P P\, u_j(r_1)\, u_j(r_2) \cdots u_j(r_{N_1}) \cdots u_k(r_\alpha)\, u_k(r_{\alpha+1}) \cdots u_k(r_{\alpha+N_k-1}) \cdots u_s(r_{\mathcal{N}})$$

where we have put the correct numbers in each state. j denotes initial state with N_1 particles and s is some final state with $\mathcal{N}$.

Dirac also shows (PQM, p. 231) that if the Hamiltonian can be written as a sum of single particle Hamiltonians, then we have:

$$H_T = \sum_{\alpha=1}^{\mathcal{N}} \underline{H}(\alpha)$$

and taking the matrix element with symmeterized wave functions to get $\bar{H}_T$, we find:

$$H_T = \sum_{mn} b_n^* \underline{H}_{nm} b_m$$

as before with the single particle case.

Suppose now we have interaction terms in H_T, That is:

$$H_T = \sum_{\alpha=1}^{n} \underline{H}(\alpha) + \sum_{\alpha \neq \beta} V(|\vec{r}_\alpha - \vec{r}_\beta|)$$

In the simplest case; $V(|\vec{r}_\alpha - \vec{r}_\beta|) = \dfrac{e^2}{2|\vec{r}_\alpha - \vec{r}_\beta|}$.

Now Dirac also has shown that when we form $\bar{H}_T$, we get (PQM, p. 231):

$$\bar{H}_T = \sum_{mn} b_n^* \underline{H}_{nm} b_m + \sum_{mnlp} b_n^* b_p^* V_{np,ml} b_m b_l$$

where:

$$V_{np,ml} = \int u_n^*(\vec{r}) u_m(\vec{r}) u_p^*(\vec{r}') u_l(\vec{r}') V(|\vec{r}-\vec{r}'|) d\vec{r} d\vec{r}'$$

and indicates a transition from an initial ml to a final np. In occupation number representation with $i\hbar \frac{\partial \Psi}{\partial t} = \bar{H}_T \Psi$:

$i\hbar \frac{\partial}{\partial t} \Psi(N_1 \cdots N_m \cdots) =$ (single particle term as before)

$$+ \sum_{mnlp} V_{np,ml} \left[N_n N_p (N_m+1)(N_l+1) \right]^{1/2} \Psi(\cdots N_n - 1 \cdots N_p - 1 \cdots N_l + 1 \cdots N_m + 1 \cdots)$$

note the importance of the arrangements of the b's. This order comes about "normally" from forming:

$$\bar{H} = \int \psi^* \underline{H} \psi \, d\vec{r} + \int d\vec{r} d\vec{r}' \psi^*(\vec{r}) \psi^*(\vec{r}') V(|\vec{r}-\vec{r}'|) \psi(\vec{r}') \psi(\vec{r})$$

sometimes, in $V_{np,ml}$, the u's are arranged to give "density"-like quantities: $\underline{u_n^*(\vec{r}) u_m(\vec{r})}$, $\underline{u_p^*(\vec{r}') u_l(\vec{r}')}$, however, the b's cannot be changed from their normal order $b_n^* b_p^* b_l b_m$ because of the commutation rules. Note that the destruction operators first chance. If only one particle is present, we get a chance to destroy it and assure no self-interaction.

LECTURE 30: 12-1-61

Second Quantization of Fermions: We do the second quantization of particles obeying Fermi-Dirac (FD) statistics by analogy to the BE case. The reference is the paper by Jordan and Wigner in Schwinger's reprints. In analogy to BE we define in place of the b's:

BE		FD	
b_n	$\longrightarrow$	c_n	: destruction
b_m^*	$\longrightarrow$	c_m^*	: creation

We develop the properties of the c's by making them satisfy some necessity conditions:

Property	Reason
$c_n c_n = 0$	At most one electron can occupy a state and it only can be destroyed once.
$c_n^* c_n^* = 0$	Since only one fermion can exist in a state, one is the most that can be created.

What are the relationships between different c_n's? Suppose we examine some properties of the Helium atom. The interaction potential is:

$$V = \frac{e^2}{|\vec{r}_{12}|} = \frac{e^2}{|\vec{r}_1 - \vec{r}_2|} = V(|\vec{r}_1 - \vec{r}_2|)$$

Suppose we now calculate the interaction perturbation term when we are connecting the state l, m to n, p via the interaction potential using antisymmetric wave functions:

$$\Psi_{lm}(12) = \frac{u_l(1)\, u_m(2) - u_l(2)\, u_m(1)}{\sqrt{2}}$$

Interaction matrix element $= \iint d\vec{r}_1\, d\vec{r}_2\ \Psi^*_{np}(12)\ V\ \Psi_{lm}(12)$

$$= \frac{1}{2} \iint d\vec{r}_1\, d\vec{r}_2\ V(|\vec{r}_1 - \vec{r}_2|) \left\{ u_n^*(1)\, u_p^*(2) - u_n^*(2)\, u_p^*(1) \right\}$$

$$\cdot \left\{ u_l(1)\, u_m(2) - u_l(2)\, u_m(1) \right\}$$

$$= V_{np,lm} - V_{np,ml}$$

where $V_{np,lm} = \iint d\vec{r}_1\, d\vec{r}_2\ V(|\vec{r}_1 - \vec{r}_2|)\ u_n^*(1)\, u_p^*(2)\, u_l(1)\, u_m(2)$

In the above the u's are the single electron wave functions that would exist precisely in the absence of the others. Suppose now we be a little more general and denote each single electron wave function as an expansion in our complete set of u's using our c's, that is, take:

$$\psi(1) = \sum_l c_l\, u_l(1)$$

We now say that the total wave function is $\psi(1)\,\psi(2)$ and take the expectation value of $V(|\vec{r}_1 - \vec{r}_2|)$ with respect to this wave function:

$$\int d\vec{r}_1\, d\vec{r}_2\ \psi^*(1)\, \psi^*(2)\ V(|\vec{r}_1 - \vec{r}_2|)\ \psi(1)\, \psi(2)$$

$$= \sum_{nplm} c_n^*\, c_p^*\ V_{np,lm}\ c_l\, c_m$$

where $V_{np,lm}$ is the same as before. We now pick out of the sum the terms that consist of all the permutations of different np, lm: These are:

$$c_n^* c_p^*\, V_{np,lm}\, c_l c_m + c_n^* c_p^*\, V_{np,ml}\, c_m c_l$$

$$+ c_p^* c_n^*\, V_{pn,lm}\, c_l c_m + c_p^* c_n^*\, V_{pn,ml}\, c_m c_l$$

Now, to satisfy the requirements of antisymmetry as required for the functions $u_n(1)\, u_p(2)$, etc., as displayed in the previous simple calculation, we must choose the following conditions on the c's.

$$c_l c_m = - c_m c_l \; ; \quad \{c_l, c_m\} = 0$$

$$c_p^* c_n^* = - c_n^* c_p^* \; ; \quad \{c_p^*, c_n^*\} = 0$$

Suppose that in analogy to the BE case we choose:

$$c_n^* c_n = N_n \; ; \quad c_n^* c_n |1\rangle = |1\rangle \; ; \quad c_n^* c_n |0\rangle = 0$$

Then, when we operate with $c_n c_n^* + c_n^* c_n$ on a state with one electron in it, the first term gives zero as the attempt to create another gives zero while it the second term we get 1 as c_n destroys but c_n^* creates. That is:

$$(c_n c_n^* + c_n^* c_n)|1\rangle = 0 + c_n^* |0\rangle = |1\rangle$$

For no particles: $(c_n c_n^* + c_n^* c_n)|0\rangle = c_n |1\rangle + 0 = |0\rangle$

Hence we have: $\{c_n, c_n^*\} = 1$

We now make the assumption: $\{c_n, c_m^*\} = \delta_{nm}$
Although this must be an
assumption, it turns out to work and gives a good description of fermions.

Note that in the BE case $[b_n, b_n^+] = N_n + 1 - N_n$ since $b_n^* b_n = N_n$ and $[b_n, b_m^*] = \delta_{mn}$. In the limit of large occupation numbers, the b's commute and we have c-number (classical) behaviour. However, no such analogy exists for fermions because for them there is no such thing as a high occupation number limit. Even if there were, the anti-commutation property would persist and we know that classical quantities commute. Hence anything that anticommutes is very far from ever having a classical limit

The only non-vanishing matrix elements are:

$$\boxed{\begin{aligned} \langle N_i' \cdots 0_n \cdots | C_n | N_i' \cdots 1_n \cdots \rangle &= (-1)^{\sum_{j=1}^{n-1} N_j'} = \prod_{j=1}^{n-1} (1-2N_j') \\ \langle N_i' \cdots 1_n \cdots | C_n^* | N_i' \cdots 0_n \cdots \rangle &= (-1)^{\sum_{j=1}^{n-1} N_j'} = \prod_{j=1}^{n-1} (1-2N_j') \end{aligned}}$$

For details, see Schweber's book.

Jordan and Wigner originally wrote C_n, C_n^* as the product of two Hermitean matrices with N_n:

$$C_n = v_n \Gamma_n N_n \qquad ; \qquad C_n^* = N_n \Gamma_n v_n$$

v_n and Γ_n are Hermitean and v_n, Γ_n, N_n are defined by:

$$\langle N_i' \cdots 1_n \cdots | N_n | N_i' \cdots 1_n \cdots \rangle = 1$$

$$\langle N_i' \cdots 1_s \cdots | \Gamma_s | N_i' \cdots 0_s \cdots \rangle$$

$$= \langle N_i' \cdots 0_s \cdots | \Gamma_s | N_i' \cdots 1_s \cdots \rangle = 1$$

(We have switched notation from n to s, otherwise no difference. Do same for N_n; $N_n \to N_s$)

$$\langle N_i' \cdots | v_s | N_i' \cdots \rangle = (-1)^{\sum_{j=1}^{s-1} N_j'}$$

To get an idea of the meaning of these operators, imagine a long row of disks on a table with two sides, one side labelled 1 and the other side marked 0:

N_s means look at sth disc and write down what it is.

Γ_s means turn sth disc over.

v_s means write down the factor:

$$\begin{cases} +1 & \text{for even number of 1's to the left of the } s\text{th disc.} \\ -1 & \text{for an odd number of 1's to the left of the } s\text{th disc.} \end{cases}$$

All counting is done left to right beginning with N_1. Hence we have a model against which we can check the action of the c's. For example:

$$c_s c_s = v_s \Gamma_s N_s v_s \Gamma_s N_s$$

This reads, beginning with the right hand N_s: Read sth disk and suppose 1 (if zero, problem trivial since we immediately have a zero in the product); write down, turn sth disk over, write down ± 1 for v_s, and now second N_s gives a 0 in the product since the first Γ_s turned over the disc. Hence $c_s c_s = 0$ and the model checks. In the same way so would $c_s^* c_s^*$ check out.

What about:

$$c_r c_s + c_s c_r = v_r \Gamma_r N_r v_s \Gamma_s N_s + v_s \Gamma_s N_s v_r \Gamma_r N_s$$

now the effect of ΓN in each term is the same as always look at before turn over, hence, something must happen in the v's. We see if $r > s$, s has been turned when we count to the left of r in the first, while it has not been turned when we do the same thing in the second term. Hence they cancel again checking the model. A similar thing happens for $r < s$.

Consider:

$$c_r c_s^* + c_s^* c_r = v_r \Gamma_r N_r N_s \Gamma_s v_s + N_s \Gamma_s v_s v_r \Gamma_r N_r$$

for $r = s$. Of course $N_s N_s = N_s$. However, if we turn over before looking, we have $N_s \Gamma_s = -N_s + 1$, and this is the result of the first term. In the second term, we turn twice before the last look so get N_s. Hence, the model checks for this too.

LECTURE 31 : 12-4-61

We now form the Hamiltonian-like expression as we did in the BE case, but now we use the FD operators:

$$\bar{H} = \int \psi^* \underline{H}\, \psi\, d\vec{r} = \sum_{rs} H_{rs}\, C_r^* C_s \quad ; \quad H_{rs} = \int u_r^* \underline{H}\, u_s\, d\vec{r}$$

We have the identification:

$$\left.\begin{aligned} i\hbar C_s &\rightarrow q \\ C_s^* &\rightarrow p \end{aligned}\right\} \text{ implied from:} \qquad \begin{aligned} i\hbar \dot{C}_s &= \frac{\partial \bar{H}}{\partial C_s^*} = \sum_p H_{sp} C_p \\ \dot{C}_s^* &= \frac{-\partial \bar{H}}{\partial (i\hbar C_s)} = \sum_p C_p^* H_{ps} \end{aligned}$$

as required by the Dirac analogy to Hamiltons equations. Since we consider the c's as q-numbers, we can now equally write:

$$i\hbar \dot{c}_s = -[\bar{H}, c_s] = -\sum_{rp} H_{rp}\, [C_r^* C_p, C_s]$$

$$[C_r^* C_p, C_s] = C_r^* [C_p, C_s] + [C_r^*, C_s]\, C_p$$

$$= \underbrace{C_r^* C_p C_s}_{-C_r^* C_s C_p} - C_r^* C_s C_p + C_r^* C_s C_p - C_s C_r^* C_p = -\{C_s, C_r^*\}\, C_p$$

$$= -\delta_{sr}\, C_p$$

Then: $i\hbar \dot{c}_s = \sum_p H_{sp}\, C_p$ as before.

However, we will see that this is only true (strictly) for the system whose Hamiltonian is separable. Suppose one has an interaction term, however:

$$\bar{H}_T = \int \psi^*(\vec{r})\, \underline{H}\, \psi(\vec{r})\, d\vec{r} + \int \psi^*(\vec{r})\, \psi(\vec{r})\, \psi^*(\vec{r}')\, \psi(\vec{r}')\, V(|\vec{r} - \vec{r}'|)\, d\vec{r}\, d\vec{r}'$$

or:

$$\bar{H}_T = \sum_{rs} H_{rs}\, C_r^* C_s + \sum_{nplm} V_{np,lm}\, C_n^* C_l C_p^* C_m$$

Note that we have written the interaction term in the "density product" form instead of the "normal product" form. However, since the usual way of taking matrix elements is by coming in on the left with the conjugate wave function and on the right with the unconjugated, it seems as if this is the "natural" or "normal" way. The "density product" form seems to be a rearrangement held over from "pre-second quantization" which is no longer valid. It appears to have been useful to have the interaction term in the form:

$$\int \rho(\vec{r})\, V(|\vec{r}-\vec{r}'|)\, \rho(\vec{r}')\, d\vec{r}\, d\vec{r}'$$

What is the difference between the "normal product" and "density product" forms?

$$\begin{aligned} C_n^* C_\ell C_p^* C_m &= -C_n^* C_p^* C_\ell C_m + C_n^* \delta_{\ell p} C_m \\ &= C_n^* C_p^* C_m C_\ell + C_n^* \delta_{\ell p} C_m \end{aligned}$$

Then the "density product" form contains the "normal product" and the term:

$$\sum_{np\ell m} V_{np,\ell m}\, C_n^* \delta_{\ell p} C_m = \sum_{npm} V_{np,pm}\, C_n^* C_m$$

which in turn contains the sum:

$$\sum_p V_{np,pn}\, C_n^* C_n = \sum_p V_{np,pn}\, N_n$$

so, if there is one particle in the state n, this single particle interaction or self-interaction term is infinite. Thus the "normal product" scheme rids us of the self-interaction term. However, this is not the whole picture.

"Density Product Picture"

$$\dot{C}_r^* \sim \frac{\partial \bar{V}}{\partial C_r} = \sum_{np\ell m} V_{np,\ell m} \Big[C_n^* \underbrace{\frac{\partial C_\ell}{\partial C_r}}_{\delta_{\ell r}} C_p^* C_m + C_n^* C_\ell C_p^* \underbrace{\frac{\partial C_m}{\partial C_r}}_{\delta_{mr}} \Big]$$

How does this compare with $\dot{C}_r^*$ calculated by the Heisenberg commutators in both the "density product" and "normal product" pictures?

"density product"

$$\dot{C}_r^* \sim [C_n^* C_l C_p^* C_m, C_r^*] = \underbrace{C_n^* C_l C_p^* C_m C_r^*}_{} - C_r^* C_n^* C_l C_p^* C_m$$
$$\underbrace{- C_n^* C_l C_p^* C_r^* C_m}_{} + C_n^* C_l C_p^* \delta_{mr}$$
$$\underbrace{C_n^* C_l C_r^* C_p^* C_m}_{}$$
$$- C_n^* C_r^* C_l C_p^* C_m + C_n^* C_p^* C_m \delta_{lr}$$

"normal product"

$$\dot{C}_r^* \sim \frac{\partial \bar{V}}{\partial C_r} = \sum_{nplm} V_{np,lm} \Big[C_n^* C_p^* \underset{\delta_{lr}}{\underset{\downarrow}{\frac{\partial C_l}{\partial C_r}}} C_m + C_n^* C_p^* C_l \underset{\delta_{mr}}{\underset{\downarrow}{\frac{\partial C_m}{\partial C_r}}} \Big]$$

$$\dot{C}_r^* \sim [C_n^* C_p^* C_l C_m, C_r^*] = \underbrace{C_n^* C_p^* C_l C_m C_r^*}_{} - C_r^* C_n^* C_p^* C_l C_m$$
$$\underbrace{- C_n^* C_p^* C_l C_r^* C_m}_{} + C_n^* C_p^* C_l \delta_{mr}$$
$$\underbrace{C_n^* C_p^* C_r^* C_l C_m}_{} + C_n^* C_p^* C_m \delta_{lr}$$
$$C_r^* C_n^* C_p^* C_l C_m$$

So we see that $\frac{\partial \bar{V}}{\partial C_r} = [\ ]$ within each picture, but within the "cross-picture", $\frac{\partial \bar{V}}{\partial C_r} \neq [\ ]$. Since we are dealing in quantum field theory or quantum mechanics we will always use $[\ ]$ to calculate time derivatives, whether in the "density product" or "normal product" pictures. It is usual to take the "density product" picture as representing the pre-second quantization scheme.

Professor Furry says that there is trouble in keeping the theory Lorentz invariant when it is in the "normal product" form.

Now in the Heisenberg representation, the equation of motion is:

$$\dot{A} = \frac{i}{\hbar} [\bar{H}, A]$$

while in the Schroedinger representation the equation of motion is:

$$i\hbar \frac{\partial}{\partial t} \Psi = \bar{H} \Psi$$

For a non-interaction problem:

$$i\hbar \frac{\partial}{\partial t} \Psi(N_1 \cdots) = \sum_{rs} H_{rs} C_r^* C_s \Psi(N_1 \cdots)$$

$$= \sum_s H_{ss} N_s \Psi(N_1 \cdots) + \sum_{r \neq s} H_{rs} C_r^* C_s \Psi(N_1 \cdots)$$

We proceed as in the boson case, now using our fermion matrix elements:

$$c_s \langle \cdots 0_s \cdots | \rangle = \langle \cdots 0_s \cdots | c_s | \cdots 1_s \cdots \rangle \langle \cdots 1_s \cdots | \rangle$$
$$= (-1)^{\sum_{j=1}^{s-1} N_j} (1 - N_s) \langle \cdots 1_s \cdots | \rangle$$

denotes occupation before transition

$$C_r^* \langle \cdots 1_r \cdots | \rangle = \langle \cdots 1_r \cdots | C_r^* | \cdots 0_r \cdots \rangle \langle \cdots 0_r \cdots | \rangle$$
$$= (-1)^{\sum_{j=1}^{r-1} N_j} N_r$$

Now, for $\begin{matrix} r > s \\ \{r < s\} \end{matrix}$: $\displaystyle\sum_{j=1}^{s-1} N_j + \sum_{j=1}^{r-1} N_j = \underbrace{2 \sum_{j=1}^{\substack{s-1 \\ \{r-1\}}} N_j}_{\text{always even}} + \sum_{\substack{j=s \\ \{r\}}}^{\substack{r-1 \\ \{s-1\}}}$

Then:

$$\boxed{\begin{aligned} i\hbar \frac{\partial}{\partial t} \Psi(N_1 \cdots) = {} & \sum_s H_{ss} N_s \Psi(N_1 \cdots) \\ & + \sum_{r \neq s} H_{rs} \left[(1 - N_s) N_r\right] (-1)^{\sum_{j=s\{r\}}^{r-1\{s-1\}} N_j} \Psi(N_1 \cdots 0_r \cdots 1_s \cdots) \end{aligned}}$$

When we write any equations in configuration coordinates we must use an antisymmetric wave function:

$$\psi(r_1 r_2 \cdots r_N) = \frac{1}{\sqrt{N!}} \sum_P (-1)^{\sigma_P} P\, u_{s1}(r_1)\, u_{s2}(r_2) \cdots u_{sN}(r_N)$$

σ_P is the number of changes of pairs.

In NR applications, the number of particles in a given volume may easily be computed by:

$$N_v = \int_v \psi^* \psi \, d\vec{r}$$

However, in a relativistic theory, the number of particles of one type may not be conserved, but the total charge in any given volume will always be the same, hence in this case we can write:

$$Q_v = \int_v e\, \psi^* \psi \, d\vec{r}$$

Hence we can know the net charge without knowing the total number of electrons and positrons present

LECTURE 32 : 12-6-61

Quantitization of the Radiation Field

This topic was first treated by Heisenberg and Pauli. The usual starting point (not ours) is from a Lagrangian variation introducing the concepts of a Lagrangian and Hamiltonian density:

$$\delta \int L\,dt = 0 \; ; \quad L = \int \mathcal{L}\, d\vec{r} \; ; \quad \text{then: } \delta \int \mathcal{L}\, d^4x = 0$$

where $\mathcal{L}$ is the Lagrangian density and the Hamiltonian density can be introduced in a similar manner:

$$H = \int \mathcal{H}\, d\vec{r}$$

In our field theory, we will consider only free space, use vector potentials only, and apply the fact that only transverse modes are known to exist experimentally. This is not Lorentz-invariant, or, at least not "visibly" so.

In what follows, we will not use Gaussian units but rather the Heaviside-Lorentz system. In this system:

$$V = \frac{e^2}{4\pi r} \quad \text{instead of} \quad V = \frac{e^2}{r}$$

It appears that e is modified by $\sqrt{4\pi}$. Look at the fine structure constant to get the value of e in Heaviside-Lorentz (HL) units.

$$\text{CGS:} \quad \alpha = \frac{e^2}{\hbar c}$$

$$\text{HL:} \quad \alpha = \frac{e^2}{4\pi\hbar c} = \frac{1}{137} \; ; \quad \therefore \; e = \sqrt{4\pi\alpha\hbar c}$$

The reason we use the HL system of units is that it puts the energy densities and Maxwell's equations into convenient form.

Some of the classical equations of electrodynamics now become:

Maxwell's Equations:
$$\nabla \times \vec{\mathcal{H}} = \frac{1}{c}\frac{\partial \vec{\mathcal{E}}}{\partial t} \quad ; \quad \nabla \cdot \vec{\mathcal{H}} = 0$$
$$\nabla \times \vec{\mathcal{E}} = -\frac{1}{c}\frac{\partial \vec{\mathcal{H}}}{\partial t} \quad ; \quad \nabla \cdot \vec{\mathcal{E}} = 0$$

Energy Density: $H = \frac{1}{2}(\mathcal{E}^2 + \mathcal{H}^2)$; no 8π as involved in CGS.

Maxwell-Helmholtz Equation for a Field Quantity:
$$\nabla^2 u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0 \qquad (1)$$

Field-Vector Potential Relations:
$$\vec{\mathcal{E}} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} \quad ; \quad \vec{\mathcal{H}} = \nabla \times \vec{A}$$

Transverse Mode Criterion: $\nabla \cdot \vec{A} = 0$

The goal of the entire development is to bring about the quantitization of the field through the boson operators:
$$[b_s, b^*_{s'}] = \delta_{ss'}$$

We now choose to place the system in a large box and subject the field quantities to the BVK or periodic boundary conditions. This device lets us use traveling waves yet permits expansion of the field quantities in a Fourier series, that is, in the normal modes of the box. Recall that the propagation vector or wave vector $\vec{k}$ is made quasi-continuous by the box and has the form:
$$\vec{k} = \left(n_x \frac{2\pi}{L},\ n_y \frac{2\pi}{L},\ n_z \frac{2\pi}{L}\right)$$

We do a standard expansion of $\vec{A}$ in the normal modes of the box:
$$\vec{A} = \sum_{\vec{k},\lambda} \hat{e}_{\vec{k},\lambda}\left\{ a_{\vec{k},\lambda}\, e^{i\vec{k}\cdot\vec{r}} + a^*_{\vec{k},\lambda}\, e^{-i\vec{k}\cdot\vec{r}} \right\}$$

λ indexes the transverse modes.

Using both $e^{i\vec{k}\cdot\vec{r}}$ and $e^{-i\vec{k}\cdot\vec{r}}$ assures that $\vec{A}$ will be real. We shorten the notation, writing:

$$\boxed{\vec{A} = \sum_s \hat{e}_s \left\{ a_s e^{i\vec{k}\cdot\vec{r}} + a_s^* e^{-i\vec{k}\cdot\vec{r}} \right\} \quad \text{with } s = \vec{k}, \lambda} \tag{2}$$

We see now that $\nabla\cdot\vec{A}$ gives $\vec{k}\cdot\hat{e}_{\vec{k},\lambda} = 0$ or definitely transverse modes.

Note that each term in $e^{i\vec{k}\cdot\vec{r}}$ appears twice since $\sum_{\vec{k}}$ runs over both $\pm\vec{k}$. For some fixed $\vec{k}'$, say, we would get:

$$a_{\vec{k}',\lambda}\, e^{i\vec{k}'\cdot\vec{r}} \quad ; \quad a^*_{-\vec{k}',\lambda}\, e^{i\vec{k}'\cdot\vec{r}}$$

$$a^*_{\vec{k}',\lambda}\, e^{-i\vec{k}'\cdot\vec{r}} \quad ; \quad a_{-\vec{k}',\lambda}\, e^{-i\vec{k}'\cdot\vec{r}}$$

We now substitute $\vec{A}$ in the Maxwell-Helmholtz equation and equate coefficients of $e^{\pm i\vec{k}\cdot\vec{r}}$ to zero to satisfy the equation and get:

$$-k^2 a_{\vec{k},\lambda} - \frac{1}{c^2}\ddot{a}_{\vec{k},\lambda} = 0$$

The solution to this equation is:

$$a_s = a_s(0)\, e^{\pm ikct}$$

Recall that for a wave to travel in the + direction, we must have the time part and the displacement part of the argument of the oscillating term differ in sign. Since the coefficient of a_s in $\vec{A}$ is $e^{i\vec{k}\cdot\vec{r}}$ we thus choose:

$$a_s = a_s(0)\, e^{-ikct}$$

$$a_s^* = a_s^*(0)\, e^{ikct}$$

Thus for $+\vec{k}$, both terms in $\vec{A}$ run in the + direction while for $-\vec{k}$, both run in the minus direction since $k = |\vec{k}|$ doesn't change.

Under this choice, we then have:

$$\boxed{\begin{aligned} \dot{a}_s &= -ikc\, a_s \\ \dot{a}_s^* &= ikc\, a_s^* \end{aligned}} \qquad (3)$$

Consider now the energy or Hamiltonian density. We must compute $\mathcal{H}^2$ in terms of $\vec{A}$ and also $\mathcal{E}^2$.

$$\vec{\mathcal{H}} = \nabla \times \vec{A} \qquad \text{or:} \quad \mathcal{H}_i = \epsilon_{ijk} \frac{\partial A_k}{\partial x_j} \qquad (\text{sum convention})$$

$$\mathcal{H}^2 = \epsilon_{ijk}\, \epsilon_{ilm} \frac{\partial A_k}{\partial x_j} \frac{\partial A_m}{\partial x_l}$$

$$\epsilon_{ijk}\, \epsilon_{ilm} = \delta_{jl}\, \delta_{km} - \delta_{jm}\, \delta_{kl}$$

Then:

$$\boxed{\begin{aligned} \mathcal{H}^2 &= \frac{\partial A_l}{\partial x_j} \frac{\partial A_l}{\partial x_j} - \frac{\partial A_l}{\partial x_j} \frac{\partial A_j}{\partial x_l} \\ \mathcal{E}^2 &= \frac{\partial A_l}{c\,\partial t} \frac{\partial A_l}{c\,\partial t} \end{aligned}}$$

We now substitute these into the Hamiltonian, watching the order of the products of the a's, and writing the Hamiltonian as $\bar{H}$ for reasons which will become clear:

$$\bar{H} = \frac{1}{2} \int_V d\vec{r}\ (\mathcal{H}^2 + \mathcal{E}^2)$$

$$= \frac{1}{2} \int_V d\vec{r} \left[\frac{\partial A_l}{\partial x_j} \frac{\partial A_l}{\partial x_j} - \frac{\partial A_l}{\partial x_j} \frac{\partial A_j}{\partial x_l} + \frac{\partial A_l}{c\,\partial t} \frac{\partial A_l}{c\,\partial t} \right]$$

$$\frac{\partial A_l}{\partial x_j} = \frac{\partial}{\partial x_j} \sum_s e_{sl} \left\{ a_s\, e^{i\vec{k}\cdot\vec{r}} + a_s^*\, e^{-i\vec{k}\cdot\vec{r}} \right\}$$

$$= \sum_s i\, e_{sl}\, k_j \left\{ a_s\, e^{i\vec{k}\cdot\vec{r}} - a_s^*\, e^{-i\vec{k}\cdot\vec{r}} \right\}$$

$$\frac{\partial A_l}{c\,\partial t} = \sum_s -i\, k\, e_{sl} \left\{ a_s\, e^{i\vec{k}\cdot\vec{r}} - a_s^*\, e^{-i\vec{k}\cdot\vec{r}} \right\}$$

$$\bar{H} = \frac{1}{2}\int d\vec{r} \sum_{ss'} \left\{ a_s e^{i\vec{k}\cdot\vec{r}} - a_s^* e^{-i\vec{k}\cdot\vec{r}} \right\} \left\{ a_{s'} e^{i\vec{k}'\cdot\vec{r}} - a_{s'}^* e^{-i\vec{k}'\cdot\vec{r}} \right\}$$

$$\cdot \left\{ - e_{se} k_y e_{s'e} k_y' + e_{se} k_y e_{s'y} k_e' - e_{se} k e_{s'e} k' \right\}$$

We now use the orthogonality relation of the normal mode expansion, namely:

$$\int_V d\vec{r}\, e^{i(\vec{k}'-\vec{k})\cdot\vec{r}} = V \delta_{\vec{k},\vec{k}'}$$

Hence:

$$\bar{H} = \frac{V}{2} \sum_{ss'} \left[a_s a_{s'} \delta_{\vec{k},-\vec{k}'} - a_s a_{s'}^* \delta_{\vec{k},\vec{k}'} - a_s^* a_{s'} \delta_{\vec{k},\vec{k}'} + a_s^* a_{s'}^* \delta_{\vec{k},-\vec{k}'} \right]$$

$$\cdot \left[\underbrace{e_{se} e_{s'e} \left(-k_y k_y' - k k'\right)}_{\substack{0 \text{ for } \delta_{\vec{k},-\vec{k}'} \\ -2k^2 \text{ for } \delta_{\vec{k},\vec{k}'}}} + \underbrace{e_{se} e_{s'y} k_y k_e'}_{\substack{0 \text{ for } \delta_{\vec{k}',-\vec{k}'} \text{ (and } \delta_{\vec{k},\vec{k}'}) \\ \text{since } e_{se}(-k_e)\, e_{-sy}(k_y) \\ = 0 \text{ because only deal} \\ \text{with those waves that} \\ \text{are transverse.}}} \right]$$

Hence :

$$\boxed{\bar{H} = V \sum_s k^2 \left(a_s a_s^* + a_s^* a_s \right)} \qquad (4)$$

We assume here that there is something special about the a's that they do not commute. Classically, we would have:

$$\bar{H} = 2V \sum_s k^2 a_s^* a_s$$

The form of (4) and the fact that we are working with photons (bosons) leads to the speculation that the a's might be the boson operators.

LECTURE 33 : 12-8-61

If we assume in:

$$\bar{H} = V \sum_{s} (a_s^* a_s + a_s a_s^*) k^2 \qquad (4)$$

that the a's do not commute, it is suggested that we might take the a's to be linear functions of the b's, that is:

$$a_s = K b_s \quad ; \quad a_s^* = K b_s^* \quad (K \text{ real without loss of generality})$$

Dirac's b's are the boson operators. We use these operators although it is the "first time" we are quantizing the radiation field. Recall that:

$$[b_s, b_{s'}^*] = \delta_{ss'} \qquad (5)$$

Now also, the equations (3) for the a' are also reminiscent of similar equations for the b's (from the harmonic oscillator problem). We can find K by requiring that $\dot{b}_s$, $\dot{b}_s^*$ satisfy equations (3) and the Hiesenberg equation of motion, that is:

$$\left.\begin{aligned} \dot{b}_s &= \frac{i}{\hbar}[\bar{H}, b_s] = -i k c\, b_s \\ \dot{b}_s^* &= \frac{i}{\hbar}[\bar{H}, b_s^*] = i k c\, b_s^* \end{aligned}\right\} \quad (5')$$

We could use the a's in terms of the b's in (4) and find K. Let's do that:

$$-i k c\, b_s = \frac{i}{\hbar}\sum_{s'} V k'^2 \Big\{ \underbrace{[b_{s'}^* b_{s'}, b_s]}_{-\delta_{ss'} b_{s'}} + \underbrace{[b_{s'} b_{s'}^*, b_s]}_{-b_{s'}\delta_{ss'}} \Big\} K^2$$

$$= \frac{i V k^2 2K^2}{\hbar} \quad ; \quad K = \sqrt{\frac{\hbar c}{2 k V}} \qquad (6)$$

The same happens using b_s^*.

Thus we have:

$$\bar{H} = \frac{1}{2}\sum_s \hbar k c\,(b_s^* b_s + b_s b_s^*) = \frac{1}{2}\sum_s \hbar\omega_s\,(N_s + N_s + 1)$$

$$= \sum_s \hbar\omega_s\,(N_s + 1/2) \qquad (7)$$

Thus we have the important fact that the photon field behaves as a collection of harmonic oscillators. Now the presence of the zero point energy proves a little embarrassing as this leads to an infinite vacuum state energy when the volume is taken to be all space. However, we are free to choose a reference point for measuring energies and we choose to make the vacuum state energy zero. Another way to argue this is to say that equations (5') are also satisfied if we choose for $\bar{H}$ some linear function of $b_s^* b_s$. Then

$$\bar{H} = \sum_{s'} L\, b_{s'}^* b_{s'}$$

$$\frac{i}{\hbar}[\bar{H}, b_s] = \frac{-i}{\hbar} L\, b_s = -i\, k c\, b_s \quad ; \quad \therefore\ L = \hbar k c$$

Then:

$$\boxed{\bar{H} = \sum_s \hbar k c\, b_s^* b_s = \sum_s \hbar\omega_s N_s} \qquad (8)$$

and by comparing with the classical expression for $\bar{H}$ we find again $K = \sqrt{\frac{\hbar c}{2kV}}$.

At any rate, the vector potential becomes:

$$\boxed{\vec{A} = \sum_s \sqrt{\frac{\hbar c}{2kV}}\ \hat{e}_s \left\{ b_s e^{i\vec{k}\cdot\vec{r}} + b_s^* e^{-i\vec{k}\cdot\vec{r}} \right\}} \qquad (9)$$

When we use this form for $\vec{A}$ and work consistently as before, the zero-pt. energy term appears, but we choose it to be zero and use (8) as the Hamiltonian. That is, we assign zero energy to the vacuum state.

The vacuum state is defined by:

$$b_s \Psi(0) = 0 \quad ; \quad b_s |0\rangle = 0$$

and where we have said $\bar{H}|0\rangle = 0$, thus leaving off the zero point energy. However, it is not this easy to do away with all the effects of the vacuum state as the quantity $\overline{\mathcal{E}^2}$ is not stationary in the absence of particles (photons) hence giving fluctuations in the vacuum field which lead to spontaneous emission.

$$\overline{\mathcal{E}^2} = V\int_V d\vec{r} \sum_{ss'} \frac{\hbar c}{2kV}\left[b_s b_{s'}\delta_{k,-k'} - b_s b_{s'}^* \delta_{kk'} - b_s^* b_{s'}\delta_{kk'} + b_s^* b_{s'}^* \delta_{k,-k'}\right] \cdot \left[-e_{s\ell}\, e_{s'\ell}\, k k'\right]$$

$$\underset{\substack{\uparrow \\ \text{no particles} \\ \text{and no 0-pt} \\ \text{energy}}}{=} \sum_s \frac{\hbar\omega_s}{2}\left[-b_s b_s - b_s^* b_s^*\right] = -\sum_s \frac{\hbar\omega_s}{2}\left[b_s^2(0)\, e^{-i2\omega_s t} + b_s^{*2}(0)\, e^{i2\omega_s t}\right]$$

Hence we see that $\overline{\mathcal{E}^2}$ is not stationary.

