ASTRONOMY 230

CELESTIAL MECHANICS

INSTRUCTOR: LAUTMAN   OBSERVATORY CLASSROOM: MW 3:45-5:15

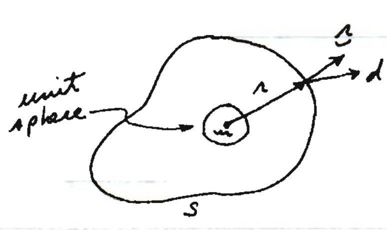
LECTURE 1: 2-6-62

Texts:
- Plummer: Dynamical Astronomy } Dover
- Brown: Lunar Theory } Dover
- Smart: Celestial Mechanics (Camb.)
- Brouwer & Clemence: Methods of Celestial Mechanics (AP)

Course Outline:
1. Two Body Problem: Elliptical motion
2. Expansions of elliptic functions
3. N-body problem
4. Perturbation: method of Lagrange and Variation of Parameters
5. Expansions of disturbing functions for moon and planets
6. Gravitational Potential: Gravitational potential of an non-spherical body; disturbing functions for artificial satellites due to oblateness of earth.
7. Planetary orbits
8. Variational Principle: Hamiltonian equations of motion and Hamilton-Jacobi equations.
9. Two-body problem in Hamilton-Jacobi contexts.
10. Motion of artificial satellites.
11. Lunar Theory: theory of Delauney; Ziepels modification.

Two Body Problem

Gauss' Theorem:



We want to get:

$$\iint \underline{E} \cdot d\underline{s}$$

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with the attractive force: $\underline{E} = -\dfrac{Gm}{r^2}\,\underline{\hat{r}}$

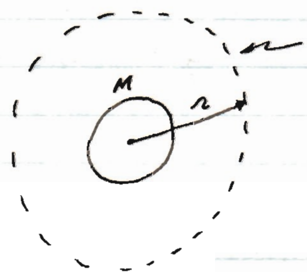
Hence, changing to the variable: $\dfrac{\underline{\hat{r}} \cdot d\underline{S}}{r^2} = d\Omega$

$$\iint \underline{E} \cdot d\underline{S} = -\iint \frac{Gm}{r^2}\,\underline{\hat{r}} \cdot d\underline{S} = -4\pi G m$$

G is the gravitational constant.

$$\boxed{\iint \underline{E} \cdot d\underline{S} = -4\pi G m}$$

Consider a spherical shell:



Gaussian surface on which F is constant

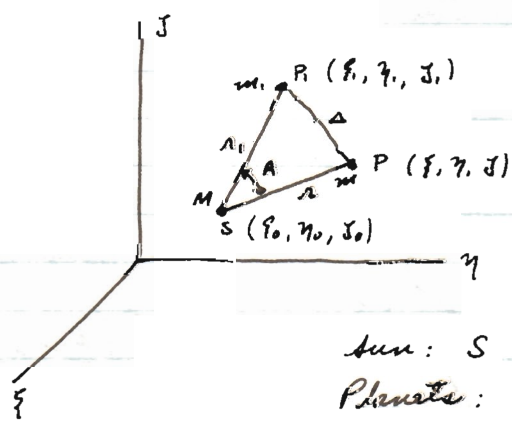
Hence:

$$\iint \underline{E} \cdot d\underline{S} = 4\pi r^2 F = -G\cancel{m}\,4\pi$$

and $\boxed{F = -\dfrac{GM}{r^2}}$ Thus spherical mass attracts as if all mass were at the center.

Because of the distance between planets, effects of oblateness and inhomogeniety are neglected.

We now find the equations of motion of a system of planets orbiting around the sun. Refer all motion in solar system to center of gravity of sun.
Go to inertial coordinate system:



Sun: S : M

Planets: P, P_1 ; m, m_1

$$r = \sqrt{(\xi-\xi_0)^2 + (\eta-\eta_0)^2 + (\zeta-\zeta_0)^2}$$

$$\Delta = \sqrt{(\xi-\xi_1)^2 + (\eta-\eta_1)^2 + (\zeta-\zeta_1)^2}$$

$$r_1 = \sqrt{(\xi_1-\xi_0)^2 + (\eta_1-\eta_0)^2 + (\zeta_1-\zeta_0)^2}$$

The potential at P is:

$$V_P = \frac{GM}{r} + \frac{Gm_1}{\Delta}$$

$$\ddot{\xi} = \frac{\partial V_P}{\partial \xi} = GM \frac{\partial}{\partial \xi}\left(\frac{1}{r}\right) + Gm_1 \frac{\partial}{\partial \xi}\left(\frac{1}{\Delta}\right)$$

$$= -GM \frac{(\xi-\xi_0)}{r^3} - Gm \frac{(\xi-\xi_1)}{\Delta^3}$$

We want motion with respect to sun, so find acceleration due to sun:

$$V_S = \frac{Gm}{r} + \frac{Gm_1}{r_1}$$

$$\ddot{\xi}_0 = Gm \frac{\partial}{\partial \xi_0}\left(\frac{1}{r}\right) + Gm_1 \frac{\partial}{\partial \xi_0}\left(\frac{1}{r_1}\right)$$

$$= Gm \frac{(\xi-\xi_0)}{r^3} + Gm_1 \frac{(\xi_1-\xi_0)}{r_1^3}$$

Now transform to sun coordinates:

$$\left.\begin{array}{ll} \text{Sun}: & 0,0,0 \\ P: & x, y, z \\ P_1: & x_1, y_1, z_1 \end{array}\right\} \quad \begin{array}{l} x = \xi - \xi_0 \\ x_1 = \xi_1 - \xi_0 \\ r = \sqrt{x^2+y^2+z^2}\ ; \quad r_1 = \sqrt{x_1^2+y_1^2+z_1^2} \end{array}$$

We find acceleration of planet P:

$$\ddot{x} = \ddot{\xi} - \ddot{\xi}_0$$

$$\ddot{x} = -\frac{GM(\xi-\xi_0)}{r^3} - \frac{Gm_1(\xi-\xi_1)}{\Delta^3} - \frac{Gm(\xi-\xi_0)}{r^3} - \frac{Gm_1(\xi_1-\xi_0)}{r_1^3}$$

$$\ddot{x} = -\frac{GMx}{r^3} - \frac{Gmx}{r^3} - \frac{Gm_1(x-x_1)}{\Delta^3} - \frac{Gm_1 x_1}{r_1^3}$$

Let: $\mu = G(M+m)$, then:

$$\ddot{x} = -\frac{\mu x}{r^3} + G m_1 \frac{\partial}{\partial x}\left(\frac{1}{\Delta}\right) - G m_1 \frac{\partial}{\partial x}\left\{\frac{x x_1 + y y_1 + z z_1}{r_1^3}\right\}$$

The addition of the extra factors makes no difference.

$$\ddot{x} = -\frac{\mu x}{r^3} + G m_1 \frac{\partial}{\partial x}\left\{\frac{1}{\Delta} - \frac{x x_1 + y y_1 + z z_1}{r_1^3}\right\}$$

and similarly for $\ddot{y}$ and $\ddot{z}$.

Considering $\ddot{x} = \frac{\partial V}{\partial x}$, we see that we can write:

$$V = \frac{\mu}{r} + R$$

where $R = G m_1 \left(\frac{1}{\Delta} - \frac{x x_1 + y y_1 + z z_1}{r_1^3} \right)$

Note that because $M >> m_1$, $R << \frac{\mu}{r}$ so that it is natural to consider this term as a perturbation.

If we add more planets, we can immediatly write:

$$R = G \sum_{i=1}^{n} m_i \left\{ \frac{1}{\Delta_i} - \underbrace{\frac{x x_i + y y_i + z z_i}{r_i^3}} \right\}$$

This can be put in cosine form.

$$\cos A_i = \frac{x x_i + y y_i + z z_i}{r r_i} \quad \text{(from dot product)}$$

Then:

$$\boxed{R = G \sum_{i=1}^{n} m_i \left\{ \frac{1}{\Delta_i} - \frac{r \cos A_i}{r_i^2} \right\}}$$

For the two body problem, $R=0$ and we have:

$$\ddot{x} = -\frac{\mu x}{r^3} \; ; \; \ddot{y} = -\frac{\mu y}{r^3} \; ; \; \ddot{z} = -\frac{\mu z}{r^3} \; ; \; \mu = G(M+m)$$

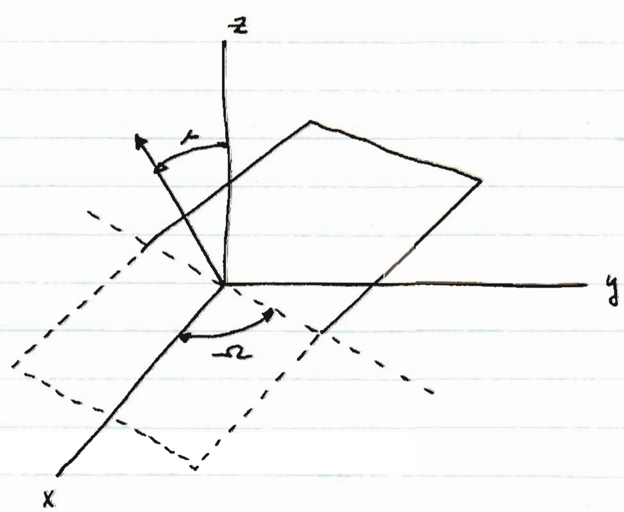
Form: $\ddot{y} z - \ddot{z} y = 0$

Integrate:

$$\left.\begin{array}{lll} \dot{y} z - \dot{z} y = A & \{ & x \\ z \dot{x} - \dot{z} x = B & \} & y \\ x \dot{y} - y \dot{x} = C & \} & z \end{array}\right\} \quad Ax + By + Cz = 0$$

which is the equation of a plane going thru the sun. Motion of planet is confined to this plane. The above equations also represent the components of angular momentum which are constants of the motion. The form can be seen from:

$$\underline{\Omega} \times \underline{V} = \begin{vmatrix} i & j & k \\ x & y & z \\ \dot{x} & \dot{y} & \dot{z} \end{vmatrix}$$



The position of the plane is given by i and Ω:

$$\cos i = \frac{C}{\sqrt{A^2 + B^2 + C^2}} \quad ; \quad \tan \Omega = -\frac{A}{B}$$

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We have determined two of the constants of integration, i, Ω, and now we find the motion in the plane:

$$\ddot{\underline{r}} = -\frac{\mu}{r^2}\,\underset{\smile}{\rho} \quad ; \quad \underline{r} = r\,\underset{\smile}{\rho}$$

$$\dot{\underline{r}} = r\,\dot{\underset{\smile}{\rho}} + \dot{r}\,\underset{\smile}{\rho} \quad ; \quad \dot{\underset{\smile}{\rho}} = \dot{\theta}\,\underline{n}$$

$\underline{n}$ is unit vector in θ direction.

$$\dot{\underline{r}} = r\dot{\theta}\,\underline{n} + \dot{r}\,\underset{\smile}{\rho}$$

$$\ddot{\underline{r}} = r\dot{\theta}\,\dot{\underline{n}} + (r\ddot{\theta} + \dot{r}\dot{\theta})\,\underline{n} + \ddot{r}\,\underset{\smile}{\rho} + \dot{r}\,\dot{\underset{\smile}{\rho}}$$

$$\dot{\underline{n}} = -\dot{\theta}\,\underset{\smile}{\rho}$$

Hence: $\ddot{\underline{r}} = (-r\dot{\theta}^2 + \ddot{r})\,\underset{\smile}{\rho} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\underline{n}$

now: $\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0$ since there is no normal component in the acceleration.

Furthermore:

$$\ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2}$$

We have then: $r^2\dot{\theta} = h$ (angular momentum) and we hence have one more constant, the magnitude of the angular momentum.

We attempt to eliminate time. Let $r = \frac{1}{u}$.

$$\dot{r} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{d}{d\theta}\left(\frac{1}{u}\right)hu^2 = -\frac{\frac{du}{d\theta}}{u^2}\,hu^2$$

$$= -h\,\frac{du}{d\theta} \quad , \text{ and:}$$

$$\ddot{r} = -h\,\frac{d}{d\theta}\left(\frac{du}{d\theta}\right)\dot{\theta} = -h^2u^2\,\frac{d^2u}{d\theta^2}$$

Plug in radial equation:

$$-h^2 u^2 \frac{d^2 u}{d\theta^2} - \frac{1}{u} h^2 u^4 = -\mu u^2$$

or:
$$\frac{d^2 u}{d\theta^2} + u = \frac{\mu}{h^2}$$

Essentially a driven harmonic oscillator whose solution is:

$$u = \frac{\mu}{h^2} \left\{ 1 + e \cos(\theta - \omega) \right\}$$

$$r = \frac{h^2/\mu}{1 + e \cos(\theta - \omega)} \qquad (\text{ellipse})$$

e and ω are constants of integration, e will be eccentricity. Also:

$$r = \frac{a(1-e^2)}{1 + e \cos\theta} \quad ; \quad a(1-e^2) = h^2/\mu$$

The radius vector sweeps out the area of the ellipse:

$$h = \frac{2A}{T} = \frac{2\pi a b}{P}$$

P, T = period

Let $n = \frac{2\pi}{P}$ = mean angular motion, then:

$$h = n a^2 \sqrt{1-e^2}$$

$$h^2 = \mu a (1-e^2) = n^2 a^4 (1-e^2) \quad ; \quad \mu = n^2 a^3$$

so $n^2 a^3$ is a constant for each planet. We have above verified Keplers Laws.

Kepler's Laws:

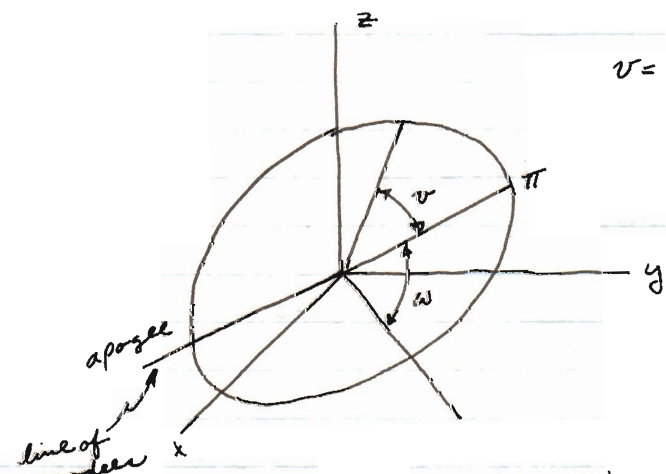
1. Planets move in ellipses with the sun at one focus
2. Equal areas are described in equal times.
3. $\frac{a^3}{P^2}$ = constant

The 3rd law can be seen from $\frac{a^3}{P^2} = G(M+m)$

or for another planet: $\frac{a_i^3}{P_i^2} = G(M+m_i)$

so the 3rd law must be modified somewhat. The original observations of Tycho were just enough to give ellipses but not good enough to give perturbations.

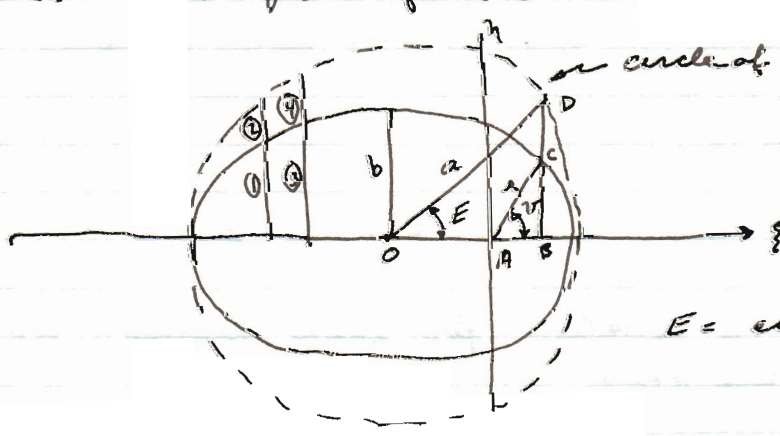
What is ω?



v = true anomaly

$$r = \frac{a(1-e^2)}{1+e\cos v}$$

We now find last constant of integration, the time element. Go to plane of orbit:



or circle of radius a

E = eccentric anomaly

$$\left.\begin{aligned} OB &= a \cos E \\ BD &= a \sin E \\ AB &= r \cos v \\ CB &= r \sin v \end{aligned}\right\}$$ In an ellipse, $OA = ae$

Another property of ellipses : $\frac{①}{②} = \frac{③}{④}$, hence $CB = b \sin E$

Then, we get the coordinates of any point on the ellipse:

$$\xi = a \cos E - ae = r \cos v$$
$$\eta = b \sin E \quad = r \sin v$$

We can now get equation for r in terms of E:

$$r^2 = a^2 \cos^2 E - 2b^2 e \cos E + a^2 e^2 + b^2 \sin^2 E$$

or: $r^2 = a^2 (1 - 2e \cos E + e^2 \cos^2 E)$

$$r = a(1 - e \cos E)$$

which is somewhat simpler than the other equations.
We also have relation for v from:

$$r = \frac{a(1-e^2)}{1+e \cos v}$$

Another relation can be gotten from:

$r \cos v = a(\cos E - e)$; use $\cos^2 \frac{x}{2} = \frac{1}{2} + \frac{1}{2} \cos x$

$$\cos v = 2 \cos^2 \frac{v}{2} - 1$$

Then: $2r \cos^2 \frac{v}{2} - a\left[1 - e\left(2\cos^2 \frac{E}{2} - 1\right)\right] = a\left(2 \cos^2 \frac{E}{2} - 1 - e\right)$

and: $r \cos^2 \frac{v}{2} = a(1-e) \cos^2 \frac{E}{2}$

now use: $\sin^2 \frac{x}{2} = \frac{1}{2} - \frac{1}{2} \cos x$ and get:

$$r \sin^2 \frac{v}{2} = a(1+e) \sin^2 \frac{E}{2}$$

Divide and get:

$$\boxed{\tan\frac{v}{2} = \sqrt{\frac{1+e}{1-e}}\,\tan\frac{E}{2}}$$

We now make use of E:

$$r = a(1-e\cos E)\ ,\quad \frac{1}{r} = \frac{1+e\cos v}{a(1-e^2)}$$

Then: $\dot{r} = ae\sin E\,\dot{E}$

$$-\frac{\dot{r}}{r^2} = -\frac{e\sin v\,\dot{v}}{a(1-e^2)}$$

and: $r^2\dot{v} = h$

$$\dot{r} = \frac{eh\sin v}{a(1-e^2)} = ae\sin E\,\dot{E}$$

Finally: $rae\dot{E} = \frac{ehb}{a(1-e^2)}$; use $h = n^{?}a^2\sqrt{1-e^2}$

or: $\boxed{r\dot{E} = na}$ Also: $\boxed{r^2\dot{v} = h}$

Now we look at: $(1-e\cos E)\dot{E} = n$ and get:

$$E - e\sin E = nt + c = n(t-\tau) = M$$

so that c is the last integration constant.

τ = time of perihelion passage

M = mean anomaly.

$$E - e\sin E = M$$

is known as Keplers equation (transcendental). Easy to solve if e is small.

$E \approx M$ if eccentricity is small and the solution can be approached by iteration:

$$E = M + e \sin\left(M + e \sin(M \cdots)\right)$$

Let us write down the elements (constants of integration). These of course are not unique. They are: $a, e, \tau, \omega, \Omega, i$. One could use any functions of these as well, for example, energy instead of a and angular momentum instead of e.

We now calculate the energy integral:

$$\left.\begin{aligned} \dot{x}\ddot{x} &= \frac{\partial U}{\partial x}\dot{x} \\ \dot{y}\ddot{y} &= \frac{\partial U}{\partial y}\dot{y} \\ \dot{z}\ddot{z} &= \frac{\partial U}{\partial z}\dot{z} \end{aligned}\right\} \qquad \ddot{x}\dot{x} + \ddot{y}\dot{y} + \ddot{z}\dot{z} = \frac{\partial U}{\partial x}\dot{x} + \frac{\partial U}{\partial y}\dot{y} + \frac{\partial U}{\partial z}\dot{z}$$

$$U = \frac{\mu}{r}$$

Then: $\frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) = U + C \quad ; \quad U = -V$

and: $T + V = C$ or energy is conserved and constant.

In the plane: $v^2 = \dot{x}^2 + \dot{y}^2$ (not same x, y as above)

$$\left.\begin{aligned} x &= a(\cos E - e) \\ y &= b \sin E \end{aligned}\right\} \quad \dot{x}^2 + \dot{y}^2 = \left(a^2 \sin^2 E + b^2 \cos^2 E\right)\dot{E}^2$$

$$= a^2\left(1 - e^2\cos^2 E\right)\dot{E}^2$$

$$= a^2(1 - e\cos E)\left[2 - (1 - e\cos E)\right] \quad ; \quad \text{use } \dot{E} = \frac{na}{r}$$

$$\cdot \frac{n^2 a^2}{r a(1 - e\cos E)}$$

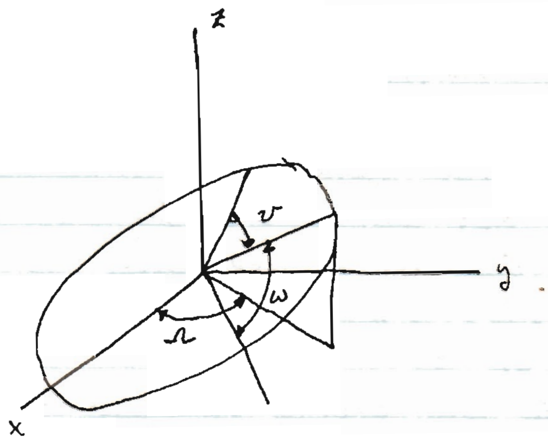
Then:

$$\dot{x}^2 + \dot{y}^2 = v^2 = \left(2 - \frac{r}{a}\right)\frac{n^2 a^3}{r}$$

and: $v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$

Finally: $\boxed{T + V = -\frac{\mu}{2a}}$ (depends only on semi-major axis)

LECTURE 3 : 2-12-62



The previously chosen elements are not necessarily unique.

Elements:

a	a	a	$M = n(t-\tau)$
e	e	e	$M = nt + \chi$
τ	χ	ϵ	$\chi = -n\tau$
i	i	i	
ω	ω	$\tilde{\omega}$	
Ω	Ω	Ω	

$\tilde{\omega} = \omega + \Omega$: longitude of perihelion

$L = \tilde{\omega} + v$: true longitude

$l = \tilde{\omega} + M$: mean longitude

$\quad = \tilde{\omega} + nt - n\tau$

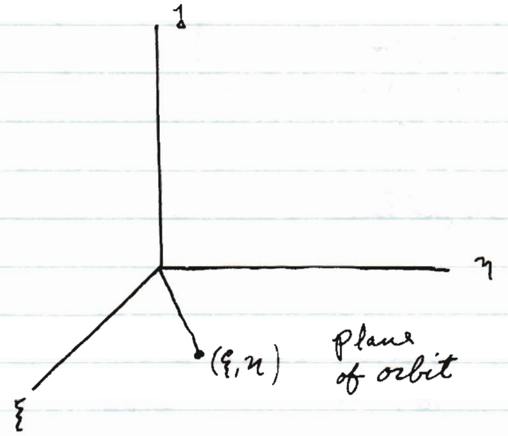
$\epsilon = \tilde{\omega} - n\tau$; mean longitude at the epoch

$M = nt + \epsilon - \tilde{\omega}$

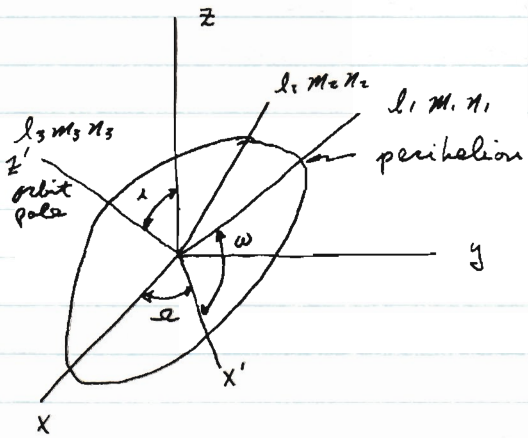
The first set is usually used in theoretical derivations while the last is used in qualitative discussions. The 6 elements can be determined by a knowledge of $x, y, z, \dot{x}, \dot{y}, \dot{z}$.

If ξ axis goes thru perihelion, so that in the orbit plane:

$$\xi = a(\cos E - e)$$
$$\eta = b \sin E$$



Now goto xyz coordinates:



Do the rotations on the Euhler angles: first thru Ω, then i, which brings into orbit plane and then ω to supperpose.

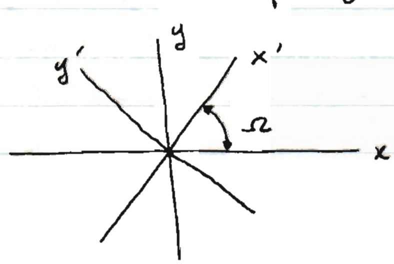
That is, perform the matrix rotations:

$$\underline{x}' = A\,\xi \;;\quad \underline{x}'' = B\,\underline{x}' \;;\quad \underline{x} = C\,\underline{x}'' \;;\quad \underline{x} = CBA\,\underline{\xi}$$

Then we will get from CBA:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ \zeta = 0 \end{pmatrix}$$

A is seen to be: $\quad A = \begin{pmatrix} \cos\Omega & \sin\Omega & 0 \\ -\sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{pmatrix}$



$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\Omega & \sin\Omega \\ -\sin\Omega & \cos\Omega \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Furthermore: $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos i & \sin i \\ 0 & -\sin i & \cos i \end{pmatrix}$; $C = \begin{pmatrix} \cos\omega & \sin\omega & 0 \\ -\sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Upon multiplying together, we get matrix of direction cosines.

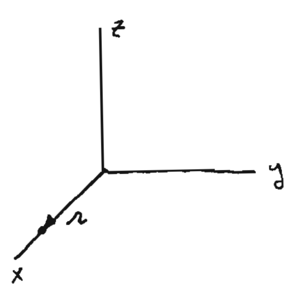
$$l_1 = \cos\Omega \cos\omega - \sin\Omega \sin\omega \cos i$$
$$m_1 = \sin\Omega \cos\omega + \cos\Omega \sin\omega \cos i$$
$$n_1 = \sin\omega \sin i$$

$$l_2 = -\cos\Omega \sin\omega - \sin\Omega \cos\omega \cos i$$
$$m_2 = -\sin\Omega \sin\omega + \cos\Omega \cos\omega \cos i$$
$$n_2 = \cos\omega \sin i$$

$$l_3 = \sin\Omega \sin i$$
$$m_3 = -\cos\Omega \sin i$$
$$n_3 = \cos i$$

Finally: $x = l_1 \xi + l_2 \eta$
$y = m_1 \xi + m_2 \eta$
$z = n_1 \xi + n_2 \eta$

We can also get $x y z$ in terms of mean anomaly. Use coordinate system that travels with planet. Can do same rotations thru angle $\omega + v$, or $\omega \to \omega + v$



$\begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix}$, $\therefore$ $x = l_1' r$
$y = m_1' r$
$z = n_1' r$

The x-y plane above is the plane of the earth's orbit or ecliptic. The inclination of the earth's orbit is zero and all other planets are measured from this. The ascending node is undefined. This ends discussion of two-body problem.

The next step is to obtain expansions in periodic functions of the various two-body elements.

Suppose we have the function: $f(r, v)$ which we want to expand in the time or mean anomaly. First due in terms of eccentric anomaly:

$$f(r, v) = \sum_{n=0}^{\infty} \left[A_n \cos nE + B_n \sin nE \right]$$

Then:
$$\cos nE = \sum_{n=0}^{\infty} C_n \cos nM$$

$$\sin nE = \sum_{n=1}^{\infty} D_n \sin nM$$

and then substitute above.

First consider Lagrange's Expansion Theorem. This has to do with functions like: $y = x + \alpha \phi(y)$

A function of this form is Kepler's equation: $E = M + e \sin E$ where e is the small parameter.

This theorem says:

$$F(y) = F(x) + \sum_{n=1}^{\infty} \frac{\alpha^n}{n!} \frac{\partial^{n-1}}{\partial x^{n-1}} \left\{ \varphi^n(x) \frac{\partial F(x)}{\partial x} \right\}$$

To prove, assume $y = x + \alpha\phi(y)$ has been solved for y, that is, $y = f(x, \alpha)$. Expand in MacLaurin series in α.

$$F(y) = F_{\alpha=0} + \sum_{n=1}^{\infty} \frac{\alpha^n}{n!} \left\{ \frac{\partial^n F}{\partial \alpha^n} \right\}_{\alpha=0}$$

Now compare two equations; $F(x) = F_{\alpha=0}$ is OK.
For other term, must show:

$$\left(\frac{\partial^n F}{\partial \alpha^n} \right)_0 = \frac{\partial^{n-1}}{\partial x^{n-1}} \left\{ \phi^n(x) \frac{\partial F(x)}{\partial x} \right\}$$

Define the operators: $D = \dfrac{\partial}{\partial x}$; $A = \dfrac{\partial}{\partial \alpha}$

Then: $Dy = 1 + \alpha \dfrac{d\phi}{dy} Dy$ from $y = x + \alpha\phi(y)$

Operate with A: $Ay = \alpha \dfrac{d\phi}{dy} Ay + \phi$

Now form: $Ay - \phi Dy = \alpha \frac{d\phi}{dy} Ay - \alpha\phi \frac{d\phi}{dy} Dy$

$$= \alpha \frac{d\phi}{dy}(Ay - \phi Dy)$$

But since ϕ is arbitrary function, $\alpha \frac{d\phi}{dy}$ is not necessarily 1, so $Ay = \phi Dy$ must be true.
Now consider F:

$$AF = \frac{dF}{dy} Ay \quad ; \quad DF = \frac{dF}{dy} Dy$$

Since $Ay = \phi Dy$: $\quad AF = \phi DF$

But we want to show: $A^n F = D^{n-1}(\phi DF)$
Do by mathematical induction. It is true for $n=1$ show that it is true for $n+1$ by operating with A^n on $n=1$ case:

$$A^{n+1} F = A D^{n-1}(\phi^n DF) = D^{n-1}\left[A(\phi^n DF)\right]$$

Want:

$$A^{n+1} F = D^n(\phi^{n+1} DF) \quad ; \quad D(\phi^{n+1} DF) = A(\phi^n DF)$$

Consider and Expand:

$D(\phi^n AF)$	$A(\phi^n DF)$
$\phi^n DAF + AF D\phi^n$	$\phi^n ADF + DF A\phi^n$
$\phi DF D\phi^n$	$DF \phi D\phi^n$

We now see $\alpha = 0$ after differentiation, Then $y = x$, and we have shown the Lagrange Expansion Theorem. We can now use this to expand $E = M + e \sin E$. This is kind of inconvenient for a multiply-periodic representation as we obtain powers of the trig functions. We will not use this now.

Suppose we have the set of equations:

$$y_i = x_i + \alpha a_i \phi(y_1, \cdots y_n) \quad ; \quad i = 1, \cdots, m$$

We want to show that the following expansion is possible:

$$F(y_i) = F(x_i) + \sum_{n=1}^{\infty} \frac{\alpha^n}{n!} D^{n-1} \left\{ \phi^n(x_i)\, D F(x_i) \right\}$$

$$D^n = \sum_{i=1}^{m} a_i \frac{\partial^n}{\partial x_i^n}$$

Differentiate y_i with respect to x_j:

$$\frac{\partial y_i}{\partial x_j} = \delta_{ij} + \alpha a_i \sum_k \frac{\partial \phi}{\partial y_k} \frac{\partial y_k}{\partial x_j} \quad \Big\} \text{ multiply by } \sum_j a_j :$$

$$\sum_j a_j \frac{\partial y_i}{\partial x_j} = \sum_j a_j \delta_{ij} + \alpha a_i \sum_{jk} a_j \frac{\partial \phi}{\partial y_k} \frac{\partial y_k}{\partial x_j}$$

Thus we see: $D y_i = a_i + \alpha a_i \sum_k \frac{\partial \phi}{\partial y_k} D y_k$

Now do with respect to α:

$$A y_i = a_i \phi + \alpha a_i \sum_k \frac{\partial \phi}{\partial y_k} A y_k$$

and form:

$$A y_i - \phi D y_i = \alpha a_i \sum_k \frac{\partial \phi}{\partial y_k} \left(A y_k - \phi D y_k \right)$$

This is a homogeneous matrix equation of the form:

$$\begin{pmatrix} & \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ \vdots \end{pmatrix} \quad ; \text{ hence: } A y_i = \phi D y_i$$

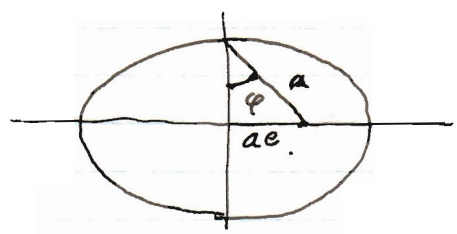
Form AF, DF, on some arbitrary F; $AF = \phi DF$ and the proof follows thru just as before.

LECTURE 4 : 2-14-62

(1) $\tan\frac{v}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}$

$E = e\sin E + M$

$r = \frac{a(1-e^2)}{1+e\cos v} = a(1-e\cos E)$



$\beta = \frac{e}{2} + \frac{e}{2}\beta^2$

$\beta \approx \frac{e}{2}$; $e = \sin\varphi$, $\beta = \tan\frac{1}{2}\varphi$; $e = \frac{2\beta}{1+\beta^2}$

$\beta = \frac{1}{e}\left[1-\sqrt{1-e^2}\right]$

$y = x + \alpha\,\varphi(y)$

$F(\beta) = \beta^K$

$$F(y) = F(x) + \sum \frac{\alpha^n}{n!}\frac{\partial^{n-1}}{\partial x^{n-1}}\left\{\varphi^n(x)\frac{\partial F(x)}{\partial x}\right\}$$

$$\beta^K = \left(\frac{e}{2}\right)^K + \sum \frac{\left(\frac{e}{2}\right)^n}{n!}K\frac{\partial^{n-1}}{\partial\left(\frac{e}{2}\right)^{n-1}}\left\{\left(\frac{e}{2}\right)^{2n}\left(\frac{e}{2}\right)^{K-1}\right\}$$

$$= \left(\frac{e}{2}\right)^K\left[1 + \sum_n \left(\frac{e}{2}\right)^{2n} K\frac{(2n+K-1)(2n+K-2)\cdots(n+K+1)(n+K)\cdots(1)}{n!\,(n+K)!}\right.$$

or: $$\beta^K = \left(\frac{e}{2}\right)^K\left[1 + \sum_{n=1}^{\infty}\left(\frac{e}{2}\right)^{2n}\frac{K(2n+K-1)!}{n!\,(n+K)!}\right]$$

This is example of Lagrange expansion.

Change to the variables: $\xi = e^{iV}$; $\eta = e^{iE}$

substitute in (1):

$$\frac{e^{i\frac{V}{2}} - e^{-i\frac{V}{2}}}{e^{i\frac{V}{2}} + e^{-i\frac{V}{2}}} = \frac{1+\beta}{1-\beta} \; \frac{e^{i\frac{E}{2}} - e^{-iE/2}}{e^{iE/2} + e^{-iE/2}}$$

or:

$$\frac{\xi - 1}{\xi + 1} = \frac{1+\beta}{1-\beta} \; \frac{\eta - 1}{\eta + 1}$$

solve for ξ: $$\xi = \frac{\eta(1 - \beta\eta^{-1})}{1 - \beta\eta}$$

solve for η: $$\eta = \frac{\xi(1 + \beta\xi^{-1})}{1 + \beta\xi}$$

Take logarithms to get true anomaly in terms of eccentric anomaly:

$$iV = iE + \ln(1 - \beta\eta^{-1}) - \ln(1 - \beta\eta)$$

Use $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^2}{3} - \cdots$

$$iV = iE - \beta\eta^{-1} - \frac{\beta^2\eta^{-2}}{2} - \cdots + \beta\eta + \frac{\beta^2\eta^2}{2} + \cdots$$

$$= iE + \beta(\eta - \eta^{-1}) + \frac{\beta^2}{2}(\eta^2 - \eta^{-2}) + \cdots$$

or:

$$V = E + \sum_{n=1}^{\infty} \frac{2\beta^n}{n} \sin nE$$

This is our first expansion in periodic functions. The other relation gives:

$$E = V + \sum_{n=1}^{\infty} \frac{(-1)^n \, 2\beta^n}{n} \sin nV$$

Now work on equation for r:

$$\frac{r}{a} = 1 - \frac{\beta}{1+\beta^2} \; \eta + \eta^{-1} = \frac{1 + \beta^2 - \beta\eta - \beta\eta^{-1}}{1 + \beta^2}$$

$$\frac{r}{a} = \frac{(1-\beta\eta)(1-\beta\eta^{-1})}{1+\beta^2}$$

$$\frac{r}{a} = \frac{(1-\beta^2)^2}{1+\beta^2} \frac{1}{(1+\beta\xi)(1+\beta\xi^{-1})}$$

Furthermore: $r\xi = \frac{a\eta(1-\beta\eta^{-1})}{(1+\beta^2)} (1-\beta\eta^{-1})$

Form: $r^p\xi^p = \frac{a^p}{(1+\beta^2)^p} \eta^p (1-\beta\eta^{-1})^{2p}$

$$= \frac{a^p}{(1+\beta^2)^p} \eta^p \left\{1 - C_1^{2p}\beta\eta^{-1} + C_2^{2p}\beta^2\eta^2 - \cdots\right\}$$

$$= \frac{a^p}{(1+\beta^2)^p} \left\{\eta^p - C_1^{2p}\beta\eta^{p-1} + C_2^{2p}\beta^2\eta^{p-2} - \cdots\right\}$$

Take real and imaginary parts:

$$r^p \cos pV = \frac{a^p}{(1+\beta^2)^p}\left[\cos pE - C_1^{2p}\beta\cos(p-1)E + \cdots\right]$$

where: $C_j^k = \frac{k(k-1)\cdots(k-j+1)}{j!}$

Now form:

$$r^p\xi^q = \frac{a^p}{(1+\beta^2)^p}(1-\beta\eta)^p(1-\beta\eta^{-1})^p\eta^q\frac{(1-\beta\eta^{-1})^q}{(1-\beta\eta)^q}$$

$$= \frac{a^p}{(1+\beta^2)^p}\eta^q(1-\beta\eta)^{p-q}(1-\beta\eta^{-1})^{p+q}$$

$$= \left[1 + \sum_{n=1}^{\infty}(-1)^n C_n^{p-q}(\beta\eta)^n\right]\left[1 + \sum_{n=1}^{\infty}(-1)^n C_n^{p+q}(\beta\eta^{-1})^n\right]$$

$$1 - C_1^{p-q}\beta\eta + C_2^{p-q}\beta^2\eta^2 - C_3^{p-q}\beta^3\eta^3$$

$$1 - C_1^{p+q}(\beta\eta^{-1}) + C_2^{p+q}(\beta\eta^{-1})^2 - C_3^{p+q}(\beta\eta^{-1})^3$$

We collect terms in orders of η:

$$1 + \sum_{K=1}^{\infty} C_K^{p+q} C_K^{p-q} \beta^{2K} + \sum_{n=1}^{\infty} (\beta\eta)^n \Big[C_n^{p-q}$$

$$+ \sum_{K=1}^{\infty} C_K^{p+q} C_{n+K}^{p-q} (\beta)^{2K} \Big] (-1)^n$$

$$+ \sum_{n=1}^{\infty} (-\beta\eta^{-1})^n \Big[C_n^{p+q} + \sum_{K=1}^{\infty} C_K^{p-q} C_{n+K}^{p+q} (\beta)^{2K} \Big]$$

We now make manipulations of the C's: We want to show:

$$C_{n+K}^{j} = C_n^j C_K^{j-n} \frac{K!}{(n+1)(n+2)\cdots(n+K)}$$

$$C_{n+K}^{j} = C_K^{j-n} C_n^j \frac{j(j-1)\cdots(j-n-K+1)\, n!\, K!}{(n+K)!\; j(j-1)\cdots(j-n+1)(j-n)(j-n-1)(j-n-K+1)}$$

$$= C_K^{j-n} C_n^j \frac{K!}{(n+1)(n+2)\cdots(n+K)}$$

We now show that this leads to a hypergeometric series:

$$F(a,b,c;x) = 1 + \frac{ab}{c}x + \frac{a(a+1)b(b+1)}{2c(c+1)}x^2$$

$$+ \cdots + \frac{a(a+1)\cdots(a+K-1)\,b(b+1)\cdots(b+K-1)}{c(c+1)\cdots(c+K-1)\,K!}x^K$$

$$F(-a,-b,c;x) = \sum_K \frac{a(a-1)\cdots(a-K+1)\,b(b-1)\cdots(b-K+1)}{c(c+1)\cdots(c+K-1)\,K!}x^K$$

We now identify with binomial coefficients C.