What are some handy commutation rules, particularly between $\vec{\mathcal{E}}$ and $\vec{H}$? Form:

$$A_\ell = \sum_s \sqrt{\frac{\hbar c}{2kV}}\, e_{s\ell}\left\{b_s e^{i\vec{k}\cdot\vec{r}} + b_s^* e^{-i\vec{k}\cdot\vec{r}}\right\}$$

$$[A_\ell, A_{\ell'}] = \sum_{ss'} \frac{\hbar c}{2V}\frac{1}{\sqrt{kk'}}\, e_{s\ell}\, e_{s'\ell'}\left[\left\{b_s e^{i\vec{k}\cdot\vec{r}} + b_s^* e^{-i\vec{k}\cdot\vec{r}}\right\}, \left\{b_{s'} e^{i\vec{k}'\cdot\vec{r}'} + b_{s'}^* e^{-i\vec{k}'\cdot\vec{r}'}\right\}\right]$$

$$[\{\},\{\}] = \underbrace{[b_s, b_{s'}^*]}_{\delta_{ss'}} e^{i\vec{k}\cdot\vec{r} - i\vec{k}'\cdot\vec{r}'} + \underbrace{[b_s^*, b_{s'}]}_{-\delta_{ss'}} e^{-i\vec{k}\cdot\vec{r} + i\vec{k}'\cdot\vec{r}'}$$

Then:

$$\boxed{[A_\ell, A_{\ell'}] = \frac{i\hbar c}{V}\sum_s \frac{\sin[\vec{k}(\vec{r}-\vec{r}')]}{|k|} = 0}$$

since the summand is an odd function of $\vec{k}$.

now consider $[\mathcal{H}_i, \mathcal{E}_{i'}]$: It is obvious that $[\mathcal{H}_i, \mathcal{H}_{i'}]$ $= [\mathcal{E}_i, \mathcal{E}_{i'}] = 0$ because of the same reason that $[A_l, A_{l'}] = 0$.

$$\mathcal{H}_i = \epsilon_{ijl} \frac{\partial A_l}{\partial x_j} = \epsilon_{ijl} \sum_s i\, e_{sl}\, k_j \sqrt{\frac{\hbar c}{2kV}} \left\{ b_s e^{i\vec{k}\cdot\vec{r}} - b_s^* e^{-i\vec{k}\cdot\vec{r}} \right\}$$

$$\mathcal{E}_i = -\frac{1}{c}\frac{\partial A_i}{\partial t} = \sum_s i\, k\, e_{si} \sqrt{\frac{\hbar c}{2kV}} \left\{ b_s e^{i\vec{k}\cdot\vec{r}} - b_s^* e^{-i\vec{k}\cdot\vec{r}} \right\}$$

$$[\{\},\{\}] = -\underbrace{[b_s, b_{s'}^*]}_{\delta_{ss'}} e^{i\vec{k}\cdot\vec{r} - i\vec{k}'\cdot\vec{r}'} - \underbrace{[b_s^*, b_{s'}]}_{-\delta_{ss'}} e^{-i\vec{k}\cdot\vec{r} + i\vec{k}'\cdot\vec{r}'}$$

$$[\mathcal{H}_i, \mathcal{E}_{i'}] = +\epsilon_{ijl} \sum_{ss'} e_{sl}\, e_{s'i'}\, k_j\, k' \frac{\hbar c}{2V} \frac{1}{\sqrt{kk'}} \left[\delta_{ss'} e^{-i\vec{k}\cdot\vec{r} + i\vec{k}'\cdot\vec{r}'} - \delta_{ss'} e^{i\vec{k}\cdot\vec{r} - i\vec{k}'\cdot\vec{r}'} \right]$$

$$= -2i\, \epsilon_{ijl} \sum_s \underbrace{e_{sl}\, e_{si'}}_{\delta_{li'}} k_j \frac{\hbar c}{2V} \sin[\vec{k}\cdot(\vec{r}-\vec{r}')]$$

$$= -\frac{i\hbar c}{V} \epsilon_{iji'} \sum_s k_j \sin[\vec{k}\cdot(\vec{r}-\vec{r}')]$$

$$= \frac{i\hbar c}{V} \epsilon_{iji'} \frac{\partial}{\partial x_j} \sum_s \cos[\vec{k}\cdot(\vec{r}-\vec{r}')]$$

We now use the closure rule of Fourier analysis, namely:

$$\sum_{\vec{k}} \cos[\vec{k}\cdot(\vec{r}-\vec{r}')] = V\, \delta(\vec{r}-\vec{r}')$$

and get finally:

$$\boxed{[\mathcal{H}_i, \mathcal{E}_{i'}] = i\hbar c\, \epsilon_{ii'j} \frac{\partial}{\partial x_j'} \delta(\vec{r}-\vec{r}')}$$

We have been working in the Heisenberg picture where the time dependence has been contained in the operator and the states where they occurred were stationary. In the Schroedinger picture all the time dependence is in the wave function.

To discuss the interaction between matter and the radiation field, we will go to the Dirac picture or interaction picture where the perturbed time dependence is in the wave function and the operator is transformed by unitary transformation containing the unperturbed Hamiltonian acting on the interaction potential.

The total Hamiltonian of the combined radiation-matter field is:

$$\overline{H} = H_M + H_R + H_I$$

Matter: $H_M = \sum_{pr} H_{pr} C_p^* C_r + \sum_{prqu} V_{pr,qu} C_p^* C_r^* C_q C_u$

$V_{pr,qu}$ is different from the radiation interaction and is like the coulomb interaction more or less. We will work with one electron.

Radiation: $H_R = \sum_s \hbar\omega_s b_s^* b_s$

Interaction: $H_I = \int \psi^*(\vec{r}) \left[\text{terms in the Hamiltonian } \underline{H}\text{, which is the complete } \underline{H}\text{, that contain } \vec{A} \right] \psi(\vec{r}) d\vec{r}$

$\underline{H}$ is the single electron Hamiltonian and:

$$\psi(\vec{r}) = \sum_p C_p(t)\, u_p(\vec{r})$$

H_{rp} above contains the kinetic energy term and the nuclear field term. $\overline{H}$ was formed by taking the total Hamiltonian including the radiation Hamiltonian and sandwiching it between $\int \psi^*(\vec{r}) \cdots \psi(\vec{r})\, d\vec{r}$.

As far as H_I goes, we have:

Dirac Electron: $\underline{H} = c\,\vec{\alpha}\cdot\vec{\pi} + \beta mc^2 + V$; V = nuclear field

and: $[\ ] = -e\,(\vec{\alpha}\cdot\vec{A})$ in H_I.

$$H_I = -e \int \psi^* (\vec{\alpha}\cdot\vec{A})\, \psi\, d\vec{r}$$

Schroedinger Electron: $\underline{H} = \frac{1}{2m} \vec{\pi} \cdot \vec{\pi} + V$

$$\text{and: } [\;] = \frac{ie\hbar}{2mc} (\nabla \cdot \vec{A} + \vec{A} \cdot \nabla) + \frac{e^2}{2mc^2} A^2 \quad \text{in } H_I$$

One would classically expect the interaction energy between the field and the motion of the particle to be of the form:

$$H_I = -\frac{1}{c} \int \vec{A} \cdot \vec{j} \, d\vec{r}$$

This form checks form of H_I for the Dirac equation since $\vec{j} = ec\, \psi^* \vec{\alpha}\, \psi$ but is off for the Schroedinger electron because of 2 in the $\vec{A}$ term of $\vec{j}$:

$$\vec{j} = \frac{e\hbar}{2mi} (\psi^* \nabla \psi - \psi \nabla \psi^*) - \frac{e^2}{mc} \psi^* \vec{A} \psi$$

This $\vec{A}$ term in $\vec{j}$ lacks a 2 in the denominator to make $\frac{\vec{j} \cdot \vec{A}}{c}$ equal to [ ] above.

However, in sketch form (as long as we remember the exception above) we may write:

$$H_I = -\frac{1}{c} \int \vec{j} \cdot \vec{A} \, d\vec{r}$$

$$\vec{j} \downarrow \; C_p^* C_e \qquad \vec{A} \downarrow \; \begin{cases} b_s \leftarrow \sqrt{N_s} & \text{photon absorbed in process} \\ b_s^* \leftarrow \sqrt{N_s + 1} & \text{photon emitted in process} \end{cases}$$

Now in the interaction picture, the C's carry the time dependence of the wave functions, that is:

$$C_e = C_e(0)\, e^{-iE_e t/\hbar}$$

Then: $C_p^* C_e b_s \sim e^{i(E_p - E_e)t/\hbar - i\omega_s t}$ describes the transition which absorbs a photon of ω_s, destroys the electron in e and creates one in p all while conserving energy. Note how the energy conservation fits in with positive going wave convention. This particular transition is given by the matrix element in the "occupation" representation:

$$\langle \cdots 1_p \cdots 0_e \cdots N_s - 1 \,|\, C_p^* C_e b_s \,|\, \cdots 0_p \cdots 1_e \cdots N_s \cdots \rangle$$

LECTURE 34: 12-11-61

Free Electron Scattering in The Quantized Field

We first show that a free electron cannot emit nor absorb radiation. We write momentum as:

$$p_\mu = (\vec{p}, i\tfrac{E}{c})$$

Since $E = \hbar k c$, the wave vector can now be written:

$$k_\mu = (\vec{k}, ik)$$

We now introduce natural units: $\hbar = m = c = 1$ and write:

Electron at rest: $p_\mu = (0, i)$
Photon: $k_\mu = (\vec{k}, ik)$

We write the conservation of momentum equation for the absorption of a photon by a free electron at rest, the prime will always refer to the scattered system.

$$p_\mu + k_\mu = p'_\mu \quad \text{(conservation of momentum and energy)}$$

and take square: $(p_\mu p_\mu = p'_\mu p'_\mu = -m^2c^2 = -1)$

$$p_\mu p_\mu + k_\mu k_\mu + 2 p_\mu k_\mu = p'_\mu p'_\mu$$

$$k_\mu k_\mu = \vec{k}\cdot\vec{k} - k^2 = 0$$

$$p_\mu k_\mu = (0,i)(\vec{k}, ik) = -k$$

Thus we have: $1 + 2k = 1$ or $k = 0$ which is impossible if there is to be an incident photon at all.

So just by considering the absorption via the conservation laws one can find that free electrons cannot absorb and a similar statement holds for emission.

Compton Scattering:

We now derive the Compton scattering law for photons by free electrons. The conservation equation is:

$$p_\mu + k_\mu = p'_\mu + k'_\mu \quad ; \quad p_\mu = (\vec{p}, iE) \; ; \; p'_\mu = (\vec{p}', iE')$$
$$k_\mu = (\vec{k}, ik) \; ; \; k'_\mu = (\vec{k}', ik')$$

or: $p_\mu - p'_\mu = k'_\mu - k_\mu$

Form $-(p_\mu - p'_\mu) = -(k'_\mu - k_\mu)$ and multiply into above:

$$\underbrace{-p_\mu p_\mu}_{1} \underbrace{- p'_\mu p'_\mu}_{1} + 2 p'_\mu p_\mu = \underbrace{-k'_\mu k'_\mu}_{0} - \underbrace{k_\mu k_\mu}_{0} + 2 k'_\mu k_\mu$$

Take the electron initially at rest: $p_\mu = (0, i)$, then:

$$p_\mu p'_\mu = (0, i)(\vec{p}', iE') = -E'$$

$$k_\mu k'_\mu = \vec{k} \cdot \vec{k}' - k k'$$

and $1 - E' = -kk' + \vec{k} \cdot \vec{k}'$

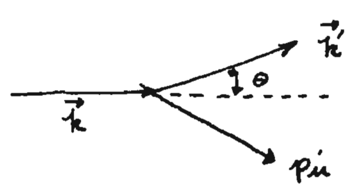
now $E' = mc^2 + (\hbar c k - \hbar c k') \rightarrow 1 + k - k'$ (conservation of energy)

Also: $\vec{k} \cdot \vec{k}' = kk' \cos\theta$

$$\boxed{k - k' = kk'(1 - \cos\theta) \quad ; \quad \frac{1}{k'} - \frac{1}{k} = 1 - \cos\theta}$$

In our natural units: $k = \frac{h\nu}{mc^2} = \frac{h/mc}{\lambda}$, hence we get the usual Compton Formula:

$$\boxed{\lambda' - \lambda = \frac{h}{mc}(1 - \cos\theta)}$$



We now inquire into the intensity of the scattering. Consider soft light incident on a Schroedinger electron; $\hbar k << mc$ or $k << 1$. Recall the interaction term:

$$H_I = \int \psi^* \left[\frac{i e \hbar}{2mc} (\nabla \cdot \vec{A} + \vec{A} \cdot \nabla) + \frac{e^2}{2mc} A^2 \right] \psi \, d\vec{r}$$

This is not quite equal to $-\frac{1}{c} \int \vec{j} \cdot \vec{A} \, d\vec{r}$ because of the 2 in the A^2 term. In field theory, $\vec{j}$ can also be found via the relation: $\vec{j} = c \frac{\delta \mathcal{L}}{\delta \vec{A}}$.

We will see that the A^2 term in H_I can give scattering by itself as:

$$\psi^* \sim c^* e^{-i\vec{p}' \cdot \vec{r}} \; ; \; \psi \sim c e^{i\vec{p} \cdot \vec{r}} \; ; \; A^2 \sim b_{s'}^* e^{-i\vec{k}' \cdot \vec{r}} \, b_s e^{i\vec{k} \cdot \vec{r}}$$

and: $\psi^* A^2 \psi \sim e^{i(\vec{p} - \vec{p}' + \vec{k} - \vec{k}') \cdot \vec{r}} \; c^* b_{s'}^* b_s c_s$

It is seen that momentum is conserved and the incident electron and photon is destroyed while the scattered electron and photon are created thus the A^2 term can give the conditions for scattering all by itself. Now, in the first order, the $\nabla \cdot \vec{A}$ terms give no scattering as they contain only b^* or b and we know that a single photon process (emission or absorption) is impossible with free electrons. The A^2 term contains $b^* b$ or $b b^*$ which are two photon processes so scattering can occur in the first order, but with this term only. On the other hand, the $\nabla \cdot \vec{A}$ terms do contribute in the second order through the action of intermediate states as we will see. Let us now form the quantity $\psi^* A^2 \psi$ where:

$$\boxed{\begin{aligned} \psi &= \frac{c_{\vec{p}}}{\sqrt{V}} e^{i\vec{p} \cdot \vec{r}} \; ; \quad \psi^* = \frac{c_{\vec{p}'}}{\sqrt{V}} e^{-i\vec{p}' \cdot \vec{r}} \\ \vec{A} &= \sqrt{\frac{\hbar c}{2kV}} \, \hat{e}_s \left(b_s e^{i\vec{k} \cdot \vec{r}} + b_s^* e^{-i\vec{k} \cdot \vec{r}} \right) \end{aligned}}$$

We are using plane waves for both the electron and the phonon as only one at a time are involved, that is, before and after the collision there is only one of each around.

$$A^2 = \frac{\hbar c}{2V} \frac{1}{\sqrt{k k'}} \hat{e}_{s'} \cdot \hat{e}_s \Big[b_{s'} b_s \, e^{i(\vec{k}+\vec{k}')\cdot\vec{r}} + b_{s'} b_s^* \, e^{i(\vec{k}'-\vec{k})\cdot\vec{r}}$$

$$+ b_{s'}^* b_s \, e^{-i(\vec{k}'-\vec{k})\cdot\vec{r}} + b_{s'}^* b_s^* \, e^{-i(\vec{k}+\vec{k}')\cdot\vec{r}} \Big]$$

$$\psi^* A^2 \psi = \frac{\hbar c}{2V^2} \frac{1}{\sqrt{k k'}} \hat{e}_{s'} \cdot e_s \Big[C_{\vec{p}'}^* b_{s'} b_s C_{\vec{p}} \, e^{i(\vec{k}+\vec{k}'+\vec{p}-\vec{p}')\cdot\vec{r}}$$

$$+ C_{\vec{p}'}^* b_{s'} b_s^* C_{\vec{p}} \, e^{i(\vec{k}'-\vec{k}+\vec{p}-\vec{p}')\cdot\vec{r}} + C_{\vec{p}'}^* b_{s'}^* b_s C_{\vec{p}} \, e^{-i(\vec{k}'-\vec{k}-\vec{p}+\vec{p}')\cdot\vec{r}}$$

$$+ C_{\vec{p}}^* b_{s'}^* b_s^* C_p \, e^{-i(\vec{k}+\vec{k}'-\vec{p}+\vec{p}')\cdot\vec{r}} \Big]$$

Note that s and s' can be interchanged in any of the terms as it is indifferente to the way A^2 was formed, that is, $A^2 = \vec{A}(s')\cdot\vec{A}(s) = \vec{A}(s)\cdot\vec{A}(s')$. Also, the 2nd term has $\vec{k}'$ for the incident photon wave (as can be seen by conservation of momentum in the exponent) whereas the 3rd term has $\vec{k}$ and we want these two terms to be equivalent in form. Now in the 2nd term we can then write $b_s b_{s'}^* = b_{s'}^* b_s$ as the wave vectors of the incident and scattered photons will never be quite identical. We are now in the position where in the 2nd and 3rd terms unprimed means incident particles and primed means scattered particles. The appropriate matrix element we want to form is then:

$$\langle \cdots 1_{\vec{p}'} \cdots 0_{\vec{p}} \cdots 0_s \cdots 1_{s'} \cdots | \psi^* A^2 \psi | \cdots 0_{\vec{p}'} \cdots 1_{\vec{p}} \cdots 1_s \cdots 0_{s'} \cdots \rangle$$

We see that the first and fourth terms give zero because they do not give non-vanishing matrix elements. The only non-vanishing are the 2nd and 3rd terms which are now identical. They contain $C_{\vec{p}'}^* b_{s'}^* b_s C_{\vec{p}}$ which do what is wanted.

Then: $\langle | \psi^* \vec{A}^2 \psi | \rangle = \frac{\hbar c}{V^2} \frac{1}{\sqrt{k k'}} \hat{e}_s \cdot \hat{e}_{s'} \, e^{-i(\vec{k}' - \vec{k} - \vec{p} + \vec{p}') \cdot \vec{r}}$

Performing the integration gives:

$$\frac{\hbar c}{V} \frac{1}{\sqrt{k k'}} \quad \hat{e}_s \cdot \hat{e}_{s'} \quad \delta_{\vec{k}', \vec{k} + \vec{p} - \vec{p}'}$$

We now assume that the incident light is "soft", that is, infrared where the wave vector "momentum" does not have a "relativistic" value. Then the change in wave length is not much and we put $k \approx k'$. We write in place of $\hat{e}_s \cdot \hat{e}_{s'} \, \delta_{\vec{k}', \vec{k} + \vec{p} - \vec{p}'}$ the equivalent quantity $\hat{e} \cdot \hat{e}'$. Cannot put $\vec{k}' = \vec{k}$ as their directions can be vastly different. Finally we have for the matrix element of the relevant part of H_I:

$$\langle | \int d\vec{r} \, \psi^* A^2 \psi | \rangle = \frac{e^2}{mc^2} \frac{\hbar}{2Vk} \quad \hat{e} \cdot \hat{e}'$$

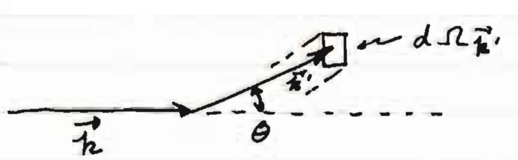
Recall from time-dependent perturbation theory that the transition probability per unit time in the first order is:

$$\frac{dP}{dt} = \frac{2\pi}{\hbar} \left| \langle \text{final} | \quad | \text{initial} \rangle \right|^2 \rho_E (\text{final})$$

Here:

$$\frac{dP}{dt} = \frac{2\pi}{\hbar} \left[\frac{e^2}{mc^2} \frac{\hbar c}{2kV} (\hat{e} \cdot \hat{e}') \right]^2 \rho_E \ (\text{final})$$

Now $\rho_E \, dE$ = number of states in the solid angle $d\Omega_{\vec{k}'}$ of the scattered wave and in energy range dE.



We work with the $\frac{dP}{dt}$ of the scattered photon as we are interested in the beam intensity of the scattered beam.

The number of states in $d\vec{k}'$ $(= dk_x' dk_y' dk_z')$ is, using the BVK condition that;

$$k_x' = \frac{n_x 2\pi}{L} \text{, etc.,}$$

equal to:

$$dn_x\, dn_y\, dn_z = \left(\frac{L}{2\pi}\right)^3 dk_x'\, dk_y'\, dk_z'$$

$$= \left(\frac{L}{2\pi}\right)^3 k'^2\, dk'\, d\Omega_{\vec{k}'}$$

Hence:

$$\rho_E(\text{final})\, dE = \frac{V}{8\pi^3} k'^2\, d\Omega_{\vec{k}'} \frac{dk'}{dE}\, dE \qquad (E \to \text{final energies})$$

Now: $k' = \frac{\omega'}{c} = \frac{E'}{\hbar c}$; $\frac{dk'}{dE'} = \frac{1}{\hbar c}$

We actually should have $\rho_{E'} dE'$ above but we leave as is.
Then:

$$\rho_E(\text{final})\, dE = \frac{V}{8\pi^3 \hbar c} k'^2\, d\Omega_{\vec{k}'} \approx \frac{V}{8\pi^3 \hbar c} k^2\, d\Omega_{\vec{k}'}\, dE$$

And:

$$\frac{dP}{dt} = \frac{c}{16\pi^2 V} \frac{e^4}{m^2 c^4} (\hat{e}\cdot\hat{e}')^2\, d\Omega_{\vec{k}'}$$

We have for the definition of the differential scattering cross-section:

$$\sigma(\theta,\varphi)\, d\Omega_{\vec{k}'} = \frac{dP/dt}{c/V}$$

where c/V is the flux incident on the scattering center per unit area per unit time.

We have, using the HL units of $e'^2 = \frac{e^2}{4\pi}$: e' is the CGS charge:

$$\boxed{\sigma(\theta,\varphi)\, d\Omega_{\vec{k}'} = \frac{e'^4}{m^2 c^4} (\hat{e}\cdot\hat{e}')^2\, d\Omega_{\vec{k}'}}$$

The above is the differential form of Thompson's Scattering Formula. Recall the radiation from an accelerated electron (classical):

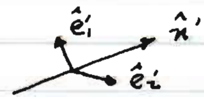
$$\frac{dE}{dt} = \text{Power radiated} = \frac{2e'^2}{3c^3}\,\overline{(a)^2}$$

But $\overline{(a)^2} = \dfrac{e'^2\,\overline{\mathcal{E}_0^2}}{m^2}$ and the incident power per unit area is of course $c\,\dfrac{\overline{\mathcal{E}_0^2}}{4\pi}$, hence:

$$\sigma = \frac{\dfrac{2e'^2}{3c^3}\,\dfrac{e'^2\,\overline{\mathcal{E}_0^2}}{m^2}}{c\,\dfrac{\overline{\mathcal{E}_0^2}}{4\pi}} = \frac{8\pi}{3}\,\frac{e'^4}{m^2c^4}$$

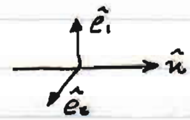
This is the total cross-section so we see we need to developed $\frac{8\pi}{3}$ in going from $\sigma(\theta,\varphi)$ to σ. To get from σ to $\sigma(\theta,\varphi)$ we must sum over polarizations and integrate over angles:

$$\sum_{\hat{e}'}(\hat{e}\cdot\hat{e}')^2 = 1-(\hat{e}\cdot\hat{n}')^2$$



because: $e^2 = (\hat{n}'\cdot\hat{e})^2 + (\hat{e}_1'\cdot\hat{e})^2 + (\hat{e}_2'\cdot\hat{e})^2 = 1$

We now average over the incident polarizations:



$$\frac{1}{2}\sum_{\hat{e}}\left[1-(\hat{e}\cdot\hat{n}')^2\right] = 1-\frac{1}{2}\left\{1-(\hat{n}\cdot\hat{n}')^2\right]$$

$$= \frac{1}{2}\left[1+(\hat{n}\cdot\hat{n}')^2\right] = \frac{1}{2}\left\{1+\cos^2\theta\right]$$

now integrate over angles: $\frac{1}{2}\int_0^{2\pi}d\varphi\int_{-1}^{1}d\mu\,(1+\mu^2) = \pi\left[\mu+\frac{\mu^3}{3}\right]_{-1}^{1} = \frac{8\pi}{3}$

Then: Thompson Scattering Formula

$$\boxed{\sigma = \frac{8\pi}{3}\,\frac{e^4}{m^2c^4}}$$

LECTURE 35 : 12-13-61

Klein - Nishina Formula

Collision of Photon - Free Dirac Electron

We have for the interaction term:

$$H_I = -e \int \psi^* (\vec{\alpha} \cdot \vec{A}) \psi \, d\vec{r}$$

We work in the interaction picture which means that the destruction - creation operators carry the time dependence in the operator H_I :

$$\psi = \frac{C_{\vec{p},r} \, a^{(r)}(\vec{p})}{\sqrt{V}} e^{i\vec{p}\cdot\vec{r}} \quad ; \quad \psi^* = \frac{C^*_{\vec{p}',r'} \, a^{(r')*}(\vec{p}')}{\sqrt{V}} e^{-i\vec{p}'\cdot\vec{r}}$$

$$\vec{A} = \sqrt{\frac{\hbar c}{2kV}} \, \hat{e}_s \left[b_s \, e^{i\vec{k}\cdot\vec{r}} + b_s^* \, e^{-i\vec{k}\cdot\vec{r}} \right]$$

$$\boxed{\begin{aligned} H_I = -e\sqrt{\frac{\hbar c}{2kV}} \Big\{ & \left[a^{*(r')}(\vec{p}') \, \vec{\alpha}\cdot\hat{e}_s \, a^{(r)}(\vec{p}) \right] C^*_{\vec{p}',r'} \, b_s \, C_{\vec{p}r} \; \delta_{\vec{p}-\vec{p}',-\vec{k}} \\ & + \left[a^{*(r')}(\vec{p}') \, \vec{\alpha}\cdot\hat{e}_s \, a^{(r)}(\vec{p}) \right] C^*_{\vec{p}'r'} \, b_s^* \, C_{\vec{p}r} \; \delta_{\vec{p}-\vec{p}',\vec{k}} \Big\} \end{aligned}}$$

We see that any matrix elements of H_I must involve the destruction or creation of the photon involved in the process, hence we must have no first order scattering. However, we can have scattering in the second order because here we use the device of the intermediate state to get to which requires the absorption or emission of a photon but the whole process preserves the presence of the photon.

Therefore, we must now consider second-order time-dependent perturbation theory. We can use successive approximations to get to the second order term.

Generally we have:

$$i\hbar \dot{a}_n = \sum_l V_{nl} \, e^{i(E_n - E_l)t/\hbar} a_l$$

or, taking V as a perturbation:

$$i\hbar \dot{a}_n^{(\nu)} = \sum_l V_{nl} \, e^{i(E_n - E_l)t/\hbar} a_l^{(\nu-1)}$$

In the zeroth order, at $t=0$, we put $a_l^{(0)} = \delta_{ln_0}$ or we are definitely in the state n_0 so $|a_n^{(1)}|^2$ gives the transition probability to state n. Hence:

$$a_n^{(1)} = V_{nn_0} \frac{e^{i(E_n - E_{n_0})t/\hbar} - 1}{E_{n_0} - E_n}$$

which leads to the first order transition rate. Suppose however, that $V_{nn_0} = 0$ as it does for free electrons. On the other hand, one can imagine some "discrete" state for which $V_{ln_0} \neq 0$. We can then write for the "transition" from n_0 to l, in the first order:

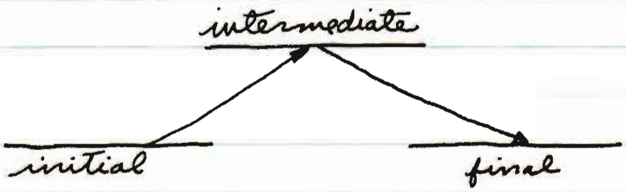
$$a_l^{(1)} = V_{ln_0} \frac{e^{i(E_l - E_{n_0})t/\hbar} - 1}{E_{n_0} - E_l}$$

We resubstitute this above and get:

$$i\hbar \dot{a}_n^{(2)} = \sum_l \frac{V_{nl} V_{ln_0}}{E_{n_0} - E_l} \left[e^{i(E_n - E_{n_0})t/\hbar} - e^{i(E_n - E_l)t/\hbar} \right]$$

Integrating and squaring, we readily obtain:

$$\boxed{\frac{dP}{dt} = \frac{2\pi}{\hbar} \left| \sum_{\substack{\text{intermediate} \\ \text{states}}} \frac{\langle \text{final} | V | \text{intermediate} \rangle \langle \text{intermediate} | V | \text{initial} \rangle}{E_{\text{initial}} - E_{\text{intermediate}}} \right|^2 \rho_E(\text{final})}$$

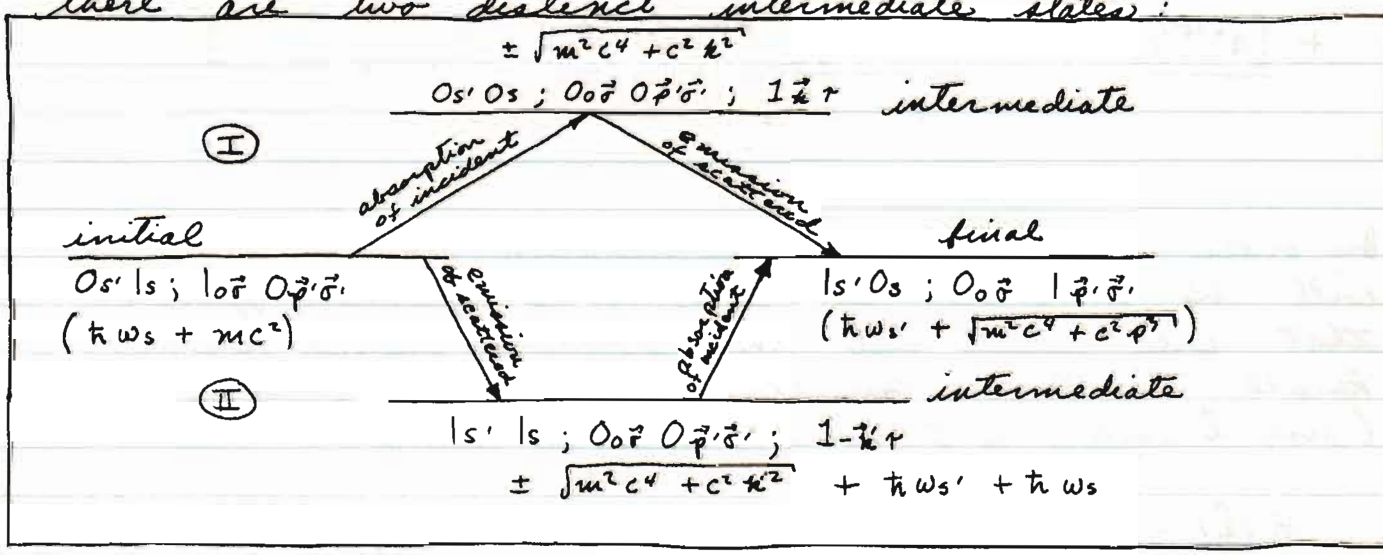


physical picture where $\langle f | V | i \rangle$ is forbidden.

Hence, through the fixture of the intermediate state, we are able to effect scattering by the Dirac electrons.

We will now consider the physics of the scattering. We take an electron with spin $\vec{\sigma}$ at rest and bombard it with a photon of wave vector and polarization s. The final result will be an electron with momentum $\vec{p}'$ and spin $\vec{\sigma}'$. Both final and initial states are taken to be either both + or both − energy states. There can be no pair production as this would require the absorption or emission of a photon. However, the intermediate states, which will be labeled by r with runs over both spins of both + and − energy states, can exist in energy opposite to that of the initial and final states.

Putting aside the energy question for a while and talking as if all states were positive energy (initial and final will always be chosen thus), we see that, also putting spin aside as it has the same energy either way, there are two distinct intermediate states:



The appropriate terms in $\frac{dP}{dt}$ now are:

I $$\frac{\langle\cdots 1_{s'} 0_s\,;\ 0_{0\vec{\sigma}}\ 1_{\vec{p}'\vec{\sigma}'}\cdots|\cdots C_{\vec{p}'\vec{\sigma}'}\, b^*_{s'}\, C_{\vec{k}r}\cdots|\cdots 0_{s'} 0_s\,;\ 0_{0\vec{\sigma}}\ 0_{\vec{p}'\vec{\sigma}'}\,;\ 1_{\vec{k}r}\cdots\rangle\langle\cdots|\cdots C^*_{\vec{k}r}\, b_s\, C_{0\vec{\sigma}}\cdots|\cdots 0_{s'} 1_s\,;\ 1_{0\vec{\sigma}}\ 0_{\vec{p}'\vec{\sigma}'}\rangle}{\hbar\omega_s + mc^2 \mp \sqrt{m^2c^4 + c^2k^2}}$$

II $$\frac{\langle \text{final}|\cdots C^*_{\vec{p}'\vec{\sigma}'}\, b_s\, C_{-\vec{k}'r}\cdots|\cdots 1_{s'} 1_s\,;\ 0_{0\vec{\sigma}}\ 0_{\vec{p}'\vec{\sigma}'}\,;\ 1_{-\vec{k}'r}\cdots\rangle\langle\cdots|\cdots C^*_{-\vec{k}'r}\, b^*_{s'}\, C_{0\vec{\sigma}}\cdots|\text{initial}\rangle}{-\hbar\omega_{s'} + mc^2 \mp \sqrt{m^2c^4 + c^2k'^2}}$$

We define the new notation:

$\langle$ final $| H_I |$ intermediate $\rangle\langle$ intermediate $| H_I |$ initial $\rangle$

$$\equiv \left(\cdots 1_{s'} 0_s ; 1_{\vec{p}'\sigma'} 0_{0\vec{\sigma}} \cdots | H_I | \cdots 0_{s'} 1_s ; 0_{\vec{p}'\sigma'} 1_{0\vec{\sigma}}\right)^2$$

$$= \left(\text{final} | H_I | \text{initial}\right)^2$$

Then, using the results for I and II and the formula for H_I, we have; using natural units where convenient:

$$\boxed{\begin{aligned}&\sum_{\text{intermediate}} \frac{(\text{final} | H_I | \text{initial})^2}{E_i - E_{\text{intermediate}}}\\ &= \frac{2\pi e^2}{\sqrt{kk'}\,V} \sum_{\tau} \left\{ \frac{\left[a^{*(\sigma')}(\vec{p}')\,(\vec{\alpha}\cdot\hat{e}')\,a^{(\tau)}(\vec{k})\right]\left[a^{*(\tau)}(\vec{k})\,(\vec{\alpha}\cdot\hat{e})\,a^{(\sigma)}(0)\right]}{1 + k \mp \sqrt{1+k^2}} \right.\\ &\left. + \frac{\left[a^{*(\sigma')}(\vec{p}')\,(\vec{\alpha}\cdot\hat{e})\,a^{(\tau)}(-\vec{k}')\right]\left[a^{*(\tau)}(-\vec{k}')\,(\vec{\alpha}\cdot\hat{e}')\,a^{(\sigma)}(0)\right]}{1 - k' \mp \sqrt{1+k'^2}} \right\} \qquad (A)\end{aligned}}$$

In order to perform the summation over τ, it will be convenient to construct a selection operator that will select out the positive energy states. Recall the Dirac Hamiltonian for a free particle (use $\vec{k}$ instead of $\vec{p}$, no difference in natural units).

$$\underline{H}(\vec{k}) = \vec{\alpha}\cdot\vec{k} + \beta$$

We have:

$$\underline{H}(\vec{k})\, u^{\binom{1}{2}}(\vec{k}) = \sqrt{1+k^2}\, u^{\binom{1}{2}}(\vec{k})$$

$$\underline{H}(\vec{k})\, u^{\binom{3}{4}}(\vec{k}) = -\sqrt{1+k^2}\, u^{\binom{3}{4}}(\vec{k})$$

Hence we easily see thant:

$$\frac{\underline{H}(\vec{k}) + \sqrt{1+k^2}}{2\sqrt{1+k^2}}\, a^{(\tau)}(\vec{k}) = \begin{cases} a^{(\tau)}(\vec{k}) \; ; \; \tau = 1,2 \\ 0 \; ; \; \tau = 3,4 \end{cases}$$

does the job. The operator to pick out $\tau = 3,4$ is obvious.

LECTURE 36 : 12-15-61

We digress for a while to consider the second order transition probability due to the $\nabla \cdot \vec{A}$ terms of H_I for the Schroedinger electron with soft light incident upon it. By soft light we mean:

$$k << \frac{mc}{\hbar} \quad \text{or} \quad \hbar k << mc$$

This means that the electron scattered by the light only attains non-relativistic velocities, that is, $mv \lesssim \hbar k << mc$

Recall:

$$H_I \sim \frac{ie\hbar}{2mc}\left(\nabla \cdot A + A \cdot \nabla\right) + \frac{e^2}{2mc^2} A^2$$

Now the $\nabla \cdot A$ brings out a k from $\vec{A}$, so, only considering $\nabla \cdot A$ and not A^2, we have:

$$H_I \sim e\left(\frac{\hbar k}{mc}\right)\frac{\hbar c}{k}$$

Now: Matrix Element (ME) $= \sum_{I_n} \frac{\langle f \mid H_I \mid I_n \rangle \langle I_n \mid H_I \mid i \rangle}{E_i - E_{I_n}}$

$$E_i - E_{I_n} = mc^2 + \hbar kc - \sqrt{m^2c^4 + (\hbar kc)^2}$$

$$\simeq \hbar kc - \frac{\hbar k}{mc}\frac{mc^2}{2} \simeq \hbar kc$$

Then: $(ME) = \frac{e^2 \left(\frac{\hbar k}{mc}\right)^2}{\hbar kc} \cdot \sqrt{\frac{\hbar^2 c^2}{k^2}}$

Finally: $\frac{dP}{dt} \sim \frac{\rho_E}{\hbar}(ME)^2 \sim \frac{k^2}{\hbar^2 c} \frac{e^4 \left(\frac{\hbar k}{mc}\right)^4}{k^4} = \frac{e^4}{m^2c^3}\left(\frac{\hbar k}{mc}\right)^2$

Now for the first order $\frac{dP}{dt}$ using the A^2 term, we obtained:

$$\frac{dP}{dt} \sim \frac{e^4}{m^2c^3}$$

so we see that the second order $\frac{dP}{dt}$ using $\nabla \cdot \vec{A}$ makes practically no contribution to the scattering and when it does, we are beyond the applicability of the Schroedinger electron theory.