We can write the coefficient of the general term in the hypergeometric series as:

$$\frac{C_K^a \, C_K^b \, K!}{c\,(c+1)\cdots(c+K-1)}$$

We can now write:

$$C_{n+K}^{p-q} = C_n^{p-q} \, C_K^{p-q-n} \, \frac{K!}{(n+1)(n+2)\cdots(n+K)}$$

$$C_n^{p-q}\left[1 + \sum_K C_K^{p+q} \, C_K^{p-q-n} \, \frac{K!}{(n+1)\cdots(n+K)} \, (\beta)^{2K}\right]$$

So we can write the coefficients of each term in the expansion:

$$F(p+q,\, p-q,\, 1;\, \beta^2)$$
$$F(p+q,\, p-q-n,\, n+1;\, \beta^2)$$
$$F(p-q,\, p+q-n,\, n+1;\, \beta^2)$$

We can use the identity:

$$F(a,b,c;x) = (1-x)^{-a} F\left(a,\, c-b,\, c;\, -\frac{x}{1-x}\right)$$

to improve the convergence. The final form is:

$$\left(\frac{r}{a}\right)^p \begin{smallmatrix}\sin\\ \cos\end{smallmatrix}(qv) = \sum_{n=0}^{\infty}\left[A_n \begin{smallmatrix}\sin\\ \cos\end{smallmatrix}(q+n)E + B_n \begin{smallmatrix}\sin\\ \cos\end{smallmatrix}(q-n)E\right]$$

where:

$$B_0 = 0$$
$$A_0 = (1-e^2)^{p/2}(1-\beta^2)^q \, T_0(p,q)$$
$$A_n = (-1)^n (1-e^2)^{p/2}(1-\beta^2)^q \, C_n^{p-q} \, T_n(p,q)\, \beta^n$$
$$B_n = (-1)^n (1-e^2)^{p/2}(1-\beta^2)^{-q} \, C_n^{p+q} \, T_n(p,-q)\, \beta^n$$
$$T_n(p,q) = F\left(-p-q,\, p-q+1,\, n+1;\, -\frac{\beta^2}{1-\beta^2}\right)$$

LECTURE 5: 2-21-62

Using the previous results, we can form all sorts of desired expansions, letting p and q take on desired values.

We will now show that our coefficients can be expressed as Bessel functions:

Properties of Bessel Functions:

Generating Function:

$$u = e^{\frac{x}{2}(z - z^{-1})} = e^{\frac{x}{2}z} e^{-\frac{x}{2}z^{-1}}$$

$$= \sum_{n=0}^{\infty} \frac{(\frac{x}{2})^n z^n}{n!} \sum_{m=0}^{\infty} \frac{(\frac{x}{2})^m (-z)^{-m}}{m!} = \sum_{n=-\infty}^{\infty} J_n(x) z^n$$

Set $-z^{-1} \to z$, which shows: $J_{-n}(x) = (-1)^n J_n(x)$
$J_n(-x) = (-1)^n J_n(x)$

We get further relations on differentiating:

$$\frac{\partial u}{\partial x} = \frac{1}{2}(z - z^{-1})\, u = \sum J_n'(x) z^n$$

$$= \frac{1}{2} \sum \left[J_n(x) z^{n+1} - J_n(x) z^{n-1} \right]$$

or: $J_n'(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x) \right]$

Take $\frac{\partial u}{\partial z}$:

$$\frac{\partial u}{\partial z} = u\, \frac{x}{2}(1 + z^{-2}) = \sum J_n(x)\, n\, z^{n-1}$$

$$= \frac{x}{2} \sum \left(J_n(x) z^n + J_n(x) z^{n-2} \right)$$

or: $\frac{x}{2}\left[J_n(x) + J_{n+2}(x) \right] = J_{n+1}(x)\,(n+1)$

or: $n\, J_n(x) = \frac{x}{2}\left[J_{n-1}(x) + J_{n+1}(x) \right]$

Integral Form: Let $z = e^{i\theta}$:

$$u = e^{\frac{x}{2}(e^{i\theta} - e^{-i\theta})} = e^{ix\sin\theta}$$

$$= \sum_{n=-\infty}^{\infty} J_n(x)\, e^{in\theta}$$

Multiply by $\int_0^{2\pi} d\theta\, e^{-in\theta}$: Get:

$$J_n(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{-i(n\theta - x\sin\theta)}\, d\theta$$

Let $\int_0^{2\pi} \to \int_0^{\pi} + \int_{\pi}^{2\pi}$:

$$J_n(x) = \frac{1}{2\pi}\left\{ \int_0^{\pi} e^{-i(n\theta - x\sin\theta)} d\theta + \int_{\pi}^{2\pi} e^{-i(n\theta - x\sin\theta)} d\theta \right.$$

Let $\theta \to 2\pi - \theta$; the second integral is:

$$\int_0^{\pi} e^{-i(2\pi n - n\theta + n\sin\theta)}\, d\theta = \int_0^{\pi} e^{i(n\theta - x\sin\theta)}\, d\theta$$

Then:

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - x\sin\theta)\, d\theta$$

We can apply this to the Fourier analysis if we limit ourselves to domains of $0 \to \pi$, or we keep to even or odd functions, that is:

$$F_e(M) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos nM$$

$$F_o(M) = \sum_{n=1}^{\infty} B_n \sin nM$$

In the usual way, we have; using the orthogonality of trig functions.

$$A_0 = \frac{2}{\pi}\int_0^{\pi} F_e(M)\, dM$$

$$A_n = \frac{2}{\pi}\int_0^{\pi} F_e(M)\cos nM\, dM$$

$$B_n = \frac{2}{\pi}\int_0^{\pi} F_o(M)\sin nM\, dM$$

Since $M = E - e \sin E$, the symmetry properties of the expansion are preserved when using E.

Write:

$$A_{n,K} = \frac{2}{\pi}\int_0^\pi \cos KE \cos nM \, dM$$

$$\underset{\text{by parts}}{=} \underbrace{\frac{2}{\pi} \cos KE \sin nM \Big|_0^\pi}_{0} + \underbrace{\frac{2K}{\pi n}\int_0^\pi \sin nM \sin KE \, dE}_{\frac{K}{\pi n}\int_0^\pi \left[-\cos(nM+KE) + \cos(nM-KE)\right] dE}$$

$$= \frac{K}{\pi n}\int_0^\pi -\cos\left[(n+K)E - ne\sin E\right] + \cos\left[(n-K)E - ne\sin E\right] dE$$

which we identify with the integral definition of the Bessel functions. Then:

$$A_{K,n} = \frac{K}{n}\left[J_{n-K}(ne) - J_{n+K}(ne)\right]$$

$$A_{0K} = \frac{2}{\pi}\int_0^\pi \cos KE \, dM = \frac{2}{\pi}\int_0^\pi \cos KE (1 - e\cos E)\, dE$$

$$= \frac{2}{\pi}\int_0^\pi (\cos KE - e \cos KE \cos E)\, dE$$

$$A_{0K} = -e \, \delta_{K1}$$

If $K = 1$, we immediately see:

$$\cos E = -e + \sum \frac{1}{n}\left[J_{n-1}(ne) - J_{n+1}(ne)\right] \cos nM$$

$$= -e + \sum \frac{1}{2n^2} \frac{\partial}{\partial e} J_n(ne) \cos nM$$

This also gives r from $r = a(1 - e\cos E)$, so we now have r in terms of M. For $\sin KE$, we have:

$$\sin KE = K \sum \frac{1}{n}\left[J_{(n-K)}(ne) + J_{(n+K)}(ne)\right] \sin nM$$

For $k=1$:

$$\sin E = \sum \frac{1}{n} \left\{ J_{n-1}(ne) + J_{n+1}(ne) \right] \sin nM$$

Use: $n J_n(x) = \frac{x}{2} \left[J_{n-1} + J_{n+1} \right]$, then:

$$\sin E = \frac{2}{e} \sum_{n=1}^{\infty} \frac{1}{n} J_n(ne) \sin nM$$

Note this gives $E = f(M)$ because $E = M + e \sin E$.
Also:

$$1 = \frac{dE}{dM} - e \cos E \frac{dE}{dM}$$

or: $\frac{dE}{dM} = \left(\frac{r}{a}\right)^{-1}$

Hence: $\frac{a}{r} = 1 + 2 \sum J_n'(ne) \cos nM$

We now see that it is possible to express the motion of a planet completely in terms of the time:

$$\xi = a(\cos E - e) \; ; \; \eta = b \sin E :$$

Then: $\left.\begin{matrix} x \\ y \\ z \end{matrix}\right\} = f(\text{elements}, M)$

so it is not necessary to have Kepler's equations.

Approximations for small e. Useful forms to $O(e)$:

$$\sin nv = -ne \sin(n-1)M + \sin nM + ne \sin(n+1)M$$
$$\cos nv = \text{same}$$

$$\left(\frac{a}{r}\right)^n = 1 + ne \cos M$$

$$\left(\frac{a}{r}\right)^n \sin v = \sin M + \left(\frac{n}{2}+1\right) e \sin 2M$$

$$\left(\frac{a}{r}\right)^n \cos v = \left(\frac{n}{2}-1\right)e + \cos M + \left(\frac{n}{2}+1\right) e \cos 2M$$

This completes the treatment of expansions.

LECTURE 6 : 2-26-62

The N-body Problem

Coordinates of each body are x_i, y_i, z_i and is of mass m_i. The potential at point P_i is:

$$V_i = \frac{G m_1}{\Delta_{i1}} + \frac{G m_2}{\Delta_{i2}} + \cdots = \sum_j \frac{G m_j}{\Delta_{ij}} \quad ; \; i \neq j$$

$$m_i \ddot{x}_i = \sum_j G m_i m_j \frac{\partial}{\partial x_i}\left(\frac{1}{\Delta_{ij}}\right)$$

where:

$$\Delta_{ij} = \left\{ (x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 \right\}^{1/2}$$

and similarly for y_i, z_i. Write out for a particular planet:

$$m_1 \ddot{x}_1 = G m_1 m_2 \frac{\partial}{\partial x_1}\left(\frac{1}{\Delta_{12}}\right) + G m_1 m_3 \frac{\partial}{\partial x_1}\left(\frac{1}{\Delta_{13}}\right) + \cdots + G m_1 m_n \frac{\partial}{\partial x_1}\left(\frac{1}{\Delta_{1n}}\right)$$

We can add with no change:

$$G m_1 m_3 \frac{\partial}{\partial x_1}\left(\frac{1}{\Delta_{23}}\right), \quad G m_2 m_n \frac{\partial}{\partial x_1}\left(\frac{1}{\Delta_{2n}}\right), \quad G m_{n-1} m_2 \frac{\partial}{\partial x_1}\left(\frac{1}{\Delta_{n-1,n}}\right)$$

This allows us to write:

$$U = \sum_{i=1}^{n-1} \sum_{j>i} \frac{G m_i m_j}{\Delta_{ij}}$$

in which the distances between every pair of points. U is called the force function. We can then write for all the equations of motion:

$$\frac{\partial U}{\partial x_i} = m_i \ddot{x}_i$$

The force function is the work done in bringing all bodies in from infinity to their present positions. The potential is then the negative of this.

For two bodies:

$$U_2 = \frac{G m_1 m_2}{\Delta_{12}}$$

The work done is:

$$\int_A^B (F_x dx + F_y dy + F_z dz) = \int_A^B \frac{\partial U_2}{\partial x_1} dx + \frac{\partial U_2}{\partial y} dy + \frac{\partial U_2}{\partial z} dz$$

$$= \int_A^B dU_2 = (U_2)_B - (U_1)_A = U_2 \text{ , taking } m_1 \text{ fixed.}$$

For three bodies:

$$U_3 = \frac{G m_1 m_3}{\Delta_{13}} + \frac{G m_2 m_3}{\Delta_{23}}$$

If we integrate as before, and add on more bodies, we indeed get:

$$V = -U$$

We obtain our integrals of the motion from U. Form:

$$\frac{\partial U}{\partial x_i} = \sum_j G m_i m_j \frac{\partial}{\partial x_i}\left(\frac{1}{\Delta_{ij}}\right) \quad ; \quad i \neq j$$

$$= -\sum_j G m_i m_j \frac{(x_i - x_j)}{\Delta_{ij}^3} = m_i \ddot{x}_i$$

Sum on i:

$$\sum_i \frac{\partial U}{\partial x_i} = -\sum_i \sum_j G m_i m_j \frac{(x_i - x_j)}{\Delta_{ij}^3} = 0$$

or $\sum_i m_i \ddot{x}_i = 0$ and the same for y, z.

Then: $\sum_i m_i \dot{x}_i = a_1$ or the momentum is constant.

Further: $\sum_i m_i x_i = a_1 t + a_2 \quad ; \quad \bar{X} = \dfrac{\sum m_i x_i}{\sum m_i}$

We now have six integrals. However, they can be eliminated by the proper choice of coordinates.

next form: $\dfrac{\partial U}{\partial y_i} = -\sum_j G m_i m_j \dfrac{(y_i - y_j)}{\Delta_{ij}^3}$

now form:

$$x_i \frac{\partial U}{\partial y_i} - y_i \frac{\partial U}{\partial x_i} = \sum_j G m_i m_j (x_i y_j - x_j y_i)$$

sum on i:

$$\sum_i (m_i x_i \ddot{y}_i - m_i \ddot{x}_i y_i) = 0$$

Integrating, we find:

$$\sum_i m_i (x_i \dot{y}_i - \dot{x}_i y_i) = C_3$$

which corresponds to a constant angular momentum component. Similarly, we can find C_1 and C_2, the rest of the angular momentum. These could be used for reference but usually are not. 85% of the angular momentum of the solar system comes from Jupiter and Saturn. Jupiter is inclined 1° to the ecliptic while Saturn is 2°. The planets contain 98% of the angular momentum in the solar system, the sun contributing the rest.

We now have 9 constants, the last is the energy:
Form:

$$\sum_i m_i (\ddot{x}_i \dot{x}_i + \ddot{y}_i \dot{y}_i + \ddot{z}_i \dot{z}_i) = \sum_i \left(\frac{\partial U}{\partial x_i} \dot{x}_i + \frac{\partial U}{\partial y_i} \dot{y}_i + \frac{\partial U}{\partial z_i} \dot{z}_i \right)$$

$$\frac{d}{dt} \left(\sum_i \tfrac{1}{2} m_i (\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2) \right) = \frac{dU}{dt}$$

$$\frac{dT}{dt} = \frac{dU}{dt} \; ; \quad T - U = C, \text{ or } \quad T + V = C$$

It was shown by Bruns and Poincaré that these are the only 10 independent constants of the motion.

We now show the Virial Theorem.

Form:

$$\sum_i \left(\frac{\partial U}{\partial x_i} x_i + \frac{\partial U}{\partial y_i} y_i + \frac{\partial U}{\partial z_i} z_i \right)$$

$$= \sum_i m_i \left(x_i \ddot{x}_i + y_i \ddot{y}_i + z_i \ddot{z}_i \right) = -U$$

Euler's Theorem: If we have a homogeneous function $F(x_i)$ of order n, we have:

$$\sum_i \frac{\partial F}{\partial x_i} x_i = n F$$

because: $F(\lambda x_i) = \lambda^n F(x_i)$ and:

$$\sum_i \frac{\partial F}{\partial(\lambda x_i)} \frac{\partial(\lambda x_i)}{\partial \lambda} = n \lambda^{n-1} F(x_i) = \sum_i \frac{\partial F}{\partial(\lambda x_i)} x_i$$

and let $\lambda \to 1$. This justifies $-U$ above.

Now note that $x\ddot{x} = \frac{d}{dt}(x\dot{x}) - \dot{x}^2 = \frac{1}{2}\frac{d^2}{dt^2}(x^2) - \dot{x}^2$.
Hence:

$$-U = \sum_i \frac{1}{2} m_i \frac{d^2 R_i^2}{dt^2} - 2T$$

where $R_i^2 = x_i^2 + y_i^2 + z_i^2$.

$$2T - U = \sum_i \frac{1}{2} m_i \frac{d^2 R_i^2}{dt^2}$$

For a closed system (no escape of particles), we see that:

$2T - U < 0$. On the average, $T = \frac{1}{2} U = -\frac{1}{2} V$.

Also, from $2T - U = T + C < 0$, so we see that the total energy must be negative. For the two body problem the average of $r = a$, so that a is sometimes called the mean axis.

This is about as far as the N body problem can be carried.