We now discuss soft light scattering by the Dirac electron. Recall that terms in the matrix element appear of the form:

$$\frac{\left(a_f^*\,(\vec{\alpha}\cdot\vec{e}\,')\,a_{In}\right)\left(a_{In}^*\,(\vec{\alpha}\cdot\vec{e}\,)\,a_i\right)}{1+k\mp\sqrt{1-k^2}}$$

For positive energy intermediate states: $1+k-\sqrt{1+k^2}\sim k$
For negative energy intermediate states: $1+k+\sqrt{1+k^2}\sim 2$

In the Dirac representation $\vec{\alpha}\sim\begin{pmatrix}0&\vec{\sigma}\\ \vec{\sigma}&0\end{pmatrix}$. The a's are of the form:

$$(\text{large}\ \ \text{small})\quad\text{or:}\quad\begin{pmatrix}\text{large}\\ \text{small}\end{pmatrix}\quad(+\text{ energy})$$

Then, for positive energy states, we have matrix products of the form:

$$(\text{large}\ \ \text{small})\begin{pmatrix}0&\vec{\sigma}\\ \vec{\sigma}&0\end{pmatrix}\begin{pmatrix}\text{large}\\ \text{small}\end{pmatrix}$$

But this is merely the expectation value of the velocity operator $\vec{\alpha}=\frac{\vec{v}}{c}$. Hence for + energy states, the scattering goes as:

$$\frac{dP}{dt}\sim\rho_E\,(ME)^2\sim k^2\left[\frac{\frac{\vec{v}}{c}\cdot\frac{\vec{v}}{c}\ \ \frac{\hbar c}{kV}}{k}\right]^2\sim\frac{\left(\frac{v}{c}\right)^4}{k^2}\sim\left(\frac{v}{c}\right)^2$$

What about − energy intermediate states? For these states the large and small components reverse so we have matrix products of the form:

$$(\text{large}\ \ \text{small})\begin{pmatrix}0&\vec{\sigma}\\ \vec{\sigma}&0\end{pmatrix}\begin{pmatrix}\text{small}\\ \text{large}\end{pmatrix}$$

$$=\ (\text{large}\,|\,\vec{\sigma}\,|\,\text{large})+(\text{small}\,|\,\vec{\sigma}\,|\,\text{small})$$

Looking back at the a's for the free electron and also the fact that this is the expectation value of the spin operator, we see that the above product is of order unity.

Thus for transitions involving negative energy states as intermediate states:

$$\frac{dP}{dt} \sim \rho_E (ME)^2 \sim k^2 \left[\frac{1 \cdot 1}{2} \; \frac{\hbar c}{\hbar v} \right]^2 \sim 1$$

Thus we arrive at the remarkable fact that for collisions involving soft photons and the Dirac electron, almost all scattering takes place through the negative energy intermediate states.

We now move on to complete the derivation of the Klein - Nishina Formula. This formula is good even for collisions with bound electrons if the photon energy is much higher than the electron binding energy. However, if the photon energy is around 1MEV and the target is a heavy atom, pair production can result as the nucleus will carry away some of the momentum and hence it will be conserved and absorption or emission can take place.

We wish to perform the sum over r in expression (A) for the total matrix element. Because of the orthonormality of the $a^{(r)}$'s this would be a trivial operation were it not for the denominators changing energy when one goes from $r=1,2$ to $r=3,4$. It is for this reason, then, that we define the projection operators:

$$\frac{\underline{H}(\vec{k}) + \sqrt{1+k^2}}{2\sqrt{1+k^2}} \; a^{(r)}(\vec{k}) = \begin{cases} a^{(r)}(\vec{k}) \; ; \; r=1,2 \\ 0 \; ; \; r=3,4 \end{cases} \qquad \underline{H}(\vec{k}) = \vec{\alpha}\cdot\vec{k} + \beta$$

$$H^2 = k^2 + 1$$

$$\frac{\underline{H}(\vec{k}) - \sqrt{1+k^2}}{-2\sqrt{1+k^2}} \; a^{(r)}(\vec{k}) = \begin{cases} 0 \; ; \; r=1,2 \\ a^{(r)}(\vec{k}) \; ; \; r=3,4 \end{cases}$$

These projection or selection operators enable us to split up up each term in (A) into a + energy and a - energy part but still remaining able to sum over all r in each part and thus taking advantage of the orthonormality condition on the $a^{(r)}$'s. These operators need only to be applied to one of the $a^{(r)}$'s at a time, but here we apply to both to show that the proper result is obtained anyway.

For the positive energy part of the first term in Ⓐ we have:

$$\frac{H^2 + 2H\sqrt{1+k^2} + 1+k^2}{4(1+k^2)\left[1+k-\sqrt{1+k^2}\right]} = \frac{1+(\vec{\alpha}\cdot\vec{k}+\beta)\frac{1}{\sqrt{1+k^2}}}{2\left[1+k-\sqrt{1+k^2}\right]}$$

Including the negative energy part gives:

$$\frac{1+(\vec{\alpha}\cdot\vec{k}+\beta)\frac{1}{\sqrt{1+k^2}}}{2\left[1+k-\sqrt{1+k^2}\right]} + \frac{1-(\vec{\alpha}\cdot\vec{k}+\beta)\frac{1}{\sqrt{1+k^2}}}{2\left[1+k+\sqrt{1+k^2}\right]} = \frac{(1+k)+(\vec{\alpha}\cdot\vec{k}+\beta)}{2k}$$

$$= \frac{k(1+\vec{\alpha}\cdot\hat{n})+1+\beta}{2k} \text{ ; using } \vec{k}=k\hat{n} \text{ ; } \vec{k}'=k'\hat{n}'$$

For the second term in Ⓐ just put $k \to -k'$ and we have altogether:

$$(ME) = \sum_{\text{intermediate}} \frac{(\text{final}\,|H_I|\,\text{initial})^2}{E_i - E_{\text{intermediate}}}$$

$$= \frac{2\pi e^2}{\sqrt{kk'}\,V}\, a^{*(\vec{\sigma}')}(\vec{k}-\vec{k}')\left\{ \vec{\alpha}\cdot\hat{e}'\, \frac{k(1+\vec{\alpha}\cdot\hat{n})+1+\beta}{2k}\, \vec{\alpha}\cdot\hat{e} \right.$$

$$\left. + \vec{\alpha}\cdot\hat{e}\, \frac{k'(1+\vec{\alpha}\cdot\hat{n}')-1-\beta}{2k'}\, \vec{\alpha}\cdot\hat{e}' \right\} a^{(\vec{\sigma})}(0)$$

We now note that $\beta(\vec{\alpha}\cdot\hat{e}) = -(\vec{\alpha}\cdot\hat{e})\beta$; $\beta a^{(\vec{\sigma})}(0) = a^{(\vec{\sigma})}(0)$, which results in the following simplification:

$$\boxed{\begin{aligned}(ME) = \frac{\pi e^2}{\sqrt{kk'}\,V}\, a^{*(\vec{\sigma}')}(\vec{k}-\vec{k}')\Big\{ &(\vec{\alpha}\cdot\hat{e}')(1+\vec{\alpha}\cdot\hat{n})(\vec{\alpha}\cdot\hat{e}) \\ &+ (\vec{\alpha}\cdot\hat{e})(1+\vec{\alpha}\cdot\hat{n}')(\vec{\alpha}\cdot\hat{e}')\Big\}\, a^{(\vec{\sigma})}(0)\end{aligned}}$$

What one now must do to get the differential cross-section is to square (ME), multiply by $\frac{2\pi}{1}\cdot\rho_E\cdot\frac{1}{1/V}$ (in natural units), sum over the scattered spins $\vec{\sigma}'$ and average over the initial spins $\vec{\sigma}$ as we lack information as to their exact direction.

We will have then found $\sigma(\hat{n}', \hat{e}')\, d\Omega_{\hat{n}'}$. Note that we have been using e in CGS, not HL, units. We now want to calculate $\rho_E(\text{final})\, dE$. E, although unprimed, refers to the energy of the final state.

$$\rho_E\, dE = \frac{V}{8\pi^3} k'^2 dk'\, d\Omega_{\hat{n}'} = \frac{V}{8\pi^3} k'^2\, d\Omega_{\hat{n}'} \frac{1}{\frac{dE}{dk'}}\, dE$$

The total final energy of the system can be written in two ways. One is to add together the values of the final energy of both both particles and another is to write the initial energy (by conservation):

$$E = \sqrt{1 + |\vec{k} - \vec{k}'|^2} + k' = \underbrace{\sqrt{1 + k^2 + k'^2 - 2kk'(\hat{n}\cdot\hat{n}')}}_{\text{electron}} + \underbrace{k'}_{\text{photon}}$$

or: $$E = 1 + k = \underbrace{1 + k - k'}_{\text{electron}} + \underbrace{k'}_{\text{photon}}$$

$1 + k - k' = \sqrt{1 + k^2 + k'^2 - 2kk'(\hat{n}\cdot\hat{n}')}$ as can be seen when the Compton relation is used, namely:

$$k - k' = kk'(1 - \hat{n}\cdot\hat{n}')$$

which makes $\sqrt{} = \sqrt{(1 + k - k')^2}$. Using all these relations:

$$\frac{dE}{dk'} = 1 + \frac{k' - k(\hat{n}\cdot\hat{n}')}{\sqrt{1 + |\vec{k} - \vec{k}'|^2}} = \frac{k}{k'(1 + k - k')}$$

upon using the Compton relation. This could also be written:

$$\frac{dE}{dk'} = \frac{k}{k' p_0'}\ ; \text{ where } p_0' = \sqrt{1 + |\vec{k} - \vec{k}'|^2} = \text{energy of final electron}$$

Thus we finally get:

$$\sigma(\hat{n}', \hat{e}')\, d\Omega_{\hat{n}'} = \frac{2\pi}{1} \cdot \frac{V}{8\pi^3} k'^2 \frac{k' p_0'}{k} d\Omega_{\hat{n}'} \cdot V \cdot \frac{\pi^2 e^4}{kk' V^2} \cdot \frac{1}{2} \sum_{\zeta} \sum_{\zeta'} |a^* \{\cdots\} a|^2$$

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We have arrived at:

$$\boxed{\begin{aligned}\sigma(\hat{n}',\hat{e}')\,d\Omega_{\hat{n}'} &= \frac{e^4}{8}\frac{k'^2}{k^2}\,p_0'\,d\Omega_{\hat{n}'}\sum_{\vec{\sigma}}\sum_{\vec{\sigma}'}\\ &\left|a^{*(\vec{\sigma}')}(\vec{p}')\left\{(\vec{\alpha}\cdot\hat{e}')(1+\vec{\alpha}\cdot\hat{n})(\vec{\alpha}\cdot\hat{e})+(\vec{\alpha}\cdot\hat{e})(1+\vec{\alpha}\cdot\hat{n}')(\vec{\alpha}\cdot\hat{e}')\right\}a^{(\vec{\sigma})}(0)\right|^2\end{aligned}}$$

The spin summation would be easy to do if we were going over both + and - energy initial and final states. However we have stipulated that $\vec{\sigma},\vec{\sigma}'$ belong to + energy states so that we again find use for the selection operators. We want to choose them such that we can sum over all states while just picking out non-zero values for the + energy states. This enables us to use the orthonormality of the $a^{(\tau)}$'s. Here we have:

$$\frac{1+\beta}{2}\,a^{(\tau)}(0) = \begin{cases} a^{(\tau)}(0) & ;\ \tau=\vec{\sigma}_0\ ,\ +\text{ energy}\\ 0 & ;\ \tau=3,4\ ,\ -\text{ energy}\end{cases}$$

$$\frac{p_0'+\vec{\alpha}\cdot\vec{p}'+\beta}{2p_0'}\,a^{(\tau')}(\vec{p}') = \begin{cases} a^{(\tau')}(\vec{p}') & ;\ \tau'=\vec{\sigma}'\ ,\ +\\ 0 & ;\ \tau'=3,4\ ,\ -\end{cases}$$

We use: $\quad p_0' = 1+k-k'\ ;\quad \vec{p}' = \vec{k}-\vec{k}'\ ;\ \vec{\alpha}\cdot\vec{p}' = k\,\vec{\alpha}\cdot\hat{n}-k'\,\vec{\alpha}\cdot\hat{n}'$

Then:

$$\frac{p_0'+\vec{\alpha}\cdot\vec{p}+\beta}{2p_0'} = \frac{\left[k\,(1+\vec{\alpha}\cdot\hat{n})-k'\,(1+\vec{\alpha}\cdot\hat{n}')+1+\beta\right]}{2p_0'}$$

We can write the $|\ \ |^2$ term in $\sigma(\hat{n}',\hat{e}')$ as:

$$\begin{aligned}|\ \ |^2 &= \left[a^*(\vec{p}')\{\ \}a(0)\right]\left[a^*(\vec{p}')\{\ \}a(0)\right]^\dagger\\ &= a^*(\vec{p}')\{\ \}a(0)\,a^*(0)\,\{\ \}^\dagger\,a(\vec{p}')\end{aligned}$$

We now put in the proper selection operators and replace $\vec{\sigma}$'s by τ's and write in matrix element form.

We have:

$$\sum_{\vec{\sigma}} \sum_{\vec{\sigma}'} |\quad|^2 = \sum_{\tau,\tau'} \sum_{j\ell n s} \left[a_j^{*(\tau')}(\vec{p}') \left(\{\ \} \frac{1+\beta}{2} \right)_{j\ell} a_\ell^{(\tau)}(0)\, a_n^{*(\tau)}(0) \left(\{\ \}^+ \frac{[\cdots]}{2p_0'} \right)_{ns} a_s^{(\tau')}(\vec{p}') \right]$$

The orthonormality of the a's give us $\delta_{\ell n}$ and δ_{js} thus resulting in the trace of [ ].

$$\sum_{\vec{\sigma}} \sum_{\vec{\sigma}'} |\quad|^2 = \frac{1}{4p_0'} \operatorname{Tr} \left[\{\ \}(1+\beta)\{\ \}^+ [\cdots] \right]$$

$$= \frac{1}{4p_0'} \operatorname{Tr} \left[[\cdots] \{\ \} (1+\beta) \{\ \}^+ \right]$$

The expression for the differential cross-section then becomes:

$$\boxed{\begin{aligned} \sigma(\hat{n}',\hat{e}')\, d\Omega_{\hat{n}'} = \frac{e^4}{8} \frac{k'^2}{k^2} d\Omega_{\hat{n}'} \cdot \frac{1}{4} \operatorname{Tr} \Big[& \underbrace{\{ k(1+\vec{\alpha}\cdot\hat{n}) - k'(1+\vec{\alpha}\cdot\hat{n}') + 1 + \beta \}}_{\{\mathrm{I}\}} \\ & \cdot \underbrace{\{ (\vec{\alpha}\cdot\hat{e}')(1+\vec{\alpha}\cdot\hat{n})(\vec{\alpha}\cdot\hat{e}) + (\vec{\alpha}\cdot\hat{e})(1+\vec{\alpha}\cdot\hat{n}')(\vec{\alpha}\cdot\hat{e}') \}}_{\{\mathrm{II}\}} \\ & \cdot (1+\beta) \underbrace{\{ (\vec{\alpha}\cdot\hat{e})(1+\vec{\alpha}\cdot\hat{n})(\vec{\alpha}\cdot\hat{e}') + (\vec{\alpha}\cdot\hat{e}')(1+\vec{\alpha}\cdot\hat{n}')(\vec{\alpha}\cdot\hat{e}) \}}_{\{\mathrm{III}\}} \Big] \end{aligned}}$$

We now develop some convenient commutation rules:

$$(\vec{\alpha}\cdot\vec{a})(\vec{\alpha}\cdot\vec{b}) = \alpha_k a_k \alpha_\ell b_\ell = a_k b_\ell \alpha_k \alpha_\ell$$

$$= a_k b_\ell \left[\delta_{k\ell} + i \sigma_m \epsilon_{k\ell m} \right] = (\vec{a}\cdot\vec{b}) + i\vec{\sigma}\cdot[\vec{a}\times\vec{b}]$$

$$(\vec{\alpha}\cdot\vec{b})(\vec{\alpha}\cdot\vec{a}) = (\vec{a}\cdot\vec{b}) - i\vec{\sigma}\cdot[\vec{a}\times\vec{b}]$$

Hence:

$$\boxed{(\vec{\alpha}\cdot\vec{a})(\vec{\alpha}\cdot\vec{b}) + (\vec{\alpha}\cdot\vec{b})(\vec{\alpha}\cdot\vec{a}) = 2(\vec{a}\cdot\vec{b})}$$

In the notation to be used in what follows, the $\rightarrow$ will be dropped and $\vec{\alpha}$ will be implicit in $\hat{\sigma}, \hat{\pi}$.

Let us first consider the term:

$$\frac{1}{4} \operatorname{Tr}\left[(1+\beta)\{\mathrm{II}\}(1+\beta)\{\mathrm{III}\}\right]$$

with:

$$\{\mathrm{II}\} = \underbrace{\{(e')(e) + (e)(e')\}}_{2\,e\cdot e'} + \underbrace{\{\overset{①}{(e')(n)(e)} + \overset{②}{(e)(n')(e')}\}}_{Ⓐ}$$

$$\{\mathrm{III}\} = \underbrace{\{(e)(e') + (e')(e)\}}_{2\,e\cdot e'} + \underbrace{\{\overset{①^+}{(e)(n)(e')} + \overset{②^+}{(e')(n')(e)}\}}_{Ⓐ^+}$$

Consider first the β cross-terms:

$$\frac{1}{4} \operatorname{Tr}\left[\beta\,(2e\cdot e')(2e\cdot e')\right] = \frac{1}{4} \operatorname{Tr}\left[(2e\cdot e')\,\beta\,(2e\cdot e')\right]$$

$$= (e\cdot e')^2 \operatorname{Tr}\beta = 0$$ since $\operatorname{Tr}\beta = 0$ in any representation.

$$\frac{1}{4} \operatorname{Tr}\left[\beta\,(2e\cdot e')\,Ⓐ^+\right] = \frac{1}{4} \operatorname{Tr}\left[(2e\cdot e')\,\beta\,Ⓐ^+\right] = 0$$

because $\beta Ⓐ^+ = -Ⓐ^+\beta$ but the trace must remain invariant giving quantities equal to their negatives which can only be zero. The same thing happens for the terms in $\beta Ⓐ$. Hence the trace of all the β cross-terms vanish. Hence we have left:

$$\frac{1}{4} \operatorname{Tr}\left[(1+\beta)\{\mathrm{II}\}(1+\beta)\{\mathrm{III}\}\right] = \frac{1}{4} \operatorname{Tr}\left[\{\mathrm{II}\}\{\mathrm{III}\} + \beta\{\mathrm{II}\}\beta\{\mathrm{III}\}\right]$$

$$= \frac{1}{4} \operatorname{Tr}\Big[4(e\cdot e')^2 + 2(e\cdot e')Ⓐ^+ + 2(e\cdot e')Ⓐ + ⒶⒶ^+$$
$$+ 4(e\cdot e')^2 + 2(e\cdot e')Ⓐ^+ - 2(e\cdot e')Ⓐ - ⒶⒶ^+\Big]$$

Now $Ⓐ^+$ is comprised of the products of three of the α_n, β whose products form a group. Now it is known that the trace of the elements of the group vanishes except for the unit matrix.

now it is impossible to form the unit matrix out of the product of three of the α_i, β. Since the product of these three form an element of the group, $Tr\, Ⓐ^{+} = 0$. Knowing that $Tr\, 1 = 4$, we finally have:

$$\boxed{\frac{1}{4} Tr\left[(1+\beta)\{II\}(1+\beta)\{III\}\right] = 8\,(e\cdot e')^2}$$

We must now calculate the terms:

$$\frac{1}{4} Tr\left[k\,(1+\alpha\cdot n)\{II\}(1+\beta)\{III\}\right]$$

and: $\frac{1}{4} Tr\left[-k'\,(1+\alpha\cdot n')\{II\}(1+\beta)\{III\}\right]$

These two terms give the same result except that in the second the primes will be switched and the sign reversed. The terms with $\{II\}\beta\{III\}$ vanish as shown, while the terms with $(\alpha\cdot n)\{II\}\beta\{III\}$ vanish because all of the terms contain products of α_i and β which can never give the unit matrix and hence their trace vanishes, thus we need only consider:

$$\frac{1}{4} Tr\left[k\,(1+\alpha\cdot n)\{II\}\{III\}\right]$$

We disregard the k untill the end. Note that the traces of an odd number of α_k's or σ_k's always vanish so we need only ourselves with even products. Consider first:

$$\frac{1}{4} Tr\left[\{II\}\{III\}\right] = 4(e\cdot e')^2 + \frac{1}{4} Tr\left[Ⓐ Ⓐ^{+}\right]$$

$$\frac{1}{4} Tr\left[Ⓐ Ⓐ^{+}\right] = \frac{1}{4} Tr\left[①①^{+} + ①②^{+} + ②①^{+} + ②②^{+}\right]$$

$$= 2 + \frac{1}{2} Tr\left[①②^{+}\right]$$

because: $①①^{+} = (e')(n)(e)(e)(n)(e') = 1 = ②②^{+}$

and $①②^{+} = (②①^{+})^{+}$ so $Tr\left[①②^{+}\right] = Tr\left[②①^{+}\right]$

Now: $\mathrm{Tr}\left[① ②^+ \right] = \mathrm{Tr}\left\{ (e')(n)(e)\underbrace{(e')(n')}_{-(n')(e')}(e) \right]$

$$= - \mathrm{Tr}\left[\underbrace{(e)(e')}_{-(e')(e)\ +\ 2(e\cdot e')}(n)(e)(n')(e') \right] = -2(e\cdot e')\ \mathrm{Tr}\left[(n)(e)(n')(e') \right]$$

$$+ \underbrace{\mathrm{Tr}\left[(e')(e)(n)(e)(n')(e') \right]}$$

$$\mathrm{Tr}\left\{ (e)\underbrace{(n)(e)}_{-(n)(e)}(n') \right] = - \mathrm{Tr}\left[(n)(n') \right] = -4\,(n\cdot n')$$

Therefore:

$$\frac{1}{4}\mathrm{Tr}\left\{ \{\mathrm{II}\}\{\mathrm{III}\} \right\} = 4(e\cdot e')^2 + 2 - 2(n\cdot n') - (e\cdot e')\ \mathrm{Tr}\left[(n)(e)(n')(e') \right]$$

Now consider:

$$\frac{1}{4}\mathrm{Tr}\left[(n)\{\mathrm{II}\}\{\mathrm{III}\} \right] = \frac{(e\cdot e')}{2}\mathrm{Tr}\left[(n)\,Ⓐ^+ + (n)\,Ⓐ \right]$$

$$= \frac{(e\cdot e')}{2}\mathrm{Tr}\left[(n)(e')(n)(e) + (n)(e)(n')(e') + (n)(e)(n)(e') + (n)(e')(n')(e) \right]$$

Then:

$$\mathrm{Tr}\left\{ (n)(e')\underbrace{(n)(e)}_{-(e)(n)} \right] = -4(e\cdot e')$$

$$\mathrm{Tr}\left[\underbrace{(n)(e)}_{-(e)(n)}(n)(e') \right] = -4(e\cdot e')$$

$$\mathrm{Tr}\left[\underbrace{(n)(e)}_{-(e)(n)}\underbrace{(n')(e')}_{-(e')(n')} \right] = \mathrm{Tr}\left[(e)(n)(e')(n') \right] = \mathrm{Tr}\left[(n)(e')(n')(e) \right]$$

Hence:

$$\frac{1}{4}\mathrm{Tr}\left[(n)\{\mathrm{II}\}\{\mathrm{III}\} \right] = -4(e\cdot e')^2 + (e\cdot e')\ \mathrm{Tr}\left\{ (n)(e)(n')(e') \right]$$

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We have obtained:

$$\boxed{\frac{1}{4}\,\mathrm{Tr}\left[k(1+\alpha\cdot n)\{\mathrm{II}\}\{\mathrm{III}\}\right] = 2k(1-n\cdot n')}$$

Finally we have:

$$\boxed{\frac{1}{4}\,\mathrm{Tr}\left[\{\mathrm{I}\}\{\mathrm{II}\}(1+\beta)\{\mathrm{III}\}\right] = 8(e\cdot e')^2 + 2k(1-n\cdot n') - 2k'(1-n\cdot n')}$$

Threading everything together, we find for the cross-section (differential); re-introducing CGS units:

$$\sigma(\vec{k}',\hat{e}')\,d\Omega_{\vec{k}',\hat{e}'} = \frac{e^4}{m^2c^4}\left(\frac{k'}{k}\right)^2\left[(\hat{e}\cdot\hat{e}')^2 + \frac{1}{4}(k-k')(1-\hat{n}\cdot\hat{n}')\right]d\Omega_{\vec{k}'}$$

We use the Compton result (conservation law):

$$k - k' = \lambda k'(1-\hat{n}\cdot\hat{n}')$$

Then we have:

$$\frac{1}{4}(k-k')(1-\hat{n}\cdot\hat{n}') = \frac{1}{4}(k-k')\left(\frac{1}{k'}-\frac{1}{k}\right) = \frac{1}{4}\left(\frac{k}{k'}+\frac{k'}{k}\right)-\frac{1}{2}$$

and:

$$\boxed{\sigma(\vec{k}';\hat{e}')\,d\Omega_{\vec{k}'} = \frac{e^4}{m^2c^4}\left(\frac{k'}{k}\right)^2\left[(\hat{e}\cdot\hat{e}')^2 + \frac{1}{4}\left(\frac{k}{k'}+\frac{k'}{k}\right)-\frac{1}{2}\right]d\Omega_{\vec{k}'}}$$

If we go to the classical limit, $k \sim k'$, we have:

$$\sigma(\vec{k}',\hat{e}')\,d\Omega_{\vec{k}'} = \frac{e^4}{m^2c^4}(\hat{e}\cdot\hat{e}')^2$$ which is precisely our previous result. Now average over incident polarizations since they are uncertain:

$$\bar{\sigma}(\vec{k}';\hat{e}')\,d\Omega_{\vec{k}'} = \frac{e^4}{m^2c^4}\left(\frac{k'}{k}\right)^2\left[\frac{1}{2}\left(1-(\hat{n}\cdot\hat{e}')^2\right)+\frac{1}{4}\left(\frac{k}{k'}+\frac{k'}{k}\right)-\frac{1}{2}\right]$$

using the same relations as for the Thompson scattering case.

We now sum over the scattered polarizations $\hat{e}'$ and obtain:

$$\boxed{\bar{\sigma}(\vec{k}')\, d\Omega_{\vec{k}'} = \frac{e^2}{m^2c^4}\left(\frac{k'}{k}\right)^2 \left[\frac{1}{4}\left(\frac{k}{k'}+\frac{k'}{k}\right)-\frac{1}{2}\left(1-(\hat{n}\cdot\hat{n}')^2\right)\right] d\Omega_{\vec{k}'}}$$

We can now carry out the integration over $\Omega_{\vec{k}'}$. This is a long but trivial operation as long as one uses for the variable of integration k' and not $\cos\theta$. The change is made by using Compton's law for $(\hat{n}\cdot\hat{n}')$ to relate $\cos\theta$ and k'. We have:

$$\frac{1}{k'}-\frac{1}{k} = 1-\cos\theta \quad ; \quad \cos^2\theta = (\hat{n}\cdot\hat{n}')^2 = 1+2\left(\frac{1}{k}-\frac{1}{k'}\right)+\left(\frac{1}{k}+\frac{1}{k'}\right)^2$$

$$d(\cos\theta) = \frac{dk'}{k'^2} \quad ; \quad \int_{-1}^{+1} \rightarrow \int_{\left(\frac{2k+1}{k}\right)^{-1}}^{k}$$

The result is, for the total cross-section:

$$\boxed{\sigma = \frac{2\pi e^4}{m^2c^4}\left[\frac{1+k}{k^2}\left\{\frac{2(1+k)}{1+2k}-\frac{1}{k}\ln(1+2k)\right\}+\frac{1}{2k}\ln(1+2k) - \frac{1+3k}{(1+2k)^2}\right] \quad ; \quad k = \frac{h\nu}{mc^2}}$$

By carefully expanding the first $\ln(1+2k)$ out to the fourth power we see that as $k\to 0$, we have in the limit exactly the Thompson result:

$$\sigma = \frac{8\pi}{3}\frac{e^4}{m^2c^4}$$

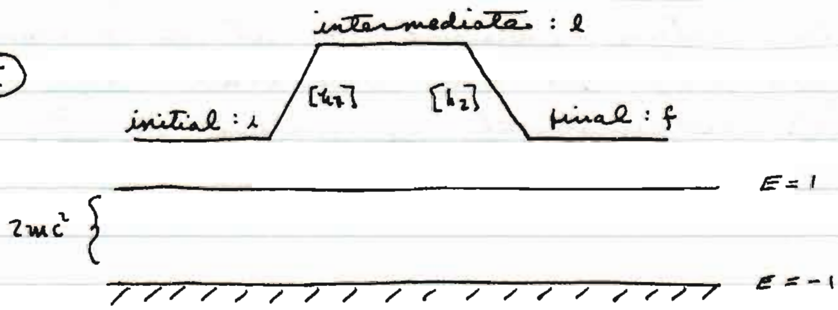
LECTURE 39 : 1-8-62

We now consider the physical implications of the negative energy intermediate states in the Compton scattering process.

Consider the following situations:

Positive Energy Intermediate States

I



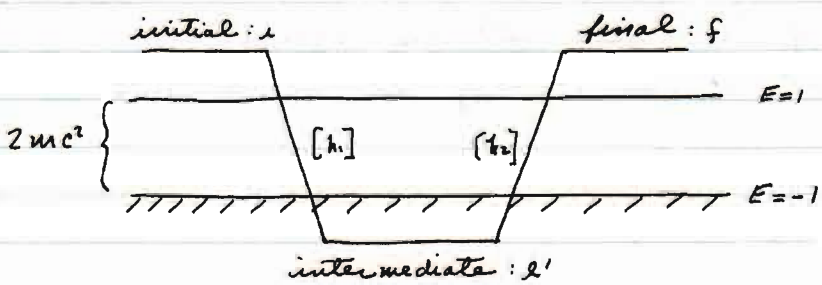
[k] denotes corresponding transitions in each intermediate state scheme.

This transition is physically allowed as the intermediate states are unoccupied.

Negative Energy Intermediate States

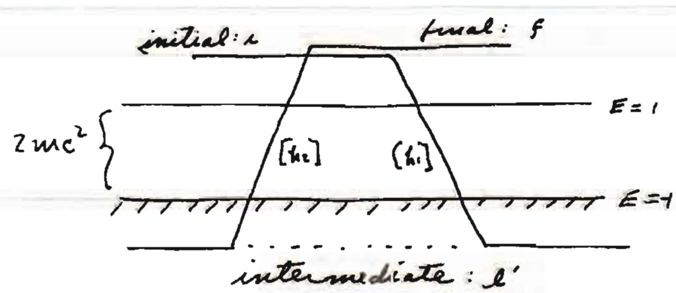
[k1] denotes absorption of k. [k2] denotes emission of k'

II



This transition must be physically disallowed as the intermediate state are filled and transition to them cannot occur.

III



The physical picture here is as follows: an electron in the negative energy state $(-p_0)l'$ jumps over into a positive energy state producing the final photon k', the final electron p_0' and a positron $(p_0)_{e'}$. This intermediate state still contains the initial photon and electron. The last transition is one where the initial photon and electron disappear by annihilating the positron. Energy need not be conserved between the initial and intermediate states but momentum must be.

We see then that process (III) is the one that is physically possible while (II) is not. However, in the development of the matrix element (A) it is process (II) that appears, not (III), so we had better hope that we can get (III) mathematically equivalent to (II). Examine the pertinent parts of the matrix element for (II) and (III):

Numerator in (II): $\left(u_f^* \vec{\alpha}\cdot\hat{e}_2\, u_{e'}\right)\left(u_{e'}^* \vec{\alpha}\cdot\hat{e}_1\, u_i\right)$

Numerator in (III): $\left(u_{e'}^* \vec{\alpha}\cdot\hat{e}_1\, u_e\right)\left(u_f^* \vec{\alpha}\cdot\hat{e}_2\, u_i\right)$

The 1, 2 refer to absorption, emission, respectively. Since ( ) are merely numbers and can be exchanged the numerators of (II) and (III) are equal.

We now look at the energy denominators of (II) and (III):

(II): $E_{initial} = E_{final}$; $p_0 + k = p_0' + k'$

$E_{initial} = p_0 + k$

$E_{intermediate} = (-p_0)_{e'}$ = (electron energy in $-E$ state)

$E_{initial} - E_{intermediate} = p_0 + k + (p_0)_{e'}$

(III) $E_{intermediate} = \underbrace{p_0 + k}_{\substack{\text{initial}\\ \text{photon}\\ \text{and}\\ \text{electron}}} + \underbrace{p_0' + k'}_{\substack{\text{final}\\ \text{photon}\\ \text{and}\\ \text{electron}}} + \underbrace{(p_0)_{e'}}_{\text{positron}}$

$E_{initial} - E_{intermediate} = E_{final} - E_{intermediate}$

$= -p_0 - k - (p_0)_{e'}$

Hence we see that the energy denominators of II and III are negatives of each other and that then the pertinent parts of the matrix element A are negatives of each other for II and III.

This would be disasterous where it not for the fact that III is an exchange process between two fermions which, because of the antisymmetric nature of the wave functions involved, changes the sign of the whole matrix element term representing III. That is, III is essentially an exchange between the initial positive energy electron and an electron in a negative energy state.

Process III in matrix element A

$$= - \frac{(u_{e'}^{*}\, \vec{\alpha}\cdot\hat{e}_1\, u_i)(u_f^{*}\, \vec{\alpha}\cdot\hat{e}_2\, u_i)}{-p_0 - k - (p_0)_{e'}}$$

which is mathematically, but not physically, identical to II.

Let us look into the details of the exchange process that changes the sign. We write product wave functions, properly antisymmeterized, where α, β denote electrons not taking part in the process. Although the number of electrons in the negative energy states would be infinite, we choose a very large number N.

Empty Intermediate State Picture II:

We consider this even though it is not physically possible. The appropriate wave functions are:

$$\psi_i = \frac{1}{\sqrt{N!}} \sum_P (-1)^{\sigma_P} P\, u_i(1)\, u_{e'}(2)\, u_\alpha(3)\, u_\beta(4) \cdots$$

$$\psi_{int} = \frac{1}{\sqrt{N!}} \sum_{P'} (-1)^{\sigma_{P'}} P'\, u_{e'}(1)\, u_{e'}(2)\, u_\alpha(3)\, u_\beta(4) \cdots$$

$$\psi_f = \frac{1}{\sqrt{N!}} \sum_{P''} (-1)^{\sigma_{P''}} P''\, u_f(1)\, u_{e'}(2)\, u_\alpha(3)\, u_\beta(4) \cdots$$

If we begin in ψ_i with a certain arrangement of the u functions before beginning the permutation, the initial arrangement is preserved throughout from ψ_i to ψ_{int} to ψ_f, that is, $P'' = P' = P$ and we can write:

$$(\psi_f^* \cdots \hat{e}_2 \cdots \psi_{int})(\psi_{int}^* \cdots \hat{e}_1 \cdots \psi_i) \sim (u_f^*(1) \cdots \hat{e}_2 \cdots u_{e'}(1))(u_{e'}^*(1) \cdots \hat{e}_1 \cdots u_i(1))$$

$$= (u_{e'}^*(1) \cdots \hat{e}_1 \cdots u_i(1))(u_f^*(1) \cdots \hat{e}_2 \cdots u_{e'}(1))$$

Pair Intermediate State Picture III:

The appropriate wave functions are:

$$\psi_i = \frac{1}{\sqrt{N!}} \sum_P (-1)^{\sigma_P} P \; u_i(1) \, u_{e'}(2) \, u_\alpha(3) \, u_\beta(4) \cdots$$

$$\psi_{int} = \frac{1}{\sqrt{N!}} \sum_{P'} (-1)^{\sigma_{P'}} P' \; u_i(1) \, u_f(2) \, u_\alpha(3) \, u_\beta(4) \cdots$$

$$\psi_f = \frac{1}{\sqrt{N!}} \sum_{P''} (-1)^{\sigma_{P''}} P'' \; u_f(1) \, u_{e'}(2) \, u_\alpha(3) \, u_\beta(4) \cdots$$

If we took $P'' = P' = P$, we would get:

$$(\psi_f^* \cdots \hat{e}_1 \cdots \psi_{int})(\psi_{int}^* \cdots \hat{e}_2 \cdots \psi_i) \sim (u_{e'}^*(2) \cdots \hat{e}_1 \cdots u_i(1))(u_f^*(2) \cdots \hat{e}_2 \cdots u_{e'}(2))$$

But this is incorrect so we must pick out the $P''(12)$ term. But this reverses the sign so we have:

$$(\psi_f^* \cdots \hat{e}_1 \cdots \psi_{int})(\psi_{int}^* \cdots \hat{e}_2 \cdots \psi_i) \sim -(u_{e'}^*(1) \cdots \hat{e}_1 \cdots u_i(1))(u_f^*(2) \cdots \hat{e}_2 \cdots u_{e'}(2))$$

which is the desired result so that the minus signs cancel and under the process III we have mathematically the same term as under process II.

Let us look at the same problem in the electron second quantitization scheme.

Empty Intermediate State Picture II:

$$\underbrace{C_f^* C_{e'} C_{e'}^* C_0}_{\text{physically untrue}} = C_f^* C_0 C_{e'} C_{e'}^* = C_f^* C_0 (1 - C_{e'}^* C_{e'}) = C_f^* C_0$$

as there is no electron initially in the intermediate state.

Pair Intermediate State Picture (III):

$$c_{e'}^{*}\, c_0\, c_f^{*}\, c_{e'} \;=\; c_0\, c_f^{*}\, c_{e'}^{*}\, c_{e'} \;=\; -c_f^{*}\, c_0\, c_{e'}^{*}\, c_{e'} \;=\; -c_f^{*}\, c_0$$

as there is an electron initially in the intermediate state so we again witness the change of sign under the exchange of fermions in this process.

Concluding Remarks

The modern way of treating quantum electrodynamics (visibly Lorentz invariant) does not bring in intermediate state wave functions or selection operators. It uses operators called "propagators", actually Green's functions. In this theory both energy and momentum are conserved in transitions to the intermediate states, but the relation between E and p is not the same for intermediate states.