Lagrange's Planetary Theory:

Recall we have found:

$$\ddot{x} = -\frac{\mu x}{r^3} + \frac{\partial R}{\partial x}$$

where: $R = G\, m_1 \left(\frac{1}{\Delta} - \frac{x x_1 + y y_1 + z z_1}{r_1^{\,3}} \right)$

and is called the disturbing function due to a third body. We have solved $\ddot{x} = -\frac{\mu x}{r^3}$ and found 6 elements. We now consider these 6 elements to be slowly varying. The motion of the two body problem can be expressed in terms of the 6 elements and the time:

$$x = l_1 a \cos E + l_2 b \sin E - e a l_1$$

and E can be expressed in terms of M. Let us write:

$$a, e, \chi \to \alpha_1, \alpha_2, \alpha_3 \quad ; \quad i, \omega, \Omega \to \beta_1, \beta_2, \beta_3$$

$$x = F_1(\alpha_i, \beta_i, t) \; ; \; y = F_2(\alpha_i, \beta_i, t) \; ; \; z = F_3(\alpha_i, \beta_i, t)$$

$$\dot{x} = G_1 \; ; \; \dot{y} = G_2 \; ; \; \dot{z} = G_3$$

We want to transform to 6 new variables. We keep the functional form of F, but not G because the elements now vary with time. Before the perturbation we could write: $\dot{x} = \frac{\partial x}{\partial t}$. Under the perturbation, we can write the same form $\frac{dx}{dt} = \frac{\partial x}{\partial t}$ and call it the velocity. This means that at any instant, x is given by F_1, or that at any instant the orbit is given by an ellipse. This also means $\left(\frac{dx}{dt} = \frac{\partial x}{\partial t}\right)$ that the same functional form is kept for the velocities. Write:

$$\frac{dx}{dt} = G_1(\alpha, \beta, t) \; ; \; \frac{d^2 x}{dt^2} = \frac{\partial G_1}{\partial t} + \sum_i \frac{\partial G_1}{\partial \alpha_i}\dot{\alpha}_i + \sum_i \frac{\partial G}{\partial \beta_i}\dot{\beta}_i$$

$$\frac{d^2 x}{dt^2} = \frac{\partial^2 x}{\partial t^2} + \sum_i \frac{\partial \dot{x}}{\partial \alpha_i}\dot{\alpha}_i + \sum_i \frac{\partial \dot{x}}{\partial \beta_i}\dot{\beta}_i \; ; \; \frac{\partial^2 x}{\partial t^2} = -\frac{\mu x}{r^3}$$

$$\sum \frac{\partial \dot{x}}{\partial \alpha}\dot{\alpha} + \sum \frac{\partial \dot{x}}{\partial \beta}\dot{\beta} = \frac{\partial R}{\partial x} \text{. From } \frac{dx}{dt} = \frac{\partial F_1}{\partial t} + \sum \frac{\partial F_1}{\partial \alpha_i}\dot{\alpha}_i + \sum \frac{\partial F_1}{\partial \beta_i}\dot{\beta}_i$$

or: $\sum \frac{\partial x}{\partial \alpha_i}\dot{\alpha}_i + \sum \frac{\partial x}{\partial \beta_i}\dot{\beta}_i = 0$

LECTURE 7 : 2-28-62

Recall:

$$\sum \frac{\partial \dot{x}}{\partial \alpha_i} \dot{\alpha}_i + \sum \frac{\partial \dot{x}}{\partial \beta_i} \dot{\beta}_i = \frac{\partial R}{\partial x} \qquad \Big\} \frac{\partial x}{\partial \alpha_1}$$

$$\sum \frac{\partial x}{\partial \alpha_i} \dot{\alpha}_i + \sum \frac{\partial x}{\partial \beta_i} \dot{\beta}_i = 0 \qquad \Big\} \frac{\partial \dot{x}}{\partial \alpha_1}$$

Then:

$$\sum_i \left[\frac{\partial x}{\partial \alpha_1} \frac{\partial \dot{x}}{\partial \alpha_i} - \frac{\partial x}{\partial \alpha_i} \frac{\partial \dot{x}}{\partial \alpha_1} \right] \dot{\alpha}_i + \sum_i \left[\frac{\partial x}{\partial \alpha_1} \frac{\partial \dot{x}}{\partial \beta_i} - \frac{\partial x}{\partial \beta_i} \frac{\partial \dot{x}}{\partial \alpha_1} \right] \dot{\beta}_i = \frac{\partial R}{\partial x} \frac{\partial x}{\partial \alpha_1}$$

or:
$$\sum_i \frac{\partial (x, \dot{x})}{\partial (\alpha_1, \alpha_i)} \dot{\alpha}_i + \sum_i \frac{\partial (x, \dot{x})}{\partial (\alpha_1, \beta_i)} \dot{\beta}_i = \frac{\partial R}{\partial x} \frac{\partial x}{\partial \alpha_1}$$

We can do the same for the other components and then add together: We will get the Lagrange Bracket:

$$[\alpha_1, \alpha_i] = \left[\frac{\partial (x, \dot{x})}{\partial (\alpha_1, \alpha_i)} + \frac{\partial (y, \dot{y})}{\partial (\alpha_1, \alpha_i)} + \frac{\partial (z, \dot{z})}{\partial (\alpha_1, \alpha_i)} \right]$$

and thus:

$$\sum_i [\alpha_1, \alpha_i] \dot{\alpha}_i + \sum_i [\alpha_1, \beta_i] \dot{\beta}_i = \frac{\partial R}{\partial x} \frac{\partial x}{\partial \alpha_1} + \frac{\partial R}{\partial y} \frac{\partial y}{\partial \alpha_1} + \frac{\partial R}{\partial z} \frac{\partial z}{\partial \alpha_1}$$
$$= \frac{\partial R}{\partial \alpha_1}$$

or in general:

$$\sum_i [\alpha_r, \alpha_i] \dot{\alpha}_i + \sum_i [\alpha_r, \beta_i] \dot{\beta}_i = \frac{\partial R}{\partial \alpha_r} \quad ; \quad r = 1, 2, 3$$

and for β:

$$\sum_i [\beta_r, \alpha_i] \dot{\alpha}_i + \sum_i [\beta_r, \beta_i] \dot{\beta}_i = \frac{\partial R}{\partial \beta_r}$$

These 6 equations can now in principle be solved for the elements. We write in matrix form:

$$L \dot{e} = \Delta R \quad ; \quad \Delta = \begin{pmatrix} \partial \alpha_1 \\ \vdots \\ \partial \beta_3 \end{pmatrix}$$

$\dot{e}$ is the time derivatives of all the elements previously denoted by α, β.

Because $[p,p] = 0$, $[p,q] = -[q,p]$, we eliminate many of the 36 elements in L and have left a skew symmetric matrix. Also note that the Lagrange bracket is independent of time, $\frac{\partial}{\partial t}[p,q] = 0$. Define $x_1 = x$, $x_2 = y$, $x_3 = z$. p, q are some elements.

$$\frac{\partial}{\partial t}\left[\sum_i \left(\frac{\partial x_i}{\partial p}\frac{\partial \dot{x}_i}{\partial q} - \frac{\partial x_i}{\partial q}\frac{\partial \dot{x}_i}{\partial p}\right)\right]$$

$$= \sum_i \frac{\partial x_i}{\partial p}\frac{\partial \ddot{x}_i}{\partial q} + \frac{\partial \dot{x}_i}{\partial p}\frac{\partial \dot{x}_i}{\partial q} - \frac{\partial x_i}{\partial q}\frac{\partial \ddot{x}_i}{\partial p} - \frac{\partial \dot{x}_i \partial \dot{x}_i}{\partial p\, \partial q}$$

$$= \sum_i \frac{\partial x_i}{\partial p}\frac{\partial}{\partial q}\frac{\partial V}{\partial x_i} \qquad \text{since} \qquad \ddot{x}_i = \frac{\partial V}{\partial x_i}$$

V does not contain R, or at any rate, V is the potential of the solvable exact part of the problem which may contain part of R. Also, the last term on the RHS above must be added:

$$-\frac{\partial x_i}{\partial q}\frac{\partial}{\partial p}\left(\frac{\partial V}{\partial x_i}\right)$$

Continuing:

$$\sum_i \left[\frac{\partial}{\partial q}\left(\frac{\partial V}{\partial x_i}\frac{\partial x_i}{\partial p}\right) - \frac{\partial}{\partial p}\left(\frac{\partial V}{\partial x_i}\frac{\partial x_i}{\partial q}\right)\right]$$

$$= \frac{\partial}{\partial q}\frac{\partial V}{\partial p} - \frac{\partial}{\partial p}\frac{\partial V}{\partial q} = 0$$

Now recall:

$$x = \xi l_1 + \eta l_2 \qquad \dot{x} = \dot{\xi} l_1 + \dot{\eta} l_2$$
$$y = \xi m_1 + \eta m_2 \qquad \dot{y} = \dot{\xi} m_1 + \dot{\eta} m_2$$
$$z = \xi n_1 + \eta n_2 \qquad \dot{z} = \dot{\xi} n_1 + \dot{\eta} n_2$$

$$\xi, \eta = f(a, e, \tau) \quad ; \quad l, m, n = f(i, \omega, \Omega)$$

We are now in a formal position to evaluate the brackets. However, we will not approach this tedium, but use a method due to Campbell.

Start with the energy integral:

$$T = \frac{\mu}{r} - \frac{\mu}{2a} = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = V + V_0$$

$$\frac{\partial T}{\partial p} = \frac{\partial V}{\partial p} + \frac{\partial V_0}{\partial p} = \sum_i \frac{\partial V}{\partial x_i}\frac{\partial x_i}{\partial p} + \frac{\partial V_0}{\partial p}$$

$$= \sum_i \ddot{x}_i \frac{\partial x_i}{\partial p} + \frac{\partial V_0}{\partial p} = \sum_i \dot{x}_i \frac{\partial \dot{x}_i}{\partial p}$$

Form:

$$2\frac{\partial T}{\partial p} = \sum_i \left[\dot{x}_i \frac{\partial \dot{x}_i}{\partial p} + \ddot{x}_i \frac{\partial x_i}{\partial p} \right] + \frac{\partial V_0}{\partial p}$$

$$= \sum_i \frac{\partial}{\partial t}\left[\dot{x} \frac{\partial x_i}{\partial p} \right] + \frac{\partial V_0}{\partial p}$$

Do a time integration from τ to t because of the simplicity at τ:

$$2\int_\tau^t \frac{\partial T}{\partial p}\,dt = \left[\sum_i \dot{x}_i \frac{\partial x_i}{\partial p}\right]_t - \left[\sum_i \dot{x}_i \frac{\partial x_i}{\partial p}\right]_\tau + (t-\tau)\frac{\partial V_0}{\partial p}$$

Consider: $\displaystyle \frac{\partial}{\partial p}\int_\tau^t T\,dt = \int_\tau^t \frac{\partial T}{\partial p}\,dt - T(\tau)\frac{\partial \tau}{\partial p}$

which we can substitute above and rearrange:

$$2\frac{\partial}{\partial p}\int_\tau^t T\,dt - \sum_i \dot{x}_i \frac{\partial x_i}{\partial p} - t\frac{\partial V_0}{\partial p}$$

$$= -\sum_i \dot{x}_i \frac{\partial x_i}{\partial p}\bigg|_\tau - \tau\frac{\partial V_0}{\partial p} - 2T(\tau)\frac{\partial \tau}{\partial p} = C_p$$

We form the same thing with respect to another element, say q, and then take $\frac{\partial}{\partial q}$ of the C_p one and vice versa and subtract:
The mixed partials drop out and we are left with a Lagrange bracket term:

$$[p,q] = \frac{\partial C_q}{\partial p} - \frac{\partial C_p}{\partial q} \quad ; \text{ we now look at } C_p.$$

We have $2T = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$; at τ, $r = a(1-e)$

so:

$$2T(\tau) = \frac{\mu}{a}\,\frac{1+e}{1-e}$$

For $\sum x_i \frac{\partial x_i}{\partial p}\Big|_\tau$, we have the forms:

$$\sum \left(\dot{\xi} l_1 + \dot{\eta} l_2\right)\left(l_1 \frac{\partial \xi}{\partial p} + \xi \frac{\partial l_1}{\partial p} + l_2 \frac{\partial \eta}{\partial p} + \eta \frac{\partial l_2}{\partial p}\right)$$

At $t = \tau$, $\dot{\xi} = 0$ since this is a turning point. 

$\sum \dot{\eta}\, l_1 l_2 \frac{\partial \xi}{\partial p} = 0$ since $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$ (right angles) (at perigee)

$\sum \dot{\eta}\, \xi\, l_2 \frac{\partial l_1}{\partial p}$: $l_2^2 + m_2^2 + n_2^2 = 1$

$\sum \dot{\eta}\, l_2^2 \frac{\partial \eta}{\partial p} = \dot{\eta} \frac{\partial \eta}{\partial p}$

$\sum \dot{\eta}\, \eta\, l_2 \frac{\partial l_2}{\partial p} = 0$ since $l_2 \frac{\partial l_2}{\partial p} + m_2 \frac{\partial m_2}{\partial p} + n_2 \frac{\partial n_2}{\partial p} = 0$

Then: $\underbrace{\dot{\eta}\xi}_{h} \sum l_2 \frac{\partial l_1}{\partial p} + \dot{\eta} \frac{\partial \eta}{\partial p}$

the angular momentum

Since $l_1 = f(i, \omega, \Omega)$;

$$\frac{\partial l_1}{\partial p} = \frac{\partial l_1}{\partial i}\frac{\partial i}{\partial p} + \frac{\partial l_1}{\partial \omega}\frac{\partial \omega}{\partial p} + \frac{\partial l_1}{\partial \Omega}\frac{\partial \Omega}{\partial p}$$

It turns out that the above $h \sum l_2 \frac{\partial l_1}{\partial p}$ is:

$h \frac{\partial \omega}{\partial p} + h \cos i \frac{\partial \Omega}{\partial p}$ from various identities

among the direction cosines and i, ω, Ω.

Since $\eta = 0$ at perihelion, we must expand before taking derivative:

$$\eta = b \sin E \quad ; \quad E - e \sin E = n(t-\tau)$$

$$\sin E = \frac{n(t-\tau)}{1-e} + (t-\tau)^3$$

$$\eta = \frac{bn}{1-e}(t-\tau) + A(t-\tau)^3$$

$$\frac{\partial \eta}{\partial p} = \frac{bn}{1-e}\frac{\partial \tau}{\partial p} + (t-\tau)\frac{\partial}{\partial p}\left(\frac{bn}{1-e}\right) + \cdots$$

or: $$\left.\frac{\partial \eta}{\partial p}\right|_\tau = \frac{bn}{1-e}\frac{\partial \tau}{\partial p}$$

and $$\dot{\eta}\frac{\partial \eta}{\partial p} = -\frac{na\sqrt{1-e^2}}{1-e}\sqrt{\frac{\mu}{a}\frac{1+e}{1-e}}\;\frac{\partial \tau}{\partial p}$$

Now $n = \mu^3 a^3$, then $\dot{\eta}\dfrac{\partial \eta}{\partial p} = -\dfrac{\mu}{a}\dfrac{1+e}{1-e}\dfrac{\partial \tau}{\partial p}$

which cancels against $2T(\tau)\dfrac{\partial \tau}{\partial p}$. Then:

$$C_p = -\tau\frac{\partial}{\partial p}\left(-\frac{\mu}{2a}\right) - h\frac{\partial \omega}{\partial p} - h\cos i\,\frac{\partial \Omega}{\partial p}$$

$$C_q = -\tau\frac{\partial}{\partial q}\left(-\frac{\mu}{2a}\right) - h\frac{\partial \omega}{\partial q} - h\cos i\,\frac{\partial \Omega}{\partial q}$$

and:
$$[p,q] = \frac{\partial(-\tau, -\mu/2a)}{\partial(p,q)} + \frac{\partial(\omega,h)}{\partial(p,q)} + \frac{\partial(\Omega, h\cos i)}{\partial(p,q)}$$

We now have the equations for the elements by computing all the brackets. We see $[a,a]=0$, $[a,e]=0$
and:
$$[a,\tau] = \frac{\partial(-\tau)}{\partial a}\frac{\partial(-\mu/2a)}{\partial \tau} - \frac{\partial(-\tau)}{\partial \tau}\frac{\partial(-\mu/2a)}{\partial a} = +\frac{n^2a^3}{2a^2} = \frac{1}{2}n^2 a$$

LECTURE 8: 3-5-62

Absent

LECTURE 9: 3-7-62

Perturbations By Jupiter and Saturn: To orders of e:

$2n - 5n_1$: 890 years : 3rd order
$29n - 72n_1$: 1800 years : 43rd order
$60n - 149n_1$: 36,000 years : 89th order

The analysis we have used diverges after a long period of time, but is good over ranges of several hundred years.

Recapitulation:

$$\left.\begin{aligned} a &= a_0 + \Sigma J \cos\theta \\ e &= e_0 + dt + \Sigma J \cos\theta \\ i &= i_0 + dt + \Sigma J \cos\theta \\ \Omega &= \Omega_0 + dt + \Sigma J \sin\theta \\ \tilde{\omega} &= \tilde{\omega}_0 + dt + \Sigma J \sin\theta \\ \epsilon &= \epsilon_0 + dt + \Sigma J \sin\theta \end{aligned}\right\}$$

J's are different for each one, and so are d's. J's give the periodic perturbation terms, while d gives the secular terms.

$$\dot{\epsilon} = -\frac{2}{na}\frac{\partial R}{\partial a} \quad ; \quad l = nt + \epsilon$$

$$R = \Sigma c \cos\theta_0 + \Sigma C \cos\theta$$

$$\frac{\partial R}{\partial a} = \Sigma \frac{\partial c}{\partial a}\cos\theta_0 + \Sigma \frac{\partial C}{\partial a}\cos\theta + \Sigma C \sin\theta \, i \frac{dl}{da}$$

$$+ \frac{\partial n}{\partial a}\Sigma i C \sin\theta$$

$$\frac{\partial l}{\partial a} = \frac{\partial l}{\partial n}\frac{dn}{da}$$

Now write: $\dfrac{\partial R}{\partial a} = \underbrace{\left(\dfrac{\partial R}{\partial a}\right)} + \dfrac{\partial R}{\partial l}\dfrac{\partial l}{\partial a}$

means only differentiate coefficients in R

Now $\dfrac{\partial R}{\partial l} = \dfrac{\partial R}{\partial \epsilon}$; $\dfrac{\partial l}{\partial a} = \dfrac{\partial l}{\partial n}\dfrac{dn}{da}$

Then: $\frac{\partial R}{\partial a} = \left(\frac{\partial R}{\partial a}\right) + \frac{\partial R}{\partial \epsilon}\, t \,\frac{dn}{da}$

and:

$$\dot{\epsilon} = -\frac{2}{na}\left(\frac{\partial R}{\partial a}\right) - \frac{2}{na}\, t\, \frac{dn}{da}\, \frac{na}{2}\, \dot{a}$$

or: $\dot{\epsilon} = -\frac{2}{na}\left(\frac{\partial R}{\partial a}\right) - t\,\frac{dn}{dt}$

Define a new element ϵ' such that:

$$\dot{\epsilon}' = \dot{\epsilon} + t\,\frac{dn}{dt} = -\frac{2}{na}\left(\frac{\partial R}{\partial a}\right)$$

Write: $\dot{\epsilon}' = \dot{\epsilon} + \frac{d}{dt}(nt) - n$

$$\epsilon' = \epsilon + nt - \int n\, dt$$

or: $l = nt + \epsilon = \int n\,dt + \epsilon'$; Call $\rho = \int n\,dt$

Then: $l = \rho + \epsilon'$ (from now on, we let $\epsilon' \to \epsilon$)

We now substitute this into the disturbing function, whenever $nt + \epsilon$ appears. What we have done is add a 7th element, ρ, called the mean motion. This does not change:

$$\frac{\partial R}{\partial \epsilon} = \frac{\partial R}{\partial \epsilon'}$$

For $\dot{\epsilon}$: $\dot{\epsilon} = -\frac{2}{na}\left(\frac{\partial R}{\partial a}\right) + P\,\frac{\partial R}{\partial e} + Q\,\frac{\partial R}{\partial i}$

Consider $m \approx m_1$: $R = \underbrace{m_1 R_1}_{\text{1st order perturbation}} + \underbrace{m_1^2 R_2}_{\text{2nd order}}$

In general, one of the elements can be written as:

$a = a_0 + \Delta' a + \Delta'' a + \cdots$; $\Delta' a$ means 1st order in a.

Also:

$n = n_0 + \Delta' n + \Delta'' n + \cdots$

Now we have: $n = \mu^{1/2} a^{-3/2}$

Hence: $n = \mu^{1/2}\left[a_0 + \Delta' a + \Delta'' a\right]^{-3/2}$; $n_0 = \mu^{1/2} a_0^{-3/2}$

$$\Delta' n = \mu^{1/2} a_0^{-3/2}\left(-\frac{3}{2}\frac{\Delta' a}{a_0}\right) = -\frac{n_0}{a_0}\frac{3}{2}\Delta' a$$

and:

$$\Delta'' n = n_0\left[\frac{15}{8}\left(\frac{\Delta' a}{a_0}\right)^2 - \frac{3}{2}\frac{\Delta'' a}{a_0}\right]$$

Now consider $\rho = \int n\,dt = \rho_0 + \Delta'\rho + \Delta''\rho + \cdots$

$$\rho_0 = \int n_0\,dt = n_0 t$$

$$\Delta'\rho = -\frac{3}{2}\frac{n_0}{a_0}\int \Delta' a\,dt \quad ; \text{ and } \quad \Delta''\rho = \int \Delta'' n\,dt$$

Now; $l = \rho + \epsilon$; $\Delta' l = \Delta'\rho + \Delta'\epsilon$

$$\Delta' a = \frac{A\cos\theta}{i n + i_1 n_1} \quad ; \quad \Delta'\rho = \frac{A\sin\theta}{(i n + i_1 n_1)^2}$$

It turns out that the mean longitude of Saturn is perturbed by 48′ (minutes) of arc. We have used: $\Delta' a = \sum J\cos\theta$. The perturbations in ρ are periodic.

$$l = n_0 t + \Delta'\rho + \epsilon_0 + \epsilon_1 t + \Delta'\epsilon$$

The secular part is: $l = (n_0 + \epsilon_1)t + \epsilon_0$. l can be observed. What is $\dot\rho$?

$$\dot\rho = \mu^{1/2} a^{-3/2} \quad ; \quad \ddot\rho = \frac{3}{2}\mu^{1/2} a^{-5/2}\frac{2}{na}\frac{\partial R}{\partial\epsilon} = -\frac{3}{a^2}\frac{\partial R}{\partial\epsilon}.$$

The above covers all the first order terms.

Now go to second Order:

Form; $\frac{d}{dt}$ of some element, say Ω :

$$\frac{d\Omega}{dt} = \frac{d}{dt}\left[\Omega_0 + \Delta'\Omega + \Delta''\Omega\right] = \sum \frac{\partial c}{\partial i}\cos\theta_0 + \sum\frac{\partial C}{\partial i}\cos\theta$$

$$= m_1 F(\text{elements})$$

Expand in a Taylor Series about unperturbed values of elements:

$$m_1 F_0 + m_1 \left[\Delta' \rho \left(\frac{\partial F}{\partial \rho}\right)_0 + \Delta' \rho_1 \left(\frac{\partial F}{\partial \rho_1}\right) + \sum \Delta' \alpha_i \left(\frac{\partial F}{\partial \alpha_i}\right)_0 \right]$$

We can pick out:

$$\frac{d}{dt} \Delta' \Omega = m_1 F_0$$

$$\frac{d}{dt} \Delta'' \Omega = m_1 \left[\quad \right]$$

Write: $F = A + \sum B \cos\theta \quad ; \left(\frac{\partial F}{\partial e}\right)_0 = C + \sum D \cos\theta$

Then: $\Delta' e \left(\frac{\partial F}{\partial e}\right)_0 = Et + t \sum F \cos\theta$ + Periodic terms + constant terms

The other elements give similar results, except for $\cos\theta \to \sin\theta$ for $\Omega, \tilde{\omega}, \epsilon$.

For the ρ terms:

$$\left.\begin{aligned} \Delta' \rho &= \sum Q \sin\theta \\ \left(\frac{\partial F}{\partial \rho}\right)_0 &= \sum R \sin\theta \end{aligned}\right\}$$ these combine to give secular and periodic terms.

Then, $\Delta'' \Omega$ behaves as:

$$\Delta'' \Omega = Gt + \underbrace{\tfrac{1}{2} E t^2}_{\text{called}} + P.T. + t \left[\sum (U \sin\theta + V \cos\theta) \right]$$

called secular acceleration, however, so far unobservable.

All the other elements have this form in the second order except a: Here we have $F = \sum A \sin\theta$, then $\left(\frac{\partial F}{\partial \epsilon}\right)_0 = \sum B \sin\theta$.

Then:

$\Delta' e \left(\frac{\partial F}{\partial e}\right) = t \sum C \sin\theta + P.T.$. For $\Omega, \tilde{\omega}, \epsilon$, let $\sin\theta \to \cos\theta$

Also; since $\left(\frac{\partial F}{\partial \rho}\right)_0 \sim \cos\theta$; $\Delta' \rho \left(\frac{\partial F}{\partial \rho}\right)_0 = P.T.$

$$\Delta'' a = P.T. + t \left[\sum D \sin\theta + \sum E \cos\theta \right]$$

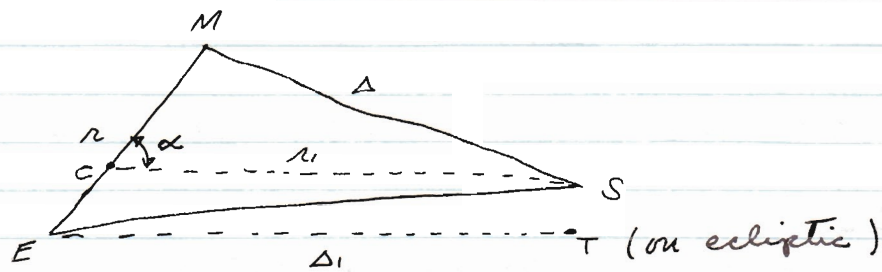
We see now that a is no longer bounded because of the secular-periodic terms. We can get rid of these by a transformation. Often find secular terms are part of long-periodic terms.

LECTURE 10 : 3-17-62

Absent

LECTURE 11: 3-19-62

Recapitulation: Lunar Theory:

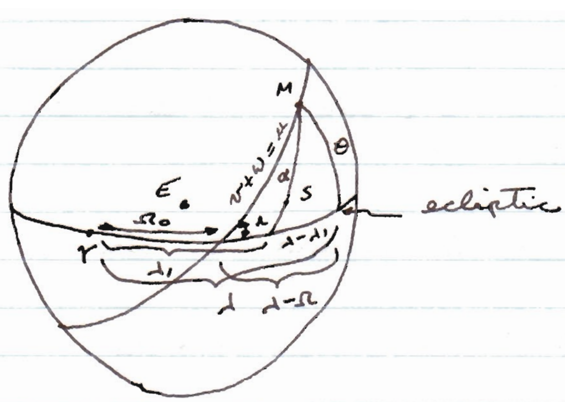


$$R = R_2 + R_4$$

$$R_2 = m^2 n^2 r^2 \left(\frac{a_1}{r_1}\right)^3 P_2(\cos\alpha)$$

$$R_4 = \underbrace{m^2 n^2 \underset{\sim a^2}{r^2} \; \frac{a}{a_1} \; \underset{\sim 1}{\frac{r}{a}} \; \underset{\sim 1}{\left(\frac{a_1}{r_1}\right)^4}}_{\text{in zeroth order}} P_3(\cos\alpha)$$

Find a relation between α and elements. We must use spherical trigonometry. Center a celestial sphere on the earth:



We use the identity: $\cos\alpha = \cos\theta \cos(\lambda - \lambda_1)$

and: $P_2 = \frac{1}{2}\left(3\cos^2\alpha - 1\right)$

$$P_3 = \frac{1}{2}\left(5\cos^3\alpha - 3\cos\alpha\right)$$

Let $s = \tan\theta$, then $\cos\alpha = \dfrac{\cos(\lambda-\lambda_1)}{\sqrt{1+s^2}}$

$$= \cos(\lambda-\lambda_1)\left(1-\tfrac{1}{2}s^2+\ldots\right)$$

Use another spherical identity:

$$s = \tan\theta = \tan i \sin(\lambda-\Omega) \quad ; \quad s = \gamma \sin(\lambda-\Omega)$$

Then:

$$P_2 = \tfrac{1}{2}\left[3\cos^2(\lambda-\lambda_1)(1-s^2)-1\right]$$

$$= \tfrac{1}{2}\left[3(1-s^2)\left(\tfrac{1}{2}+\tfrac{1}{2}\cos 2(\lambda-\lambda_1)\right)-1\right]$$

$$= \tfrac{1}{4}\left[1-3s^2+3(1-s^2)\cos 2(\lambda-\lambda_1)\right]$$

and: $P_3 = \tfrac{1}{8}\left[3\left(1-\tfrac{11}{2}s^2\right)\cos(\lambda-\lambda_1) + 5\left(1-\tfrac{3}{2}s^2\right)\cos 3(\lambda-\lambda_1)\right]$

To zeroeth order:

$$P_3 = \tfrac{1}{8}\left[3\cos(nt+\epsilon-n_1t-\epsilon_1)+5\cos 3\xi\right]$$

and:

$$R_4 = \frac{m^2n^2a^3}{8a_1}\left[3\cos\xi+5\cos 3\xi\right]$$

We now want to expand s. Use:

$$\tan(\lambda-\Omega) \equiv \cos i \tan(v+\omega)$$

Define: $x \equiv \tan\tfrac{1}{2}i$

Then: $\tan(\lambda-\Omega) = \dfrac{1-x^2}{1+x^2}\tan(v+\omega)$

We know previously:

$$\tan\frac{E}{2} = \frac{1-\beta}{1+\beta}\tan\frac{v}{2}$$

$$E = v + 2 \sum \frac{(-1)^n}{n} \beta^n \sin nv$$

$$2(\lambda - \Omega) = 2(v+\omega) + 2 \sum \frac{(-1)^n}{n} x^{2n} \sin 2n(v+\omega)$$

We want to expand to 2nd order:

$$\lambda - \Omega = v + \omega - \frac{\gamma^2}{4} \sin 2(v+\omega)$$

Recall:

$$\frac{r}{a} = 1 + \tfrac{1}{2} e^2 - e \cos M - \tfrac{1}{2} e^2 \cos 2M$$

$$v = M + 2e \sin M + \frac{5}{4} e^2 \sin 2M$$

$$\lambda = V + \omega + \Omega = nt + \epsilon - \tilde{\omega} + \omega + \Omega = nt + \epsilon$$

Substituting:

$$2(v+\omega) = 2M + 2\omega + 4e \sin M + \cdots$$

$$\sin(2M + 2\omega + \Sigma \sin nM)$$

$$\sin(x + \Delta x) = \sin x \left(1 - \frac{\Delta x^2}{2}\right) + \cos x \, \Delta x$$

The final result for λ will be:

$$\lambda = nt + \epsilon + \sum \sin nM + \sum \sin m\eta \quad ; \quad \eta = nt + \epsilon - \Omega$$

Now expand s; which we can find from the expression for $\lambda - \Omega$; we will get a sine series for s and a cosine series for s^2.

Form:

$$\lambda - \lambda_1 = \xi + \text{sine terms}$$

$$\cos 2(\lambda - \lambda_1) = \cos(\underbrace{2\xi}_{x} + \underbrace{\text{sine terms}}_{\Delta x})$$

$$\rightarrow \cos 2\xi \left(1 - \frac{\Delta x^2}{2}\right) - \sin 2\xi \, \Delta x$$

This means we will get secular terms in R_2 but not R_4, generally true when have even and odd Legendre polynomials.

We have for the secular part of R_2:

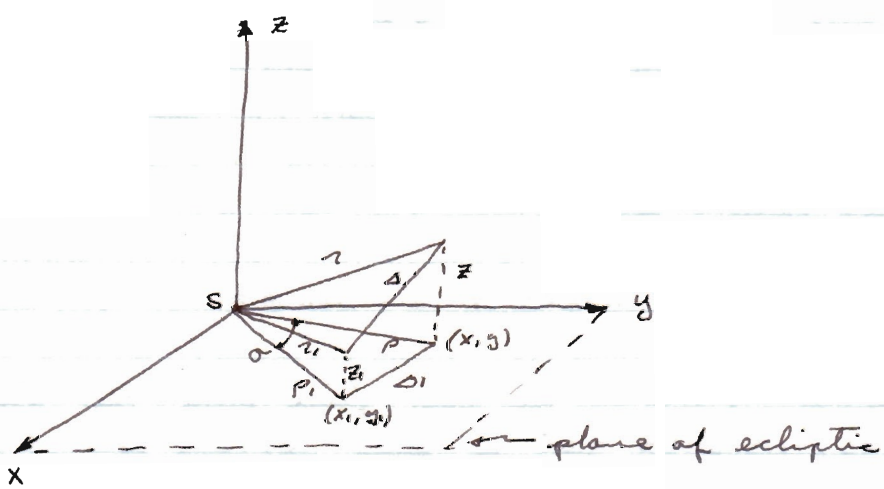
$$R_2 = m^2 n^2 a^2 \left[\frac{1}{4} + \frac{3}{8} e^2 + \frac{3}{8} \epsilon_1^2 - \frac{3}{8} \gamma^2 \right]$$

It turns out that there are no secular terms in a, e, i for the moon, but there are for Ω, ω, τ. This means that we can use planetary theory to find the motion of the moon to the first order approximately but this is not good enough.

We now do the problem in terms of the disturbing function:

$$R = G m_1 \left(\frac{1}{\Delta} - \frac{x x_1 + y y_1 + z z_1}{r_1^3} \right)$$

We take as 1st order quantities: $e, e_1, \gamma, \gamma_1, \rho, \rho_1, \sigma$



$$x x_1 + y y_1 = \rho \rho_1 \cos \sigma$$

$$\Delta^2 = (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2$$

$$\Delta_1^2 = (x - x_1)^2 + (y - y_1)^2 \quad ; \quad \Delta^2 = \Delta_1^2 + (z - z_1)^2$$

$$\frac{1}{\Delta} = \frac{1}{\Delta_1} \left[1 + \frac{(z - z_1)^2}{\Delta_1^2} \right]^{1/2} = \frac{1}{\Delta_1} - \frac{1}{2} \frac{(z - z_1)^2}{\Delta_1^3}$$

We can now use: $\Delta_1^2 = \rho^2 + \rho_1^2 - 2 \rho \rho_1 \cos \sigma$

This gives Δ in terms of the 1st order quantities.

Also: $r_1^2 = \rho_1^2 + z_1^2$, which gives:

$$-\frac{xx_1+yy_1+zz_1}{r_1^3} = \frac{-\rho\rho_1\cos\sigma + zz_1}{(\rho_1^2+z_1^2)^{3/2}}$$

$$= -\frac{\rho\rho_1\cos\sigma + zz_1}{\rho_1^3\left(1+\frac{z_1^2}{\rho_1^2}\right)^{3/2}} = -\frac{\rho\rho_1\cos\sigma + zz_1}{\rho_1^3}\left(1-\frac{3}{2}\frac{z_1^2}{\rho_1^2}\right)$$

$$= -\frac{\rho\cos\sigma}{\rho_1^2} + \frac{3}{2}\frac{\rho z_1^2\cos\sigma}{\rho_1^4} - \frac{zz_1}{\rho_1^3}$$

$$R = Gm_1\left[\frac{1}{\Delta_1} - \frac{1}{2}\frac{(z-z_1)^2}{\Delta_1^3} - \frac{\rho\cos\sigma}{\rho_1^2} + \frac{3}{2}\rho z_1^2\frac{\cos\sigma}{\rho_1^4} - \frac{zz_1}{\rho_1^3}\right]$$

We have now neglected all terms of higher than second order in the inclination.

Furthermore, $\rho = r\cos\theta$, and $\sin\theta = \sin i\,\sin(v+\omega)$, so that, if $\sin\theta$ is small; $\cos\theta \doteq 1-\frac{1}{2}\sin\theta$, and:

$$\rho = r\left\{1-\frac{1}{2}\gamma^2\left(\frac{1}{2}-\frac{1}{2}\cos(v+\omega)\right)\right] \quad ; \quad \gamma^2 = \sin^2 i$$

$$= r - \frac{1}{4}a\gamma^2 + \frac{1}{4}a\gamma^2\cos(M+\omega)$$

We can now expand r in ω, so that finally:

$$\rho = a\,[1 + \text{first order terms}]$$

with ρ_1 of the same structure. We can expand $\sigma = \lambda - \lambda_1$ and obtain as before: $\sigma = nt+\epsilon-(n_1t+\epsilon_1)$ + first order terms. or: $\sigma = \phi+\omega$. Hence: $\rho = a(1+u)$; $\rho_1 = a_1(1+u_1)$ where we know what u, u_1 are from above. Finally we have the expansion for z: $z = r\sin\theta = r\sin i\,\sin(v+\omega)$ $= a\gamma\sin(M+\omega)$, which we now can put into the disturbing function.

LECTURE 12 : 3-21-62

Motion of Longitude and Line of Apsides

$$R_S = \frac{3}{8} m^2 n^2 a^2 (e^2 - \tan^2 i)$$

$$\dot{\Omega} = \frac{1}{na^2 \sqrt{1-e^2} \sin i} \frac{\partial R}{\partial i}$$

??

$$\dot{\omega} = \frac{\sqrt{1-e^2}}{na^2 e} \frac{\partial R}{\partial e} - \frac{\cot i}{na^2\sqrt{1-e^2}} \frac{\partial R}{\partial i}$$

$$\dot{\Omega} = -\frac{3}{8} \frac{n_1^2 a^2 \, 2 \tan i \sec^2 i}{n a^2 \sin i} = -\frac{3}{4} \frac{n_1^2}{n}$$

$$\dot{\omega} = \frac{3}{8} \frac{n_1^2 a^2 \, 2e}{na^2 e} + \frac{3}{4} \frac{n_1^2}{n} = \frac{3}{2} \frac{n_1^2}{n}$$

These results can be correlated with observation.