Also in this theory, each particle and antiparticle has its own set of creation-destruction operators. For example, the correspondence between electrons and positrons is:

	Electrons		Positrons	
(destruction)	$c_{e'}$	$\longleftrightarrow$	$d_{e'}^{*}$	(creation)
(creation)	$c_{e'}^{*}$	$\longleftrightarrow$	$d_{e'}$	(destruction)

The ends the formal lectures.

Physics 251b, 1961
and P253

Material to supplement reading of Dirac, Chapter X, 'Theory of Radiation.'

The basic arguments are handled beautifully in Dirac, but both the notation and the emphasis in the statement of results are out of step with the great bulk of the literature.

Notation

The operator that Dirac calls η is commonly called $a^\dagger$, and his $\bar{\eta}$ is commonly called a ($b^\dagger$, b and $c^\dagger$, c are also often used). The boson commutation relations ((11) in Dirac) now become

$$a^\dagger_r\, a^\dagger_s - a^\dagger_s\, a^\dagger_r = 0$$

$$a_r\, a_s - a_s\, a_r = 0$$

$$a_r\, a^\dagger_s - a^\dagger_s\, a_r = \delta_{rs}$$

and the occupation-number observables are

$$N_r = a^\dagger_r\, a_r$$

(cf. (12),(13) in Dirac),

$$a_r\, a_r^\dagger = N_r + 1$$

For fermions these relations ((11')-(13') in Dirac) are:

$$a^\dagger_r\, a^\dagger_s + a^\dagger_s\, a^\dagger_r = 0$$

$$a_r\, a_s + a_s\, a_r = 0$$

$$a_r\, a^\dagger_s + a^\dagger_s\, a_r = 1$$

$$N_r = a^\dagger_r\, a_r$$

$$a_r\, a^\dagger_r = 1 - N_r$$

What Dirac calls the kets $\rangle_S$ (or $\rangle_A$ for fermions) are commonly called state-vectors Ψ or Φ; if arguments are specified for the state-vectors, they are most frequently the occupation numbers N'_r.

Form of the Results

Instead of Dirac's U in (22)-(29) we commonly see H_0, the

Hamiltonian without interaction, The V of (30)-(35) is the perturbing term for interactions between pairs of particles. (29) becomes

$$\mathcal{H}_o = \sum_{nm} H^{(o)}_{nm} \, a^{\dagger}_n \, a_m$$

and since

$$H^{(o)}_{nm} = \int u^*_n \, H_o \, u_m \, d\vec{r}$$

we have

$$\mathcal{H}_o = \int \psi^{\dagger} \, H_o \, \psi \, dr$$

where

$$\psi = \sum_m a_m \, u_m$$

is the quantized wave function, and

$$\psi^{\dagger} = \sum_n a^{\dagger}_n \, u^*_n$$

is its Hermitian adjoint. The basis functions u_m are ordinary 'c-numbers.' ψ and $\psi^{\dagger}$ get their operator character from the coefficients a_m and $a^{\dagger}_m$.

From the commutation relations for a_m and $a^{\dagger}_m$ we get those for ψ and $\psi^{\dagger}$: for bosons,

$$\psi^{\dagger}(\vec{r}) \, \psi^{\dagger}(\vec{r}\,') - \psi^{\dagger}(\vec{r}\,') \, \psi^{\dagger}(\vec{r}) = 0$$

$$\psi(\vec{r}) \, \psi(\vec{r}\,') - \psi(\vec{r}\,') \psi(\vec{r}) = 0$$

$$\psi(\vec{r}) \, \psi^{\dagger}(\vec{r}\,') - \psi^{\dagger}(\vec{r}\,') \, \psi(\vec{r}) = \delta(\vec{r}-\vec{r}\,')$$

For fermions the - signs in the left numbers are replaced by + signs. The proof of the last equation (with the δ function) is:

$$\text{left number} = \sum_{mn} u_m(\vec{r}) u^*_n(\vec{r}\,') \left(a_m \, a^{\dagger}_n - a^{\dagger}_n \, a_m \right)$$

Then

$$\text{left number} = \sum_{mn} u_m(\vec{r}) u^*_n(\vec{r}\,') \delta_{mn}$$

$$= \sum_n u_n(\vec{r}) u^*_n(\vec{r}\,')$$

$$= \delta(\vec{r}-\vec{r}\,')$$

by the completeness of the u_n.

For bosons there is no difference between (35) and (35'). Thus (35') holds for either bosons or fermions. In more usual notation, (35') is

$$\mathcal{V} = \sum_{nmsr} V_{ns;mr}\, a^{\dagger}_n\, a^{\dagger}_s\, a_r\, a_m$$

Since

$$V_{ns;mr} = \iint u^*_n(\vec{r})\, u_m(\vec{r})\, u^*_s(\vec{r}\,')u_r(\vec{r}\,')V(\vec{r},\vec{r}\,')\cdot d\vec{r}\, d\vec{r}\,'$$

we have

$$\mathcal{V} = \iint \Psi^{\dagger}(\vec{r})\Psi^{\dagger}(\vec{r}\,')V(\vec{r},\vec{r}\,')\Psi(\vec{r}\,')\Psi(\vec{r})d\vec{r}\, d\vec{r}\,'$$

The forms we have written for $\mathcal{H}_o$ and $\mathcal{V}$ are just what we would write for <u>expectation</u> <u>values</u> of the ordinary operators, except that the particular wave functions $u_m(\vec{r})$, etc. have been replaced by quantized (operator) wave functions $\Psi(\vec{r})$, etc. The quantum-mechanical convention that one uses u^*_s for final states, u_r for initial states, has been extended so that $\Psi^{\dagger}$ means appearance of a particle and Ψ its disappearance.

In the expressions we have written from Dirac, operators $\Psi^{\dagger}\ldots\Psi$ mean disappearance from one state and appearance in another - transition of a particle from one state to another. It is a characteristic of <u>relativistic</u> theories that the number of particles need not be constant. Light quanta, which are always relativistic (having zero rest mass) are continually being created (emitted) and destroyed (absorbed). Other kinds of particles can be created when enough energy is available. For example, high energy light quanta can produce electron-positron pairs; π mesons can disappear(decay) with the production of μ mesons and neutrinos; and so on. In the terms in the Hamiltonian that describe such processes, $\Psi^{\dagger}$ means <u>creation</u> of a particle (or, for fermions, <u>destruction</u> of the <u>antiparticle</u> - see last page of chapter in <u>4th edition</u>), and Ψ can mean <u>destruction</u> of a particle (or, for fermions, creation of the antiparticle).

Note that the order of the operators in our $\mathcal{V}$ on page 3 has been changed from the 'natural' order that would be found from

$$\overline{V} = \int \rho(\vec{r})V(\vec{r},\vec{r}\,')\rho(\vec{r}\,')d\vec{r}\, d\vec{r}\,'$$

by replacing ρ by $\Psi^{\dagger}\Psi$. This would give

$$\sum_{nmsr} V_{ns;mr}\, a^{\dagger}_n\, a_m\, a^{\dagger}_s\, a_r\ ,$$

which differs from our $\mathcal{V}$ by terms

$$\sum_{nmsr} V_{ns;mr}\, a^{\dagger}_n\, \delta_{ms} a_r = \sum_{nmr} V_{nm;mr}\, a^{\dagger}_n\, a_r$$

(this can be worked out from the relations on the a's and $a^{\dagger}$'s, for either the boson on the fermion case).

These terms give the non-vanishing expectation value

$$\sum_m V_{nm;mr}$$

for a state in which there is just one particle and it is in the state n. This is the <u>self-energy</u> of the particle given by the interaction V. The choice we have found for $\mathcal{V}$, from Dirac's (35'), avoids the appearance of self-energy by putting all creation operators on the left, so that the <u>two-particle</u> interaction can contribute only for states where there <u>are two</u> particles. The products of operators in $\mathcal{V}$ are examples of '<u>ordered products</u>' or '<u>normal products</u>,' with all creation operators on the left and destruction operators on the right (where they get first chance at the state function.) Such products play a special role in quantum field theories.

<u>Additional Argument on Meaning of Operator Ψ</u>

Consider the operator

$$N_V = \int_V \Psi^{\dagger}(\vec{r}\,')\Psi(\vec{r}\,')d\vec{r}\,'$$

where V is <u>any fixed volume</u>. Because $\Psi^{\dagger}$ is the Hermitian adjoint of Ψ, we have for any state Φ :

$$\overline{N}_V = (\Phi, N_V \Phi) = \int (\Phi, \Psi^{\dagger}(\vec{r}\,')\Psi(\vec{r}\,')\Phi)d\vec{r}\,'$$
$$= \int (\Psi(\vec{r}\,')\Phi, \Psi(\vec{r}\,')\Phi)d\vec{r}\,' \geqslant 0$$

N_V can't have negative exp. value, hence can't have negative eigenvalue.

Using <u>either</u> the boson <u>or</u> the fermion commutation relations, we find that

$$N_V \Psi(\vec{r}) - \Psi(\vec{r})N_V = -\int_V \delta(\vec{r}-\vec{r}\,')\Psi(\vec{r}\,')dr' = \begin{cases} -\Psi(\vec{r}), \vec{r} \text{ in } V \\ 0, \vec{r} \text{ not in } V \end{cases}$$

$$N_V \Psi^{\dagger}(\vec{r}) - \Psi^{\dagger}(\vec{r})N_V = \begin{cases} \Psi^{\dagger}(\vec{r}), \vec{r} \text{ in } V \\ 0, \vec{r} \text{ not in } V \end{cases}$$

$\therefore$ For $\vec{r}$ in V, $N_V \Psi(\vec{r}) = \Psi(\vec{r})(N_V - 1)$

and

$$N_V \Psi^\dagger(\vec{r}) = \Psi^\dagger(\vec{r})(N_V+1)$$

Then if Ψ is an eigenvector of N_V,

$$N_V \Psi = N_V' \Psi$$

we have (for $\vec{r}$ in V):

$$N_V \Psi(\vec{r})\Psi = \Psi(\vec{r})(N_V-1)\Psi = (N_V'-1)\Psi(\vec{r})\Psi$$

Thus if N_V' is an eigenvalue, then <u>either</u>

N_V' is an eigenvalue, with eigenvectors

$\Psi(\vec{r})\Psi$ for (one or more) $\vec{r}$'s in V

<u>or else</u> $\Psi(\vec{r})\Psi = 0$ for all $\vec{r}$ in V.
In the latter case, we get by multiplying by $\Psi^\dagger(\vec{r})$ and integrating:

$$N_V \Psi = 0, \text{ or } N_V' = 0.$$

From these facts, and the fact that no eigenvalue can be negative, we see that the possible eigenvalues of N_V are 0,1,2,......, whatever the size and shape of V.

The occurrence of $\delta(\vec{r}-\vec{r}')$ in the commutation relations corresponds to a quantum field theory with <u>point particles</u>. Every particle is either completely inside or completely outside any given volume.

Reading Period Assignment : P253

Mandl

Introduction to Quantum Field Theory

Chapters I - IV

Chapter I:

This chapter is devoted to providing a discription of the spinless meson field thru occupation number state vectors. The wave functions of the individual mesons are solution of the KG equation. The essential results are the definition of the destruction - creation operators:

$$a(k_\alpha)\,|n_1 \cdots n_\alpha \cdots\rangle = \sqrt{n_\alpha}\,|n_1 \cdots n_\alpha - 1, \cdots\rangle \qquad (1.10)$$

$$a^+(k_\alpha)\,|n_1 \cdots n_\alpha \cdots\rangle = \sqrt{n_\alpha + 1}\,|n_1 \cdots n_\alpha + 1 \cdots\rangle \qquad (1.11)$$

$$a^+(k_\alpha)\,a(k_\alpha)\,|n_1 \cdots n_\alpha \cdots\rangle = n_\alpha\,|n_1 \cdots n_\alpha \cdots\rangle \qquad (1.12)$$

$$\left.\begin{aligned} [a(k_\alpha), a(k_\beta)] &= [a^+(k_\alpha), a^+(k_\beta)] = 0 \\ [a(k_\alpha), a^+(k_\beta)] &= \delta(k_\alpha, k_\beta) \end{aligned}\right\} \qquad (1.13)$$

Problems:

1.1 $\quad a(k_\alpha)\,|n_1 \cdots n_\alpha \cdots\rangle = \sqrt{n_\alpha}\,|n_1 \cdots n_\alpha - 1 \cdots\rangle$

$$\langle n_1 \cdots n_\alpha - 1 \cdots |\, a(k_\alpha)\,| n_1 \cdots n_\alpha \cdots\rangle = \sqrt{n_\alpha}$$

$$\langle n_1 \cdots n_\alpha \cdots |\, a^+(k_\alpha)\,| n_1 \cdots n_\alpha - 1 \cdots\rangle = \sqrt{n_\alpha}$$

or: $\quad a^+(k_\alpha)\,|n_1 \cdots n_\alpha \cdots\rangle = \sqrt{n_\alpha + 1}\,|n_1 \cdots n_\alpha + 1 \cdots\rangle$

1.2 $\quad a^+(k_\alpha)\,a(k_\alpha)\,|n_1 \cdots n_\alpha \cdots\rangle = \sqrt{n_\alpha}\,a^+(k_\alpha)\,|n_1 \cdots n_\alpha - 1 \cdots\rangle$

$$= n_\alpha\,|n_1 \cdots n_\alpha \cdots\rangle$$

(1.3) $a(k_i)\, a^+(k_i)\, | n_1 \cdots n_i \cdots \rangle = \sqrt{n_i+1}\; a(k_i)\, | n_1 \cdots n_i + 1 \cdots \rangle$
$= (n_i+1)\, | n_1 \cdots n_i \cdots \rangle$

hence: $\left[a(k_i), a^+(k_i) \right] = 1$

$a^+(k_i)\, a(k_j)\, | n_1 \cdots n_i \cdots n_j \cdots \rangle = \sqrt{n_j(n_i+1)}\; | \cdots n_i+1, n_j - 1 \cdots \rangle$
$= a(k_j)\, a^+(k_j)\, | \cdots n_i \cdots n_j \cdots \rangle \quad ; \; i \neq j$

hence $\left[a(k_i), a^+(k_j) \right] = \delta(k_i, k_j)$

Chapter II:

We now devellop a classical field theory in the Hamiltonian form. Postulate the Lagrangian density:

$$\mathcal{L} = \mathcal{L}(\phi^\alpha, \phi^\alpha_{,\nu})$$

where ϕ^α is a field component (like $\mathcal{E}$ or $\mathcal{H}$) or refers to a particular field in a collection of fields (that is, boson or fermion fields, meson or nucleon fields, etc.)

$$\phi^\alpha_{,\nu} = \frac{\partial \phi^\alpha}{\partial x_\nu}$$

We work in relativistic 4-space. The action integral is:

$$I(\Omega) = \int d^4x\; \mathcal{L}(\phi^\alpha, \phi^\alpha_{,\nu}) \quad ; \quad d^4x = d^3x\, dt$$

We now vary the fields : $\phi^\alpha \to \phi^\alpha + \delta\phi^\alpha$ in the region Ω sothat the variation vanishes on the surface Γ of Ω, then:

$$\delta I(\Omega) = 0$$

$$\delta I(\Omega) = \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi^\alpha} \delta\phi^\alpha + \frac{\partial \mathcal{L}}{\partial \phi^\alpha_{,\nu}} \delta\phi^\alpha_{,\nu} \right\}$$

$$= \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi^\alpha} + \frac{\partial \mathcal{L}}{\partial \phi^\alpha_{,\nu}} \frac{\partial}{\partial x_\nu} \right\} \delta\phi^\alpha$$

$$= \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi^\alpha} - \frac{\partial}{\partial x_\nu} \frac{\partial \mathcal{L}}{\partial \phi^\alpha_{,\nu}} \right\} \delta\phi^\alpha + \underbrace{\int d^4x \frac{\partial}{\partial x_\nu} \left\{ \frac{\partial \mathcal{L}}{\partial \phi^\alpha_{,\nu}} \delta\phi^\alpha \right\}}_{\int_\Gamma dS \left\{ \frac{\partial \mathcal{L}}{\partial \phi^\alpha_{,\nu}} \delta\phi^\alpha \right\} = 0}$$

Hence we have the Euhler - Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial \phi^\alpha} - \frac{\partial}{\partial x_\nu} \frac{\partial \mathcal{L}}{\partial \phi^\alpha_{,\nu}} = 0$$

The Lagrangian density is invariant under Lorentz transformations. We want to go to the Hamiltonian form, but must reckon with the infinite # of degrees of freedom. We thus fix time and write the space as small units of volume $\delta \vec{x}^{(s)}$, s indexing each element. We can now count the variables describing the system and work toward the canonical conjugate variables of the Hamiltonian formalism.

Take: $q_s^\alpha = \phi^\alpha(s,t)$, $\alpha = 1, \cdots N$; $s = 1, 2, \cdots$

We can now define the Lagrangian in terms of the Lagrangian density by:

$$L(t) = \sum_s \delta\vec{x}^{(s)} \mathcal{L}^{(s)}$$

We then define the momentum p_s^α conjugate to q_s^α by:

$$p_s^\alpha = \frac{\partial L}{\partial \dot{q}_s^\alpha} = \frac{\partial L}{\partial \dot{\phi}^\alpha(s,t)} = \frac{\partial \mathcal{L}^{(s)}}{\partial \dot{\phi}^\alpha(s,t)} \delta\vec{x}^{(s)}$$

In view of going to the limit $\delta\vec{x}^{(s)} \to 0$, we define as conjugate to ϕ^α:

$$\pi^\alpha = \frac{\partial \mathcal{L}}{\partial \dot{\phi}^\alpha}$$

The Hamiltonian is:

$$H(t) = \sum_s p_s^\alpha \dot{q}_s^\alpha - L$$

or $$H = \sum_s \delta\vec{x}^{(s)} \left\{ \pi^\alpha(s,t)\, \dot{\phi}^\alpha(s,t) - \mathcal{L}^{(s)} \right\}$$

and going to the limit:

$$H(t) = \int d^3\vec{x}\ \mathcal{H}(\vec{x},t)$$

or $$\mathcal{H}(x) = \pi^\alpha \dot{\phi}^\alpha - \mathcal{L}(x)$$

Consider now, the $\mathcal{L}$:

$$\mathcal{L}(\phi, \phi_\nu) = -\frac{1}{2}\left\{ \phi_{,\nu}\, \phi_{,\nu} + m^2\phi^2 \right\}$$

which is the Lagrangian for spinless mesons of mass m.

Now:

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi} \quad \text{since in } \phi_{,\nu}\,,\ \nu = \vec{x}, ict\,, \text{ but } c = 1.$$

Finally:

$$\mathcal{H} = \frac{1}{2}\left\{ \pi^2 + \sum_i \left(\frac{\partial \phi}{\partial x_i}\right)^2 + m^2 \phi^2 \right\}$$

Problems

(2.1) $$\frac{\partial \mathcal{L}}{\partial \phi^\alpha} - \frac{\partial}{\partial x^\nu}\left(\frac{\partial \mathcal{L}}{\partial \phi^\alpha_{,\nu}}\right) = 0$$

$$\mathcal{L}(x) = -\frac{1}{2}\sum_{\mu=1}^{4}\sum_{\nu=1}^{4} \frac{\partial A_\mu}{\partial x_\nu}\frac{\partial A_\mu}{\partial x_\nu} = -\frac{1}{2}\frac{\partial A_{\mu'}}{\partial x_{\nu'}}\frac{\partial A_{\mu'}}{\partial x_{\nu'}}$$

(2.3) Show that $T_{\mu\nu} = -\phi^\alpha_{,\nu}\dfrac{\partial \mathcal{L}}{\partial \phi^\alpha_{;\mu}} + \mathcal{L}\,\delta_{\mu\nu}$,

$\mathcal{L} = \mathcal{L}(\phi^\alpha, \phi^\alpha_{,\nu})$ satisfies the equation

$$\frac{\partial T_{\mu\nu}}{\partial x_\mu} = 0$$

(2.4) Derive $T_{\mu\nu}$ for the meson field and show $-T_{44} = \mathcal{H}$

(2.2) Show that the substitution:

$$\mathcal{L}' = \mathcal{L} + \frac{\partial \Lambda_\nu}{\partial x_\nu}\,, \quad \Lambda = \Lambda(\phi)$$

does not alter the Euler-Lagrange equations.

(2.5) Define $P_\lambda(t) = -i\int d^3x\, T_{4\lambda}$. Show that $P_\lambda(t)$ is constant in time if $\phi(x) \to 0$ as $|x| \to \infty$ and interpret.

(2.6) Show that if $T_{\mu\nu}$ is symmetric, $\dfrac{\partial M_{\lambda\mu\nu}}{\partial x_\lambda} = 0$ where $M_{\lambda\mu\nu} = T_{\lambda\mu}x_\nu - T_{\lambda\nu}x_\mu$ and hence the angular momentum of the field is conserved.

Chapter III

Because ϕ and π of chapter II can be interpreted as canonically conjugate coordinates, we define commutation relations between them to quantize the field:

$$[\phi^\alpha(s,t),\ \phi^\beta(s',t)] = [\pi^\alpha(s,t),\ \pi^\beta(s',t)] = 0$$

$$[\phi^\alpha(s,t),\ \pi^\beta(s',t)] = i\,\delta_{\alpha\beta}\,\frac{\delta_{ss'}}{\delta \vec{x}^{(s)}}$$

In the limit:

$$[\phi^\alpha(\vec{x}),\ \phi^\beta(\vec{x}')] = [\pi^\alpha(\vec{x}),\ \pi^\beta(\vec{x}')] = 0$$

$$[\phi^\alpha(\vec{x}),\ \pi^\beta(\vec{x}')] = i\,\delta_{\alpha\beta}\,\delta(\vec{x}-\vec{x}')$$

since $\delta(\vec{x}-\vec{x}') = \lim\limits_{\delta\vec{x}^{(s)}\to 0} \dfrac{\delta_{ss'}}{\delta\vec{x}^{(s)}}$

We Fourier analyze the field operators $\phi(x)$ and $\pi(x)$ in the usual way:

$$\phi(x) = \frac{1}{\sqrt{V}} \sum_k \frac{1}{\sqrt{2k_0}} \left\{ a(\vec{k})\, e^{ikx} + a^*(\vec{k})\, e^{-ikx} \right\}$$

$$\pi(x) = \frac{i}{\sqrt{V}} \sum_k \sqrt{\frac{k_0}{2}} \left\{ -a(\vec{k})\, e^{ikx} + a^*(\vec{k})\, e^{-ikx} \right\}$$

where $kx = \vec{k}\cdot\vec{x} - k_0 t$ and $k_0 = \omega_{\vec{k}}$

In the Schroedinger picture ($t=0$):

$$\phi(\vec{x}) = \frac{1}{\sqrt{V}} \sum_k \frac{1}{\sqrt{2\omega_{\vec{k}}}} \left\{ a(\vec{k})\, e^{i\vec{k}\cdot\vec{x}} + a^\dagger(\vec{k})\, e^{-i\vec{k}\cdot\vec{x}} \right\}$$

$$\pi(\vec{x}) = \frac{i}{\sqrt{V}} \sum_k \sqrt{\frac{\omega_{\vec{k}}}{2}} \left\{ -a(\vec{k})\, e^{i\vec{k}\cdot\vec{x}} + a^\dagger(\vec{k})\, e^{-i\vec{k}\cdot\vec{x}} \right\}$$

From the commutation relations for ϕ, π, we find for the a's the destruction - creation operators of Chapter I.

$$[a(\vec{k}), a(\vec{k}')] = [a^+(\vec{k}), a^+(\vec{k}')] = 0$$
$$[a(\vec{k}), a^+(\vec{k}')] = \delta(\vec{k}, \vec{k}')$$

If we now use π and ϕ as above and plug these into the Hamiltonian of Ch. II and the Lagrangian for the spinless meson, we obtain:

$$H = \sum_k \left(a^+(\vec{k})\, a(\vec{k}) + \tfrac{1}{2}\right) \omega_{\vec{k}}$$

Although this leads to an infinite zero-point energy, it can be done away with by shifting the zero of the scale on which we measure energies which cannot lead to any physically observable consequences.

Problems

(3.1) Derive the commutation rules for the a's from those for ϕ, π.

(3.2) Derive $H = \sum_k \left(a^+(\vec{k})\, a(\vec{k}) + \frac{1}{2}\right) \omega_{\vec{k}}$

(3.3) If the energy-momentum tensor of the quantised meson field is given by $T_{\mu\nu} = \frac{1}{2}\left\{\frac{\partial\phi}{\partial x_\mu}\frac{\partial\phi}{\partial x_\nu} + \frac{\partial\phi}{\partial x_\nu}\frac{\partial\phi}{\partial x_\mu}\right\} + \mathcal{L}\,\delta_{\mu\nu}$, $\mathcal{L} = -\frac{1}{2}\left\{\phi_{,\nu}\,\phi_{,\nu} + m^2\phi^2\right\}$, show that the momentum of the field is: $\vec{P} = \sum_k a^+(\vec{k})\, a(\vec{k})\, \vec{k}$

Chapter IV

Consider the interaction of pions with nucleons, taking for the interaction Hamiltonian:

$$H_I = g\,\rho(\vec{x})\,\phi(\vec{x})$$

$\phi(\vec{x})$ is the quantized meson field, $\rho(\vec{x})$ is the nucleon density at point $\vec{x}$, g is characteristic of the strength of the interaction. This is analogous to:

$$\mathcal{H}_I = e\,S_\mu(x)\,A_\mu(x)$$

in the EM field case.

Take the nucleons as point particles of infinite mass, so that:

$$\rho(\vec{x}) = \sum_n \delta(\vec{x}-\vec{x}_n)$$

There is no recoil.

The interaction $\mathcal{H}_I$ is:

$$\mathcal{H}_I = \frac{g}{\sqrt{V}} \sum_k \sum_n \frac{1}{\sqrt{2\omega_k}} \left\{ a(\vec{k})\, e^{i\vec{k}\cdot\vec{x}_n} + a^+(\vec{k})\, e^{-i\vec{k}\cdot\vec{x}_n} \right\}$$

Now, the only non-zero matrix elements for single meson processes are:

$$\langle \cdots n_k+1 \cdots | \mathcal{H}_I | \cdots n_k \cdots \rangle = \frac{g}{\sqrt{V}} \sqrt{\frac{n_k+1}{2\omega_k}} \sum_n e^{-i\vec{k}\cdot\vec{x}_n}$$

$$\langle \cdots n_k-1 \cdots | \mathcal{H}_I | \cdots n_k \cdots \rangle = \frac{g}{\sqrt{V}} \sqrt{\frac{n_k}{2\omega_k}} \sum_n e^{i\vec{k}\cdot\vec{x}_n}$$

or $\langle \cdots n_k \cdots | \mathcal{H}_I | \cdots n_k+1 \cdots \rangle = \frac{g}{\sqrt{V}} \sqrt{\frac{n_k+1}{2\omega_k}} \sum_n e^{i\vec{k}\cdot\vec{x}_n}$

We are only quantizing the meson field, not the nucleon field so that the vacuum state has N nucleons present but no mesons.

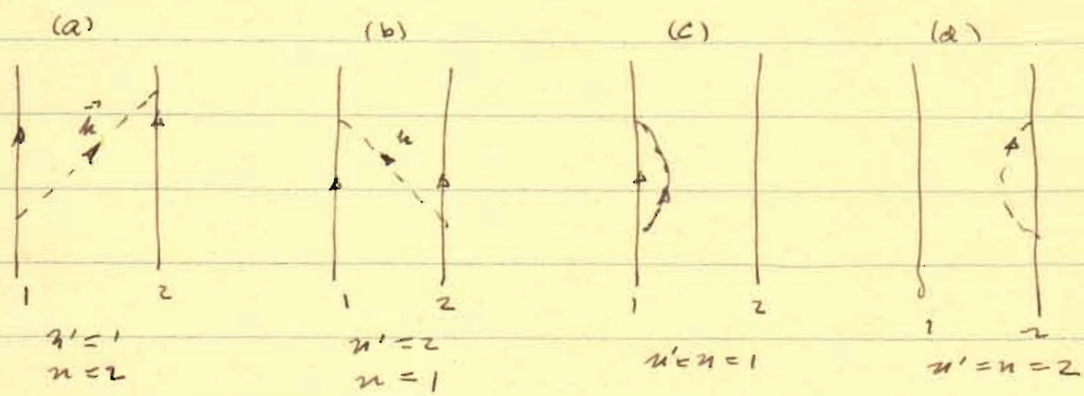
First Example: Nuclear-Force Problem

Calculate change in energy ΔE between two nucleons at $\vec{x}_1$, $\vec{x}_2$ due to their interaction through the meson field. The unperturbed problem in the vacuum state and in the first order we have: $\Delta E = \langle 0 | H_I | 0 \rangle = 0$
We hence must go to second order to get nonvanishing results. We have:

$$\Delta E = \sum_k \frac{\langle 0 | H_I | 1_k \rangle \langle 1_k | H_I | 0 \rangle}{-\omega_k}$$

$$= -\frac{g^2}{V} \sum_k \frac{1}{2\omega_k^2} \sum_{\substack{n n' \\ =1}}^{2} e^{i \vec{k}\cdot(\vec{x}_n - \vec{x}_{n'})}$$

Possible Interactions



(c) and (d) represent self-interaction effects, the interaction of a nucleon with its own meson field. We are dealing with bare nucleons without their meson clouds.

The self-meson field converts the bare nucleon into a physical nucleon.

These self-energy terms diverge in the limit $V \to \infty$ because then we have:

$$\Delta E = \frac{(2)^{-}g^2}{V} \sum_k \frac{1}{2\omega_k^2} \to \frac{-g^2}{(2\pi)^3} \int \frac{2\, d^3k}{2\omega_k^2}$$

$$= \frac{-g^2}{(2\pi)^3} \int \frac{d^3k}{m^2+k^2} = -\frac{g^2}{2\pi^2} \int_0^\infty \frac{x^2\,dx}{m^2+x^2}$$

$$= -\frac{g^2}{2\pi^2} \int_0^\infty \left\{ 1 - \frac{m^2}{m^2+x^2} \right\} dx$$

which we see diverges linearly. However, these divergent self-energy effects can be included in the properties of the physical nucleon thru the process of mass renormalization.

For the nucleon-nucleon, $n \neq n'$, interaction, we get:

$$\Delta E = -\frac{g^2}{V} \sum_k \frac{1}{2\omega_k^2} \left\{ e^{i\vec{k}\cdot\vec{x}} + e^{-i\vec{k}\cdot\vec{x}} \right\} \quad ; \quad \vec{x} = \vec{x}_1 - \vec{x}_2$$

$$= \frac{-g^2}{2(2\pi)^3} \int \frac{d^3k}{m^2+k^2} \left\{ e^{i\vec{k}\cdot\vec{x}} + e^{-i\vec{k}\cdot\vec{x}} \right\}$$

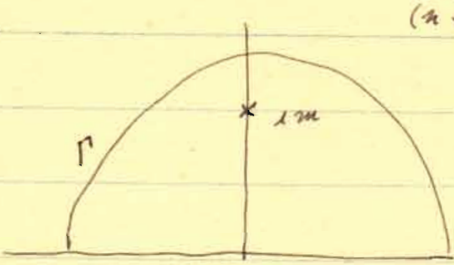
$$= \frac{-g^2 \cdot 2\pi}{(2\pi)^3} \int_{-1}^{1}\int_0^\infty \frac{\cos\{kr\cos\theta\}}{m^2+k^2}\, k^2\,dk\, d(\cos\theta)$$

$$= -\frac{g^2}{4\pi^2} \int_0^\infty \frac{k^2\,dk}{m^2+k^2} \cdot \underbrace{2\int_{-1}^{1} \cos kr\mu\, d\mu}_{\frac{2\cdot 1}{kr} \sin kr}$$

$$\Delta E = \frac{-g^2}{2\pi^2 r} \int_0^\infty \frac{k^3 dk \ \sin kr}{m^2+k^2}$$

$$= \frac{-g^2}{4\pi^2 r} \int_{-\infty}^\infty \frac{k \sin kr \ dk}{m^2+k^2}$$

$$\int_\Gamma \frac{z \cancel{\sin zr}\, e^{izr}\, dz}{m^2+k^2} = \int_{-\infty}^\infty \frac{k\, e^{ikr}\, dk}{\underbrace{m^2+k^2}_{(k+im)(k-im)}} = 2\pi i \sum (R)$$



$$R = \frac{e^{-mr}}{2}$$

$$\therefore \int_{-\infty}^\infty \frac{k \sin kr \, dk}{m^2+k^2} = \pi e^{-mr}$$

$$\Delta E = \frac{-g^2}{4\pi r} e^{-mr}$$

This is the Yukawa potential between two nucleons a distance r apart. It is very short range: of the order of Compton $\lambda \sim 1.4 \cdot 10^{-13}$ cm for pions. This potential does not reproduce the observed forces. We should have taken into account relativity, recoil and neutral pions. It is peculiar, however, that this second order result is exact which is peculiar to the spin zero and recoilless theory. The Hamiltonian can be rewritten by means of a canonical transformation as:

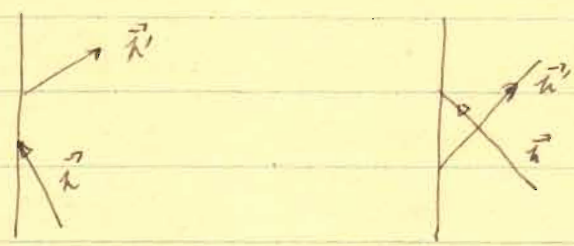
$$\sum_k \omega_k a^+(k) a(k) - g^2 e^{-mr}/4\pi r$$

Second Example: Pion-Nucleon Scattering

We consider scattering of a meson off of a fixed nucleon. The meson is scattered from the state $1_{\vec{k}}$ to the state $1_{\vec{k}'}$. Again the lowest order matrix element vanishes: $\langle 1_{\vec{k}'} | H_I | 1_{\vec{k}} \rangle = 0$
so we must go to intermediate states of which there are two possible:

(a) The no meson state: $| 0_{\vec{k}}, 0_{\vec{k}'} \rangle$

(b) The two meson state: $| 1_k, 1_{k'} \rangle$



(a) (b)

In (a), $\vec{k}$ is first absorbed and then $\vec{k}'$ emitted while in (b) $\vec{k}'$ is first emitted and then $\vec{k}$ absorbed. The second order matrix element is then:

$$M = \frac{\langle 1_{k'} | H_I | 0_k 0_{k'} \rangle \langle 0_k 0_{k'} | H_I | 1_k \rangle}{\omega_k - 0} + \frac{\langle 1_{k'} | H_I | 1_k 1_{k'} \rangle \langle 1_k 1_{k'} | H_I | 1_k \rangle}{\omega_k - 2\omega_k}$$

ω_k is conserved since there is no energy transfer to a recoilless nucleon.

$$\langle 1_n \, 1_{k'} | H_I | 0_k \, 0_{k'} \rangle \langle 0_k \, 0_{k'} | H_I | 1_k \rangle = \frac{g^2}{2V\omega_k} e^{i(\vec{k}-\vec{k}')\cdot\vec{x}}$$

$$\langle 1_{k'} | H_I | 1_k \, 1_{k'} \rangle \langle 1_k \, 1_{k'} | H_I | 1_k, 0_{k'} \rangle = \frac{g^2}{2V\omega_k} e^{i(\vec{k}-\vec{k}')\cdot\vec{x}}$$
0_k

Hence $M = 0$ to order g^2 and is in fact an exact result due to the neutral scalar no-recoil theory. The vanishing is due to the exact cancellation of (a) and (b) which would not occur under recoil conditions and/or charged mesons.

Note that the transformed Hamiltonian is the sum of two terms, one for mesons, one for nucleons, hence there is no interaction term and hence there cannot be any scattering.