Recall our previous expansion of the planetary disturbing function:

$$R = G m_1 \left[\frac{1}{\Delta_1} - \frac{1}{2} \frac{(z - z_1)^2}{\Delta_1^3} - \frac{\rho \cos\sigma}{\rho_1^2} + \frac{3}{2} \frac{\rho z_1^2 \cos\sigma}{\rho_1^4} - \frac{z z_1}{\rho_1^3} \right]$$

where:

$$\rho = a(1+u)$$
$$\rho_1 = a_1(1+u_1)$$
$$\sigma = \phi + \omega$$
$$\phi = nt + \epsilon - (n_1 t + \epsilon_1)$$
$$z = a\gamma \sin(M+\omega)$$

We expand the relevant parts of the disturbing function in a Taylor series: First consider:

$$-\frac{1}{2} \frac{(z - z_1)^2}{\Delta_1^3} + \frac{3}{2} \frac{\rho z_1^2 \cos\sigma}{\rho_1^4} - \frac{z z_1}{\rho_1^3}$$

Define : $\Delta_0^2 = a^2 + a_1^2 - 2 a a_1 \cos\varphi$

Then we can write approximately:

$$-\frac{1}{2}\frac{(z-z_1)^2}{\Delta_0^3} + \frac{3}{2}\frac{a\,z_1^2\cos\phi}{a_1^4} - \frac{z z_1}{a_1^3}$$

We now expand R in a Taylor series:

$$R = R_0 + \Delta\rho\left(\frac{\partial R}{\partial\rho}\right)_0 + \Delta\rho_1\left(\frac{\partial R}{\partial\rho_1}\right)_0 + \Delta\sigma\left(\frac{\partial R}{\partial\sigma}\right)_0$$

$$+ \frac{1}{2}\Delta\rho^2\left(\frac{\partial^2 R}{\partial\rho^2}\right)_0 + \frac{1}{2}\Delta\rho_1^2\left(\frac{\partial^2 R}{\partial\rho_1^2}\right)_0 + \frac{1}{2}\Delta\sigma^2\left(\frac{\partial^2 R}{\partial\sigma^2}\right)_0$$

$$+ \Delta\rho\,\Delta\rho_1\left(\frac{\partial^2 R}{\partial\rho\,\partial\rho_1}\right)_0 + \Delta\rho\,\Delta\sigma\left(\frac{\partial^2 R}{\partial\rho\,\partial\sigma}\right)_0 + \Delta\rho_1\Delta\sigma\left(\frac{\partial^2 R}{\partial\rho_1\partial\sigma}\right)_0$$

set: $\left(\frac{\partial R}{\partial\rho}\right)_0 = \frac{\partial R_0}{\partial a}$

$$\Delta\rho = a\mu \quad ; \quad \Delta\rho^2 = (a\mu)^2$$

$$R_0 = G m_1\left[\frac{1}{\Delta_0} - \frac{a\cos\phi}{a_1^2}\right]$$

$\frac{\partial R_0}{\partial a}$, $\frac{\partial^2 R_0}{\partial a^2}$ involve $\frac{1}{\Delta_0^3}$, $\frac{1}{\Delta_0^5}$ so that we can express the result in a Fourier series in $\cos\phi$. That is, we can generally write:

$$\frac{1}{\Delta_0^n} = \frac{1}{2}B_0 + \sum B_n\cos n\phi$$

We can write; for $a > a_1$:

$$\Delta_0^2 = a^2\left[1+\alpha^2 - 2\alpha\cos\phi\right] \quad ; \quad \alpha = \frac{a_1}{a}$$

$$D \equiv (1+\alpha^2 - 2\alpha\cos\phi)$$

Then: $$\frac{1}{\Delta_0^n} = \frac{1}{a^n D^{n/2}}$$

Let $s = n + 1/2$; $z = e^{i\phi}$, then:

$$D^{-s} = (1+\alpha^2 - \alpha z - \alpha z^{-1})^{-s} = (1-\alpha z)^{-s}(1-\alpha z^{-1})^{-s}$$

Much of the expansion follows from work before, so we just write down result:

$$D^{-s} = \tfrac{1}{2} B_0^s + \Sigma\, B_n^s \cos n\phi$$

where: $$\tfrac{1}{2} B_n^s = \frac{s(s+1)\ldots(s+n-1)}{n!} \frac{\alpha^n}{(1-\alpha^2)^s} \cdot F\left(s, 1-s, n+1, \frac{-\alpha^2}{1-\alpha^2}\right)$$

The B's are called the Laplace coefficients. There are various recurrence relations among them:

$$B_n^s = \frac{n-1}{n-s}(\alpha - \alpha^{-1}) B_{n-1}^s - \frac{n+s-2}{n-s} B_{n-2}^s$$

$$B_n^{s+1} = \frac{(n+s)(1+\alpha^2) B_n^s - 2(n-s+1)\alpha B_{n+1}^s}{s(1-\alpha^2)^2}$$

The B's can also be obtained from elliptic integrals.

$$B_0^{1/2} = \frac{4}{\pi}\int_0^{\pi/2} \frac{d\phi}{\sqrt{1-\alpha^2 \sin^2\phi}}$$

$$B_1^{1/2} = \frac{4}{\pi\alpha}\left[\int_0^{\pi/2} \frac{d\phi}{\sqrt{1-\alpha^2\sin^2\phi}} - \int_0^{\pi/2}\sqrt{1-\alpha^2\sin^2\phi}\, d\phi\right]$$

For the secular part of R, called N, we obtain:

$$N = Gm_1\Big[4C + \{e^2 + e_1^2 - \gamma^2 - \gamma_1^2 - 2\gamma\gamma_1\cos(\Omega - \Omega_1)\}D$$
$$- 2ee_1 E \cos(\tilde{\omega} - \tilde{\omega}_1)\Big]$$

where: $C = \frac{1}{8a} B_0^{1/2}$; $D = \frac{\alpha}{8a} B_1^{3/2}$; $E = \frac{\alpha}{8a} B_2^{3/2}$

In the disturbing function:

$$R = Gm_1 \Big[\underbrace{\frac{1}{\Delta}}_{\substack{\text{gives}\\ \text{secular}\\ \text{part}}} - \underbrace{\frac{x x_1 + y y_1 + z z_1}{r_1^3}}_{\text{gives periodic part}} \Big]$$

Call the periodic part R':

$$R' = \sum_p \sum_q \left[C_{pq} \cos (pN + q M_1) + S_{pq} \sin (pM + q M_1) \right]$$

Now form:

$$\int_0^{2\pi} R' \, dM_1 = \sum_p \left[C_{p0} \cos pM + S_{p0} \sin pM \right]$$

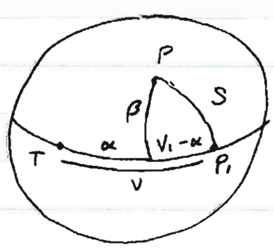
We can also write R' as:

$$R' = \frac{r \cos S}{r_1^2}$$

Then: $$r \int_0^{2\pi} \frac{\cos S}{r_1^2} \, dM_1 \longrightarrow C \int_0^{2\pi} \cos S \, dv_1$$

$$r_1^2 \dot{v} = h_1 = n_1 C$$
$$\dot{n}_1 = M \quad , \quad dM = r^2 \, dv_1$$

Consider a celestial sphere:



$$\cos S = \cos \beta \cos (V_1 - \alpha)$$

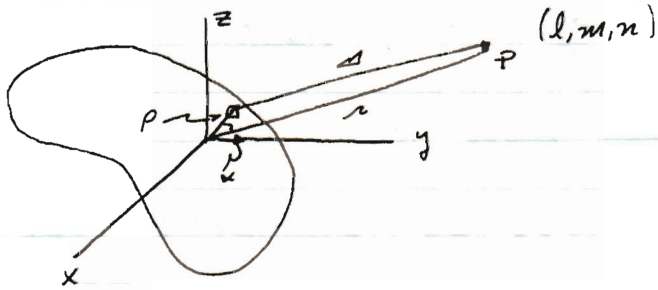
hence: $$\int_0^{2\pi} R' \, dM_1 = 0 \quad , \quad C_{00} = S_{00} = 0$$

and the R' is thus shown to be purely periodic.

This concludes the treatment of the disturbance from a third body.

Disturbance of Artificial Satellites Due to a Non-Spheroidal Body.

Potential Due to an Irregular Body:



Axis taken thru CG.

We have: $dU = \dfrac{G\,dM}{\Delta}$; $U = G\displaystyle\int_M \dfrac{dM}{\Delta}$

Or:

$$U = \frac{G}{r}\int \Sigma \left(\frac{\rho}{r}\right)^n P_n(\cos\alpha)\, dM$$

$$= \frac{G}{r}\int \Big\{ \underbrace{1}_{\substack{\text{0th} \\ \text{harmonic}}} + \underbrace{\frac{\rho}{r}\cos\alpha}_{\substack{\text{1st} \\ \text{harmonic}}} + \left(\frac{\rho}{r}\right)^2 \frac{1}{2}\left(3\cos^2\alpha - 1\right) + \cdots \Big]\, dM$$

Define a new coordinate system ξ, η, ζ such that $\xi = \rho\cos\alpha = lx + ny + mz$. Since we have taken the CG as origin, we have:

$$\int x\, dM = \int y\, dM = \int z\, dM = 0$$

so that the 1st harmonic vanishes.

Consider:

$$3\rho^2\cos^2\alpha - \rho^2 \;\rightarrow\; 3\xi^2 - \rho^2$$

We know $\rho^2 = x^2 + y^2 + z^2 = \xi^2 + \eta^2 + \zeta^2$
so that we can write:

$$3(\xi^2 - \rho^2) + 2\rho^2$$

$$-3(\eta^2 + \zeta^2) + 2x^2 + 2y^2 + 2z^2$$

Also: $\int (y^2+z^2)\,dM = A$

$$\left.\begin{aligned}\int (y^2+z^2)\,dM &= A\\ \int (z^2+x^2)\,dM &= B\\ \int (x^2+y^2)\,dM &= C\end{aligned}\right\}\ \text{principle moments of inertia}$$

$$\int (\eta^2+\zeta^2)\,dM = I \ \}\ \text{moment about } \xi$$

Hence we get for the 2nd harmonic term:

$$\frac{G}{2r^3}\,(A+B+C-3I)$$

Consider again: $3\xi^2-\rho^2 \rightarrow 3(lx+my+nz)^2-\rho^2$

Because we have chosen principle axes, xy, yz, zx products will vanish and we need only consider:

$$3l^2x^2+3m^2y^2+3n^2z^2-\rho^2$$

or: $3(1-m^2-n^2)x^2+3(1-l^2-n^2)y^2+3(1-l^2-m^2)z^2-\rho^2$

Since $l^2+m^2+n^2=1$; We get: $A+B+C-3n^2(x^2+y^2)$

or: $A+B+C-3l^2A-3m^2B-3n^2C$

Now let $A=B$ to introduce cylindrical symmetry:

$$2A+C-3A(l^2+n^2)-3n^2C \quad = \quad (C-A)(1-3n^2)$$

Altogether, to the 2nd harmonic:

$$U = \frac{GM}{r} + \frac{G}{r^3}(C-A)\,\frac{1}{2}\,\underbrace{(1-3\cos^2\theta)}_{1-3\sin^2\phi'}$$

or: $U = \frac{GM}{r}\left[1+\frac{(C-A)}{MR^2}\left(\frac{R}{r}\right)^2 P_2(\sin\phi')\right]$

where R = equatorial radius. $\frac{C-A}{MR^2}$ is now a pure number about the order of 10^{-3}.

LECTURE 13 : 4-9-62

Recall the 2nd harmonic potential of last time. The same results can be obtained from Laplace's equation: $\nabla^2 V = 0$. The solutions are in the form of spherical harmonics.

International Standard Form of Potential:

1) Axially Symmetric

$$U = \frac{\mu}{r}\left[1 - \sum_{n=1}^{\infty} J_n \left(\frac{R}{r}\right)^n P_n(\sin\beta)\right]$$

$$\mu = G M_{\oplus}$$

2) non-axially symmetric:

$$U = \frac{\mu}{r}\left[1 + \sum_{n=1}^{\infty} \sum_{m=0}^{n} \left(\frac{R}{r}\right)^n P_n^m(\sin\beta)\,\lambda \cdot \left(C_{n,m}\cos m\lambda + S_{n,m}\sin m\lambda\right)\right]$$

$$C_{n,0} = -J_n \quad ; \quad S_{n,0} = 0$$

Note: $$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

$$P_n^m(x) = (1-x^2)^{\frac{m}{2}}\frac{d^m}{dx^m}P_n(x)$$

$$\int_{-1}^{1}\left[P_n^m(x)\right]^2 dx = \frac{2}{2n+1}\,\frac{(n+m)!}{(n-m)!}$$

Definition: $$p_{n,m}(x) = \left[\frac{(n-m)!}{(n+m)!}\right]^{1/2} P_n^m(x)$$

Geodesic Form of Potential:

$$U = \frac{GM}{r}\left[1 + J\left(\frac{R}{r}\right)^2\left(\frac{1}{3} - \sin^2\phi'\right)\right]$$

$$J = \frac{3}{2}\,\frac{C-A}{R^2 M}$$

Shape of the Earth:

The centripetal potential can be written as: $\frac{1}{2}\omega^2(x^2+y^2)$
Then on the earth's surface we have for the geopotential:

$$\psi = U + \tfrac{1}{2}\omega^2(x^2+y^2) = \text{constant}$$

Now: $x^2+y^2 = r^2(1-\sin^2\varphi')$

Let: $r = R(1-Y)$:

$$\psi = \frac{GM}{R(1-Y)} + \frac{GMJR^2}{R^3(1-Y)^3}\left(\tfrac{1}{3}-\sin^2\varphi'\right)$$

$$+ \tfrac{1}{2}\omega^2R^2(1-Y)^2(1-\sin^2\varphi') = C$$

Expand to first order in Y:

$$\psi = \frac{GM}{R}(1+Y) + \frac{GMJ}{R}(1+3Y)\left(\tfrac{1}{3}-\sin^2\varphi'\right) + \tfrac{1}{2}\omega^2R^2(1-2Y)(1-\sin^2\varphi')$$

$$= C$$

Solving for the φ' dependence of Y:

$$Y\frac{GM}{R} = \frac{GMJ}{R}\sin^2\varphi' - \tfrac{1}{2}\omega^2R^2\sin^2\varphi'$$

or: $$Y = \left(J + \tfrac{1}{2}\frac{\omega^2R^3}{GM}\right)\sin^2\varphi'$$

If we write: $$\frac{\omega^2R^3}{GM} = \frac{\omega^2R}{\frac{GM}{R^2}} = m$$

we have the ratio of the centripetal to the gravitational force at the equator:

Finally:

$$r = R\left[1-(J+\tfrac{1}{2}m)\sin^2\varphi'\right]$$

$$r = R\left[1-f\sin^2\varphi'\right] \quad ; \quad f = J+\tfrac{1}{2}m = \text{flattening}$$

This of course is a "dynamical" flattening. The dynamical shape of the earth is an ellipsoid of revolution.

The acceleration on the surface of the earth is given by:

$$g = \frac{\partial \psi}{\partial r} = -\frac{GM}{r^2} - \frac{GMJ}{r^4} 3R^2\left(\tfrac{1}{3} - \sin^2\varphi'\right) + \omega^2 r\left(1 - \sin^2\varphi'\right)$$

At the equator:

$$g_e = -\frac{GM}{R^2} - \frac{GMJ}{R^2} + \omega^2 R$$

$$= -\frac{GM}{R^2}\left[1 + J - m\right]$$

Hence:

$$g = g_e\left[1 + \left(\tfrac{5}{2}m - f\right)\sin^2\varphi'\right]$$

We now go on to find the effect of the potential second harmonic as a disturbing function: Take:

$$R = -\frac{J_2}{r}\left(\frac{R}{r}\right)^2 \frac{1}{2}\left(3\sin^2\varphi' - 1\right)$$

$$\sin\varphi' = \sin i \sin(v + \omega)$$

$$R = -\frac{J_2}{r}\left(\frac{R}{r}\right)^2 \frac{1}{2}\left[3\sin^2 i \sin^2(v+\omega) - 1\right] \Rightarrow H \text{ (instead of } R)$$

We can expand in terms of M:

$$H = \sum_{ij} A_{ij} \cos(iM + j\omega)$$

We can form, over one revolution,

$$\int_0^{2\pi} H\,dM = 2\pi\left[A_{00} + \sum_j A_{0j}\cos j\omega\right]$$

Thus we have eliminated all short period terms and are left with the secular and long-term perturbation terms.

We can find these Lagrange planetary equations that Ω, ω, χ have secular perturbations while those of a, e, i are periodic.

$$\frac{H}{J} = \frac{\left(1 - \frac{3}{2}\sin^2 i\right)}{r^3} + \frac{3}{2}\,\frac{\sin^2 i \cos 2(v+\omega)}{r^3}$$

where: $\quad \frac{1}{r^3} = A_0 + \sum A_n \cos M$

$$\left.\begin{array}{lll} a & \text{involves} & \frac{\partial H}{\partial \chi} \\ e & \text{"} & \frac{\partial H}{\partial \chi}, \frac{\partial H}{\partial \omega} \\ i & \text{"} & \frac{\partial H}{\partial \omega}, \frac{\partial H}{\partial \Omega} \end{array}\right\} \quad \text{no secular terms}$$

$$\left.\begin{array}{lll} \omega & \text{"} & \frac{\partial H}{\partial e}, \frac{\partial H}{\partial i} \\ \Omega & \text{"} & \frac{\partial H}{\partial i} \\ \chi & \text{"} & \frac{\partial H}{\partial e}, \frac{\partial H}{\partial a} \end{array}\right\} \quad \text{no period terms}$$

We can write for the secular part of H:

$$\bar{H} = A_{00}$$

Long-period part: $\quad H^* = \sum_j A_{0j} \cos j\omega$

For the short period part:

$$H_{sp} = H - \bar{H} - H^*$$

We now depart from celestial mechanics to treat the canonical form of classical mechanics.

Canonical Formulation of Mechanics

Conservative systems can be based on a variational principle (Hamilton's Principle):

$$\delta \int_{t_1}^{t_2} L\, dt = 0 , \quad L = T - V$$

Consider the coordinates of configuration space (N-fold) subject to S holonomic constraints:

$$x_i, y_i, z_i \quad ; \quad i = 1 \cdots N$$

$$f_j(x_i\, y_i\, z_i) = 0 \quad ; \quad j = 1, \cdots, S$$

Then we can form $3N - S = n$ relations among the coordinates which involve a system of generalized coordinates:

$$\left.\begin{aligned} x_1 &= f_1(q_1 \cdots q_n, t) \\ &\vdots \\ z_n &= f_{3N}(q_1 \cdots q_n, t) \end{aligned}\right\} \text{ equations of transformation}$$

Time derivatives can be expressed as:

$$\dot{x}_1 = \sum_i \frac{\partial f_1}{\partial q_i} \dot{q}_i$$

Consider the work done by a small displacement:

$$đW = \sum_i \left[X_i\, dx_i + Y_i\, dy_i + Z_i\, dz_i \right]$$

$$= \sum_{i=1}^{n} F_i\, dq_i \quad ; \quad F_i \text{ are the generalized forces}$$

Assume $đw$ is an exact differential, hence:

$$đW = dU = \sum_{i=1}^{n} \frac{\partial U}{\partial q_i} dq_i$$

Hence: $F_i = \dfrac{\partial U}{\partial q_i}$

Hence the force is derivable from a potential.
Note: $U = -V$.

We will derive the Lagrange equations from the principle of virtual work (most general). This says that a system is in equilibrium if and only if the virtual work of the impressed forces is zero.

Call the virtual displacements δx. The virtual work is:

$$dW = \sum \underline{F} \cdot \delta \underline{r} = 0$$

$$= \sum F_i \, \delta q_i = 0$$

We now use the D'Alembert's principle to convert a dynamical system to a system in equilibrium:

$$\underline{F} - m\underline{a} = 0$$

Apply the principle of virtual work:

$$(\underline{F} - m\underline{a}) \cdot \delta \underline{r} = 0$$

Integrate this with respect to time:

$$\int \sum \left(F_i - m_i \frac{dv_i}{dt} \right) \cdot \delta r_i \, dt$$

Now: $F_i \cdot \delta r_i = \delta U = -\delta V$

Consider: $\frac{d}{dt}(v \cdot \delta r) = \underline{v} \cdot \frac{d}{dt} \delta r + \delta r \cdot \frac{dv}{dt}$

Assume that $\frac{d}{dt}$ and δ are independent operations.
Also: $v \cdot \delta v = \frac{1}{2} \delta v^2$: Threading all this together:

$$\int_{t_1}^{t_2} [-\delta V + \delta T] \, dt + v \cdot \delta r \Big|_{t_1}^{t_2} = 0$$. Assume no variation of the end points.

Hence: $\delta \int_{t_1}^{t_2} L \, dt = 0$

LECTURE 14: 4-11-62

Consider the variational problem:

$$\delta \int_{x_1}^{x_2} F(y, y'; x)\, dx = 0$$

Find the path $y = f(x)$ that satisfies above. Take a path $\overline{f(x)}$ an infinitesimal amount away from $f(x)$:

$$\delta y = \overline{f(x)} - f(x) = \epsilon \phi(x)$$

$$\overline{f(x)} = f(x) + \epsilon \phi(x)$$

This will allow us to interchange variation and differentiation:

$$y' = f'(x) \; ; \quad \delta y' = \overline{f'(x)} - f'(x)$$

$$= f'(x) + \epsilon \phi'(x) - f'(x) = \epsilon \phi'(x)$$

Hence we can write: $\frac{d}{dx}(\delta y) = \epsilon \phi'(x)$

This can also be done with integration:

$$\delta \int F dx = \int \bar{F} dx - \int F dx = \int (\bar{F} - F) dx = \int \delta F \cdot dx$$

Furthermore:

$$\int \delta F\, dx = \int \left[F(y + \epsilon\phi, y' + \epsilon\phi', x) - F(y, y'; x) \right] dx$$

$$= \int \left[\epsilon \phi \frac{\partial F}{\partial y} + \epsilon \phi' \frac{\partial F}{\partial y'} \right] dx = 0$$

Integrate second term by parts:

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y'} \phi'\, dx = \frac{\partial F}{\partial y'} \phi \Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \phi \frac{d}{dx} \frac{\partial F}{\partial y'}\, dx$$

And: $\int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} \right] \delta y \, dx = 0$

Finally:

$$\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} = 0$$

which is the Euhler-Lagrange equation.