Problems for Chapter II

(2.1) $$\frac{\partial \mathcal{L}}{\partial \phi^\alpha} - \frac{\partial}{\partial x_\nu}\left(\frac{\partial \mathcal{L}}{\partial \phi^\alpha_{,\nu}}\right) = 0$$

$$\mathcal{L}(x) = -\frac{1}{2}\sum_{\mu,\nu=1}^{4} \frac{\partial A_\mu}{\partial x_\nu}\frac{\partial A_\mu}{\partial x_\nu} = -\frac{1}{2}\frac{\partial A_{\mu'}}{\partial x_{\nu'}}\frac{\partial A_{\mu'}}{\partial x_{\nu'}}$$

$$\frac{\partial \mathcal{L}}{\partial \phi^\alpha} = -\frac{\partial A_\mu}{\partial x_\nu}\frac{\partial A_\mu}{\partial \phi^\alpha}$$

$$\frac{\partial \mathcal{L}}{\partial \phi^\alpha_{,\nu}} = -\frac{\partial A_\mu}{\partial x_\nu}\frac{\partial A_\mu}{\partial \phi^\alpha_{,\nu}}$$

$$\frac{\partial}{\partial x_\nu}\left(\frac{\partial \mathcal{L}}{\partial \phi^\alpha_{,\nu}}\right) = -\frac{\partial^2 A_\mu}{\partial x_\nu \partial x_\nu}\frac{\partial A_\mu}{\partial \phi^\alpha_{,\nu}} - \frac{\partial A_\mu}{\partial x_\nu}\underbrace{\frac{\partial}{\partial x_\nu}\left(\frac{\partial A_\mu}{\partial \phi^\alpha_{,\nu}}\right)}_{\frac{\partial A_\mu}{\partial \phi^\alpha}}$$

since $\partial \phi^\alpha_{,\nu} = \partial \frac{\partial \phi^\alpha}{\partial x_\nu}$

Therefore, if $\frac{\partial A_\mu}{\partial \phi^\alpha_{,\nu}} \neq 0$, we have maxwell's equations:

$$\boxed{\frac{\partial^2 A_\mu}{\partial x_\nu \partial x_\nu} = \Box^2 A_\mu = 0}$$

(2.2) $$I(\Omega) = \int d^4x\, \mathcal{L}(\phi^\alpha, \phi^\alpha_{,\nu}) \quad ; \quad \delta I(\Omega) = 0$$

Let $\mathcal{L}' = \mathcal{L} + \frac{\partial \Lambda_\nu}{\partial x_\nu}$, $\Omega = \Lambda(\phi^\alpha)\phi^\alpha_{,\nu})$

$$\delta I(\Omega) = \underbrace{\delta\int d^4x\, \mathcal{L}(\phi^\alpha, \phi^\alpha_{,\nu})}_{\text{gives Euler equations}} + \delta\int d^4x \frac{\partial \Lambda_\nu(\phi^\alpha)}{\partial x_\nu}$$

We then consider: $\delta \int d^4x \, \dfrac{\partial \Lambda_\nu(\phi^\alpha; \phi^\alpha_{,\nu})}{\partial x_\nu}$

$$= \int d^4x \left[\frac{\partial}{\partial \phi_\alpha} \left(\frac{\partial \Lambda_\nu}{\partial x_\nu} \right) \delta\phi^\alpha + \frac{\partial}{\partial \phi^\alpha_{,\nu}} \left(\frac{\partial \Lambda_\nu}{\partial x_\nu} \right) \delta \phi^\alpha_{,\nu} \right]$$

$$\frac{\partial}{\partial \phi^\alpha} \left(\frac{\partial \Lambda_\nu}{\partial x_\nu} \right) \delta\phi^\alpha = \frac{\partial}{\partial x_\nu} \left(\frac{\partial \Lambda_\nu}{\partial \phi^\alpha} \right) \delta\phi^\alpha$$

$$= \frac{\partial}{\partial x_\nu} \left\{ \left(\frac{\partial \Lambda_\nu}{\partial \phi^\alpha} \right) \delta\phi^\alpha \right\} - \left[\frac{\partial \Lambda_\nu}{\partial \phi^\alpha} \frac{\partial}{\partial x_\nu} \delta\phi^\alpha = \frac{\partial \Lambda_\nu}{\partial \phi^\alpha} \delta \phi^\alpha_{,\nu} \right.$$

$$\left. = \frac{\partial}{\partial x_\nu} \frac{\partial \Lambda_\nu}{\partial \phi^\alpha_{,\nu}} \delta \phi^\alpha_{,\nu} = \frac{\partial}{\partial \phi^\alpha_{,\nu}} \frac{\partial \Lambda_\nu}{\partial x_\nu} \delta \phi^\alpha_{,\nu} \right]$$

Hence $\delta \int d^4x \, \dfrac{\partial \Lambda_\nu}{\partial x_\nu} = \int d^4x \, \dfrac{\partial}{\partial x_\nu} \left\{ \left(\dfrac{\partial A_\nu}{\partial \phi^\alpha} \right) \delta\phi^\alpha \right\}$

$$= \int_S dS \left\{ \frac{\partial \Lambda_\nu}{\partial \phi^\alpha} \delta\phi^\alpha \right\} = 0$$

so the Euler-Lagrange equations are invariant under the above transformation of the Lagrangian.

(2.3) $\quad T_{\mu\nu} = -\phi^\alpha_{,\nu} \dfrac{\partial \mathcal{L}}{\partial \phi^\alpha_{,\mu}} + \mathcal{L}\,\delta_{\mu\nu} \; ; \quad \mathcal{L} = \mathcal{L}(\phi^\alpha; \phi^\alpha_{,\nu})$

$$\frac{\partial T_{\mu\nu}}{\partial x_\mu} = -\frac{\partial \phi^\alpha_{,\nu}}{\partial x_\mu} \frac{\partial \mathcal{L}}{\partial \phi^\alpha_{,\mu}} - \phi^\alpha_{,\nu} \frac{\partial}{\partial x_\mu} \left(\frac{\partial \mathcal{L}}{\partial \phi^\alpha_{,\mu}} \right) + \frac{\partial \mathcal{L}}{\partial x_\mu} \delta_{\mu\nu}$$

$$= -\frac{\partial^2 \phi^\alpha}{\partial x_\nu \partial x_\mu} \frac{\partial \mathcal{L}}{\partial \phi^\alpha_{,\mu}} - \frac{\partial \phi^\alpha}{\partial x_\mu} \frac{\partial \mathcal{L}}{\partial \phi^\alpha} + \frac{\partial \mathcal{L}}{\partial x_\mu} \delta_{\mu\nu}$$

$$\frac{\partial T_{\mu\nu}}{\partial x_\mu} = -\frac{\partial \phi^\alpha_{,\nu}}{\partial x_\mu}\frac{\partial \mathcal{L}}{\partial \phi^\alpha_{,\mu}} - \phi^\alpha_{,\nu}\left\{\frac{\partial^2 \mathcal{L}}{\partial\phi^\alpha \partial\phi^\alpha_{,\mu}}\frac{\partial \phi^\alpha}{\partial x_\mu} + \frac{\partial^2 \mathcal{L}}{\partial^{\cancel{2}}\phi^\alpha_{,\mu}\,\partial\phi^\alpha_{,\nu}}\frac{\partial \phi^\alpha_{,\nu}}{\partial x_\mu}\right\}$$

$$+\left\{\frac{\partial \mathcal{L}}{\partial \phi^\alpha}\frac{\partial \phi^\alpha}{\partial x_\mu} + \frac{\partial \mathcal{L}}{\partial \phi^\alpha_{,\nu}}\frac{\partial \phi^\alpha_{,\nu}}{\partial x_\mu}\right\}\delta_{\mu\nu}$$

$\mu = \nu$: $$-\phi^\alpha_{,\mu}\left\{\frac{\partial^2 \mathcal{L}}{\partial\phi^\alpha\partial\phi^\alpha_{,\mu}}\frac{\partial\phi^\alpha}{\partial x_\mu} + \frac{\partial^2\mathcal{L}}{\partial^2\phi^\alpha_{,\mu}}\frac{\partial\phi^\alpha_{,\mu}}{\partial x_\mu}\right\} + \frac{\partial\mathcal{L}}{\partial\phi^\alpha}\frac{\partial\phi^\alpha}{\partial x_\mu}$$

$$\frac{\partial\mathcal{L}}{\partial\phi^\alpha}\frac{\partial\phi^\alpha}{\partial x_\mu} = \left(\frac{\partial}{\partial x_\mu}\frac{\partial\mathcal{L}}{\partial\phi^\alpha_{,\mu}}\right)\frac{\partial\phi^\alpha}{\partial x_\mu}$$ (by Euler's equation)

$$= \left(\frac{\partial^2\mathcal{L}}{\partial\phi^\alpha\partial\phi^\alpha_{,\mu}}\frac{\partial\phi^\alpha}{\partial x_\mu} + \frac{\partial^2\mathcal{L}}{\partial^2\phi^\alpha_{,\mu}}\frac{\partial\phi^\alpha_{,\mu}}{\partial x_\mu}\right)\phi^\alpha_{,\mu}$$

So $\frac{\partial T_{\mu\nu}}{\partial x_\mu} = 0$ for $\mu = \nu$

For $\mu \neq \nu$: $$-\frac{\partial\phi^\alpha_{,\nu}}{\partial x_\mu}\frac{\partial\mathcal{L}}{\partial\phi^\alpha_{,\mu}} - \phi^\alpha_{,\nu}\left\{\frac{\partial^2\mathcal{L}}{\partial\phi^\alpha\partial\phi^\alpha_{,\mu}}\frac{\partial\phi^\alpha}{\partial x_\mu} + \frac{\partial^2\mathcal{L}}{\partial\phi^\alpha_{,\mu}\partial\phi^\alpha_{,\nu}}\frac{\partial\phi^\alpha_{,\nu}}{\partial x_\mu}\right\}$$

$$-\frac{\partial\phi^\alpha_{,\nu}}{\partial x_\mu}\frac{\partial\mathcal{L}}{\partial\phi^\alpha_{,\mu}} = -\frac{\partial}{\partial x_\mu}\left[\phi^\alpha_{,\nu}\frac{\partial\mathcal{L}}{\partial\phi^\alpha_{,\mu}}\right] + \phi^\alpha_{,\nu}\underbrace{\frac{\partial}{\partial x_\mu}\frac{\partial\mathcal{L}}{\partial\phi^\alpha_{,\mu}}}_{\frac{\partial\mathcal{L}}{\partial\phi^\alpha}}$$

$$\frac{\partial}{\partial x_\mu}\left[\phi^\alpha_{,\nu}\frac{\partial\mathcal{L}}{\partial\phi^\alpha_{,\mu}}\right] = \frac{\partial}{\partial\phi^\alpha}\left[\phi^\alpha_{,\nu}\frac{\partial\mathcal{L}}{\partial\phi^\alpha_{,\mu}}\right]\frac{\partial\phi^\alpha}{\partial x_\mu} + \frac{\partial}{\partial\phi^\alpha_{,\nu}}\left[\phi^\alpha_{,\nu}\frac{\partial\mathcal{L}}{\partial\phi^\alpha_{,\mu}}\right]\frac{\partial\phi^\alpha_{,\nu}}{\partial x_\mu}$$

Try:

$$\frac{\partial}{\partial x_\mu}\left\{\phi^\alpha_{,\nu}\frac{\partial\mathcal{L}}{\partial\phi^\alpha_{,\mu}}\right\} = \frac{\partial}{\partial\phi^\alpha}\left\{\phi^\alpha_{,\nu}\frac{\partial\mathcal{L}}{\partial\phi^\alpha_{,\mu}}\right\}\frac{\partial\phi^\alpha}{\partial x_\mu}$$

$$+\frac{\partial}{\partial\phi^\alpha_{,\nu}}\left\{\phi^\alpha_{,\nu}\frac{\partial\mathcal{L}}{\partial\phi^\alpha_{,\mu}}\right\}\frac{\partial\phi^\alpha_{,\nu}}{\partial x_\mu} = \phi^\alpha_{,\nu}\frac{\partial^2\mathcal{L}}{\partial\phi^\alpha\partial\phi^\alpha_{,\mu}}\frac{\partial\phi^\alpha}{\partial x_\mu} + \frac{\partial\mathcal{L}}{\partial\phi^\alpha_{,\mu}}\frac{\partial\phi^\alpha_{,\nu}}{\partial x_\mu}$$

$$+\phi^\alpha_{,\nu}\frac{\partial^2\mathcal{L}}{\partial\phi^\alpha_{,\nu}\partial\phi^\alpha_{,\mu}}\frac{\partial\phi^\alpha_{,\nu}}{\partial x_\mu}$$

$$\phi^\alpha_{,\nu}\left\{ \frac{\partial^2 \mathcal{L}}{\partial\phi^\alpha\, \partial\phi^\alpha_{,\mu}} \frac{\partial\phi^\alpha}{\partial x_\mu} + \frac{\partial^2\mathcal{L}}{\partial\phi^\alpha_{,\nu}\,\partial\phi^\alpha_{,\mu}} \frac{\partial\phi^\alpha_{,\nu}}{\partial x_\mu}\right\}$$

$$= \phi^\alpha_{,\nu}\left\{\frac{\partial}{\partial\phi^\alpha_{,\mu}}\frac{\partial\mathcal{L}}{\partial\phi^\alpha}\frac{\partial\phi^\alpha}{\partial x_\mu} + \cdots\right\}$$

$$= \phi^\alpha_{,\nu}\left\{\frac{\partial}{\partial\phi^\alpha_{,\mu}}\underbrace{\left(\frac{\partial}{\partial x_\nu}\frac{\partial\mathcal{L}}{\partial\phi^\alpha_{,\nu}}\right)}_{\frac{\partial^2\mathcal{L}}{\partial\phi^\alpha\,\partial\phi^\alpha_{,\nu}}\frac{\partial\phi^\alpha}{\partial x_\nu} + \frac{\partial^2\mathcal{L}}{\partial^2\phi^\alpha_{,\nu}}\frac{\partial\phi^\alpha_{,\nu}}{\partial x_\nu}}\frac{\partial\phi^\alpha}{\partial x_\mu}+\cdots\right\}$$

Wait! For $\mu\neq\nu$ $\quad \frac{\partial\mathcal{L}}{\partial\phi^\alpha_{,\mu}}=0$ since $\mathcal{L} = \mathcal{L}(\phi^\alpha,\phi^\alpha_{,\nu})$

$$\therefore \quad \frac{\partial T_{\mu\nu}}{\partial x_\mu}=0$$

(2.4) $\quad T_{\mu\nu} = -\phi^\alpha_{,\nu}\frac{\partial\mathcal{L}}{\partial\phi^\alpha_{,\mu}} + \mathcal{L}\,\delta_{\mu\nu} \quad ; \quad \mathcal{L} = \mathcal{L}(\phi^\alpha,\phi^\alpha_{,\nu})$

The Lagrangian for the one ~~particle~~ component meson field is:

$$\mathcal{L} = \mathcal{L}(\phi,\phi_{,\nu}) = -\frac{1}{2}\left\{\phi_{,\nu}\phi_{,\nu}+m^2\phi^2\right\}$$

$$\frac{\partial\mathcal{L}}{\partial\phi_{,\mu}} = \frac{\partial\mathcal{L}}{\partial\phi_{,\mu}} = -\phi_{,\mu}$$

Then:

$$\cancel{T_{\mu\nu} = \left\{(-\phi_{,\mu})(-\phi_{,\mu}) + \left(-\tfrac{1}{2}\right)\left\{\phi_{,\mu}\phi_{,\mu}+m^2\phi^2\right\}\right\}}$$

$$\boxed{\cancel{T_{\mu\nu} = \tfrac{1}{2}\left\{\phi_{,\mu}\phi_{,\mu} - m^2\phi^2\right\}}}$$

$$T_{\mu\nu} = \left[\phi_{,\mu}\phi_{,\mu} - \tfrac{1}{2}\left\{ \phi_{,\lambda}\phi_{,\lambda} + m^2\phi^2 \right\} \right] \delta_{\mu\nu}$$

$$T_{44} = -\dot{\phi}\dot{\phi} + \tfrac{1}{2}\dot{\phi}\dot{\phi} - \tfrac{1}{2}\phi_{,\mu}\phi_{,\mu} - \tfrac{1}{2}m^2\phi^2$$

$$-T_{44} = \tfrac{1}{2}\left\{ \pi\pi + \phi_{,\mu}\phi_{,\mu} + m^2\phi^2 \right\} = \mathcal{H}$$

(2.6) $$M_{\lambda\mu\nu} = T_{\lambda\mu}x_\nu - T_{\lambda\nu}x_\mu$$

$$\frac{\partial M_{\lambda\mu\nu}}{\partial x_\lambda} = \underbrace{\frac{\partial T_{\lambda\mu}}{\partial x_\lambda}}_{0} x_\nu + T_{\lambda\mu}\underbrace{\frac{\partial x_\nu}{\partial x_\lambda}}_{\delta_{\nu\lambda}} - \underbrace{\frac{\partial T_{\lambda\nu}}{\partial x_\lambda}}_{0} x_\mu - T_{\lambda\nu}\underbrace{\frac{\partial x_\mu}{\partial x_\lambda}}_{\delta_{\mu\lambda}}$$

$$= \left(\underset{\delta_{\nu\lambda}}{T_{\nu\mu}} - \underset{\delta_{\mu\lambda}}{T_{\mu\nu}} \right) = 0 \text{ if } T_{\mu\nu} \text{ is symmetric}$$

(2.5) $$P_\lambda(t) = -i\int d^3x\, T_{4\lambda}$$

Problems: Chapter III

(3.3) $$T_{\mu\nu} = \frac{1}{2}\left\{ \frac{\partial\phi}{\partial x_\mu}\frac{\partial\phi}{\partial x_\nu} + \frac{\partial\phi}{\partial x_\nu}\frac{\partial\phi}{\partial x_\mu} \right\} + \mathcal{L}\,\delta_{\mu\nu}$$

$$\mathcal{L} = -\frac{1}{2}\left\{ \phi_{,\lambda}\phi_{,\lambda} + m^2\phi^2 \right\}$$

$$P_\lambda = -i\int d^3x\, T_{4\lambda}$$

$$\phi(x) = \frac{1}{\sqrt{V}}\sum_k \frac{1}{\sqrt{2k_0}}\left\{ a(\vec{k})\,e^{ikx} + a^+ e^{-ikx} \right\}$$

$$\phi_{,\lambda}(x) = \frac{1}{\sqrt{V}}\sum_k \frac{ik_\lambda}{\sqrt{2k_0}}\left\{ a(\vec{k})e^{ikx} - a^+(\vec{k})\,e^{-ikx} \right\}$$

$$T_{4\lambda} = \frac{1}{2}\left\{ \phi_{,4}\phi_{,\lambda} + \phi_{,\lambda}\phi_{,4} \right\}$$

$$\phi_{,4} = \frac{-1}{\sqrt{V}}\sum_k \frac{k_0}{\sqrt{2k_0}}\left\{ a(\vec{k})\,e^{ikx} - a^+(\vec{k})\,e^{-ikx} \right\}$$

$\phi_{,4}\,\phi_{,\lambda}$, OK, will go thru.

$$\phi_{,4}\,\phi_{,\lambda} = -\frac{1}{V}\sum_{kk'}$$

Physics 253 Final Examination

February, 1960

① Exactly same problem as had for homework in P251B, that of showing spectroscopic stability.

② Write the Dirac equation in the $\vec{\alpha}$ form. Carry out the transformation to the γ form, obtaining equations for both ψ and $\bar{\psi}$. Derive the equation of continuity from the γ-form equations.

③ Find the relations that must be satisfied by the matrix S, in the relation $\psi = S\psi'$, to make the Dirac equation Lorentz invariant. Determine S for an infinitesimal velocity transformation, and for space inversion.

Using these results, find a two-rowed representation of the proper Lorentz group. Discuss the equation of the two-component neutrino.

④ For the case of a time-independent electromagnetic field described by the potentials $\vec{A}$, A_0, write the Dirac equation for the stationary states of a particle. Use the notation E = energy $- mc^2$, and carry out the Darwin reduction to the two-component approximate equation, neglecting the terms for change of mass with velocity. Comment on the meaning of each term.

⑤ Write the Dirac equations for ψ and $\bar{\psi}$ for a particle of charge e in an electromagnetic field. Substitute $C\bar{\psi} = \psi_c$, and find the conditions imposed on C so that ψ_c will satisfy the equation of a particle of charge $-e$. How do you know that a unitary matrix C satisfying these conditions exists? Prove that with the C so chosen we have also $\psi = C\bar{\psi}_c$.

The Relativistic Quantum Mechanical Isotropic Harmonic Oscillator

Dirac Equation: $\quad i\hbar \frac{\partial \psi}{\partial t} = H\psi$

where

$$H = -c\,\vec{\alpha}\cdot\vec{p} - \beta mc^2 + V$$

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \quad ; \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; ; \; \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \; ; \; \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Take as the solution:

$$\psi = e^{-i\frac{Et}{\hbar}} \begin{pmatrix} \varphi \\ \chi \end{pmatrix}$$

$$(E - mc^2 - V)\,\varphi = (\vec{\sigma}\cdot\vec{p})\,\chi$$
$$(E + mc^2 - V)\,\chi = (\vec{\sigma}\cdot\vec{p})\,\varphi$$

$$V = \frac{1}{2} m\omega^2 (x^2 + y^2 + z^2)$$

$$\left[E - mc^2 - \frac{1}{2}m\omega^2(x^2+y^2+z^2)\right]\varphi = -i\hbar\left(\sigma_x \frac{\partial}{\partial x} + \sigma_y \frac{\partial}{\partial y} + \sigma_z \frac{\partial}{\partial z}\right)\chi$$

$$\left[E + mc^2 - \frac{1}{2}m\omega^2(x^2+y^2+z^2)\right]\chi = -i\hbar\left(\sigma_x \frac{\partial}{\partial x} + \sigma_y \frac{\partial}{\partial y} + \sigma_z \frac{\partial}{\partial z}\right)\varphi$$

$$H = -i\hbar c\,\vec{\alpha}\cdot\nabla - \beta mc^2 + \tfrac{1}{2}m\omega^2(z^2+y^2+x^2)$$

$$\Psi = e^{-i\frac{E}{\hbar}t}\,\psi$$

$$i\hbar\frac{\partial\Psi}{\partial t} = H\Psi \quad ; \quad i\hbar\frac{\partial\Psi}{\partial t} = E\Psi = H\Psi$$

or: $H\psi = E\psi$

$$i\hbar c\left(\alpha_x\frac{\partial}{\partial x} + \alpha_y\frac{\partial}{\partial y} + \alpha_z\frac{\partial}{\partial z}\right)\psi - \left(E + \beta mc^2 - \tfrac{1}{2}m\omega^2\{x^2+y^2+z^2\}\right)\psi = 0$$

Let: $\psi(x,y,z) = \psi(x)\,\psi(y)\,\psi(z)$

Then:

$$i\hbar c\,\alpha_x\frac{\partial\psi(x)}{\partial x} - \left(E_x + \tfrac{1}{3}\beta mc^2 - \tfrac{1}{2}m\omega^2x^2\right)\psi(x) = 0$$

$$\vdots$$

$$E = E_x + E_y + E_z$$

now: $\psi(x) = \begin{pmatrix}\psi_1(x)\\ \psi_2(x)\\ \psi_3(x)\\ \psi_4(x)\end{pmatrix}$; $\alpha_x = \begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}$; $\beta = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}$

$$i\hbar c\frac{\partial\psi_4}{\partial x} - \left(E_x + \tfrac{1}{3}mc^2 - \tfrac{1}{2}m\omega^2x^2\right)\psi_1 = 0$$

$$i\hbar c\frac{\partial\psi_3}{\partial x} - \left(E_x + \tfrac{1}{3}mc^2 - \tfrac{1}{2}m\omega^2x^2\right)\psi_2 = 0$$

$$i\hbar c\frac{\partial\psi_2}{\partial x} - \left(E_x \cancel{+} \tfrac{1}{3}mc^2 - \tfrac{1}{2}m\omega^2x^2\right)\psi_3 = 0$$

$$i\hbar c\frac{\partial\psi_1}{\partial x} - \left(E_x - \tfrac{1}{3}mc^2 - \tfrac{1}{2}m\omega^2x^2\right)\psi_4 = 0$$

Set: $\quad \epsilon_+ = \dfrac{E_x + \frac{1}{3}mc^2}{i\hbar c} \quad ; \quad \epsilon_- = \dfrac{E_x - \frac{1}{3}mc^2}{i\hbar c}$

$$\gamma^2 = \frac{\frac{1}{2}m\omega^2}{i\hbar c}$$

$$\psi_4' - (\epsilon_+ - \gamma^2 x^2)\psi_1 = 0$$
$$\psi_3' - (\epsilon_+ - \gamma^2 x^2)\psi_2 = 0$$
$$\psi_2' - (\epsilon_- - \gamma^2 x^2)\psi_3 = 0$$
$$\psi_1' - (\epsilon_- - \gamma^2 x^2)\psi_4 = 0$$

Asymptotically: $\quad \psi_4' = -\gamma^2 x^2 \psi_1$

$$\psi_3' = -\gamma^2 x^2 \psi_2$$
$$\psi_2' = -\gamma^2 x^2 \psi_3$$
$$\psi_1' = -\gamma^2 x^2 \psi_4$$

$$\left.\begin{aligned} \psi_4'' &= -\gamma^2 x^2 \psi_1' = \gamma^4 x^4 \psi_4 \\ \psi_3'' &= \gamma^4 x^4 \psi_3 \\ \psi_2'' &= \gamma^4 x^4 \psi_2 \\ \psi_1'' &= \gamma^4 x^4 \psi_1 \end{aligned}\right\} \quad \begin{aligned} &\text{Try } \psi \sim e^{-\frac{\gamma^4 x^4}{4}} \\ &\psi' = -\gamma^4 x^3 e^{-\frac{\gamma^4 x^4}{4}} \\ &\psi'' = \text{no good} \end{aligned}$$

Try $\psi \sim \dfrac{e^{-\frac{\gamma^4 x^4}{4}}}{\gamma x} \quad ; \quad \psi' \sim -\gamma^3 x^2 e^{-\frac{\gamma^4 x^4}{4}}$

$\psi'' \sim$ no good

Try: $\psi \sim \dfrac{e^{-\frac{\gamma^4 x^4}{4}}}{\gamma^4 x^2} \quad ; \quad \psi' \sim -x e^{-\frac{\gamma^4 x^4}{4}} \quad ; \quad \psi'' \sim \gamma^4 x^4 e^{-\frac{\gamma^4 x^4}{4}}$

no good

Try: $\psi \sim e^{-\frac{\gamma^2 x^3}{3}} \quad . \quad \psi' \sim -\gamma^2 x^2 e^{-\frac{\gamma^2 x^3}{3}}$

$\psi'' \sim \gamma^4 x^4 e^{-\frac{\gamma^2 x^3}{3}}$ OK so: $\psi = e^{-\frac{\gamma^2 x^3}{3}} F$

$$\psi' = e^{-\frac{\gamma^2 x^3}{3}} F' - \gamma^2 x^2 e^{-\frac{\gamma^2 x^3}{3}} F$$

$F_4' - \gamma^2 x^2 F_4 - (\epsilon_+ - \gamma^2 x^2) F_1 = 0$; etc.

$$F = \sum_s C^{(s)} x^s$$

$$F' = \sum_s s\, C^{(s)} x^{s-1}$$

$$(s+1)\, C_4^{(s+1)} - \gamma^2 C_4^{(s-2)} - \epsilon_+ C_1^{(s)} + \gamma^2 C_1^{(s-2)} = 0$$

$$(s+1)\, C_3^{(s+1)} - \gamma^2 C_3^{(s-2)} - \epsilon_+ C_2^{(s)} + \gamma^2 C_2^{(s-2)} = 0$$

$$(s+1)\, C_2^{(s+1)} - \gamma^2 C_2^{(s-2)} - \epsilon_- C_3^{(s)} + \gamma^2 C_3^{(s-2)} = 0$$

$$(s+1)\, C_1^{(s+1)} - \gamma^2 C_1^{(s-2)} - \epsilon_- C_4^{(s)} + \gamma^2 C_4^{(s-2)} = 0$$

Asymptotically:

$\psi_4' = -\gamma^2 x^2 \psi_1$; $\psi_4'' = \gamma^4 x^4 \psi_4 - 2\gamma^2 x\, \psi_1$

$\psi_3' = -\gamma^2 x^2 \psi_2$ $\qquad = \gamma^4 x^4 \psi_4 + \frac{2}{x} \psi_4'$

$\psi_2' = -\gamma^2 x^2 \psi_3$

$\psi_1' = -\gamma^2 x^2 \psi_4$

Recap from lecture:

$$H\psi = i\hbar \frac{\partial \psi}{\partial t}$$

$$H = -c\,\vec{\alpha}\cdot\vec{p} - \beta mc^2 + V(r)$$

$$\psi = e^{\frac{-iEt}{\hbar}} \begin{pmatrix} \varphi \\ \chi \end{pmatrix}$$

$$(E - mc^2 - V)\,\varphi = (\vec{\sigma}\cdot\vec{p})\,\chi$$

$$(E + mc^2 - V)\,\chi = (\vec{\sigma}\cdot\vec{p})\,\varphi$$

$$\psi_{l+1/2,\,l,\,\mu+1/2} = \begin{pmatrix} Y_{l+1/2,\,l,\,\mu+1/2} \; f \\ Y_{l+1/2,\,l+1,\,\mu+1/2} \; g \end{pmatrix}$$

$$\psi_{l-1/2,\,l,\,\mu+1/2} = \begin{pmatrix} f \; Y_{l-1/2,\,l,\,\mu+1/2} \\ g \; Y_{l-1/2,\,l-1,\,\mu+1/2} \end{pmatrix}$$

$$\left(E - mc^2 - V(r)\right) f(r) = \frac{\hbar c}{i}\frac{dg}{dr} + (k-1)\frac{i\hbar c}{r}\,g$$

$$\left(E + mc^2 - V(r)\right) g(r) = \frac{\hbar c}{i}\frac{df}{dr} - (k+1)\frac{i\hbar c}{r}\,f$$

$k = 1, 2, 3, \dots$ for $j = l - 1/2$

$\;\;\, = -1, -2, -3, \dots$ for $j = l + 1/2$

$$V(r) = \tfrac{1}{2} m\omega^2 r^2$$

efine: $\frac{\hbar c}{E+mc^2} = a_1$; $\frac{\hbar c}{-E+mc^2} = a_2$; $\sqrt{a_1 a_2} = a = \frac{\hbar c}{\sqrt{(mc^2)^2 - E^2}}$

$$\gamma^2 = \frac{m\omega^2}{2\hbar c}$$

$$\left[-\frac{1}{a_2} - \gamma^2 r^2\right] f = -i \frac{dg}{dr} + i(k-1)\frac{g}{r}$$
$$\left[\frac{1}{a_1} - \gamma^2 r^2\right] g = -i \frac{df}{dr} - i(k+1)\frac{f}{r}$$

asymptotically: $\frac{dg}{dr} = -i\gamma^2 r^2 f$; $g'' = -\gamma^4 r^4 g$

$\frac{df}{dr} = -i\gamma^2 r^2 g$; $f'' = -\gamma^4 r^4 f$

$$g \sim f \sim e^{i\frac{\gamma^2 r^3}{3}}$$

Take $f = r^{-1} e^{i\frac{\gamma^2 r^3}{3}} F$

$g = -i r^{-1} e^{i\frac{\gamma^2 r^3}{3}} G$

$$f' = r^{-1} e^{i\frac{\gamma^2 r^3}{3}} F' - r^{-2} e^{i\frac{\gamma^2 r^3}{3}} F + i\gamma^2 r\, e^{i\frac{\gamma^2 r^3}{3}} F$$

$$\left[-\frac{1}{a_2} - \gamma^2 r^2\right] F = G' - \frac{G}{r} + i\gamma^2 r^2 G - \frac{kG}{r} + \frac{G}{r}$$

or: $\left(\frac{1}{a_2} + \gamma^2 r^2\right) F + \left[\frac{dG}{dr} - \frac{kG}{r} + i\gamma^2 r^2 G\right] = 0$

HARVARD UNIVERSITY

Physics 253

Answer FIVE questions

1. Let γ^a $(a = 1, \cdots 5)$ be a set of 4-rowed matrices which satisfy

$$\gamma^a\gamma^b + \gamma^b\gamma^a = 2\delta_{ab} .$$

Prove that Tr $\gamma^a = 0$ and Tr $\gamma^a\gamma^b = 0$, $a \neq b$.

How do we know that there exists a matrix K such that

$$K\gamma^a = \tilde{\gamma}^a K \qquad (a = 1, \cdots 5)?$$

Prove that the matrix K is either symmetric or antisymmetric. Find out which it is.

2. Write Dirac's "α-form" of the Dirac equation. Obtain from it the Pauli "γ-form" equations for the functions ψ and $\bar{\psi}$. Discuss the Lorentz invariance of these equations for transformations which do not reverse the time.

3. Carry through the Darwin derivation from the Dirac equation of the Pauli equation with approximate relativistic corrections, in the case $v/c \ll 1$ $(|E-mc^2| \ll mc^2)$.

4. Explain why "physical time reversal" (or "motion reversal", called "Wigner" time reversal) necessarily involves complex conjugation. Discuss this time reversal for the Schrodinger electron, in the coordinate representation and in the momentum representation. Discuss it also for the Pauli electron; explain what the Kramers degeneracy is, and prove its existence.

5. Quantize the transverse electromagnetic field (radiation field) in the gauge $\nabla \cdot \vec{A} = 0$, for periodic boundary conditions in a box of volume $V = L^3$. Use the condition that the total field energy

$$\int \tfrac{1}{2}(E^2 + H^2)\, d\vec{r}$$

must be equal to

$$\sum_s a_s^* a_s \hbar \omega_s + \text{const.}$$

to get a definite form for the Fourier expansion of $\vec{A}$. Explain the physical meanings of the terms in this expansion, including their relations to the Einstein A and B coefficients.

6. Write the equations of motion that follow from the variational principle

$$\delta \int \mathcal{L}\, d^4x = ic \int \mathcal{L}\, d\vec{r}dt = 0$$

where

$$\mathcal{L} = \mathcal{L}(\varphi^\alpha, \varphi^\alpha{}_{,\mu}), \quad \varphi^\alpha{}_{,\mu} \equiv \partial\varphi^\alpha/\partial x_\mu$$

Show that it is a consequence of these equations that the tensor

$$T_{\mu\nu} = -(\partial\mathcal{L}/\partial\varphi^\alpha{}_{,\mu})\varphi^\alpha{}_{,\nu} + \mathcal{L}\delta_{\mu\nu}$$

satisfies the equation

$$\partial T_{\mu\nu}/\partial x_\mu = 0$$

Show that if all field quantities vanish sufficiently rapidly at large (spatial) distances, it follows from the last equation that the vector

$$P_\lambda(t) = -i\int d\vec{r}\, T_{4\lambda}$$

is constant (independent of t).

Final, January 1962

P253 Course Outline

I. Relativistic Electron

① KG Equation

② KG Eq. of Cont.

③ Dirac Eq.
- α form
- γ form
- gauge invariance
- eq. of cont.

④ Lorentz Inv. of D. Eq.
- $\bar{\psi}$ form
- Find B
- Find S's

⑤ γ Representation
- S
- Dirac
- Weyl
- neutrino
- Pauli Inv. Quantities

⑥ Conservation of Angular Mom.

⑦ Pauli Electron Equation

⑧ Coulomb Potential
- Dirac const. of motion
- Obtaining radial eq.
- Sommerfeld Fine Structure

⑨ Free Electrons
- Eigenvalues
- $S(\vec{p})$ matrix
- α operator ($\vec{v}$)
- Schrodinger & Dirac $E \neq 0$
- Ehrenfest's Theorem

⑩ Charge Conjugation
- Condition on C
- Physical Interpretation

⑪ Time Reversal
- Non-relativistic
- Kramer's Degeneracy
- Wigner Time Reversal
- Time reversal and charge conjugation.

II Field Theory and Radiation Theory

① Second Quantization
- Bosons
- Fermions

② Quantization of the Radiation Field
- Hamiltonian Density
- Form of $\vec{A}$

③ Free Electron Scattering
- Compton Scattering λ-shift
- Thompson scattering amplitude
- KN formula
- Interpretation of $-E$ states

III Reading Period Assignment

① Classical Relativistic Field Theory
- Euler equations
- Hamiltonian and Lagrangian

② Commutation Rules for Field Components

③ Yukawa Potential

④ Pion-Nucleon Scattering

Relativistic Electron

① KG Equation:

$$E^2 = p^2c^2 + m^2c^4 \; ; \quad p_\mu p_\mu = -m^2c^2$$

$$p_\mu = (\vec{p}, i\tfrac{E}{c}) \; ; \quad x_\mu = (\vec{x}, ict)$$

$$p_\mu \to -i\hbar \frac{\partial}{\partial x_\mu} \; ; \quad -\frac{\partial^2 \psi}{\partial x_\mu \partial x_\mu} + \frac{m^2c^2}{\hbar^2}\psi = 0$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot\vec{j} = 0 \; ; \quad \frac{\partial S_\mu}{\partial x_\mu} = 0 \; ; \quad S_\mu = (\vec{j}, ic\rho)$$

$$-\frac{\partial^2 \psi^*}{\partial x_\mu \partial x_\mu} + \frac{m^2c^2}{\hbar^2}\psi^* = 0$$

② $$\frac{\partial S_\mu}{\partial x_\mu} = \psi^*(\;) - (\;)^*\psi = \psi^*\frac{\partial^2\psi}{\partial x_\mu \partial x_\mu} - \frac{\partial^2\psi^*}{\partial x_\mu \partial x_\mu}\psi$$

$$= \frac{\partial}{\partial x_\mu}\left[\psi^*\frac{\partial\psi}{\partial x_\mu}\right] - \frac{\partial\psi^*}{\partial x_\mu}\frac{\partial\psi}{\partial x_\mu} - \frac{\partial}{\partial x_\mu}\left[\psi\frac{\partial\psi^*}{\partial x_\mu}\right] + \frac{\partial\psi}{\partial x_\mu}\frac{\partial\psi^*}{\partial x_\mu}$$

$$= \frac{\partial}{\partial x_\mu}\left\{\psi^*\frac{\partial\psi}{\partial x_\mu} - \psi\frac{\partial\psi^*}{\partial x_\mu}\right\}$$

③ $$H\psi = i\hbar\frac{\partial\psi}{\partial t} \; : \quad H = c\vec{\alpha}\cdot\vec{p} + \beta mc^2$$

$$\{\alpha_k, \alpha_\ell\} = 2\delta_{k\ell} \; , \quad \{\alpha_k, \beta\} = 0 \; ; \quad \beta^2 = 1$$

$$\left\{H + i\hbar\frac{\partial}{\partial t}\right\}\left\{H - i\hbar\frac{\partial}{\partial t}\right\}\psi = H^2\psi + \hbar^2\frac{\partial^2}{\partial t^2}\psi$$

$$H\cdot H = c^2p^2 + m^2c^4$$

$$\varphi_\mu = (\vec{A}, i\varphi) \quad ; \qquad \pi_\mu = p_\mu - \frac{e}{c}\varphi_\mu$$
$$\rightarrow -i\hbar \frac{\partial}{\partial x_\mu} - \frac{e}{c}\varphi_\mu$$

$$H = c\,\vec{\alpha}\cdot\vec{p} + \beta m c^2 \;\cancel{-i\hbar\,\partial}$$
$$\left\{ c\,\vec{\alpha}\cdot\vec{p} + \beta m c^2 - i\hbar\frac{\partial}{\partial t} \right\}\psi = 0$$

Define $\gamma^k = -i\beta\alpha_k \quad ; \quad \gamma^4 = \beta \quad ; \quad \{\gamma^\mu, \gamma^\nu\} = 2\delta_{\nu\mu}$

$$\left\{ -i\hbar c\,\vec{\alpha}\cdot\vec{\nabla} + \beta m c^2 - i\hbar\frac{\partial}{\partial t} \right\}\psi = 0$$
$$\left[\vec{\alpha}\cdot\vec{\nabla} + i\frac{\beta m c}{\hbar} + \frac{\partial}{c\,\partial t} \right]\psi = 0$$

$\times\, -i\beta$: $\quad \gamma^\mu\left[\frac{\partial}{\partial x_\mu} - \frac{ie}{\hbar c}\varphi_\mu\right]\psi + \frac{mc}{\hbar}\psi = 0$

or $\quad \left[\frac{\partial}{\partial x_\mu} - \frac{ie}{\hbar c}\varphi_\mu\right]\gamma^\mu\psi + \frac{mc}{\hbar}\psi = 0$

Now define: $\quad \bar{\psi} = \psi^*\gamma^4$

$$\left[\frac{\partial}{\partial x_\mu^*} + \frac{ie}{\hbar c}\varphi_\mu^*\right]\psi^*\gamma^\mu + \frac{mc}{\hbar}\psi^* = 0$$

$$\left[\frac{\partial}{\partial x_\mu^*} + \frac{ie}{\hbar c}\varphi_\mu^*\right]\underbrace{\psi^*\gamma^\mu\gamma^4}_{\gamma^4\gamma^\mu - 2\delta_{\mu 4}} \;\cancel{+}\; \frac{mc}{\hbar}\bar{\psi} = 0$$

$$= \left[\frac{\partial}{\partial x_\mu} + \frac{ie}{\hbar c}\varphi_\mu\right]\bar{\psi}\gamma^\mu \;\cancel{+}\; \frac{mc}{\hbar}\bar{\psi} = 0$$

Eq. of Cont.

$$H = c\vec{\alpha}\cdot\left(-i\hbar\nabla - \frac{e}{c}\vec{A}\right) + i\frac{e}{c}\varphi + \beta mc^2$$

$$\cancel{\frac{\partial}{\partial x_\mu}\bar{\psi}\gamma^\mu\psi + \frac{\partial}{\partial x_\mu}} \qquad \bar{\psi}\frac{\partial}{\partial x_\mu}\gamma^\mu\psi + \frac{\partial\bar{\psi}}{\partial x_\mu}\gamma^\mu\psi$$

$$= \frac{\partial}{\partial x_\mu}\bar{\psi}\gamma^\mu\psi$$

$$\bar{\psi}\gamma^k\psi = -i\psi^*\gamma^4\beta\alpha_k\psi = -i\psi^+\alpha_k\psi$$

$$\bar{\psi}\gamma^4\psi = \psi^*\psi$$

$$\frac{\partial}{\partial x_\mu}\bar{\psi}\gamma^\mu\psi = \frac{\partial}{ic\,\partial t}\psi^*\psi - i\nabla\cdot\psi^*\vec{\alpha}\psi$$

$$\frac{\partial S_\mu}{\partial x_\mu} = 0 = \frac{\partial}{ic\,\partial t}(ic\rho) + \nabla\cdot c\psi^*\vec{\alpha}\psi = 0$$

$$S_\mu = (\vec{j}, ic\rho)$$

Gauge Trans: $\vec{A}' = \vec{A} + \nabla\lambda$

$$\varphi' = \varphi - \frac{1}{c}\frac{\partial\lambda}{\partial t}$$

$$\boxed{\begin{aligned}\varphi_\mu' &= \varphi_\mu + \frac{\partial\lambda}{\partial x_\mu} \\ \psi' &= \psi e^{i\frac{e\lambda}{\hbar c}}\end{aligned}}$$

$$\left\{\frac{\partial}{\partial x_\mu} - \frac{ie}{\hbar c}\varphi_\mu\right]\gamma^\mu\psi + \frac{mc}{\hbar}\psi = 0$$

$$\left[\frac{\partial}{\partial x_\mu} - \frac{ie}{\hbar c}\varphi_\mu - \frac{ie}{\hbar c}\frac{\partial\lambda}{\partial x_\mu}\right]\gamma^\mu\psi e^{\frac{ie}{\hbar c}\lambda} + \frac{mc}{\hbar}\psi e^{\frac{ie}{\hbar c}\lambda} = 0$$

$$\frac{\partial}{\partial x_\mu}\psi e^{\frac{ie}{\hbar c}\lambda} = e^{\frac{ie}{\hbar c}\lambda}\frac{\partial\psi}{\partial x_\mu} + \psi e^{\frac{ie}{\hbar c}\lambda}\frac{ie}{\hbar c}\frac{\partial\lambda}{\partial x_\mu}$$

hence gauge inv.

Lorentz Inv. of Dirac Eq.

$$x_\mu = a_{\mu\nu} x'_\nu$$

a's orthogonal : $a_{\mu\nu} a_{\mu\sigma} = \delta_{\nu\sigma}$

$$a_{\nu\mu} a_{\sigma\mu} = \delta_{\nu\sigma}$$

Make Lorentz transform on γ^μ

$$\Gamma^\nu = a_{\nu\mu} \gamma^\mu$$
$$\Gamma^\lambda = a_{\lambda\sigma} \gamma^\sigma$$

$$\Gamma^\lambda \Gamma^\nu + \Gamma^\nu \Gamma^\lambda = a_{\nu\mu} a_{\lambda\sigma} \underbrace{\{\gamma^\sigma, \gamma^\mu\}}_{2\delta_{\sigma\mu}} = 2\delta_{\nu\lambda}$$

Use Pauli's theorem that only one set of γ's or those related to it by similitude transform: Hence, since Γ's commute like γ's, can form

$$S^{-1} a_{\nu\mu} \gamma^\mu S = \gamma^\nu$$

S not unitary since Γ not Hermitean.

Apply to D. eq.

$$\left[\frac{\partial}{\partial x_\mu} - \frac{ie}{\hbar c} \varphi_\mu\right] \gamma^\mu \psi + \frac{mc}{\hbar} \psi = 0$$

$$\gamma^\mu a_{\mu\nu} \left[\frac{\partial}{\partial x_{\nu'}} - \frac{ie}{\hbar c} \varphi_{\nu'}\right] \psi + \frac{mc}{\hbar} \psi = 0$$

$$S^{-1} \gamma^\mu a_{\mu\nu} S S^{-1} \left[\frac{\partial}{\partial x_{\nu'}} - \frac{ie}{\hbar c} \varphi_{\nu'}\right] \psi + \frac{mc}{\hbar} S^{-1} \psi$$

$$\left[\frac{\partial}{\partial x'_\nu} - \frac{ie}{\hbar c} \varphi_{\nu'}\right] \gamma^\nu S^{-1} \psi + \frac{mc}{\hbar} S^{-1} \psi$$

Choose: $\psi' = S^{-1}\psi$ or $\psi = S\psi'$

now look at:

$$\left[\frac{\partial}{\partial x_\mu} + \frac{ie}{\hbar c}\varphi_\mu\right]\bar{\psi}\gamma^\mu - \frac{mc}{\hbar}\bar{\psi} = 0$$

$$\left[\frac{\partial}{\partial x'_\nu} + \frac{ie}{\hbar c}\varphi'_\nu\right]\bar{\psi}\, a_{\nu\mu}\gamma^\mu - \frac{mc}{\hbar}\bar{\psi} = 0$$

$$\left[\frac{\partial}{\partial x'_\nu} + \frac{ie}{\hbar c}\varphi'_\nu\right]\bar{\psi} S \underbrace{S^{-1} a_{\nu\mu}\gamma^\mu S}_{\gamma^\nu} - \frac{mc}{\hbar}\bar{\psi} S = 0$$

Choose $\cancel{B}\bar{\psi}\cancel{S} = \bar{\psi} S$, $\bar{\psi} = \cancel{B}\bar{\psi}' S^{-1} B$

where B is a numerical matrix. Now it should be possible to get B from the fact that we should get $\bar{\psi}'$ from ψ': This will show invariance:

$\psi = S\psi'$; $\psi = \psi'\tilde{S}$; $\psi^* = \psi'^* S^+$

$$\bar{\psi} = \bar{\psi}'\gamma^4 S^+\gamma^4 = \cancel{B}\bar{\psi}' S^{-1} B$$

$\therefore \quad B = S\gamma^4 S^+\gamma^4$

Want to show that this commutes with all γ's: Schur's Lemma

$S^{-1} a_{\mu\nu}\gamma^\nu S = \gamma^\mu$; $a_{\mu\nu}\gamma^\nu = S\gamma^\mu S^{-1}$

~~$\gamma S \vec{\ } a_{\mu\nu} S$~~

$a_{\mu\nu}\gamma^\nu S = S\gamma^\mu$; $a^*_{\nu\mu} S^+\gamma^\nu = \gamma^\mu S^+$; $S^+\gamma^\nu = a^*_{\mu\nu}\gamma^\mu S^+$

$$\gamma^4 B = \gamma^4 S\gamma^4 S^+\gamma^4 = a_{4\nu}\gamma^4\gamma^\nu S S^+\gamma^4$$

$$= a_{4\nu} a^*_{\mu 4}\gamma^4\gamma^\nu S\gamma^\mu S^+$$

⑦

$B = S\gamma^4 S^+\gamma^4$; $S^{-1} a_{\nu\mu}\gamma^\mu S = \gamma^\nu$

$a_{\nu\mu}\gamma^\mu S = S\gamma^\nu$; $\gamma^\mu S = a_{\mu\nu} S\gamma^\nu$

$S^+\gamma^\mu = a^*_{\nu\mu}\gamma^\nu S^+$

$\gamma^4 B = \gamma^4 S\gamma^4 S^+\gamma^4 = a_{4\nu} S\gamma^\nu\gamma^4 S^+\gamma^4$

$= a_{4\nu} a^*_{\lambda 4} S\gamma^\nu\gamma^4\gamma^\lambda S^+$

$= a_{4\nu} a_{\lambda 4} S\gamma^\nu\gamma^\lambda\gamma^4 S^+$

$= \frac{1}{2} a_{4\nu} a_{\lambda 4} S\left[\underbrace{\gamma^\nu\gamma^\lambda + \gamma^\lambda\gamma^\nu}_{2\delta_{\nu\lambda}}\right]\gamma^4 S^+$

$B = S\gamma^4 S^+\gamma^4$; $S^{-1} a_{\nu\mu}\gamma^\mu S = \gamma^\nu$

$a_{\nu\mu}\gamma^\mu S = S\gamma^\nu$

~~$a_{\lambda\mu}\gamma^\mu S = S\gamma^\lambda$~~

~~$\gamma^\mu S = a_{\lambda\mu} S\gamma^\lambda$~~

$a_{\nu\lambda}\Big\{ S\gamma^\nu = a_{\nu\mu}\gamma^\mu S$

$a_{\nu\lambda} S\gamma^\nu = \gamma^\lambda S$

$S\gamma^\nu = a_{\nu\mu}\gamma^\mu S$

~~$\gamma^\mu S^+ = a^*_{\mu\nu} S^+\gamma^\nu$~~

$$\boxed{\begin{aligned} &\gamma^\lambda S = a_{\nu\lambda} S\gamma^\nu \\ &S^+\gamma^\lambda = a^*_{\lambda\nu}\gamma^\nu S^+ \\ &S^+\gamma^\mu = a^*_{\mu\lambda}\gamma^\lambda S^+ \end{aligned} \quad \begin{aligned} \gamma^4 B &= \gamma^4 S\gamma^4 S^+\gamma^4 = a_{\nu 4} S\gamma^\nu\gamma^4 S^+\gamma^4 \\ &= a_{\nu 4} a^*_{4\lambda} \end{aligned}}$$

$S\gamma^\nu = a_{\nu\mu}\gamma^\mu S \quad ; \quad a_{\nu\lambda} S\gamma^\nu = \gamma^\lambda S$

$\gamma^\nu S = a_{\lambda\nu} S\gamma^\lambda \quad$ or $\quad a_{\lambda\nu} S\gamma^\lambda = \gamma^\nu S$

$S^+\gamma^\nu = S^+\gamma^\lambda a^*_{\nu\lambda}$

$\gamma^4 B = \gamma^4 S\gamma^4 S^+\gamma^4 = a_{\lambda 4} S\gamma^\lambda\gamma^4 S^+\gamma^4 =$

$S^{-1} a_{\nu\mu}\gamma^\mu S = \gamma^\nu \; ; \quad S\gamma^\nu = a_{\nu\mu}\gamma^\mu S$

$\gamma^\mu S = a_{\nu\mu} S\gamma^\nu$

$S^+\gamma^\mu = a^*_{\mu\nu}\gamma^\nu S^+$

$S^{-1} a_{\nu\mu}\gamma^\mu S = \gamma^\nu \; ; \quad a_{\nu\mu}\gamma^\mu S = S\gamma^\nu$

$S\gamma^\nu = a_{\nu\mu}\gamma^\mu S \quad ; \quad \gamma^\nu S^+ = a^*_{\nu\mu} S^+\gamma^\mu \; ; \; \gamma^\nu S^+ = a^*_{\nu\lambda} S^+\gamma^\lambda$

$\cancel{\gamma^4 B = \gamma^4 S\gamma^4 S^+\gamma^4 =}$

$$\boxed{\gamma^\mu S = a_{\mu\nu} S\gamma^\nu \quad ; \quad S^+\gamma^\mu = a^*_{\mu\lambda}\gamma^\lambda S^+}$$

$\gamma^4 B = \gamma^4 S\gamma^4 S^+\gamma^4 = a_{4\nu} S\gamma^\nu\gamma^4 a^*_{4\lambda}\gamma^\lambda S^+$

$= a_{4\nu} a_{4\lambda} S\gamma^\nu\gamma^\lambda\gamma^4 S^+$

$= S\gamma^4 S^+ = B\gamma^4$

$\gamma^k B = \gamma^k S\gamma^4 S^+\gamma^4 = a_{k\nu} S\gamma^\nu\gamma^4 \cdot a^*_{4\lambda}\gamma^\lambda S^+$

$= a_{k\nu} S\gamma^\nu\gamma^4\gamma^\lambda S^+ \quad = + a_{k\nu} a^*_{4\lambda} S\gamma^4\gamma^\nu\gamma^\lambda S^+$
$\;\; a^*_{4\lambda}$

$= a_{k\nu} a_{4\lambda} S\gamma^\nu\gamma^\lambda\gamma^4 S^+$ ✓

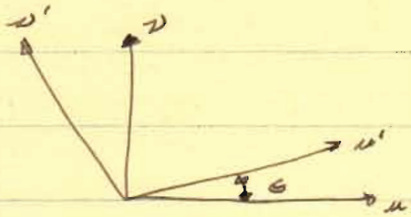
$B\gamma^k = S\gamma^4 S^+\gamma^4\gamma^k = - S\gamma^4 S^+\gamma^k\gamma^4$

$= - a^*_{k\lambda} S\gamma^4\gamma^\lambda S^+\gamma^4 = - a^*_{4\nu} a^*_{k\lambda} S\gamma^4\gamma^\lambda\gamma^\nu S^+$

$= a_{4\nu} a_{k\lambda} S\gamma^\lambda\gamma^\nu\gamma^4 S^+$ ✓

$B^{+} = B$ and hence is real and a scalar. B must be +1 for a proper Lorentz transformation because a differential change hardly matters.

What is S?



S must satisfy: $S^{-1}S = 1$

$$S^{-1} a_{\mu\nu}\gamma^{\nu} S = \gamma^{\mu}$$

Infinitesimal to 1st order: $S = 1 + \epsilon T$; $S^{-1} = 1 - \epsilon T$

$$a_{\mu\nu} = \delta_{\mu\nu} + \epsilon \; ; \; a_{\nu\mu} = \delta_{\nu\mu} - \epsilon$$

hence $a_{\mu\nu}\gamma^{\nu} = \gamma^{\mu} + \epsilon\gamma^{\nu}$

$$a_{\nu\mu}\gamma^{\mu} = \gamma^{\nu} - \epsilon\gamma^{\mu}$$

$$\left.\begin{aligned}
(1-\epsilon T)(\gamma^{\mu} + \epsilon\gamma^{\nu})(1+\epsilon T) &= \gamma^{\mu} \\
(1-\epsilon T)(\gamma^{\nu} - \epsilon\gamma^{\mu})(1+\epsilon T) &= \gamma^{\nu} \\
(1-\epsilon T)\gamma^{\lambda}(1+\epsilon T) &= \gamma^{\lambda} \\
(1-\epsilon T)\gamma^{\sigma}(1+\epsilon T) &= \gamma^{\sigma}
\end{aligned}\right\} \quad \mu \neq \nu \neq \lambda \neq \sigma$$

To first order

$$\left.\begin{aligned}
[T, \gamma^{\mu}] &= \gamma^{\nu} \\
[T, \gamma^{\nu}] &= -\gamma^{\mu} \\
[T, \gamma^{\lambda}] &= [T, \gamma^{\sigma}] = 0
\end{aligned}\right\}$$

guess: $T = a1 + b\gamma^{\mu}\gamma^{\nu}$

$$b[\gamma^{\mu}\gamma^{\nu}, \gamma^{\mu}] = b\gamma^{\mu}\underbrace{[\gamma^{\nu}, \gamma^{\mu}]}_{\gamma^{\nu}\gamma^{\mu} - \underbrace{\gamma^{\mu}\gamma^{\nu}}_{-\gamma^{\nu}\gamma^{\mu} + 2\delta_{\mu\nu}}}$$

$$= b2 \cdot -\gamma^{\mu}\gamma^{\mu}\gamma^{\nu} = \gamma^{\nu}$$

Hence $b = -1/2$

$$T = a1 - \tfrac{1}{2}\gamma^{\mu}\gamma^{\nu}$$

Then $S = 1 + \epsilon a 1 - \frac{1}{2}\epsilon \gamma^\mu \gamma^\nu$

Requirement of unimodularity on S: $\mathrm{Det}(S) = 1$

Write " $S = 1 + \epsilon a 1 - \frac{1}{2} i\epsilon (i\gamma^\mu \gamma^\nu)$

$i\gamma^\mu \gamma^\nu$ is Hermitean ($\mu \neq \nu$), hence can be diagonalized.

Note $i\gamma^\mu\gamma^\nu * i\gamma^\mu\gamma^\nu = 1$

A diagonal representation that does this is:

$$\begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$

Hence in this rep: $S = \begin{pmatrix} 1 + \epsilon a 1 - \frac{1}{2} i\epsilon & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \end{pmatrix}$

$\det(S) = 1 + 4\epsilon a = 1$ hence $a = 0$

and:

$$S = 1 - \frac{\epsilon}{2}\gamma^\mu\gamma^\nu = 1 + \frac{\epsilon}{2}\gamma^\nu\gamma^\mu$$

Finite rot: $S = \lim_{\epsilon \to 0} \left(1 - \frac{\epsilon}{2}\gamma^\mu\gamma^\nu\right)^{\frac{\theta}{\epsilon}} = e^{-\frac{\theta}{2}\gamma^\mu\gamma^\nu} = e^{\frac{\theta}{2}\gamma^\nu\gamma^\mu}$

$S^{-1} = e^{-\frac{\theta}{2}\gamma^\nu\gamma^\mu} = e^{\frac{\theta}{2}\gamma^\mu\gamma^\nu}$

$S = \cos\frac{\theta}{2} + \gamma^\nu\gamma^\mu \sin\frac{\theta}{2}$

~~Phase Conjugation~~

Space Inversion:

$$a_{\mu\nu} x_\nu = x'_\mu \quad ; \quad x_k \to -x_k \;, \quad x_4 \to x_4$$

$$\therefore a_{kk} = -1$$

$$S^{-1} a_{\mu\nu} \gamma^\nu S = \gamma^\mu$$

Hence $S^{-1} \gamma^k S = -\gamma^k$

$S^{-1} = S = \gamma^4$ is OK

$S = i\gamma^4 \;; \; S^{-1} = -i\gamma^4$ OK

In general : $S = f\gamma^4$

$B = S\gamma^4 S^+ \gamma^4 \quad : \; S = \gamma^4$, $B = 1$; $S = i\gamma^4$, $B = 1$ OK

where f is one of the 4 roots of 1 since $\operatorname{Det} S = 1$.

Consider Geometric Time Reflection

$$a_{\mu\nu} x_\nu = x'_\mu \quad : \quad x_k \to x_k \;; \; x_4 \to -x_4$$

$$a_{kk} = 1 \;; \; a_{44} = -1$$

$$S^{-1} \gamma^k S = \gamma^k$$

$$S^{-1} \gamma^4 S = -\gamma^4$$

$$S = \gamma^1\gamma^2\gamma^3 \;; \quad \gamma^3\gamma^2\gamma^1 \gamma^4 \gamma^1\gamma^2\gamma^3 = -\gamma^4$$

$$B = S\gamma^4 S^+ \gamma^4 = \gamma^1\gamma^2\gamma^3\gamma^4\gamma^3\gamma^2\gamma^1\gamma^4 = -1$$

$$\psi = S\psi' \;; \quad \bar\psi = \bar\psi' S^{-1} B$$

(10) Charge Conjugation

Definition $\psi_c = C\bar{\Psi}$

Find C's by requiring in LT system: $\psi_c' = C\bar{\Psi}'$

Now recall: under PLT and Space inv., $B=1$

so that: $\psi = S\psi'$; $\bar{\Psi} = \bar{\Psi}'S^{-1}$

$\bar{\Psi} = \tilde{S}^{-1}\bar{\Psi}'$; $\psi_c^{*} = S\psi_c'$

$S\psi_c' = C\tilde{S}^{-1}\bar{\Psi}'$; $\psi_c' = S^{-1}C\tilde{S}^{-1}\bar{\Psi}'$

Hence $C = S^{-1}C\tilde{S}^{-1}$; $\boxed{SC = C\tilde{S}^{-1}}$

Now: $S = e^{\frac{\theta}{2}\gamma^\nu\gamma^\mu} = \cos\frac{\theta}{2} + \gamma^\nu\gamma^\mu \sin\frac{\theta}{2}$

$S^{-1} = \cos\frac{\theta}{2} - \gamma^\nu\gamma^\mu \sin\frac{\theta}{2}$

$\tilde{S}^{-1} = \cos\frac{\theta}{2} - \tilde{\gamma}^\mu\tilde{\gamma}^\nu \sin\frac{\theta}{2} = \cos\frac{\theta}{2} + \tilde{\gamma}^\nu\tilde{\gamma}^\mu \sin\frac{\theta}{2}$

Hence: $\gamma^\nu\gamma^\mu C = C\tilde{\gamma}^\nu\tilde{\gamma}^\mu$

We choose Racah's form for the space inversion:

$S = i\gamma^4$; $\tilde{S}^{-1} = -i\tilde{\gamma}^4$

Hence $\gamma^4 C = -C\tilde{\gamma}^4$

$\gamma^\nu\gamma^4 C = C\tilde{\gamma}^\nu\tilde{\gamma}^4$

We get: $\boxed{\gamma^\nu C = -C\tilde{\gamma}^\nu}$ $\boxed{\gamma^\mu C = -C\tilde{\gamma}^\mu}$

Show that C.C. is self-reciprocal, ie, if $\psi_c = C\bar{\Psi}$

Then $\psi = C\bar{\Psi}_c$

$\psi_c = C\bar{\psi}$; $\bar{\psi} = C^{-1}\psi_c$; $\bar{\psi} = \psi^*\gamma^4$; $\bar{\psi} = \tilde{\gamma}^4\psi^*$

$\psi^* = \tilde{\gamma}^4 C^{-1}\psi_c$

$\tilde{\gamma}^\mu$ is just as good a representation as γ^μ, so C can be taken unitary: $\gamma^\mu C = -C\tilde{\gamma}^\mu$; $C^{-1}\gamma^\mu = -\tilde{\gamma}^\mu C^{-1}$

$C^+\gamma^\mu = -\tilde{\gamma}^\mu C^+$

Hence: $\psi^* = \tilde{\gamma}^4 C^+\psi_c = -C^+\gamma^4\psi_c$

$\psi = \gamma^4\tilde{C}\psi_c^* = -\tilde{C}\gamma^{4*}\psi_c^* = -\tilde{C}\bar{\psi}_c$

$\bar{\psi} = C^{-1}\psi_c$; $\bar{\psi} = \psi^*\gamma^4 = \tilde{\gamma}^4\psi^*$

$\psi^* = \tilde{\gamma}^4 C^{-1}\psi_c = \tilde{\gamma}^4 C^+\psi_c = -C^+\gamma^4\psi_c$

$\psi = -\tilde{C}\bar{\psi}_c$

Then for charge conj. to be 1-1: C is antisymmetric

$\boxed{C = -\tilde{C}}$

Let us show that this is so by commuting $\tilde{C}C^{-1}$ with all γ:

$\cancel{\gamma^\mu C = -C\tilde{\gamma}^\mu}$; $\cancel{\tilde{C}\gamma^\mu = -\tilde{C}C\tilde{\gamma}^\mu C^{-1}}$

$\cancel{C^{-1}\tilde{C}\gamma^\mu =}$

$\cancel{\tilde{C}\tilde{\gamma}^\mu = -\gamma^\mu\tilde{C}}$

$\tilde{C}\tilde{\gamma}^\mu = -\gamma^\mu\tilde{C}$ $\quad$ $\gamma^\mu C = -C\tilde{\gamma}^\mu$; $\quad$ $C^+\gamma^\mu = -\tilde{\gamma}^\mu C^+$

$\tilde{C}C^{-1}\gamma^\mu = \tilde{C}C^+\gamma^\mu = -\tilde{C}\tilde{\gamma}^\mu C^+$

$= \gamma^\mu\tilde{C}C^+$

so that $\tilde{C}C^{-1}$ is a multiple of 1

$\tilde{C} = bC \quad ; \quad C = b\tilde{C} \quad ; \quad \tilde{C} = b^2 C$ so $b = \pm 1$

Form products with γ group: Can only have 6 anti-sym.:

$\cancel{\ldots} \qquad C\tilde{\gamma}^\mu = -\gamma^\mu C$

$\tilde{C} = bC$ (1)

$\widetilde{(\gamma^\mu C)} = \tilde{C}\tilde{\gamma}^\mu = bC\tilde{\gamma}^\mu = -b\gamma^\mu C$ (4)

$\widetilde{(\gamma^\mu\gamma^\nu C)} = \tilde{C}\tilde{\gamma}^\nu\tilde{\gamma}^\mu = bC\tilde{\gamma}^\nu\tilde{\gamma}^\mu = -b\gamma^\nu C\tilde{\gamma}^\mu = b\gamma^\nu\gamma^\mu C$

$= -b\gamma^\mu\gamma^\nu C$ (6)

$\widetilde{(\gamma^5\gamma^\mu C)} = bC\tilde{\gamma}^\mu\tilde{\gamma}^5 = -b\gamma^\mu C\tilde{\gamma}^4\tilde{\gamma}^3\tilde{\gamma}^2\tilde{\gamma}^1 = -b\gamma^\mu\gamma^5 C$

$= b\gamma^5\gamma^\mu C$ (4)

$\widetilde{(\gamma^5 C)} = bC\tilde{\gamma}^5 = b\gamma^5 C$

Hence $b = -1$ and $\tilde{C} = -C$ and charge conj. is self reciprocal.

Perform CC operation on Dirac eqns.

(1) $\left[\frac{\partial}{\partial x_\mu} - \frac{ie}{\hbar c}\varphi_\mu\right]\gamma^\mu\psi + \frac{mc}{\hbar}\psi = 0$

(2) $\left[\frac{\partial}{\partial x_\mu} + \frac{ie}{\hbar c}\varphi_\mu\right]\bar{\psi}\gamma^\mu - \frac{mc}{\hbar}\bar{\psi} = 0$

$\gamma^\mu C = -C\tilde{\gamma}^\mu$

$\tilde{C}\tilde{\gamma}^\mu = -\gamma^\mu\tilde{C}$

$\tilde{C}\tilde{\gamma}^\mu\tilde{C}^{-1} = -\gamma^\mu$

$\tilde{\gamma}^\mu\tilde{C}^{-1} = -\tilde{C}^{-1}\gamma^\mu$

$\psi_c = C\bar{\psi} \quad ; \quad \bar{\psi} = C^{-1}\psi_c \quad ; \quad \bar{\psi} = \psi_c\tilde{C}^{-1}$

$\psi = C\bar{\psi}_c \quad ; \quad \psi = \bar{\psi}_c\tilde{C} \quad ; \quad \bar{\psi}_c = \psi\tilde{C}^{-1}$

Trans.

On (1)

$\left[\frac{\partial}{\partial x_\mu} - \frac{ie}{\hbar c}\varphi_\mu\right]\psi\tilde{\gamma}^\mu + \frac{mc}{\hbar}\psi = 0 \qquad \Big\} \tilde{C}^{-1}$

$\left[\frac{\partial}{\partial x_\mu} - \frac{ie}{\hbar c}\varphi_\mu\right]\bar{\psi}_c\gamma^\mu - \frac{mc}{\hbar}\bar{\psi}_c = 0$

On (2) $C\Big\}\ \left[\frac{\partial}{\partial x_\mu} + \frac{ie}{\hbar c}\varphi_\mu\right]\tilde{\gamma}^\mu\bar{\psi} - \frac{mc}{\hbar}\bar{\psi} = 0$

$\left[\frac{\partial}{\partial x_\mu} + \frac{ie}{\hbar c}\varphi_\mu\right]\gamma^\mu\psi_c + \frac{mc}{\hbar}\psi_c = 0$

11) Time Reversal

The TR operation can be given by: $\psi'(-t) = T\psi^*(t)$

and the physical requirements are:

$\langle \vec{r}_{rev} \rangle = \langle \vec{r} \rangle$; $\langle \vec{p}_{rev} \rangle = -\langle p \rangle$

Use the $\vec{r}$ representation:

$$\cancel{\int \psi'^*(t)\,\vec{r}\,\psi'(t)\,d\vec{r} = \int \psi^{*}\,\vec{r}}$$

$$\int \psi'^*(-t)\,\vec{r}\,\psi'(-t)\,d\vec{r} = \int \psi^*(t)\,\vec{r}\,\psi(t)\,d\vec{r}$$

$$\int \psi'^*(-t)\,\vec{p}\,\psi'(-t)\,d\vec{r} = -\int \psi^*(t)\,\vec{p}\,\psi(t)\,d\vec{r}$$

$$= \int \psi(t)\,\vec{p}\,\psi^*(t)\,d\vec{r}$$

$T_{\vec{r}} = 1$

~~What~~ What about in $\vec{p}$ rep.?

$$\varphi'(\vec{p},-t) = T_{\vec{p}}\,\varphi(\vec{p},t)$$

$$\varphi(p,t) = \langle p|\rangle = \int \langle p|r\rangle\,dr\,\langle r|\rangle$$

$$\psi(r,t) = \langle r|\rangle = \int \langle r|p\rangle\,dp\,\langle p|\rangle$$

$\cancel{\langle p'|\rangle = \langle |p\rangle}$; $\cancel{\langle p'|\rangle = \int \langle r|p^*\rangle\,dp}$

$$\langle r|\rangle' = \langle |r\rangle$$

$$\langle r|\rangle' = \int \langle r|p\rangle\,dp\,\langle p|\rangle'$$

$$\langle |r\rangle = \int \langle |p\rangle\,dp\,\langle p|r\rangle$$

$$\therefore \int \langle r|p\rangle\,dp\,\langle p|\rangle' = \cancel{\int \langle |p'\rangle\,dp'\,\langle p'|r\rangle = \int \langle |p\rangle}$$

$$= \int \langle |p\rangle\,dp\,\langle p|r\rangle$$

$$\int \langle p'|r\rangle\, dr\, \langle r|p\rangle\, dp\, \langle p|\,\rangle' = \langle p'|\,\rangle'$$

$$= \int \langle p'|r\rangle\, dr\, \langle\,|p\rangle\, dp\, \langle p|r\rangle$$

$$\langle p'|r\rangle = e^{-ip'r} = \langle r|-p'\rangle$$

$$= \int \langle\,|p\rangle\, dp\, \langle p|r\rangle\, dr\, \langle r|-p'\rangle = \langle\,|-p'\rangle$$

$$\varphi'(p,-t) = T_p\, \varphi^*(p,t) = \varphi^*(-p,t)$$

so T_p changes $p \to -p$

$$\langle r|T|r'\rangle = \delta(r-r')$$
$$\langle p|T|p'\rangle = \delta(p+p')$$

$$\psi'(-t) = T\,\psi^*(t)$$

In the Sch.-Hei. picture: $\psi^*(t) = U^*(t)\,\psi^*(0)$

hence:

$$\psi'(-t) = T\,U^*(t)\,\psi^*(0)$$

Define: $\psi'(-t) = U_{rev}(-t)\,\psi'(0) = U_{rev}(-t)\,T\,\psi^*(0)$

Then: $T\,U^*(t) = U_{rev}(-t)\,T$

$$U_{rev}(-t) = T\,U^*(t)\,T^{-1} \quad ; \quad U(t) = e^{\frac{i}{\hbar}\int H(t)\,dt}$$
$$U^*(t) = e^{-\frac{i}{\hbar}\int H^*(t)\,dt}$$

For adiabatic changes: $U^*(t) = 1 - \frac{i}{\hbar} H^*(t)\, t + \cdots$

Then we can write:

$$H_{rev}(-t) = T\, H^*(t)\, T^{-1}$$

Zeeman: $-\frac{e\hbar}{2mc}(\vec{H}\cdot\vec{\sigma})$ SO: $\vec{\sigma}\cdot\left(\frac{\vec{p}}{mc}\times\vec{\mathcal{E}}\right)$

Look at SO: $T\, H^*(t)\, T^{-1} = T\, \vec{\sigma}^*\, T^{-1}\, T\left(\frac{-\vec{p}}{mc}\times\vec{\mathcal{E}}\right)T^{-1}$

$$= -T\,\vec{\sigma}^*\,T^{-1}\left(\frac{\vec{p}}{mc}\times\vec{\mathcal{E}}\right) = \vec{\sigma}\cdot\left(\frac{\vec{p}}{mc}\times\vec{\mathcal{E}}\right)$$

$$T\,\sigma_j^*\,T^{-1} = -\sigma_j$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};\quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix};\quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_k \sigma_l = \delta_{kl} + i\, \epsilon_{klm}\, \sigma_m$$

Choose $T = \sigma_2$: $\sigma_2\sigma_1\sigma_2 = \sigma_2 \cdot i\sigma_3 = -\sigma_1$

$$-\sigma_2\sigma_2\sigma_2 = -\sigma_2$$

$$\sigma_2\sigma_3\sigma_2 = -\sigma_3$$

For one electron, $T = \sigma_2$

For n electrons: $T = \prod_{j=1}^{n} (\sigma_2)_n$

Kramers degeneracy: $H\psi = E\psi$; $H = H^*$, $TH = HT$

$$H^*\psi^* = E\psi^*;\quad H\psi^* = E\psi^*$$

$$\underbrace{T\,H^*\,T^{-1}}_{H}\,T\psi^* = E\,T\psi^*$$

so that $T\psi^*$ is another efn of E.

now is $T\psi^* = \alpha\psi$ where α is constant?

Form:

$$(\psi')' = (T\psi^*)' = TT^*\psi \Big\} = T\alpha^*\psi^* = |\alpha|^2\psi$$

only if non-degenerate

what is TT^*

$$TT^* = \prod_{j=1}^{n} \sigma_{z_j}\sigma_{z_j}^* = \prod_{j=1}^{n}(-1) = (-1)^n$$

so for odd* have Kramer's degeneracy.

<u>Wigner TR</u>

$$\psi' = S\bar{\psi} \quad ; \quad \psi' = \bar{\psi}\tilde{S}$$

$$\bar{\psi} = \tilde{S}^{-1}\psi' \quad ; \quad \bar{\psi} = \psi' S^{-1}$$

$$\bar{\psi} = \tilde{\gamma}^4\psi^* = S^{-1}\psi' \quad ; \quad \psi^* = \tilde{\gamma}^4 S^{-1}\psi'$$

or: $\psi = \gamma^4 S^{-1*}\psi^{*\prime} \quad ; \quad \psi = \gamma^4 S^{-1*}\tilde{\gamma}^4\bar{\psi}'$

$$\bar{\psi} = \psi^*\gamma^4 = \psi' S^{-1} \quad ; \quad \psi^* = \psi' S^{-1}\gamma^4$$

$$\psi = \psi'^* S^{-1*}\tilde{\gamma}^4 \quad ; \quad \boxed{\psi = \bar{\psi}'\gamma^4 S^{-1*}\tilde{\gamma}^4} \quad \boxed{\bar{\psi} = \tilde{S}^{-1}\psi'}$$

$$\boxed{\begin{aligned} &\left[\frac{\partial}{\partial x_\mu} - \frac{ie}{\hbar c}\varphi_\mu\right]\gamma^\mu\psi + \frac{mc}{\hbar}\psi = 0 \\ &\left[\frac{\partial}{\partial x_\mu} + \frac{ie}{\hbar c}\varphi_\mu\right]\bar{\psi}\gamma^\mu - \frac{mc}{\hbar}\bar{\psi} = 0 \end{aligned}}$$

$$\boxed{\begin{aligned}&\left[\frac{\partial}{\partial x_\mu} - \frac{ie}{\hbar c}\varphi_\mu\right]\psi\,\tilde\gamma^\mu + \frac{mc}{\hbar}\psi = 0\\ &\left[\frac{\partial}{\partial x_\mu} + \frac{ie}{\hbar c}\varphi_\mu\right]\tilde\gamma_\mu\bar\psi - \frac{mc}{\hbar}\bar\psi = 0\end{aligned}}$$

① $$\left\{\left[\frac{\partial}{\partial x^*_\mu} + \frac{ie}{\hbar c}\varphi^*_\mu\right]\bar\psi'\gamma^4 S^{-1*}\tilde\gamma_4\tilde\gamma_\mu + \frac{mc}{\hbar}\bar\psi'\gamma^4 S^{-1*}\tilde\gamma^4 = 0\right\} - \tilde\gamma^4 S^*\gamma^4$$

②
$$-\tilde S\left\{\left[\frac{\partial}{\partial x^{*\prime}_\mu} + \frac{ie}{\hbar c}\varphi^*_\mu\right]\tilde\gamma^\mu\tilde S^{-1}\psi' - \frac{mc}{\hbar}\tilde S^{-1}\psi' = 0\right.$$

① $$\left\{\frac{\partial}{\partial x^*_\mu} + \frac{ie}{\hbar c}\varphi^*_\mu\right]\bar\psi'\gamma^\mu\left[-\gamma^\mu\gamma^4 S^{-1*}\tilde\gamma^4\tilde\gamma_\mu\tilde\gamma^4 S^*\gamma^4\right] - \frac{mc}{\hbar}\bar\psi' = 0$$

Requires: $-\gamma^4 S^{-1*}\tilde\gamma_4\tilde\gamma_\mu\tilde\gamma^4 S^*\gamma^4 = \gamma^\mu$

or $-\gamma^4 S^{-1*}\tilde\gamma^4\tilde\gamma^\mu\tilde\gamma^4 S^*\gamma^4 = -\gamma^4\gamma^\mu\gamma^4$

$$S^{-1*}\tilde\gamma^4\tilde\gamma^\mu\tilde\gamma^4 S^* = +\gamma^\mu$$

$$\tilde\gamma^4\tilde\gamma^\mu\tilde\gamma^4 = S^*\gamma^\mu S^{-1*}$$

$$\boxed{\begin{aligned}\gamma^4\gamma^\mu\gamma^4 &= S\tilde\gamma^\mu S^{-1}\\ \tilde\gamma^4\tilde\gamma^\mu\tilde\gamma^4 &= \tilde S^{-1}\gamma^\mu\tilde S\end{aligned}}$$

②
$$\left[\frac{\partial}{\partial x^{*\prime}_\mu} - \frac{ie}{\hbar c}\varphi^*_\mu\right](-\tilde S\tilde\gamma^\mu\tilde S^{-1})\psi' + \frac{mc}{\hbar}\psi' = 0$$

Want $\boxed{\tilde S\tilde\gamma^\mu\tilde S^{-1} = \gamma^4\gamma^\mu\gamma^4}$; $\boxed{S^{-1}\gamma^\mu S = \tilde\gamma^4\tilde\gamma^\mu\tilde\gamma^4}$

This is suggestive of $\tilde{S} = -S$.