Applying this to Hamilton's Principle:

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = 0 \quad ; \quad i = 1, \cdots, n$$

$$\delta \int_{t_1}^{t_2} L \, dt = 0 \quad ; \quad L = T - V = L(q_i, \dot{q}_i, t)$$

The canonical momentum is:

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

as can be seen if the potential is velocity independent.

$$T = \frac{1}{2} \sum_i m_i (\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2)$$

$$\frac{\partial T}{\partial \dot{x}_i} = m \dot{x}_i$$

We also have then the conservation theorem:

$$\dot{p}_i = \frac{\partial L}{\partial q_i} = 0 \quad , \text{or} \quad p_i = \text{constant} \quad \text{and} \quad \frac{\partial L}{\partial \dot{q}_i} = c_i$$

If this is true, we have in principle the solution for some q_i.

That is, if: $L = L(q_1 \cdots q_{n-1}, \dot{q}_1 \cdots \dot{q}_n, t)$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_n} = 0$$

$$\frac{\partial L}{\partial \dot{q}_n} = C_n$$

$$\dot{q}_n = \dot{q}_n(q_1 \cdots q_{n-1}, C_n, \dot{q}_1 \cdots \dot{q}_n, t)$$

We can also express Hamilton's Principle in terms of some other parameter τ related to t

$$\delta \int_{\tau_1}^{\tau_2} L\left(q_1 \cdots q_n, \frac{\dot{q}_1'}{t'} \cdots \frac{\dot{q}_n'}{t'}, t\right) t' \, d\tau = 0 \; ; \; t' = \frac{d\tau}{dt}$$

Then: $p_i = \dfrac{\partial L}{\partial \dot{q}_i}$; $p_t = \dfrac{\partial (L t')}{\partial t'}$

$$= L + t' \sum \frac{\partial L}{\partial \left(\frac{q'}{t'}\right)} \frac{\partial \left(\frac{q'}{t'}\right)}{\partial t'}$$

$$= L - t' \sum \frac{\partial L}{\partial \left(\frac{q'}{t'}\right)} \frac{q'}{t'^2}$$

Then: $p_t = L - \sum p_i \dot{q}_i = -$ total system energy

Thus the conjugate momentum to the time is the total energy. Proof:

Take $p_i = \dfrac{\partial T}{\partial \dot{q}_i}$; $T = \frac{1}{2}\sum_i m_i \dot{x}_i^2$

$$x = x(q \cdots) \; ; \; \dot{x} = \sum \frac{\partial x}{\partial q}\dot{q}$$

assuming the time does not explicitly dependent on the time.

Then: $\sum \frac{\partial T}{\partial \dot{q}} \dot{q} = 2T$

and $T - V - 2T = -(T+V)$

This defines the Hamiltonian:

$$H = \sum p\dot{q} - L$$

We see that if L is independent of t, H is a constant of the motion.

The Legendre Dual Transformation:

Consider:

$$F = F(u, \omega)$$

Define: $v_i = \frac{\partial F}{\partial u_i}$

Also define: $G = G(v, \omega)$

Then we can write: $G = \sum_i u_i v_i - F$

Consider the variation of G:

$$\delta G = \sum_i u_i \,\delta v_i + \sum_i v_i \,\delta u_i - \sum_i \frac{\partial F}{\partial u_i} \delta u_i - \sum_i \frac{\partial F}{\partial \omega_i} \delta \omega_i$$

We can also write:

$$\delta G = \sum \frac{\partial G}{\partial v} \delta v + \sum \frac{\partial G}{\partial w} \delta w$$

By comparing terms:

$$u_i = \frac{\partial G}{\partial \omega_i} \quad ; \quad \frac{\partial F}{\partial \omega_i} = - \frac{\partial G}{\partial \omega_i}$$

We can immediately apply these results to the Hamiltonian: $q's, t \to w$, $H \to G$, $L \to F$, $\dot{q} \to u$, $p \to v$

Hence: $\dot{q}_i = \frac{\partial H}{\partial p_i} \quad ; \quad \dot{p}_i = \frac{\partial L}{\partial q_i} = - \frac{\partial H}{\partial q_i}$

These are Hamilton's equations of motion.

Because of the symmetry between these equations, they should be derivable from a variational principle:

$$L = \sum p\dot{q} - H$$

$$\delta L = \sum \dot{q}\,\delta p - \sum \frac{\partial H}{\partial p}\,\delta q = 0$$

Then:

$$\delta \int_{t_1}^{t_2} \left(\sum p\dot{q} - H(q\,p\,t) \right) dt = 0$$

See that here the kinetic energies are just linear functions of $\dot{q}$, the velocities.
We now use the Euler-Lagrange equations:

$$-\frac{\partial H}{\partial q_r} - \frac{d}{dt}\,p_r = 0$$

or: $\dot{p}_r = -\dfrac{\partial H}{\partial q_r}$

Similarly: $\dot{q} = \dfrac{\partial H}{\partial p}$

It is seen that the p's are independent variables in their own right.

Canonical Transformations:

$$\dot{p} = -\frac{\partial H}{\partial q} \; ; \quad \dot{q} = \frac{\partial H}{\partial p}$$

$$\left.\begin{aligned} Q_r &= Q_r(q, p, t) \\ P_r &= P_r(q, p, t) \end{aligned}\right\} \text{ Defines canonical transformation}$$

This is a canonical transformation, if we can find a K such that:

$$\dot{Q} = \frac{\partial K}{\partial P} \; ; \quad \dot{P} = -\frac{\partial K}{\partial Q}$$

We make the connection between H and K using:

$$\delta \int_{t_1}^{t_2} \left[\Sigma P\dot{Q} - K \right] dt = 0$$

$$\delta \int_{t_1}^{t_2} \left[\Sigma \dot{q} p - H \right] dt = 0$$

Now, the two integrands can differ by at most a function F such that:

$$\delta \int_{t_1}^{t_2} \frac{\partial F}{\partial t} dt = \delta \left[F(t_2) - F(t_1) \right] = 0$$

Or:

$$\Sigma p\dot{q} - H = \Sigma P\dot{Q} - K + \frac{\partial F}{\partial t}$$

F is called the transformation generating function. Since F can be a function of both old and new variables, it can have 4 possible forms for $4n$ variables:

$F_1(q, Q, t)$
$F_2(q, P, t)$
$F_3(p, Q, t)$
$F_4(p, P, t)$

Look at F_1:

$$\Sigma p\dot{q} - H = \Sigma P\dot{Q} - K + \Sigma \frac{\partial F_1}{\partial q} \dot{q} + \Sigma \frac{\partial F_1}{\partial Q} \dot{Q} + \frac{\partial F_1}{\partial t}$$

Hence:

$$p_i = \frac{\partial F_1}{\partial q_i} \; ; \; P_i = -\frac{\partial F_1}{\partial Q_i} \; ; \; K = H + \frac{\partial F_1}{\partial t}$$

Note that $K = H$ if $\frac{\partial F_1}{\partial t} = 0$. From $p = \frac{\partial F_1}{\partial q}$ we can get $Q = Q(pqt)$ and hence get $P = P(pqt)$. Now vary F_1 using the above results:

$$\delta F_1 = \Sigma p\,\delta q - \Sigma P\,\delta Q$$

LECTURE 15: 4-16-62

Recall the discussion of canonical transformations. We define a generating function $F_2(qPt)$

$$F_2(qPt) = F_1(qQt) + \sum P_i Q_i$$

then:

$$\sum p\dot{q} - H = \sum P\dot{Q} - K + \sum \frac{\partial F_2}{\partial q}\dot{q} + \sum \frac{\partial F_2}{\partial P}\dot{P} - \sum Q\dot{P} - \sum P\dot{Q}$$

Hence: $K = H + \dfrac{\partial F_2}{\partial t}$

$$p_i = \frac{\partial F_2}{\partial q_i} \quad ; \quad Q_i = \frac{\partial F_2}{\partial P_i}$$

We can further obtain:

$$F_3(pQt) = F_1 - \sum p_i q_i$$

$$F_4(pPt) = F_1 + \sum P_i Q_i - \sum p_i q_i$$

and: $q_i = -\dfrac{\partial F_3}{\partial p_i} \quad ; \quad P_i = -\dfrac{\partial F_3}{\partial Q_i}$

$$q_i = -\frac{\partial F_4}{\partial p_i} \quad ; \quad Q_i = \frac{\partial F_4}{\partial P_i}$$

We will not use F_3 and F_4.

Recapitulation:

$F_1(qQt)$: $\quad p_i = \dfrac{\partial F_1}{\partial q_i} \quad ; \quad P_i = -\dfrac{\partial F_1}{\partial Q_i}$

$F_2(qPt)$: $\quad p_i = \dfrac{\partial F_2}{\partial q_i} \quad ; \quad Q_i = \dfrac{\partial F_2}{\partial P_i}$

Point Transformations of Coordinate Transformations:

Take $F_2 = \sum_i f_i(q\,t)\,P_i$

$$Q_i = \frac{\partial F_2}{\partial P_i} = f_i(q,t)$$

so that the coordinate transformation is a canonical transformation. We now specialize to linear coordinate transformations:

$$F_2 = \sum_{ik} a_{ik}\, q_k\, P_i$$

$$Q_i = \sum_k a_{ik}\, q_k$$

$$p_k = \frac{\partial F_2}{\partial q_k} = \sum_i a_{ik}\, P_i$$

We can write these as matrix equations:

$Q = Aq \quad ; \quad \bar{p} = \bar{P} A \qquad$ ($^{-}$ means transposed)

now: $\bar{P} = \bar{p}\overline{\overline{A^{-1}}} \quad ; \quad P = \overline{A^{-1}}\, p$

If the transformation is linear and $P, Q = f(p,q)$ then we have an extended linear point transformation.

The orthogonality conditions are:

$$\sum_k a_{ik}\, a_{jk} = \delta_{ij} \quad ; \quad A\bar{A} = I \quad ; \quad \bar{A} = A^{-1}$$

Then, for an orthogonal transformation: $P = Ap$

Form the scalar product:

$$\bar{P}Q = \bar{p}A^{-1}Aq = \bar{p}q$$

or $\sum_i P_i Q_i = \sum p_i q_i$

We have not used orthogonality here.

Note that the generating function is not yet unique. Impose the condition:

$$\delta F = \sum p\,\delta q - \sum P\,\delta Q \quad = 0$$

Then: $\bar{p}\,\delta q = \bar{P}\,\delta Q$

If $Q = Aq$; $\delta Q = A\,\delta q$
and we get the above transformation.

Another transformation is the exchange transformation:

$$F_1 = \sum_i q_i Q_i$$

The identity transformation is: $F_2 = \sum q_i P_i$

The transformations have the group property in that there is an identity transformation and successive transformations give a single transformation.

Some further properties of the canonical transformation are given in Lagrange brackets. Consider the Lagrange brackets of some two elements:

$$[a_m, a_r]_{p,q} = \sum_{i=1}^{n} \left[\frac{\partial q_i}{\partial a_m} \frac{\partial p_i}{\partial a_r} - \frac{\partial q_i}{\partial a_r} \frac{\partial p_i}{\partial a_m} \right]$$

The Poisson brackets are:

$$\{a_m, a_r\} = \sum_{j=1}^{n} \left[\frac{\partial a_m}{\partial q_j} \frac{\partial a_r}{\partial p_j} - \frac{\partial a_m}{\partial p_j} \frac{\partial a_r}{\partial q_j} \right]$$

Both of these form skew-symmetric matrices such that:

$$\bar{L} = -L$$
$$\bar{P} = -P$$

Multiplying these together:

$$\sum_{m=1}^{2n} L_{mr} P_{ms}$$

$$= \sum_{m=1}^{2n} \sum_{i=1}^{n} \sum_{j=1}^{n} \left[\frac{\partial q_i}{\partial a_m} \frac{\partial p_i}{\partial a_r} \frac{\partial a_m}{\partial q_j} \frac{\partial a_s}{\partial p_j} - \frac{\partial q_i}{\partial a_m} \frac{\partial p_i}{\partial a_r} \frac{\partial a_m}{\partial p_j} \frac{\partial a_s}{\partial q_j} \right.$$

$$\left. - \frac{\partial q_i}{\partial a_r} \frac{\partial p_i}{\partial a_m} \frac{\partial a_m}{\partial q_j} \frac{\partial a_s}{\partial p_j} + \frac{\partial q_i}{\partial a_r} \frac{\partial p_i}{\partial a_m} \frac{\partial a_m}{\partial p_j} \frac{\partial a_s}{\partial q_j} \right]$$

The negative terms vanish. The final reduction is to:

$$\sum_{m=1}^{2n} L_{mr} P_{ms} = \delta_{rs} \quad ; \quad LP = -I$$

Consider the matrix equation:

$$\dot{x} = \Phi_0 D H$$

$$x = \begin{pmatrix} q_1 \\ \vdots \\ q_n \\ p_1 \\ \vdots \\ p_n \end{pmatrix} \quad ; \quad \Phi_0 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

$$D = \begin{pmatrix} \frac{\partial}{\partial q_1} \\ \vdots \\ \frac{\partial}{\partial p_n} \end{pmatrix} \quad ; \quad H = \text{Hamiltonian}$$

This gives us the usual canonical equations:

$$\dot{q}_1 = \frac{\partial H}{\partial p_1} \quad ; \quad \dot{p}_1 = \frac{-\partial H}{\partial q_1}$$

Assume the transformation $x' = x'(x)$; $\dot{x}' = J\dot{x}$ where J is the usual Jacobian. $\dot{x}'' = J'J\dot{x} = J''\dot{x}$

To transform D, note that:

$$d\bar{x}\, D = d\bar{x}'\, D' \quad ; \quad \sum \frac{\partial}{\partial x_i} dx_i = \sum \frac{\partial}{\partial x_i'} dx_i'$$

is an invariant. The transformation is called contragradient.

Suppose:

$$\bar{a} b = \bar{a}' b' \quad ; \quad a' = S a \quad , \quad b' = T b$$

$$\bar{a} b = \bar{a} \bar{S} T b \quad ; \quad \bar{S} T = I \quad ; \quad \bar{S} = T^{-1}$$

Thus: $D' = TD$ or: $D' = \bar{J}^{-1} D \quad ; \quad D = \bar{J} D'$

We can then form:

$$\dot{x}' = J \Phi_0 \bar{J} D' H$$

Define $J \Phi_0 \bar{J} = \Phi$; Φ is the Poisson matrix.

If $J \Phi_0 \bar{J} = \Phi_0$, then $\dot{x}' = \Phi_0 D' H$

and we have preserved the canonical form. Hence $J \Phi_0 \bar{J} = \Phi_0$ is a test for a canonical transformation.

Say we transform from p, q to P, Q, then:

$$J = \begin{pmatrix} \frac{\partial Q_i}{\partial q_j} & \frac{\partial Q_i}{\partial p_j} \\ \frac{\partial P_i}{\partial q_j} & \frac{\partial P_i}{\partial p_j} \end{pmatrix}$$

Then we have the structure:

$$\begin{pmatrix} Q_q & Q_p \\ P_q & P_p \end{pmatrix} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} \bar{Q}_q & \bar{P}_q \\ \bar{Q}_p & \bar{P}_p \end{pmatrix}$$

which becomes:

$$\begin{pmatrix} Q_q \bar{Q}_p - Q_p \bar{Q}_q & Q_q \bar{P}_p - Q_p \bar{P}_q \\ P_q \bar{Q}_p - P_p \bar{Q}_q & P_q \bar{P}_p - P_p \bar{P}_q \end{pmatrix} = \text{Poisson matrix}$$

Note that each element is a Poisson bracket.
For the transformation to be canonical:

$$\{Q_i, Q_j\} = 0 \quad ; \quad \{P_i, P_j\} = 0$$

$$\{Q_i, P_j\} = \delta_{ij}$$

The Lagrange brackets must also satisfy the above relation.

LECTURE 16 : 4-18-62

We now consider canonical transformations of just one pair of canonical variables:

$$
\begin{aligned}
Q_1 &= Q_1(q, p, t) \\
P_1 &= P_1(q, p, t) \\
Q_i &= q_i \\
P_i &= p_i
\end{aligned}
\Bigg\} \; i > 1
$$

We can reduce this to two dimensional form. Hence:

$$\begin{pmatrix} \dot{Q} \\ \dot{P} \end{pmatrix} = \begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial P}{\partial q} \\ \frac{\partial Q}{\partial p} & \frac{\partial P}{\partial p} \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial Q} \\ \frac{\partial H}{\partial P} \end{pmatrix}$$

For the transformation to be canonical, we require:

$$\{Q, Q\} = \{P, P\} = 0$$

$$\{Q, P\} = 1 \quad \text{or} \quad \frac{\partial(Q, P)}{\partial(q\, p)} = 1$$

If the transformation is time-dependent, we have:

$$\dot{Q} = \frac{\partial Q}{\partial q}\dot{q} + \frac{\partial Q}{\partial p}\dot{p} + \frac{\partial Q}{\partial t}$$

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{\partial H}{\partial Q}\frac{\partial Q}{\partial p} + \frac{\partial H}{\partial P}\frac{\partial P}{\partial q} \quad \text{, etc}$$

and:

$$\dot{Q} = \frac{\partial H}{\partial P}\frac{\partial(QP)}{\partial(q\,p)} + \frac{\partial Q}{\partial t}$$

$$\dot{P} = -\frac{\partial H}{\partial Q}\frac{\partial(QP)}{\partial(q\,p)} + \frac{\partial P}{\partial t}$$

To put in canonical form, we must write:

$$\dot{Q} = \frac{\partial K}{\partial P} \quad ; \quad K = H + \phi$$

Then: $\frac{\partial \phi}{\partial P} = \frac{\partial Q}{\partial t}$ and $\frac{\partial \phi}{\partial Q} = -\frac{\partial P}{\partial t}$

For all others: $\frac{\partial K}{\partial Q_i} = \frac{\partial H}{\partial Q_i}$

so we can just add ϕ to H and retain the canonical form, by finding:

$$\phi = \int \frac{\partial Q}{\partial t}\, dP = -\int \frac{\partial P}{\partial t}\, dQ$$

Recall the matrix equation of motion:

$$\dot{x} = \Phi_0 D H$$

and recall the equations of motion of the elements:

$$\dot{\alpha}_i = \frac{\partial R}{\partial \beta_i} \quad ; \quad \dot{\beta}_i = -\frac{\partial R}{\partial \alpha_i}$$

$$\alpha_1 = -\frac{\mu}{2a} \qquad \beta_1 = -\tau$$

$$\alpha_2 = \sqrt{\mu a(1-e^2)} \qquad \beta_2 = \omega$$

$$\alpha_3 = \sqrt{\mu a(1-e^2)} \cos i \qquad \beta_3 = \Omega$$

Then we could write: $\dot{x} = \Phi_0 D R$

Transforming to the original elements, we have equations of the form:

$$\dot{E} = \underbrace{J \Phi_0 \bar{J}}_{\phi} D' R$$

We have for the new coordinates $Q_i \rightarrow a, e, i$; $P_i = p_i$

$$\phi = \begin{pmatrix} 0 & Q_q \\ -\bar{Q}_q & 0 \end{pmatrix} ; \begin{pmatrix} Q_q & Q_p \\ P_q & P_p \end{pmatrix} = \begin{pmatrix} Q_q & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} \bar{Q}_q & 0 \\ 0 & I \end{pmatrix}$$

We need to form:

$$\frac{\partial Q_i}{\partial q_j} = \begin{pmatrix} \frac{\partial a}{\partial \alpha_1} & \frac{\partial a}{\partial \alpha_2} & \frac{\partial a}{\partial \alpha_3} \\ \frac{\partial e}{\partial \alpha_1} & \frac{\partial e}{\partial \alpha_2} & \frac{\partial e}{\partial \alpha_3} \\ \frac{\partial i}{\partial \alpha_1} & \frac{\partial i}{\partial \alpha_2} & \frac{\partial i}{\partial \alpha_3} \end{pmatrix}$$

On substitution, we find:

$$\frac{\partial a}{\partial \alpha_2} = \frac{\partial a}{\partial \alpha_3} = \frac{\partial e}{\partial \alpha_3} = \frac{\partial i}{\partial \alpha_1} = 0$$

so we have only to evaluate 5 Poisson brackets which upon evaluation should yield the Lagrange planetary equations.