Examine S in the Dirac Rep.

$$\gamma^4 \gamma^\mu \gamma^4 = S \tilde{\gamma}^\mu S^{-1}$$

$$\gamma^k = -i\beta\alpha_k \; ; \; \gamma^4 = \beta \; ; \; \tilde{\gamma}^4 = \gamma^4$$

$$\tilde{\gamma}^k = -i\tilde{\alpha}_k \tilde{\beta} = i\beta\tilde{\alpha}_k$$

$$\tilde{\alpha}_1 = \alpha_1 \; ; \; \tilde{\alpha}_2 = -\alpha_2 \; ; \; \tilde{\alpha}_3 = \alpha_3$$

$$\boxed{\tilde{\gamma}^1 = -\gamma^1 \; ; \; \tilde{\gamma}^2 = \gamma^2 \; ; \; \tilde{\gamma}^3 = -\gamma^3 \; ; \; \tilde{\gamma}^4 = \gamma^4}$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\rho_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \rho_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \rho_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_1 = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}$$

$$\vec{\alpha} = \rho_1 \vec{\sigma} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \quad \beta = \rho_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\gamma^4\gamma^1\gamma^4 = +\gamma^1 = +S\gamma^1 S^{-1} \;\therefore\; S\gamma^1 = \gamma^1 S$$

$$\gamma^4\gamma^2\gamma^4 = -\gamma^2 = S\gamma^2 S^{-1} \;:\; \gamma^2 S = -S\gamma^2$$

$$\gamma^4\gamma^3\gamma^4 = +\gamma^3 = +S\gamma^3 S^{-1} \;:\; \gamma^3 S = S\gamma^3$$

$$\gamma^4 = S\gamma^4 S^{-1} \;:\; \gamma^4 S = S\gamma^4$$

Choose: $S = \gamma^1\gamma^3\gamma^4$: $S = i\gamma^1\gamma^3\gamma^4$; $S^\dagger = -i\gamma^4\gamma^3\gamma^1$
$= S$

$$S^{-1} = -i\gamma^4\gamma^3\gamma^1 = S^\dagger$$

$$\tilde{S} = i\tilde{\gamma}^4\tilde{\gamma}^3\tilde{\gamma}^1 = i\gamma^4\gamma^3\gamma^1 = -S$$

Now:

$$S = i\gamma^1\gamma^3\gamma^4 = i(-i\beta\alpha_1)(-i\beta\alpha_3)(\beta)$$

$$= -i\beta\alpha_1\beta\alpha_3\beta = i\alpha_1\alpha_3\beta = \sigma_2\beta = \begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix}$$

or like N-R case.

Transformation of Pauli Covariant Quantities

$\bar{\psi}\eta\psi$

~~$\bar{\psi}\eta\psi = \psi' S^{-1}\eta\gamma^4\tilde{S}^{-1*}\tilde{\gamma}^4\bar{\psi}'$~~

~~$= \bar{\psi}'\gamma^4 S^{-1*}\tilde{\gamma}^4\tilde{\eta}\tilde{S}^{-1}\psi'$~~

~~$\bar{\psi} = \tilde{S}^{-1}\psi' = \tilde{\gamma}^4\psi^* - \quad \psi^* = \tilde{\gamma}^4\tilde{S}^{-1}\psi'$~~

~~$\psi = \gamma^4\tilde{S}^{-1*}\tilde{\gamma}^4\bar{\psi}'$~~

~~$S^{-1} = S^{+} = S$~~

$$\boxed{\begin{aligned}&\bar{\psi} = S^{-1}\psi' = \tilde{\gamma}^4\psi^* \ ; \quad \psi^* = \tilde{\gamma}^4 S^{-1}\psi'\\ &\qquad\qquad \psi = \tilde{\gamma}^4 S^{-1*}\tilde{\gamma}^4\bar{\psi}'\\ &\cancel{\psi = \tilde{\gamma}^4 S^{-1*}} \qquad \psi = \gamma^4 S^{-1*}\tilde{\gamma}^4\bar{\psi}'\\ &S^{-1} = S^{+} \ ; \quad S^{-1*} = S^{+*} = \tilde{S}\end{aligned}}$$

$$\boxed{\tilde{S}^{-1*} = \tilde{\tilde{S}} = S}$$

$$\boxed{\bar{\psi}\eta\psi = \psi' S^{-1}\eta\gamma^4\underset{\downarrow \atop S}{\tilde{S}^{-1*}}\tilde{\gamma}^4\bar{\psi}'}$$

$$\boxed{= \bar{\psi}'\gamma^4\tilde{S}\tilde{\gamma}^4\tilde{\eta}\tilde{S}^{-1}\psi'}$$

5) γ Representations

$$\sigma_k = \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix} \; ; \; \rho_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; ; \; \rho_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \; ; \; \rho_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\left.\begin{array}{ll} \text{Dirac}: & \alpha_k = \rho_1 \sigma_k \; ; \; \beta = \rho_3 \\ \text{Weyl}: & \alpha_k = \rho_3 \sigma_k \; ; \; \beta = \rho_1 \end{array}\right\} [\rho_k, \sigma_l] = 0$$

Always: $\gamma^k = -i\beta\alpha_k \; ; \; \gamma^4 = \beta$

and:

$$\gamma^k\gamma^l = \alpha_k\alpha_l = \sigma_k\sigma_l = \delta_{kl} + i\,\epsilon_{klm}\sigma_m$$

Then $S = e^{\frac{\theta}{2}\gamma^k\gamma^l} = \begin{pmatrix} e^{\frac{\theta}{2}\sigma_m} & 0 \\ 0 & e^{\frac{\theta}{2}\sigma_m} \end{pmatrix}$ Dirac & Weyl

$$\gamma^4\gamma^k = \beta(-i\beta\alpha_k) = -i\,\alpha_k = -i\,\sigma_k\rho_3 \quad (\text{Weyl})$$
$$= -i\,\sigma_k\rho_1 \quad (\text{Dirac})$$

$$S = e^{\frac{\theta}{2}\gamma^4\gamma^k} = \begin{pmatrix} e^{-i\frac{\theta}{2}\sigma_k} & 0 \\ 0 & e^{i\frac{\theta}{2}\sigma_k} \end{pmatrix} \quad (\text{Weyl})$$

$$= \begin{pmatrix} 0 & e^{-i\frac{\theta}{2}\sigma_k} \\ e^{-i\frac{\theta}{2}\sigma_k} & 0 \end{pmatrix} \quad (\text{Dirac})$$

Weyl sorts out two subsets.

Neutrino Equation:

$$H\psi = i\hbar\frac{\partial\psi}{\partial t} \quad ; \quad H = c\vec{\alpha}\cdot\vec{p} + \beta mc^2$$

Dirac Rep: $H = c\begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0\end{pmatrix}\cdot\vec{p} + \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}mc^2$

$$\psi = \begin{pmatrix}\varphi \\ \chi\end{pmatrix} \quad ; \quad \begin{aligned} c\vec{\sigma}\cdot\vec{p}\chi + mc^2\varphi &= i\hbar\frac{\partial\varphi}{\partial t} \\ c\vec{\sigma}\cdot\vec{p}\varphi - mc^2\chi &= i\hbar\frac{\partial\chi}{\partial t}\end{aligned}$$

<u>Weyl Rep.</u>

$$H = c\begin{pmatrix}\vec{\sigma} & 0 \\ 0 & -\vec{\sigma}\end{pmatrix}\cdot\vec{p} + \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}mc^2$$

$$\psi = \begin{pmatrix}\varphi \\ \chi\end{pmatrix} : \quad \begin{aligned} c\vec{\sigma}\cdot\vec{p}\,\varphi + mc^2\chi &= i\hbar\frac{\partial\varphi}{\partial t} \\ -c\vec{\sigma}\cdot\vec{p}\,\chi + mc^2\varphi &= i\hbar\frac{\partial\chi}{\partial t}\end{aligned}$$

For neutrinos: $m=0$

$$i\hbar\frac{\partial\varphi}{\partial t} - c\vec{\sigma}\cdot\vec{p}\,\varphi = 0$$

$$i\hbar\frac{\partial\chi}{\partial t} + c\vec{\sigma}\cdot\vec{p}\,\chi = 0$$

not invariant under space inv. since $\vec{p}\to-\vec{p}$

suppose $\vec{\sigma}\to-\vec{\sigma}'$

but then $\sigma_k\sigma_l = \delta_{kl} - i\epsilon_{klm}\sigma_m$

so that D. eq. not mirror inv.

$$\varphi \sim e^{-i\frac{Et}{\hbar} + i\vec{p}\cdot\vec{r}/\hbar}$$

$$\frac{E}{c} - \vec{\sigma}\cdot\vec{p} = 0 \quad ; \quad \frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|} = 1$$

Pauli Inv. Quantities

These are: $\bar{\Psi} \eta \psi$

where $\eta = 1$ (S)

$i \gamma^{\mu}$ (V)

$i \gamma^{\mu} \gamma^{\nu}, \ \mu \neq \nu$ (T)

$i \gamma^{5} \gamma^{\mu}$ (A)

$i \gamma^{5}$ (P)

(S) $\bar{\Psi} \psi = \psi^{*} \beta \psi$

$S^{-1} a_{\mu\nu} \gamma^{\nu} S = \gamma^{\mu}$; $S^{-1} \gamma^{\nu} S = a_{\nu\mu} \gamma^{\mu}$

$\psi = S \psi'$; $\bar{\Psi} = \bar{\Psi}' S^{-1}$

$$\boxed{\bar{\Psi} \psi = \bar{\Psi}' \psi'}$$

(V) $j_{\mu} = i \bar{\Psi} \gamma^{\mu} \psi = \left[(\psi^{*} \vec{\alpha} \psi), (i \psi^{*} \psi) \right]$

$i \bar{\Psi} \gamma^{\mu} \psi = i \bar{\Psi}' S^{-1} \gamma^{\mu} S \psi' = i a_{\mu\nu} \bar{\Psi}' \gamma^{\nu} \psi'$

(T) $\mu \neq \nu$: $i \bar{\Psi} \gamma^{\mu} \gamma^{\nu} \psi =$

(c) Cons. of Angular Momentum

Central Field :

$$H = c\vec{\alpha}\cdot\vec{p} + \beta mc^2 + V(r)$$

$$\vec{L} = \vec{r}\times\vec{p} \quad ; \quad L_k = \epsilon_{klm}\, x_l\, p_m$$

$$M = \vec{L} + \frac{\hbar}{2}\vec{\sigma} \qquad \cancel{\text{Ex}} \quad L_i = \epsilon_{ijk}\, x_j\, p_k$$

$$\frac{dL_x}{dt} = \frac{i}{\hbar}\left\{H, L_x\right\}$$

$$\vec{p}\, V = -i\hbar\, V'(r)\,\frac{\vec{r}}{|\vec{r}|} \quad , \quad \vec{r}\times\vec{r} = 0$$

$$\frac{dL_x}{dt} = \frac{i}{\hbar}\overset{\epsilon_{ijk}}{\left[c\,\alpha_l p_l,\ x_j p_k\right]} = \frac{ic}{\hbar}\alpha_l\,\epsilon_{ijk}\left[p_l, x_j p_k\right]$$

$$\left[p_l, x_j p_k\right] = \left[p_l, x_j\right]p_k = -i\hbar\,\delta_{lj}\,p_k$$

$$\frac{dL_x}{dt} = c\,\alpha_l\,\delta_{lj}\,\epsilon_{ijk}\,p_k = c\,\epsilon_{ilk}\,\alpha_l\,p_k = c\left[\vec{\alpha}\times\vec{p}\right]_x$$

$$\frac{d\sigma_x}{dt} = \frac{i}{\hbar}\cancel{\left[\ \right]} \quad \frac{i}{\hbar}c\left[\alpha_l p_l,\ \sigma_x\right] = \frac{i}{\hbar}c\left[\alpha_l, \sigma_x\right]p_l$$

$$= \frac{i}{\hbar}c\,\rho_1\underbrace{\left[\sigma_l, \sigma_x\right]}_{2i\,\epsilon_{lxm}\sigma_m}p_l = -\frac{c}{\hbar}\,\epsilon_{lxm}\,\alpha_m\,p_l$$

$$= -\frac{2c}{\hbar}\,\epsilon_{xml}\,\alpha_m\,p_l = -\frac{2}{\hbar}c\left[\vec{\alpha}\times\vec{p}\right]_x$$

7) Pauli Electrons

$$H\psi = i\hbar\frac{\partial\psi}{\partial t} \quad ; \quad H = c\vec{\alpha}\cdot\vec{\pi} + \beta mc^2 + e\phi$$

$$\cancel{\{c\vec{\alpha}\cdot\vec{\pi}} \quad \left\{i\hbar\frac{\partial}{\partial t} - c\vec{\alpha}\cdot\vec{\pi} - \beta mc^2 - e\phi\right\}\begin{pmatrix}\varphi\\ \chi\end{pmatrix} = 0$$

$$i\hbar\frac{\partial\varphi}{\partial t} - c\vec{\sigma}\cdot\vec{\pi}\,\chi - mc^2\varphi - e\phi\varphi = 0$$

$$i\hbar\frac{\partial\chi}{\partial t} - c\vec{\sigma}\cdot\vec{\pi}\,\varphi + mc^2\chi - e\phi\chi = 0$$

$$\psi = \begin{pmatrix}\varphi\\ \chi\end{pmatrix} e^{-i\frac{mc^2 t}{\hbar}}$$

$$i\hbar\frac{\partial\varphi}{\partial t} - e\phi\varphi = c(\vec{\sigma}\cdot\vec{\pi})\chi$$

$$i\hbar\frac{\partial\chi}{\partial t} + 2mc^2\chi = c(\vec{\sigma}\cdot\vec{\pi})\varphi$$

$$- e\phi\chi$$

$2mc^2$ enormous hence χ is small. To first order:

$$\chi \simeq \frac{(\vec{\sigma}\cdot\vec{\pi})\varphi}{2mc} \quad ; \quad \chi \approx \frac{(\vec{\sigma}\cdot\vec{\pi})\varphi}{2mc} + \frac{e\phi\,(\vec{\sigma}\cdot\vec{\pi})\varphi}{4m^2c^3}$$

$$\cancel{+ \frac{i\hbar}{4m^2c^3}\vec{\sigma}\cdot\vec{\pi}\frac{\partial\varphi}{\partial t}} \quad - \frac{i\hbar}{4m^2c^3}\frac{\partial}{\partial t}(\vec{\sigma}\cdot\vec{\pi})\varphi$$

$$i\hbar\frac{\partial\varphi}{\partial t} - e\phi\varphi = \frac{(\vec{\sigma}\cdot\pi)^2\varphi}{2m}$$

II: Field Theory & Radiation Theory

① Second Quantization

$$i\hbar \frac{\partial \psi}{\partial t} = \underline{H}\,\psi \qquad\qquad H = H_0 + V$$

$$-i\hbar \frac{\partial \psi^*}{\partial t} = (\underline{H}\,\psi)^* \qquad\qquad H_0 u_n = E_n u_n$$

$$\psi = \sum_n b_n(t)\, u_n = \sum_n b_n \,|n\rangle$$

$$\sum_n (i\hbar \dot{b}_n)\,|n\rangle = \sum_m \underline{H}\,|m\rangle\, b_m$$

$$i\hbar \dot{b}_n = \sum_m H_{nm}\, b_m \quad ; \quad H_{nm} = \langle n|\underline{H}|m\rangle$$

Introduce:

$$\bar{H} = \int \psi^* \underline{H}\, \psi = \sum_{mn} b_n^* H_{nm}\, b_m$$

$$\therefore \quad (i\hbar \dot{b}_n) = \frac{\partial \bar{H}}{\underbrace{b_n^*}_{p}}$$

$$-i\hbar \dot{b}_n^* = \frac{\partial \bar{H}}{\partial b_n} \quad ; \quad \dot{b}_n^* = -\frac{\partial \bar{H}}{\underbrace{\partial (i\hbar b_n)}_{q}}$$

Since $[q_k, p_l] = i\hbar\,\delta_{kl}$

$$[b_n, b_m^*] = \delta_{mn} \quad ; \quad [b_n, b_m] = 0 \quad ; \quad [b_n^*, b_m^*] = 0$$

$$b_n^* b_n = N_n \; : \qquad \langle \cdots N_n - 1 \cdots | b_n | \cdots N_n \cdots\rangle = \sqrt{N_n}$$

$$\langle \cdots N_n + 1 \cdots | b_n^* | \cdots N_n \cdots\rangle = \sqrt{N_n + 1}$$

$$i\hbar \frac{\partial}{\partial t} \Psi = H \Psi \quad ; \quad \Psi = \langle N \cdots | \rangle$$

$$i\hbar \frac{\partial}{\partial t} \langle N \cdots | \rangle = \sum_{mn} b_n^* H_{nm} b_m \langle N \cdots | \rangle$$

$$= \sum_{mn} H_{nm}\, b_n^* b_m \langle N_1 \cdots | \rangle$$

$$= \sum_{mn} H_{nm} \langle N_1 \cdots | b_n^* b_m | \rangle$$

$$* \quad | \rangle = \sum_{N''} | N'' \rangle \langle N'' | \rangle \quad \checkmark$$

$$H_T = \sum_{mn} b_n^* H_{nm} b_n + \sum_{mnlp} b_n^* b_m^* V_{nm,lp}\, b_l\, b_p$$

Fermions:

$$\iint dr_1 dr_2 \; \psi^*(1) \psi_2^*(2) V(1-2) \psi(1) \psi(2)$$

$$= \sum_{mnlp} C_m^* C_n^* V_{mn,lp} C_l C_p$$

To ~~sta~~ satisfy antisym

$$\left.\begin{aligned} C_l C_p &= - C_p C_l \\ C_m^* C_n^* &= - C_n^* C_m^* \end{aligned}\right\} \; \{C_l, C_p\} = \{C_m^*, C_n^*\} = 0$$

In analogy with b's:

$$C_n^* C_n = N_n \quad : \quad C_n^+ C_n |0\rangle = 0 \quad ; \quad C_n^* C_n |1\rangle = |1\rangle$$

$$\left.\begin{aligned} \left[C_n C_n^+ + C_n^+ C_n \right] |1\rangle &= 1\,|1\rangle \\ |0\rangle &= \;\; |0\rangle \end{aligned}\right\} \quad \{ C_n, C_n^* \} = 1$$

Assume: $\{ C_m, C_n^* \} = \delta_{mn}$

---

(2) Quantization of the Radiation Field

$$H = \tfrac{1}{2} (\mathcal{E}^2 + \mathcal{H}^2) \quad ; \quad \mathcal{E} = -\frac{1}{c} \frac{\partial A}{\partial t} \quad , \quad \mathcal{H} = \nabla \times A$$

$$\vec{A} = \sum_{k,\lambda} \hat{e}_{k,\lambda} \left\{ a_{k,\lambda} e^{i k \cdot r} + a^*_{k,\lambda} e^{-i k \cdot r} \right\}$$

$$= \sum_s \hat{e}_s \left\{ a_s e^{i \vec{k} \cdot r} + a_s^* e^{-i \vec{k} \cdot r} \right\}$$

$$\nabla^2 u - \frac{1}{c^2} \frac{\partial u}{\partial t} = 0$$

$$-k^2 a_s - \frac{\ddot{a}_s}{c^2} = 0 \qquad a_s(t) = a_s(0)\, e^{\pm i k c t}$$

choose $-$ for a_s
for positive wave

$$\left.\begin{aligned} \dot{a}_s &= -ikc\, a_s \\ \dot{a}_s^* &= ikc\, a_s^* \end{aligned}\right\}$$ Find:

$$\bar{H} = 2V \sum_s k^2 a_s^* a_s$$

Take: $a_s = K b_s$

$$\dot{b}_{s'} = -ikc\, b_{s'} = \frac{i}{\hbar}\left[\bar{H}, b_{s'}\right]$$

$$= \frac{i}{\hbar}\left[2V \sum_s k^2 K^2 b_s^* b_s, b_{s'}\right] = \frac{i}{\hbar} 2V \sum_s k^2 K^2 \underbrace{\left[b_s^*, b_{s'}\right]}_{-\delta_{ss'}} b_s$$

$$= -\frac{i}{\hbar} 2V k^2 K^2 b_{s'}$$

$$1 = \frac{2Vk}{\hbar c} K^2 ; \quad K = \sqrt{\frac{\hbar c}{2Vk}}$$

$$\mathcal{H}_i = \epsilon_{ijk} \frac{\partial A_k}{\partial x_j}$$

$$\mathcal{H}^2 = \underbrace{\epsilon_{ijk}\,\epsilon_{ilm}}_{= \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}} \frac{\partial A_k}{\partial x_j} \frac{\partial A_m}{\partial x_l} = \frac{\partial A_k}{\partial x_l}\frac{\partial A_k}{\partial x_l} - \cancel{\frac{\partial A_k}{\partial x_j}\frac{\partial A_j}{\partial x_k}}$$

$$\frac{\partial A_k}{\partial x_m}\frac{\partial A_m}{\partial x_l}$$

$$\mathcal{E}^2 = \frac{1}{c^2}\frac{\partial A_k}{\partial t}\frac{\partial A_k}{\partial t}$$

$\oint$

Interaction Potential: H_I $\int d\vec{x}\, \psi^{*}$ { terms in Ham that contain $\vec{A}$

$$H = c\,\vec{\alpha}\cdot(\vec{p} - \tfrac{e}{c}\vec{A}) + \beta mc^2 + V \qquad \text{Dirac}$$

$$H = \frac{(p - \frac{e}{c}A)^2}{2m} + V \quad ; \quad \frac{e^2A^2}{2mc^2} \quad \text{sch.}$$

③ Free Electron scattering:

$$p_\mu = (\vec{p}, i\tfrac{E}{c}) \; ; \; k_\mu = (\vec{k}, ik) \quad , \text{ nat } m = \hbar = c = 1$$

Absorption: $p_\mu = (\vec{p}, iE)$

$$p_\mu + k_\mu = p_\mu'$$

$$\underbrace{p_\mu p_\mu}_{-1} + 2p_\mu k_\mu + k_\mu k_\mu = \underbrace{p_\mu' p_\mu'}_{-1}$$

$$k_\mu k_\mu = k^2 - k^2 = 0$$

$$-1 + 2p_\mu k_\mu = 0 \quad ; \quad p_\mu k_\mu = (0, i)(\vec{k}, ik)$$

$$1 + 2k = 1 \;,\; k = 0 \text{ imp.} \qquad = -k$$

$$-(p_\mu - p_\mu')\{\; p_\mu + k_\mu = p_\mu' + k_\mu' \;\}-(k_\mu - k_\mu') \qquad \begin{array}{l} p_\mu = (0, i) \;;\; k_\mu = (\vec{k}, ik) \\ p_\mu' = (\vec{p}', iE) \;;\; k_\mu' = (\vec{k}', ik') \end{array}$$

$$\underbrace{-p_\mu p_\mu}_{1} \underbrace{- p_\mu k_\mu}_{-k} + \underbrace{p_\mu' p_\mu}_{-E} + \underbrace{p_\mu' k_\mu}_{\vec{p}'\cdot k - iE} = \underbrace{-p_\mu' k_\mu}_{} - k_\mu' k_\mu + p_\mu' k_\mu' + \underbrace{k_\mu' k_\mu'}_{0}$$

$$p_\mu + k_\mu = p'_\mu + k'_\mu$$

$$-(p_\mu - p'_\mu)\left\{ p_\mu - p'_\mu = k'_\mu - k_\mu \right\} - (k'_\mu - k_\mu)$$

$$\underbrace{-p_\mu p_\mu - p'_\mu p'_\mu}_{2} + 2 p_\mu p'_\mu = \underbrace{-k'_\mu k'_\mu - k_\mu k_\mu}_{0} + 2 k'_\mu k_\mu$$

$$1 + p_\mu p'_\mu = k'_\mu k_\mu$$

$$1 - E = \vec{k}\cdot\vec{k}' - k k'$$

$$E = 1 + k - k'$$

$$-k + k' = \vec{k}\cdot\vec{k}' - k k' \quad ; \quad -\frac{1}{k'} + \frac{1}{k} = (\cos\theta - 1)$$

$$\lambda' - \lambda = \frac{h}{mc}(1 - \cos\theta)$$

Thompson scattering:

$$\bar{H}_S = \int d\tau\, \psi^* \frac{e^2}{2mc^2} A^2 \psi$$

$$\psi = \frac{C_{\vec{p}}}{\sqrt{V}} e^{i p\cdot r} \qquad \psi^* = \frac{C^*_{\vec{p}'}}{\sqrt{V}} e^{-i p'\cdot r}$$

$$A = \sqrt{\frac{\hbar c}{2Vk}}\, \hat{e}_s \left(b_s e^{i k\cdot r} + b_s^* e^{-i k\cdot r} \right)$$

$$\psi^* A^2 \psi \to \frac{\hbar c}{2V^2} \frac{\hat{e}_s\cdot\hat{e}_{s'}}{\sqrt{k k'}} C^*_{\vec{p}'} \left[b^*_{s'} b_s e^{\cdots} + b_s b^*_{s'} e^{\cdots} \right] C_{\vec{p}}$$

$$\int d\tau\, \psi^* A^2 \psi = \frac{\hbar c}{V k} \hat{e}_s\cdot\hat{e}_{s'}\, C^*_{\vec{p}'} b^*_{s'} b_s C_{\vec{p}}$$

$$k \approx k'$$

$$\langle \cdots 0_s, 1_{s'}; 0_{\vec{q}}, 1_{p'} \cdots | H_I | \cdots 1_s 0_{s'}; 1_p 0_{p'} \cdots \rangle$$

$$= \frac{\hbar c}{V k} \cdot \frac{e^2}{2mc^2} \quad \hat{e}_s \cdot \hat{e}_{s'}$$

$$\frac{dP}{dt} = \frac{2\pi}{\hbar} \left| \langle \, | H_I | \, \rangle \right|^2 \rho_E(\text{final})$$

$$\sigma(\theta, \varphi) \, d\Omega_{\vec{k}} = \frac{dP/dt}{c/V}$$

$$k_x' = n_x \frac{2\pi}{L}$$

$$d^3 n = \frac{V}{8\pi^3} k^2 dk' \, d\Omega_{\vec{k}'} \; ; \quad \rho_E(\text{final}) = \frac{V}{8\pi^3} k'^2 \, d\Omega_{\vec{k}'} \frac{dk'}{dE}$$

$$E = \hbar c k', \quad \frac{dk'}{dE} = \frac{1}{\hbar c}$$

$$\rho_E(\text{final}) = \frac{V}{8\pi^3 \hbar c} k^2 \, d\Omega_{k'}$$

$$\frac{\frac{dP}{dt}}{c/V} = \frac{\cancel{2}\pi}{\cancel{\hbar}} \frac{\cancel{\hbar} \cancel{c^2}}{\cancel{V^2} \cancel{k^2}} \frac{e^4}{4m^2c^4} (\hat{e}_s \cdot \hat{e}_{s'})^2 \cdot \frac{\cancel{V}}{\underset{4}{\cancel{8}} \pi^3 \cancel{\hbar c}} \cancel{k^2} \, d\Omega_{\vec{k}} \cdot \frac{\cancel{V}}{c}$$

$$= \frac{e^4}{16\pi^2 m^2 c^4} (\hat{e}_s \cdot \hat{e}_{s'})^2 \, d\Omega_{\vec{k}'}$$

$$e'^2 = \frac{e^2}{4\pi} \qquad = \frac{e^4}{m^2 c^4} (\hat{e}_s \cdot \hat{e}_{s'})^2 \, d\Omega_{\vec{k}'}$$

$$\sum_{e_{s'}} (e \cdot e')^2 = 1 - (e \cdot n')^2$$

$$\frac{1}{2} \sum_{e} \left(1 - (e \cdot n')^2 \right) = 1 - \frac{1}{2} \left[1 - (n \cdot n')^2 \right]$$

$$= \frac{1}{2} + \frac{1}{2} \cos^2 \theta$$

III Reading Period

① CRT Fields

$$\mathcal{L} = \mathcal{L}(\phi, \phi_\nu)$$

$$I = \int d^4x \, \mathcal{L}(\phi, \phi_\nu)$$

$\delta I = 0$

$\delta I = \cdot - - -$

Define: $\pi = \dfrac{\partial \mathcal{L}}{\partial \dot\phi}$

$$H(t) = \sum p\dot q - L$$

implies:

$$\mathcal{H} = \pi\dot\phi - \mathcal{L}(\phi, \phi_\nu)$$

for a hamiltonian density:

---

② Field Components

$$[q, p] = i\hbar \quad : \quad [\phi, \pi] = i\,\delta(x - x')$$

$$\phi(\vec x) = \frac{1}{\sqrt V}\sum_k \frac{1}{\sqrt{2\omega_k}}\left\{a(\vec k)\, e^{i\vec k\cdot\vec x} + a^+(\vec k)\, e^{-i\vec k\cdot\vec x}\right\}$$

$$\pi(\vec x) = \frac{i}{\sqrt V}\sum_k \sqrt{\frac{2}{\omega_k}}\left\{-a(k)\, e^{i\vec k\cdot\vec x} + a^+(\vec k)\, e^{-i\vec k\cdot\vec x}\right\}$$

Spinless Meson:

$$\mathcal{H} = \frac{1}{2}\left\{\pi^2 + \sum_i\left(\frac{\partial\phi}{\partial x}\right)^2 + m^2\phi^2\right\}$$

$$H = \sum_k \left(a^+(k)\, a(k) + \tfrac{1}{2}\right)\omega_k$$

③ Yukawa Potential:

$$H_I = g\,\rho(\vec{x})\,\phi(\vec{x})$$

$$\rho(\vec{x}) = \sum_n \delta(\vec{x} - \vec{x}_n) \qquad \{\infty \text{ mass}\}$$

$$H_I = g \sum_n \frac{1}{\sqrt{V}} \sum_k \sqrt{\frac{1}{2\omega_k}} \left\{ a(\vec{k})\, e^{i\vec{k}\cdot\vec{x}_n} + a^+(\vec{k})\, e^{-i\vec{k}\cdot\vec{x}_n} \right\}$$

$$\langle n_k + 1 |\, H_I \,| \cdots n_k \cdots \rangle = \frac{g}{\sqrt{V}} \cancel{\sum_k} \sum_n \sqrt{\frac{1}{2\omega_k}} \sqrt{n_k + 1}\; e^{-i\vec{k}\cdot\vec{x}_n}$$

$$\langle \cdots n_k - 1 \cdots |\, H_I \,| \cdots n_k \cdots \rangle = \frac{g}{\sqrt{V}} \sum_n \sqrt{\frac{n_k}{2\omega_k}}\; e^{i\vec{k}\cdot\vec{x}_n}$$

$$\langle \cdots n_k \cdots |\, H_I \,| \cdots n_k + 1 \rangle = \frac{g}{\sqrt{V}} \sum_n \sqrt{\frac{n_k + 1}{2\omega_k}}\; e^{i\vec{k}\cdot\vec{x}_n}$$

$$\Delta E = \langle 0 |\, H_I \,| 0 \rangle = 0$$

$$\Delta E = \sum_k \frac{\langle 0 | H_I | 1_k \rangle \langle 1_k | H_I | 0 \rangle}{-\omega_k} \quad \text{, etc.}$$

9 Free Electron

$$H\psi = c\vec{\alpha}\cdot\frac{\hbar}{i}\nabla\psi + \beta mc^2\psi = i\hbar\frac{\partial\psi}{\partial t}$$

Nat. :
$$\frac{\vec{\alpha}\cdot\nabla\psi}{i} + \beta\psi = i\frac{\partial\psi}{\partial t}$$

Take $\psi = a\, e^{i\vec{p}\cdot\vec{r} - iEt}$

$$(\vec{\alpha}\cdot\vec{p} + \beta - E)a = 0$$

$$(E - \vec{\alpha}\cdot\vec{p} - \beta)a = 0$$

$E = +p_0$: Two roots

$E = -p_0$ "

We look for S to diagonalize $H(\vec{p})$

Want:

$$S^{+}H(p)\,S = \begin{pmatrix} p_0 & & & \\ & p_0 & & \\ & & -p_0 & \\ & & & -p_0 \end{pmatrix} = p_0\beta$$

Try $S = C\,(H(p) + p_0\beta)$

H is Herm, S is un. $\quad S^{+}S = 1$

Can write:

$$S^{\dagger}(\vec{p})\left[E - H(\vec{p})\right]S(\vec{p}) = E - p_0\beta$$

$$\left[E - H(\vec{p})\right]S(\vec{p}) = S(\vec{p})\left\{E - p_0\beta\right\} \checkmark$$

$$a_i^{(r)} = S_{ir}$$

finally:

$$\langle \vec{r}, i \,|\, \vec{p}, r\rangle = \frac{1}{(2\pi)^{3/2}}\, a_i^{(r)}(\vec{p})\, e^{i\vec{p}\cdot\vec{r}}$$

Physics 253 Problems, 1961

1. In Dirac's original representation three matrices $(\alpha_x, \alpha_z, \beta)$ are real and symmetric, while one (α_y) is imaginary and antisymmetric. Thus, given four Hermitian matrices γ^μ, with $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\delta_{\mu\nu}$, a representation "of type (3,1)" is possible, where "type (m,n)" means that m of the matrices are real symmetric and n are imaginary antisymmetric. What other types (m,n) are possible, and what are impossible? (Possibility can be clinched by displaying a representation. Impossibility can be proved by examining the symmetries of the 16 linearly independent products.)

2. Show that the non-Hermitian "Darwin term" in the approximate wave equation for the large components φ (charge e, potentials A, ϕ, no nonelectric force, no Pauli magnetic moment) are, in this order, just what is required to make $\varphi^*\varphi$ non-conserved in such a way that $\psi^*\psi = \varphi^*\varphi + X^*X$ is conserved. (Using the lowest order expression for X in terms of φ, calculate $\frac{d}{dt} X^*X$ and show that it consists of:

a) divergences;
b) terms of higher order in $\frac{v}{c}$ than those in $\frac{d}{dt}\varphi^*\varphi$ calculated from the approximate wave equation;
c) terms equal and opposite to the contribution to $\frac{d}{dt}\varphi^*\varphi$ from the "Darwin term".)

Problems 1-3 due 11-29-61

3. Use the free-particle Dirac Hamiltonian

$$H = c\vec{\alpha}\cdot\vec{p} + \beta mc^2$$

to calculate, by evaluating commutators, the successive quantum-mechanical derivatives $\left(\frac{d}{dt}x\right)$, $\left(\frac{d^2}{dt^2}x\right)$, ... Carry this out until you can see a relation between derivatives that enables you to write formulas for all these derivatives. Substitute in Maclaurin's series

$$x = x(0) + \sum_{n=1}^{\infty}\left(\frac{d^n}{dt^n}x\right)_0 \frac{t^n}{n!}$$

and sum the series to obtain

$$x = x(0) + \text{const}\cdot t + \text{sinusoidal terms.}$$

Examine the coefficients in this result and make what comments you can on their meaning. (A similar, but somewhat less explicit, result is obtained by a different method in Dirac, Section 69 - 3rd or 4th edition; or Section 71 in 2nd edition.)

Don't do 4 at this time.

~~10.~~ 4. Show that for $N_v' > 0$ there are an infinite number of different state vectors Ψ for which $N_v\Psi = N_v'\Psi$. (See hectographed reading-period notes for background.) If v is a finite volume in infinite space, or is only part of a "box" to which we may confine our discussion, show that this infinite degeneracy holds also for $N_v' = 0$.

If the system is confined to a "box" (so that the basis functions $u_s(\vec{r})$ are a complete orthonormal system for this finite region), and if v is the whole volume of the box, show that $N_v\Psi_0 = 0$ defines a state vector Ψ_0 that is determined uniquely apart from a phase factor.

Physics 253 - Problems 1961

[The Problem 4 printed on the previous sheet will not be assigned at present, and will perhaps be omitted altogether. We now list a Problem 4 which *is* assigned.]

4. For charge conjugation we had

$$\psi_c = C\tilde{\bar{\psi}}, \quad C^{-1}\gamma^u C = -\tilde{\gamma}^u, \quad C^+C = 1,$$

and it was proved in lecture that $\tilde{C} = -C$ and that $(\psi_c)_c = \psi$, so that we can say that the charge-conjugation transformation is "self-reciprocal."

Writing T rather than the much-used S, we had for Wigner time-reversal

$$\tilde{\bar{\psi}}.T = \psi_{rev}, \quad T^{-1}\gamma^u T = \tilde{\gamma}^4\tilde{\gamma}^u\tilde{\gamma}^4, \quad T^+T = 1.$$

a. Without using a representation of the γ^u, show that $T\tilde{T}^{-1}$ is a multiple of the unit matrix, and that in fact $\tilde{T} = -T$. Show whether or not the transformation is self-reciprocal.

b. Show that $\psi = S_g(\psi_c)_{rev}$, where S_g is, to within a phase factor, the matrix found earlier in the term for the "geometrical time reflection", $x_k = x_k'$, $x_4 = -x_4'$.

5. a. Find out what you can about the relation between $(\psi_c)_{rev}$ and $(\psi_{rev})_c$.

b. In lecture, the formula $\psi_c = C\tilde{\bar{\psi}}$ was first fixed by requiring that ψ_c is to have the same Lorentz transformation (L.T.) properties (for proper L.T.'s and for space inversion) as ψ has. After this, we found that the substitution $\tilde{\bar{\psi}} = C^{-1}\psi_c$ gives an interesting and important transformation of the Dirac equation.

In the case of time reversal, we fixed T by requiring that the substitution $\tilde{\bar{\psi}} = \psi_{rev}T^{-1}$ in the Dirac equation give an equation of the desired form. Make such investigation as you think suitable of the relation between the L.T. properties of ψ and those of ψ_{rev}.

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Problem 1

10

A 4×4 matrix group cannot contain more than 6 linearly independent imaginary and antisymmetric matrices.
We are given 4 Hermitian matrices γ^μ obeying $\{\gamma^\mu, \gamma^\nu\} = 2\delta_{\mu\nu}$.
What types of representation $\{m, n\}$ are possible?

Try $(3,1)$: say: $\tilde{\gamma}^4 = -\gamma^4$
$\tilde{\gamma}^k = \gamma^k \; ; \; k = 1, 2, 3$

$\tilde{1} = 1$	1
$\tilde{\gamma}^4 = -\gamma^4$	1
$\tilde{\gamma}^k = \gamma^k$	3
$\tilde{\gamma}^5 = -\gamma^5$	1
$\widetilde{\gamma^4\gamma^k} = \tilde{\gamma}^k\tilde{\gamma}^4 = -\gamma^k\gamma^4 = \gamma^4\gamma^k$	3
$\widetilde{\gamma^k\gamma^l} = \tilde{\gamma}^l\tilde{\gamma}^k = \gamma^l\gamma^k = -\gamma^k\gamma^l \; ; \; k \neq l$	3
$\widetilde{\gamma^5\gamma^4} = \tilde{\gamma}^4\tilde{\gamma}^5 = \gamma^4\gamma^5 = -\gamma^5\gamma^4$	1
$\widetilde{\gamma^5\gamma^k} = \tilde{\gamma}^k\tilde{\gamma}^5 = -\gamma^k\gamma^5 = \gamma^5\gamma^k$	3

There are 6 antisymmetric matrices, hence $(3,1)$ is possible. A representation would be:

$\gamma^1 = \alpha_x$
$\gamma^2 = \alpha_z$
$\gamma^3 = \beta$
$\gamma^4 = \alpha_y$

where the α's and β are the Dirac matrices

Try $(2,2)$: say: $\tilde{\gamma}^k = \gamma^k \; ; \; k = 1, 2$
$\tilde{\gamma}^l = -\gamma^l \; ; \; l = 3, 4$

$\tilde{1} = 1$	1
$\tilde{\gamma}^5 = \gamma^5$	1
$\widetilde{\gamma^1\gamma^2} = \tilde{\gamma}^2\tilde{\gamma}^1 = \gamma^1\gamma^2$	2
$\widetilde{\gamma^2\gamma^l} = \gamma^2\gamma^l$	2
$\widetilde{\gamma^k\gamma^3} = \tilde{\gamma}^3\tilde{\gamma}^k = \gamma^k\gamma^3$	2
$\widetilde{\gamma^k\gamma^4} = \gamma^k\gamma^4$	2
$\widetilde{\gamma^5\gamma^l} = \tilde{\gamma}^l\tilde{\gamma}^5 = \gamma^5\gamma^l$	2
$\widetilde{\gamma^5\gamma^k} = \tilde{\gamma}^k\tilde{\gamma}^5 = -\gamma^5\gamma^k$	2
$\tilde{\gamma}^k = \gamma^k$	2
$\tilde{\gamma}^l = -\gamma^l$	2

We see that $(2,2)$ is possible. Choose the representation:

$\gamma^1 = \beta \; ; \; \tilde{\gamma}^1 = \beta$

$\gamma^2 = -i\beta\alpha_2 = -i\rho_3\rho_1\sigma_2 \; ; \; \tilde{\gamma}^2 = -i\tilde{\sigma}_2\tilde{\rho}_1\tilde{\rho}_3 = -i\rho_3\rho_1\sigma_2$

$\gamma^3 = -i\rho_3\rho_1\sigma_3 \; ; \; \tilde{\gamma}^3 = i\rho_3\rho_1\sigma_3$

$\gamma^4 = -i\rho_3\rho_1\sigma_1 \; ; \; \tilde{\gamma}^4 = i\rho_3\rho_1\sigma_1$

Try $(4,0)$: $\tilde{\gamma}^\mu = \gamma^\mu$; $\mu = 1,2,3,4$

$\tilde{\gamma}^\mu = \gamma^\mu$ 4

$\widetilde{\gamma^\mu\gamma^\nu} = \tilde{\gamma}^\nu\tilde{\gamma}^\mu = \gamma^\nu\gamma^\mu = -\gamma^\mu\gamma^\nu$; $\mu \neq \nu$ 6

$\widetilde{\gamma^5\gamma^\mu} = \gamma^\mu\gamma^5 = -\gamma^5\gamma^\mu$ 4

$\tilde{\gamma}^5 = \gamma^5$ 1

$\tilde{1} = 1$ 1

There are 10 linearly independent antisymmetric matrices, thus $(4,0)$ not possible.

Try $(0,4)$: $\tilde{\gamma}^\mu = -\gamma^\mu$; $\mu = 1,2,3,4$

$\tilde{1} = 1$ 1

$\tilde{\gamma}^\mu = -\gamma^\mu$ 4

$\widetilde{\gamma^\mu\gamma^\nu} = \gamma^\nu\gamma^\mu = -\gamma^\mu\gamma^\nu$; $\mu \neq \nu$ 6

$\widetilde{\gamma^5\gamma^\mu} = \gamma^\mu\gamma^5 = -\gamma^5\gamma^\mu$ 4

$\tilde{\gamma}^5 = \gamma^5$ 1

This is the same situation as $(4,0)$, thus $(0,4)$ is not possible either.

Try $(1,3)$: $\tilde{\gamma}^4 = \gamma^4$
$\tilde{\gamma}^k = -\gamma^k$; $k = 1,2,3$

$\tilde{1} = 1$ 1

$\tilde{\gamma}^4 = \gamma^4$ 1

$\tilde{\gamma}^k = -\gamma^k$ 3

$\widetilde{\gamma^4\gamma^k} = \tilde{\gamma}^k\tilde{\gamma}^4 = \gamma^4\gamma^k$ 3

$\widetilde{\gamma^k\gamma^l} = \tilde{\gamma}^l\tilde{\gamma}^k = -\gamma^k\gamma^l$; $l \neq k$ 3

$\widetilde{\gamma^5\gamma^4} = \tilde{\gamma}^4\tilde{\gamma}^5 = \gamma^5\gamma^4$ 1

$\widetilde{\gamma^5\gamma^k} = \tilde{\gamma}^k\tilde{\gamma}^5 = \gamma^k\gamma^5 = -\gamma^5\gamma^k$ 3

$\tilde{\gamma}^5 = -\gamma^5$ 1

Again we have 10 linearly independent antisymmetric matrices, so $(1,3)$ is not possible.

Only types possible: $(3,1)$; $(2,2)$

Impossible: $(4,0)$; $(0,4)$; $(1,3)$

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Problem 2

From lecture, the appropriate equations for the problem are:

$$i\hbar \frac{\partial \varphi}{\partial t} = e\phi\varphi + \underbrace{\left(\frac{\pi^2}{2m} - \frac{1}{8m^3c^2}\pi^2\pi^2\right)}_{\text{mass correction}}\varphi - \underbrace{\frac{e\hbar}{2mc}(\vec{\sigma}\cdot\vec{\mathcal{H}})\varphi}_{\text{Zeeman}}$$

$$+ \underbrace{\frac{e\hbar}{4m^2c^2}\left[\vec{\sigma}\cdot(\vec{\pi}\times\vec{\mathcal{E}})\right]\varphi}_{\text{S.O. coupling}} - \underbrace{\frac{ie\hbar}{4m^2c^2}(\vec{\pi}\cdot\vec{\mathcal{E}})\varphi}_{\text{Darwin term}}$$

$$\vec{\pi} = \vec{p} - \frac{e}{c}\vec{A} = -i\hbar\nabla - \frac{e}{c}\vec{A} \quad ; \quad \vec{\mathcal{H}} = \nabla\times\vec{A} \quad ; \quad \vec{\mathcal{E}} = -\nabla\phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}$$

$$\chi \simeq \frac{1}{2mc}(\vec{\sigma}\cdot\vec{\pi})\varphi$$

Form: $$\frac{d}{dt}\varphi^{\dagger}\varphi = \varphi^{\dagger}\frac{d\varphi}{dt} + \frac{d\varphi^{\dagger}}{dt}\varphi = \varphi^{\dagger}\frac{d\varphi}{dt} + \left(\varphi^{\dagger}\frac{d\varphi}{dt}\right)^{\dagger}$$

or: $$i\hbar\frac{d}{dt}\varphi^{\dagger}\varphi = \varphi^{\dagger}\cdot i\hbar\frac{d\varphi}{dt} - \left(\varphi^{\dagger}\cdot i\hbar\frac{d\varphi}{dt}\right)^{\dagger}$$

now: $$i\hbar\,\varphi^{\dagger}\frac{d\varphi}{dt} = e\phi\varphi^{\dagger}\varphi + \varphi^{\dagger}\left(\frac{\pi^2}{2m} - \frac{1}{8m^3c^2}\pi^2\pi^2\right)\varphi - \frac{e\hbar}{2mc}\varphi^{\dagger}(\vec{\sigma}\cdot\vec{\mathcal{H}})\varphi$$

$$+ \frac{e\hbar}{4m^2c^2}\varphi^{\dagger}\left[\vec{\sigma}\cdot(\vec{\pi}\times\vec{\mathcal{E}})\right]\varphi - \frac{ie\hbar}{4m^2c^2}\varphi^{\dagger}(\vec{\pi}\cdot\vec{\mathcal{E}})\varphi$$

$$-i\hbar\frac{d\varphi^{\dagger}}{dt}\varphi = e\phi\,\varphi^{\dagger}\varphi + \varphi^{\dagger}\left(\frac{\pi^{2\dagger}}{2m} - \frac{1}{8m^3c^2}\pi^{2\dagger}\pi^{2\dagger}\right)\varphi$$

$$- \frac{e\hbar}{2mc}\varphi^{\dagger}(\vec{\sigma}\cdot\vec{\mathcal{H}})\varphi + \frac{e\hbar}{4m^2c^2}\varphi^{\dagger}\left[(\vec{\pi}\times\vec{\mathcal{E}})^{\dagger}\cdot\vec{\sigma}\right]\varphi + \frac{ie\hbar}{4m^2c^2}\varphi^{\dagger}(\vec{\pi}\cdot\vec{\mathcal{E}})^{\dagger}\varphi$$

① $$i\hbar\frac{d}{dt}\varphi^{\dagger}\varphi = \frac{1}{2m}\left[\varphi^{\dagger}\pi^2\varphi - \varphi^{\dagger}\pi^{2\dagger}\varphi\right] - \frac{1}{8m^3c^2}\left[\varphi^{\dagger}\pi^2\pi^2\varphi - \varphi^{\dagger}\pi^{2\dagger}\pi^{2\dagger}\varphi\right]$$

$$+ \frac{e\hbar}{4m^2c^2}\left[\varphi^{\dagger}\vec{\sigma}\cdot(\vec{\pi}\times\vec{\mathcal{E}})\varphi - \varphi^{\dagger}(\pi\times\varepsilon)^{\dagger}\cdot\vec{\sigma}\varphi\right]$$

$$- \frac{ie\hbar}{4m^2c^2}\left[\varphi^{\dagger}(\vec{\pi}\cdot\vec{\mathcal{E}})\varphi + \varphi^{\dagger}(\vec{\pi}\cdot\varepsilon)^{\dagger}\varphi\right]$$

Operators with $\dagger$ are taken in the Dirac sense to operate to the left with appropriate sign changes.

Note that because the operators involved in ① are Hermitean, the first three terms lead to divergences. Now find: $i\hbar \frac{d}{dt}\chi^+\chi$

$$\chi = \frac{1}{2mc}(\vec{\sigma}\cdot\vec{\pi})\varphi \; ; \; \chi^+ = \frac{1}{2mc}\varphi^+(\vec{\sigma}\cdot\vec{\pi})^+ \; ; \; \frac{d\chi}{dt} = \frac{1}{2mc}(\vec{\sigma}\cdot\vec{\pi})\frac{d\varphi}{dt} + \frac{1}{2mc}\left(\vec{\sigma}\cdot\frac{d\vec{\pi}}{dt}\right)\varphi$$

$$i\hbar\frac{d\varphi}{dt} - e\phi\varphi = c(\vec{\sigma}\cdot\vec{\pi})\chi \quad ; \quad i\hbar\frac{d\chi}{dt} + 2mc^2\chi - e\phi\chi = c(\vec{\sigma}\cdot\vec{\pi})\varphi$$

$$i\hbar\frac{d\varphi}{dt} = e\phi\varphi + \frac{1}{2m}(\vec{\sigma}\cdot\vec{\pi})(\vec{\sigma}\cdot\vec{\pi})\varphi$$

$$i\hbar\frac{d\chi}{dt} = \frac{e}{2mc}(\sigma\cdot\pi)\phi\varphi + \frac{1}{4m^2c}(\sigma\cdot\pi)^3\varphi + \frac{i\hbar e}{2mc}\left(\sigma\cdot -\frac{1}{c}\frac{dA}{dt}\right)\varphi$$

$$= \frac{i\hbar e}{2mc}(\vec{\sigma}\cdot\mathcal{E})\varphi + \frac{e\phi}{2mc}(\sigma\cdot\pi)\varphi + \frac{1}{4m^2c}(\sigma\cdot\pi)^3\varphi$$

$$i\hbar\chi^+\frac{d\chi}{dt} = \frac{i\hbar e}{4m^2c^2}\varphi^+(\sigma\cdot\pi)^+(\sigma\cdot\mathcal{E})\varphi + \frac{e}{4m^2c^2}\varphi^+(\sigma\cdot\pi)^+\phi(\sigma\cdot\pi)\varphi + \frac{1}{8m^3c^2}\varphi^+(\sigma\cdot\pi)^+(\sigma\cdot\pi)^3\varphi$$

Examine term by term: Use Hermiticity properties:

$\varphi^+(\sigma\cdot\pi)^+(\sigma\cdot\mathcal{E})\varphi = \varphi^+(\sigma\cdot\pi)(\sigma\cdot\mathcal{E})\varphi$ + divergence $= \varphi^+(\pi\cdot\mathcal{E})\varphi + i\varphi^+\sigma\cdot[\pi\times\mathcal{E}]\varphi$

+ divergence, using $(\sigma\cdot a)(\sigma\cdot b) = a\cdot b + i\sigma\cdot[a\times b]$

$\varphi^+(\sigma\cdot\pi)^+(\sigma\cdot\pi)^3\varphi = \varphi^+(\sigma\cdot\pi)^4\varphi$ + divergence $= \varphi^+\pi^2\pi^2\varphi$ + divergence

+ higher order terms in $\frac{v}{c}$, since $(\sigma\cdot\pi)^2 = \pi^2 - \frac{e\hbar}{c}\sigma\cdot\mathcal{H}$

and the cross products of $(\sigma\cdot\pi)^2(\sigma\cdot\pi)^2$ give higher terms in $\frac{v}{c}$ than $\left(\frac{v}{c}\right)^2$. Now we know: $i\hbar\frac{d}{dt}\chi^+\chi = i\hbar\chi^+\frac{d\chi}{dt} - \left(\chi^+\cdot i\hbar\frac{d\chi}{dt}\right)^+$

so we see the term involving ϕ in $i\hbar\chi^+\frac{d\chi}{dt}$ will cancel with its counterpart in $\left(\chi^+\cdot i\hbar\frac{d\chi}{dt}\right)^+$. Thus we can finally write:

$$② \quad i\hbar\frac{d}{dt}\chi^+\chi = \frac{i\hbar e}{4m^2c^2}\left[\varphi^+(\pi\cdot\mathcal{E})\varphi + \varphi^+(\pi\cdot\mathcal{E})^+\varphi\right] - \frac{e\hbar}{4m^2c^2}\left[\varphi^+\sigma\cdot[\pi\times\mathcal{E}]\varphi - \varphi^+[\pi\times\mathcal{E}]^+\cdot\sigma\varphi\right]$$

$$+ \frac{1}{8m^3c^2}\left[\varphi^+\pi^2\pi^2\varphi - \varphi^+\pi^{2+}\pi^{2+}\varphi\right] + \text{divergences} + \text{higher order}$$

terms than $\left(\frac{v}{c}\right)^2$ as shown. It is to be recognized that the second and third terms of ② while canceling their counterparts in ① lead to divergences anyway, while the first term in ② exactly cancels the last term in ① which is the Darwin non-Hermitean term. This is what was to be shown.

Not True!

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Problem 3

Commutation Rules:

$$\{\alpha_k, \alpha_\ell\} = 2\,\delta_{k\ell} \quad ; \quad \{\alpha_\ell, \beta\} = 0 \quad ; \quad \beta^2 = 1$$

$$[x_k, p_\ell] = i\hbar\,\delta_{k\ell}$$

$$\frac{dA}{dt} = \frac{1}{i\hbar}[A, H] + \frac{\partial A}{\partial t}$$

Free particle Dirac Hamiltonian: $H = c\,\vec{\alpha}\cdot\vec{p} + \beta m c^2$

$$\dot{x} = \frac{1}{i\hbar}[x, H] = c\,\alpha_x$$

$$\ddot{x} = \frac{c}{i\hbar}\left(2c\,p_x - 2H\alpha_x\right)$$

$$\dddot{x} = \frac{1}{i\hbar}[\ddot{x}, H] = \frac{-2c}{(i\hbar)^2} H\,[\alpha_x, H] = \frac{2iH}{\hbar}\,\ddot{x}$$

Since H is a constant of the motion: $x^{(n)} = \frac{2iH}{\hbar}\,x^{(n-1)}$

$$\left(\frac{d^n x}{dt^n}\right)_0 = \frac{2c^2 p_x}{i\hbar}\left(\frac{2iH}{\hbar}\right)^{n-2} + \left(\frac{2iH}{\hbar}\right)^{n-1} v_0 \quad ; \quad n = 2, \cdots \quad ; \quad v_0 = c\,\alpha_x^0$$

$$x = x(0) + \sum_{n=1}^{\infty}\left(\frac{d^n x}{dt^n}\right)_0 \frac{t^n}{n!} = x(0) + v_0 t + \sum_{n=2}^{\infty}\frac{2c^2 p_x}{i\hbar}\left(\frac{2iH}{\hbar}\right)^{n-2}\frac{t^n}{n!}$$

$$+ \sum_{n=2}^{\infty}\left(\frac{2iH}{\hbar}\right)^{n-1}\frac{t^n}{n!}\,v_0 = x(0) + \left(\frac{\hbar}{2iH}\right)\sum_{n=1}^{\infty}\left(\frac{2iH}{\hbar}\right)^n \frac{t^n}{n!}\,v_0$$

$$+ \left(\frac{2c^2 p_x}{i\hbar}\right)\left(\frac{\hbar}{2iH}\right)^2 \sum_{n=2}^{\infty}\left(\frac{2iH}{\hbar}\right)^n \frac{t^n}{n!}$$

$$= x(0) + \left[\left(\frac{\hbar}{2iH}\right)\sum_{n=1}^{\infty}\left(\frac{2iH}{\hbar}\right)^n\frac{t^n}{n!}\right]\left[v_0 + \left(\frac{2c^2 p_x}{i\hbar}\right)\left(\frac{\hbar}{2iH}\right)\right]$$

$$- \left(\frac{2c^2 p_x}{i\hbar}\right)\left(\frac{\hbar}{2iH}\right)^2\left(\frac{2iH}{\hbar}\right)t \quad , \text{ or:}$$

$$\boxed{x = x(0) + \frac{c^2 p_x}{H}\,t + \frac{\hbar}{2iH}\,e^{\frac{i2Ht}{\hbar}}\left[v_0 - \frac{c^2 p_x}{H}\right]}$$

where $x(0)$ is different than before.

This result is essentially the same obtained by Dirac. We see that there is an oscillatory part and a linear part in the time dependence of x. We examine the linear part first. Recall that classically:

$$H \rightarrow E = \frac{mc^2}{\sqrt{1-\beta^2}} \quad ; \quad p_x = \frac{m v_x}{\sqrt{1-\beta^2}}$$

so that in the classical limit, $\frac{c^2 p_x}{H} \rightarrow v_{x0}$ for a free particle which is expected. Thus the coefficient of t expresses initial velocity in the classical limit. ✓

The oscillatory term has a tremendously high frequency, at least of the order $\frac{2mc^2}{\hbar}$. Thus in a practical experiment only the first two terms in x contribute because the interval of measurement is usually much larger than $\frac{\hbar}{2mc^2}$. Notice that the coefficient of the oscillatory term vanishes in the classical limit as one expects and is likely to remain small even non-classically, or at least of the order of magnitude $\frac{\hbar}{mc}$. The oscillatory term also expresses the effect of the uncertainty principle (relativistic). Explain.

Instantaneous velocity?

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$\boxed{\text{Problem 4}}$

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a) Given: $\psi_{rev} = \bar{\psi} T$; $T^{-1}\gamma^{\mu} T = \tilde{\gamma}^4 \tilde{\gamma}^{\mu} \tilde{\gamma}^4$; $T^{+}T = 1$

$\{\gamma^{\mu}, \gamma^{\nu}\} = 2\delta_{\mu\nu}$

We have: $T^{-1}\gamma^4 T = \tilde{\gamma}^4$: $T\tilde{\gamma}^4 = \gamma^4 T$

$T^{-1}\gamma^k T = -\tilde{\gamma}^k$: $T\tilde{\gamma}^k = -\gamma^k T$

Then:

$T\tilde{\gamma}^4 = \gamma^4 T$	$T\tilde{\gamma}^k = -\gamma^k T$
$\gamma^4 \tilde{T} = \tilde{T}\tilde{\gamma}^4$	$\gamma^k \tilde{T} = -\tilde{T}\tilde{\gamma}^k$
$\tilde{\gamma}^4 = T^{-1}\gamma^4 T$	$\tilde{\gamma}^k = -T^{-1}\gamma^k T$
$\gamma^4\tilde{T} = \tilde{T}T^{-1}\gamma^4 T \{ T^{-1}$	$\gamma^k \tilde{T} = \tilde{T}T^{-1}\gamma^k T \{ T^{-1}$
$\therefore \gamma^4 \tilde{T}T^{-1} = \tilde{T}T^{-1}\gamma^4$	$\therefore \gamma^k \tilde{T}T^{-1} = \tilde{T}T^{-1}\gamma^k$

Hence $\gamma^{\mu}\tilde{T}T^{-1} = \tilde{T}T^{-1}\gamma^{\mu}$, so that we see that $\tilde{T}T^{-1}$ commutes with the complete set of γ's and hence must be some multiple of the unit matrix. That is:

$\tilde{T}T^{-1} = b$; or $\tilde{T} = bT$

Now, $T = b\tilde{T} = b^2 T$; hence $b = \pm 1$. We find which sign by transposing T combined with all the elements of the γ group.

$\tilde{T} = bT$ ①

$\widetilde{\gamma^4 T} = \tilde{T}\tilde{\gamma}^4 = bT\tilde{\gamma}^4 = bTT^{-1}\gamma^4 T = b\gamma^4 T$ ①

$\widetilde{\gamma^k T} = \tilde{T}\tilde{\gamma}^k = bT\tilde{\gamma}^k = -bTT^{-1}\gamma^k T = -b\gamma^k T$ ③

$\widetilde{\gamma^5 T} = \tilde{T}\tilde{\gamma}^5 = bT\tilde{\gamma}^5 = bT\tilde{\gamma}^4\tilde{\gamma}^3\tilde{\gamma}^2\tilde{\gamma}^1 = bTT^{-1}\gamma^4 T\tilde{\gamma}^3\tilde{\gamma}^2\tilde{\gamma}^1 = b\gamma^4 T\tilde{\gamma}^3\tilde{\gamma}^2\tilde{\gamma}^1$

$= -b\gamma^4\gamma^3\gamma^2\gamma^1 T = -b\gamma^5 T$ ①

$\widetilde{\gamma^5\gamma^4 T} = \tilde{T}\tilde{\gamma}^4\tilde{\gamma}^5 = bT\tilde{\gamma}^4\tilde{\gamma}^5 = bTT^{-1}\gamma^4 T\tilde{\gamma}^5 = -b\gamma^4\gamma^5 T = b\gamma^5\gamma^4 T$ ①

$\widetilde{\gamma^5\gamma^k T} = \tilde{T}\tilde{\gamma}^k\tilde{\gamma}^5 = b\gamma^k\gamma^5 T = -b\gamma^5\gamma^k T$ ③

$\widetilde{\gamma^4\gamma^k T} = \tilde{T}\tilde{\gamma}^k\tilde{\gamma}^4 = bT\tilde{\gamma}^k\tilde{\gamma}^4 = -b\gamma^k T\tilde{\gamma}^4 = -b\gamma^k\gamma^4 T$ ③

$l \neq k$: $\widetilde{\gamma^k\gamma^l T} = \tilde{T}\tilde{\gamma}^l\tilde{\gamma}^k = b\gamma^l\gamma^k T = -b\gamma^k\gamma^l T$ ③

If $b = +1$, there are 10 antisymmetric matrices; not possible, therefore: $b = -1$, and:

$$\boxed{\tilde{T} = -T}$$

We want to show if:

$$(\psi_{rev})_{rev} = \psi \quad ; \quad (\psi_{rev})_{rev} = \bar{\Psi}_{rev} T \quad ; \text{ or: } \quad \underbrace{\psi}_{row} = \underbrace{\bar{\Psi}_{rev}}_{row} T \quad ?$$

We are given: $\underbrace{\psi_{rev}}_{row} = \underbrace{\bar{\Psi}}_{row} T = \underbrace{\psi^*}_{row} \gamma^4 T$

now: $\bar{\Psi}_{rev} = \psi^*_{rev} \gamma^4 = \psi \gamma^{4*} T^* \gamma^4$

Then: $\bar{\Psi}_{rev} T = \psi \gamma^{4*} T^* \gamma^4 T$

What is: $\gamma^{4*} T^* \gamma^4 T$?

$$T^* = \tilde{T}^+ = \tilde{\tilde{T}}^+ = -T^+ = -T^{-1}$$

$$\gamma^{4*} = \tilde{\gamma}^{4+} = \widetilde{\gamma^{4+}} = \tilde{\gamma}^4 \quad ; \text{ since the } \gamma\text{'s are Hermitean.}$$

Then: $\gamma^{4*} T^* \gamma^4 T = -\tilde{\gamma}^4 T^{-1} \gamma^4 T = -\tilde{\gamma}^4 \tilde{\gamma}^4 = -I$

Hence:

$$\boxed{\bar{\Psi}_{rev} T = -\psi \quad ; \text{ or, } \quad (\psi_{rev})_{rev} = -\psi}$$

and the transformation is not self-reciprocal.

b) $\underbrace{\psi_c}_{column} = C \underbrace{\tilde{\Psi}}_{column}$; $\underbrace{\psi_{rev}}_{column} = \widetilde{\underbrace{\bar{\Psi}}_{row} T} = \tilde{T} \underbrace{\tilde{\bar{\Psi}}}_{column} = -T \underbrace{\tilde{\bar{\Psi}}}_{column}$

now: $(\psi_c)_{rev} = -T \tilde{\bar{\Psi}}_c$; $\tilde{\bar{\Psi}}_c = \tilde{\gamma}^4 \psi_c^*$; $\psi_c = C \tilde{\gamma}^4 \psi^*$;

$\psi_c^* = C^* \tilde{\gamma}^{4*} \psi = C^* \gamma^4 \psi$; $\tilde{\bar{\Psi}}_c = \tilde{\gamma}^4 C^* \gamma^4 \psi$

Recall: $C^{-1} \gamma^\mu C = -\tilde{\gamma}^\mu$; $C^+ C = I$; $\tilde{C} = -C$

$\therefore C^* = \tilde{C}^+ = -C^+ = -C^{-1}$

Hence: $\tilde{\gamma}^4 C^* \gamma^4 = C^{-1}$; and, $\tilde{\bar{\Psi}}_c = C^{-1} \psi$

and: $(\psi_c)_{rev} = -T C^{-1} \psi$; $\boxed{\psi = -C T^{-1} (\psi_c)_{rev} \quad ; \quad S_g = -C T^{-1}}$

What is the effect of S_g operating on the γ^μ's? That is, what is: $S_g \gamma^4 S_g^{-1}$; $S_g \gamma^k S_g^{-1}$?

$S_g \gamma^4 S_g^{-1} = C T^{-1} \gamma^4 T C^{-1} = C \tilde{\gamma}^4 C^{-1} = -\gamma^4$; $S_g \gamma^k S_g^{-1} = \gamma^k$

Recall that earlier in the term, we had for geometric time reflection: $S \gamma^k S^{-1} = \gamma^k$; $S \gamma^4 S^{-1} = -\gamma^4$, where S was an operator in the 4-space that gave $x_k \to x'_k$; $x_4 \to -x'_4$. At that time we choose: $S = \gamma^1 \gamma^2 \gamma^3$, as the time reflection matrix.

Note that: $(C T^{-1})^+ = T C^{-1}$, or $S_g^+ = S_g^{-1}$, or S_g is unitary, hence S_g is determined as being the same, to within a factor of phase, as the old geometrical time-reflection operator.

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$\boxed{\text{Problem 5}}$

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a) $\boxed{\psi = -CT^{-1}(\psi_c)_{rev} = S_g(\psi_c)_{rev}}$

What is $(\psi_{rev})_c$? $\psi_c = C\bar{\psi}$; $\psi_{rev} = -T\bar{\psi}$; $\underbrace{(\psi_{rev})_c}_{column} = C\underbrace{\bar{\psi}_{rev}}_{column}$

Recall:

$$\left.\begin{aligned} \underbrace{\bar{\psi}_{rev}}_{row} &= -\underbrace{\psi}_{row}T^{-1} = -\underbrace{\psi}_{row}T^{+} \\ \underbrace{\bar{\psi}_{rev}}_{column} &= -\widetilde{\underbrace{\psi}_{row}T^{+}} = -\tilde{T}^{+}\underbrace{\psi}_{column} \end{aligned}\right\} \quad \begin{aligned} \tilde{T}^{+} &= \tilde{T}^{+} = -T^{+} = -T^{-1} \\ \therefore\ \bar{\psi}_{rev} &= T^{-1}\psi \end{aligned}$$

$\boxed{(\psi_{rev})_c = CT^{-1}\psi = -S_g\psi}$ $\boxed{(\psi_{rev})_c = -S_gS_g(\psi_c)_{rev} = -CT^{-1}CT^{-1}(\psi_c)_{rev}}$

What is $-S_gS_g$? We saw in problem 4 that S_g was the same as the geometrical time reflection operator up to a phase factor (S_g is unitary). Hence we take for S_g: $S_g = f\gamma^1\gamma^2\gamma^3$.

If we require S_g to be unimodular, this means: $f = \pm 1, \pm i$.

Now: $\gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3 = -1$. Hence, if $S_g = f\gamma^1\gamma^2\gamma^3$, then:

$-S_gS_g = f^2 = \pm 1$, if unimodular, and we have:

$$(\psi_{rev})_c = \pm(\psi_c)_{rev}$$

or: $(\psi_{rev})_c = f^2(\psi_c)_{rev}$, if unimodularity is not required.

However, the physically meaningful quantity is the probability density, $\psi^*\psi$, and we see, for any phase,

$$(\psi_{rev})_c^*(\psi_{rev})_c = (\psi_c)_{rev}^*(\psi_c)_{rev}$$

From the physical point of view when dealing with one state, then, it makes no difference as to whether we time-reverse and then charge conjugate, or vice versa.

b) For the case of charge conjugation, C was fixed by requiring that ψ_c have the same L.T. property as ψ, that is, we have $\psi = S\psi'$ and $\psi_c = S\psi_c'$. On the other hand, for the case of Time reversal, we fixed T by requiring that $\bar{\psi} = \psi_{rev} T^{-1}$ give an equation of the desired form, hence, ψ_{rev} does not necessarily have the same L.T. properties as ψ. However, we can write $\psi_{rev} = \mathcal{S}\psi'_{rev}$ where $\mathcal{S}$ is a transformation to bring ψ'_{rev} into the unprimed system. We have then:

$$\underbrace{\psi_{rev}}_{\text{column}} = \mathcal{S}\underbrace{\psi'_{rev}}_{\text{column}} \quad ; \quad \underbrace{\psi_{rev}}_{\text{row}} = \underbrace{\psi'_{rev}}_{\text{row}}\tilde{\mathcal{S}} \quad ; \quad \underbrace{\psi_{rev}}_{\text{row}}\tilde{\mathcal{S}}^{-1} = \underbrace{\psi'_{rev}}_{\text{row}}$$

Now: $\psi = S\psi'$; $\bar{\psi} = B\bar{\psi}' S^{-1}$; $\psi_{rev} = \bar{\psi} T$

Then: $\psi_{rev}\tilde{\mathcal{S}}^{-1} = \psi'_{rev} = \bar{\psi} T \tilde{\mathcal{S}}^{-1} = B\bar{\psi}' S^{-1} T \tilde{\mathcal{S}}^{-1}$

Now, for the invariance of the inversion operation, we require: $\psi'_{rev} = \bar{\psi}' T$, that is, the time inversion operation is to remained unchanged in the new coordinate system. Then $T = BS^{-1}T\tilde{\mathcal{S}}^{-1}$ or:

$$\boxed{\tilde{\mathcal{S}} = BT^{-1}S^{-1}T}$$

Space Inversion: $S = f\gamma^4$; $S^{-1} = f^{-1}\gamma^{4^{-1}} = f^{-1}\gamma^4$; $B = 1$

Recall $T^{-1}\gamma^4 T = \tilde{\gamma}^4$; $T^{-1}\gamma^4 = \tilde{\gamma}^4 T^{-1}$; $T^{-1}\gamma^k T = -\tilde{\gamma}^k$; $T^{-1}\gamma^k = -\tilde{\gamma}^k T^{-1}$

Now: $\tilde{\mathcal{S}} = f^{-1}T^{-1}\gamma^4 T = f^{-1}\tilde{\gamma}^4$; $\mathcal{S} = f^{-1}\gamma^4 = S^{-1}$

Pauli Choice: $f = 1$: $\mathcal{S} = \gamma^4 = S$

Racah Choice: $f = i$: $\mathcal{S} = -i\gamma^4 = -S$

Space Rotation: $k \neq l$ $S = e^{\theta/2\,\gamma^k\gamma^l} = \cos\frac{\theta}{2} + \gamma^k\gamma^l \sin\frac{\theta}{2}$; $S^{-1} = \cos\frac{\theta}{2} - \gamma^k\gamma^l\sin\frac{\theta}{2}$; $B = 1$

Now: $\tilde{\mathcal{S}} = \cos\frac{\theta}{2} - T^{-1}\gamma^k\gamma^l T \sin\frac{\theta}{2} = \cos\frac{\theta}{2} - \tilde{\gamma}^k\tilde{\gamma}^l \sin\theta/2$

$\mathcal{S} = \cos\frac{\theta}{2} - \gamma^l\gamma^k \sin\theta/2 = \cos\frac{\theta}{2} + \gamma^k\gamma^l \sin\frac{\theta}{2}$

Then: $l \neq k$: $\mathcal{S} = e^{\frac{\theta}{2}\gamma^k\gamma^l} = S$

$l = k$: $\mathcal{S} = e^{-\theta/2} = S^{-1}$ $= S$ x What in the world is this tpf.?

Take: $S = e^{\frac{\theta}{2}\gamma^4\gamma^k} = \cos\frac{\theta}{2} + \gamma^4\gamma^k\sin\frac{\theta}{2}$; $S^{-1} = \cos\frac{\theta}{2} - \gamma^4\gamma^k\sin\frac{\theta}{2}$; $B = 1$

Then: $\tilde{\mathcal{S}} = \cos\frac{\theta}{2} - T^{-1}\gamma^4\gamma^k T\sin\frac{\theta}{2} = \cos\frac{\theta}{2} + \tilde{\gamma}^4\tilde{\gamma}^k\sin\frac{\theta}{2}$

$\mathcal{S} = \cos\frac{\theta}{2} - \gamma^4\gamma^k\sin\theta/2$; $\mathcal{S} = e^{-\frac{\theta}{2}\gamma^4\gamma^k} = S^{-1}$ $\mathcal{S} = S$

Geometrical Time Reflection: $S = f\gamma^1\gamma^2\gamma^3$; $S^{-1} = f^{-1}\gamma^3\gamma^2\gamma^1$; $B = -1$

Now: $\tilde{\mathcal{S}} = -f^{-1}T^{-1}\gamma^3\gamma^2\gamma^1 T = f^{-1}\tilde{\gamma}^3\tilde{\gamma}^2\tilde{\gamma}^1$

Then: $\mathcal{S} = f^{-1}\gamma^1\gamma^2\gamma^3 = f^{-1}f^{-1}S$

Therefore: $f = \pm 1$: $\mathcal{S} = S$

$f = \pm i$: $\mathcal{S} = -S$

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