Suppose we transform to new coordinates:

$$\dot{y} = J(x|y)\, \Phi_0\, \bar{J}(x|y)\, D_y R$$

Then for $\dot{e}$ we have the structure:

$$J(e|x)\; J(x|y)\, \Phi_0\, \bar{J}(x|y)\, \bar{J}(e|x)$$

$$= \quad J(e|y)\, \Phi_0\, \bar{J}(e|y) \quad = J(e|x)\, \Phi_0\, \bar{J}(e|x)$$

so that the equation of motion is independent of the original coordinate representation.

Consider the identity transformation:

$$F_2 = \sum q_i P_i$$

$$p_i = \frac{\partial F_2}{\partial q_i} = P_i \quad ; \quad Q_i = \frac{\partial F_2}{\partial P_i} = q_i$$

Now consider an infinitesimal transformation:

$$F_2 = \sum q_i P_i + \epsilon\, G(q\, P)$$

$$p_i = P_i + \epsilon \frac{\partial G}{\partial q_i}$$

$$Q_i = q_i + \epsilon \frac{\partial G}{\partial P_i}$$

$$\delta p_i = P_i - p_i = -\epsilon \frac{\partial G}{\partial q_i}$$

$$\delta q_i = Q_i - q_i = \epsilon \frac{\partial G}{\partial p_i}$$

Let $\epsilon = dt$; $G = H(q\,p)$

Then: $\delta q_i = dt \frac{\partial H}{\partial p} = \dot{q}\,dt = dq$

$$\delta p_i = \dot{p}\,dt = dp$$

Thus F_2 generates a canonical transformation in time. We can reverse the process and find the equations: $q_i = q_i(Q\,P\,t)$
$p_i = p_i(Q\,P\,t)$
so that we can find some initial configuation. F_2 generates motion in time.

We know: $\dot{q}_i = \frac{\partial H}{\partial p_i}$; $\dot{p}_i = -\frac{\partial H}{\partial q_i}$

Can we find a CT such that $\dot{P}_i = 0$ or $P_i = c$? If so, we have solved the problem. Consider $H = H(p_i)$. Then $H = H(c_i)$ and:

$$\dot{q}_i = \frac{\partial H}{\partial c_i} = b_i \quad \text{and} \quad q_i = b_i t + a_i$$

and we have the motion. Thus, we must try to eliminate the coordinate part of the Hamiltonian by a CT. To do this, choose $K = 0$ and use:

$$p_i = \frac{\partial F_2}{\partial q_i} \; ; \quad Q_i = \frac{\partial F_2}{\partial P_i}$$

Since $K = 0$, all the new coordinates and momenta are zero, or rather constants.

The equation that then gives us F is called the Hamilton-Jacobi equation:

$$H\left(q, \frac{\partial F}{\partial q}, t\right) + \frac{\partial F}{\partial t} = 0$$

A complete solution is $F = F(q, \alpha, t)$ with n constants of integration. Of course $\alpha_i = P_i$. Hence:

$$p_i = \frac{\partial F}{\partial q_i} \quad ; \quad \beta_i = \frac{\partial F}{\partial \alpha_i}(q, \alpha, t)$$

which can be solved for:

$$q_i = q_i(\alpha, \beta, t) \quad ; \quad p_i = p_i(\alpha, \beta, t)$$

Any complete solution to the HJ equation will solve the problem.

If the HJ equation can be separated, its solution is straightforward. Define $F = S$ = Hamilton's principle function. We can always separate the time part.

Set: $S = -\alpha_1 t + S'(q, \alpha)$

$$\frac{\partial S}{\partial t} = -\alpha_1 \quad ; \quad \frac{\partial S}{\partial q} = \frac{\partial S'}{\partial q}$$

Then: $H\left(q, \frac{\partial S'}{\partial q}\right) = \alpha_1$

and α_1 = total energy.

We can do this for any separable coordinate. Suppose:

$$S = -\alpha_1 t + \alpha_i q_i + S'(q, \alpha)$$

$$\frac{\partial S}{\partial t} = -\alpha_1 \quad ; \quad \frac{\partial S}{\partial q_i} = \alpha_i \quad ; \quad \frac{\partial S}{\partial q} = \frac{\partial S'}{\partial q}$$

and: $H\left(q, \frac{\partial S'}{\partial q}, \alpha_i\right) = \alpha_1$

Consider the perturbation problem:

$$\dot{q} = \frac{\partial H}{\partial p} \quad ; \quad \dot{p} = -\frac{\partial H}{\partial q} \quad ; \quad H = H_0 - H_1$$

Then for the unperturbed problem:

$$\dot{q} = \frac{\partial H_0}{\partial p} \quad ; \quad \dot{p} = -\frac{\partial H_0}{\partial q} \quad ; \quad H_0 + \frac{\partial S}{\partial t} = 0$$

$$\beta = \frac{\partial S}{\partial \alpha} \quad ; \quad p = \frac{\partial S}{\partial q}$$

Form or find S such that: $P_i = -\beta_i$; $Q_i = \alpha_i$

$$\Sigma p\,dq - \Sigma P\,dQ = \Sigma p\,dq + \Sigma \beta\,d\alpha$$

$$= \Sigma \frac{\partial S}{\partial q} dq + \Sigma \frac{\partial S}{\partial \alpha} d\alpha = dS$$

We use S to generate the appropriate canonical transformation to give the solution for H' in the same form as H_0:

$$K = H + \frac{\partial S}{\partial t} = H - H_0 = -H_1$$

and: $$\dot{\alpha} = \frac{\partial H_1}{\partial \beta} \quad ; \quad \dot{\beta} = -\frac{\partial H_1}{\partial \alpha}$$

For the Kepler problem:

$$H = T - U \quad ; \quad U = \frac{\mu}{r} + R = U_0 + R$$

Then $H_1 = R$ and $H = H_0 - R$ so that:

$$\dot{\alpha} = \frac{\partial R}{\partial \beta} \quad ; \quad \dot{\beta} = -\frac{\partial R}{\partial \alpha}$$

LECTURE 17 : 4-23-62

The Kepler Problem

$$H = T - U \quad ; \quad H_0 = T - U_0 \quad ; \quad U = \frac{\mu}{r} + R \quad ; \quad U_0 = \frac{\mu}{r}$$

$$x = r \cos\phi \cos\lambda$$
$$y = r \cos\phi \sin\lambda$$
$$z = r \sin\phi$$

$$T = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$
$$= \frac{1}{2}(\dot{r}^2 + r^2\dot{\phi}^2 + r^2\cos^2\phi\,\dot{\lambda}^2)$$

$$p_r = \frac{\partial T}{\partial \dot{r}} = \dot{r}$$
$$p_\phi = \frac{\partial T}{\partial \dot{\phi}} = r^2\dot{\phi}$$
$$p_\lambda = \frac{\partial T}{\partial \dot{\lambda}} = r^2\cos^2\phi\,\dot{\lambda}$$

Then:

$$H = \frac{1}{2}\left(p_r^2 + \frac{1}{r^2}p_\phi^2 + \frac{1}{r^2\cos^2\phi}p_\lambda^2\right) - \frac{\mu}{r}$$

The H-J equation is:

$$\frac{1}{2}\left[\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S}{\partial \phi}\right)^2 + \frac{1}{r^2\cos^2\phi}\left(\frac{\partial S}{\partial \lambda}\right)^2\right] - \frac{\mu}{r} + \frac{\partial S}{\partial t} = 0$$

We see we can substitute:

$$S = -\alpha_1 t + S'(r,\phi) + \alpha_3\lambda \quad ; \quad \frac{\partial S}{\partial t} = -\alpha_1 \quad ; \quad \frac{\partial S}{\partial \lambda} = \alpha_3$$

$$\frac{1}{2}\left[\left(\frac{\partial S'}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S}{\partial \phi}\right)^2 + \frac{\alpha_3^2}{r^2\cos^2\phi}\right] - \frac{\mu}{r} = \alpha_1$$

Multiply by $2r^2$:

$$r^2 S_r'^2 + S_\phi'^2 + \frac{\alpha_3^2}{\cos^2\phi} - 2\mu r = 2\alpha_1 r^2$$

This is immediately separable:

$$S'(r\phi) = S_1(r) + S_2(\phi)$$

Then:

$$\left(\frac{dS_2}{d\phi}\right)^2 + \frac{\alpha_3^{\,2}}{\cos^2\phi} = \alpha_2^2$$

$$-r^2\left(\frac{dS_1}{dr}\right)^2 + 2\mu r + 2\alpha_1 r^2 = \alpha_2^2$$

We obtain:

$$S_2 = \int_0^\phi \left[\alpha_2^2 - \alpha_3^2 \sec^2\phi\right]^{1/2} d\phi$$

$$S_1 = \int_{r_1}^{r} \left[2\alpha_1 r^2 + 2\mu r - \alpha_2^2\right]^{1/2} \frac{dr}{r}$$

r_1 will be the radius vector at perigee.

Now:

$$S = -\alpha_1 t + \alpha_3 \lambda + S_1 + S_2$$

and hence:

$$\frac{\partial S}{\partial \alpha_1} = \beta_1 = -t + \frac{\partial S_1}{\partial \alpha_1}$$

$$\frac{\partial S}{\partial \alpha_2} = \beta_2 = \frac{\partial S_1}{\partial \alpha_2} + \frac{\partial S_2}{\partial \alpha_2}$$

$$\frac{\partial S}{\partial \alpha_3} = \beta_3 = \lambda + \frac{\partial S_2}{\partial \alpha_3}$$

One can invert these and obtain the equations of motion:

$$q_i = q_i(\alpha, \beta, t)$$

Now examine S_1 and S_2:

$$\overset{A}{2\alpha_1 r^2} + \overset{B}{2\mu r} - \overset{C}{\alpha_2^2} = 0$$

Call the roots r_1, r_2:

$$r_1 + r_2 = -\frac{B}{A} = -\frac{\mu}{\alpha_1}$$

$$r_1 r_2 = \frac{C}{A} = -\frac{\alpha_2^2}{2\alpha_1}$$

Assume both roots positive and real, or that the orbit is elliptical. This implies:

$\alpha_1 < 0 \quad ; \quad r_1 < r < r_2$

Write the radical as $y^{1/2}$:

$$y^{1/2} = \sqrt{-2\alpha_1}\,\sqrt{(r-r_1)(r_2-r)}$$

Then:

$$\frac{\partial S_1}{\partial \alpha_1} = \int_{r_1}^{r} \frac{1}{2}\, y^{-1/2}\, 2\, \frac{r^2\, dr}{r} - \underbrace{\frac{\partial r_1}{\partial \alpha_1}\left[\frac{y^{1/2}}{r}\right]_{r=r_1}}_{0}$$

$$= \int_{r_1}^{r} \frac{r\, dr}{y^{1/2}} = \frac{1}{\sqrt{-2\alpha_1}} \int_{r_1}^{r} \frac{r\, dr}{\left[(r-r_1)(r_2-r)\right]^{1/2}}$$

Change to the variables: $r = a(1-e\cos E)$

$r_1 = a(1-e)$

$r_2 = a(1+e)$

Then $\alpha_1 = -\dfrac{\mu}{2a} \quad ; \quad \alpha_2 = \sqrt{\mu a\,(1-e^2)}$

and:

$$t + \beta_1 = \frac{a}{\sqrt{-2\alpha_1}} \int_0^E (1-e\cos E)\, dE$$

or: $$E - e\sin E = \frac{\sqrt{2\alpha_1}}{a}\,(t+\beta_1) = n\,(t-\tau) \quad \left[\text{Kepler's Equation}\right]$$

Hence: $\beta_1 = -\tau$

Now consider: $\beta_3 = \Omega + \dfrac{\partial S_2}{\partial \alpha_3}$

$$\frac{\partial S_2}{\partial \alpha_3} = -\int_0^{\phi} \frac{\alpha_1 \sec^2\phi}{\left[\alpha_2^2 - \alpha_3^2\sec^2\phi\right]^{1/2}}\, d\phi = -\int_0^{\phi} \frac{\sec^2\phi\, d\phi}{\left[\underbrace{\frac{\alpha_2^2-\alpha_3^2}{\alpha_3^2}}_{A^2} - \tan^2\phi\right]^{1/2}}$$

This of the form: $$\int \frac{dv}{\left[A^2 - v^2\right]^{1/2}} = \sin^{-1}\left(\frac{v}{A}\right)$$

Then:

$$\frac{\partial S_2}{\partial \alpha_3} = -\sin^{-1}\left(\frac{\tan\phi}{A}\right)$$

And: $A \sin(\lambda - \beta_3) = \tan\phi$

We see that β_3 is the longitude of the ascending node. Comparing with:

$$\tan i \sin(\lambda - \Omega) = \tan\phi$$

we see: $\dfrac{\alpha_2^2 - \alpha_3^2}{\alpha_3^2} = \tan^2 i$; $\alpha_3 = \alpha_2 \cos i$

This above equation says that the orbit is in a plane passing through the center of mass.

Finally:

$$\beta_2 = -\alpha_2 \int_{r_1}^{r} \frac{dr}{r\, y^{1/2}} + \alpha_2 \int_0^{\phi} \frac{d\phi}{[\alpha_2^2 - \alpha_3^2 \sec^2\phi]^{1/2}}$$

The last integral reduces to:

$$I = \int_0^{\phi} \frac{\cos\phi \, d\phi}{\left[\frac{\alpha_2^2 - \alpha_3^2}{\alpha_2^2} - \sin^2\phi\right]^{1/2}}$$

We can write: $\dfrac{\alpha_2^2 - \alpha_3^2}{\alpha_2^2} = \sin^2 i$

Then: $I = \displaystyle\int_0^{\phi} \frac{\cos\phi \, d\phi}{[\sin^2 i - \sin^2\phi]^{1/2}} = \sin^{-1}\left(\frac{\sin\phi}{\sin i}\right)$

or:

$\sin I = \dfrac{\sin\phi}{\sin i}$ or we see $I \Rightarrow v + \omega$ (angle from node) $= u$

Now:

$$-\alpha_2 \int \frac{dr}{r\, y^{1/2}} = -\frac{\alpha_2}{\sqrt{-2\alpha_1}} \int_{r_1}^{r} \frac{dr}{r\,[(r - r_1)(r_2 - r)]^{1/2}} = -v$$

Then: $\beta_2 = u - v = \omega$. We now have solved for our six elements:

$\alpha_1 = -\dfrac{\mu}{2a}$ $\qquad\qquad \beta_1 = -\tau$

$\alpha_2 = \sqrt{\mu a (1 - e^2)}$ $\qquad \beta_2 = \omega$

$\alpha_3 = \sqrt{\mu a (1 - e^2)} \cos i$ $\qquad \beta_3 = \Omega$

We now go on to solve the disturbed problem in terms of motion of the elements. Use:

$$\dot{\alpha}_i = \frac{\partial R}{\partial \beta_i} \quad ; \quad \dot{\beta}_i = -\frac{\partial R}{\partial \alpha_i}$$

Reconsider for a moment:

$$\dot{x} = \phi_0 D_x H \quad ; \quad H = H_0 - H_1$$

Now: $\dot{x}^{(0)} = \phi_0 D_x H_0$
which we solve to obtain: $x^{(0)} = x^{(0)}(e,t)$
where e are constants of integration, not necessarily canonic. We now want to find $x = x(e,t)$, e the same as before. Then:

$$\dot{x} = x_e \dot{e} + \underbrace{\frac{\partial x}{\partial t}}_{\dot{x}^{(0)}} \quad ; \quad x_e \dot{e} = \dot{x} - \dot{x}^{(0)}$$

and: $x_e \dot{e} = \phi_0 D_x (H - H_0)$. Operate with e_x:

$$\underbrace{e_x x_e}_{1} \dot{e} = e_x \phi_0 \bar{e}_x D_x (H - H_0)$$

so we have the PB form. If the e's are canonic, $e_x \phi_0 \bar{e}_x = \phi_0$. We can also form, to see the Lagrange bracket form:

$$\bar{x}_e \phi_0 x_e \dot{e} = -\bar{x}_e D_x H_1$$, or if e's are canonic:

$$\phi_0 \dot{e} = -D_e H_1$$

and we obtain to above equations of the disturbing function.

We now write the disturbing function in terms of the canonic constants:

$$R = \sum C \cos \theta$$

$$C = C(\alpha_1 \alpha_2 \alpha_3) \quad ; \quad \theta = i M + j \Omega + k \omega$$

$M = n(t+\beta_1)$

$$\dot{\beta}_1 = -\frac{\partial R}{\partial \alpha_1} = -\sum \frac{\partial C}{\partial \alpha_1}\cos\theta - \sum C \sin\theta \,(t+\beta_1)\frac{\partial n}{\partial \alpha_1}$$

To simplify problem, redefine one of canonic constants. Take the mean anomaly in place of β_1. Call this $l = n(t+\beta_1)$ and call the replacement for α_1 by L. Make a CT to bring this about. Note $l = l(\alpha_1, \beta_1, t)$ so that $L = L(\alpha, \beta, t)$. This will keep the canonic form.

$$\dot{l} = -\frac{\partial R}{\partial L} \; ; \quad \dot{L} = \frac{\partial R}{\partial l}$$

This will have the effect of eliminating all secular-periodic terms. The condition on the Jacobian is:

$$\frac{\partial(QP)}{\partial(qp)} = 1$$

Then: $$\frac{\partial L}{\partial \alpha_1}\frac{\partial l}{\partial \beta_1} - \frac{\partial L}{\partial \beta_1}\frac{\partial l}{\partial \alpha_1} = 1$$

or: $\frac{\partial L}{\partial \alpha_1} n - \frac{\partial L}{\partial \beta_1}(t+\beta_1)\frac{dn}{d\alpha_1} = 1$. Can satisfy this if $L = L(\alpha_1)$ only. Then:

$$L = \int \frac{1}{n}\, d\alpha_1 = \sqrt{\mu a}$$

We still have to modify R because under a CT $H' = H + \phi$ where $\phi = \int \frac{\partial Q}{\partial t} dP - \int \frac{dP}{dt} dQ$. Here we have $\phi = -\int n\, dL$ or: $\phi = \frac{\mu^2}{2L^2}$ and $R' = R + \frac{\mu^2}{2L^2}$ which leaves the forms of $\alpha_2, \alpha_3, \beta_2, \beta_3$ unchanged and $\dot{L} = \frac{\partial R'}{\partial l}$, $\dot{l} = -\frac{\partial R'}{\partial L}$. Redefine the canonic constants, called now the Delauney variables

$$\begin{array}{llll} \alpha_1: & \dot{L} = \frac{\partial R'}{\partial l} & \beta_1: & \dot{l} = -\frac{\partial R'}{\partial L} \\ \alpha_2: & \dot{G} = \frac{\partial R'}{\partial g} & \beta_2: & \dot{g} = -\frac{\partial R'}{\partial G} \\ \alpha_3: & \dot{H} = \frac{\partial R'}{\partial h} & \beta_3: & \dot{h} = -\frac{\partial R'}{\partial H} \end{array}$$

Note that $\dot{l}$ has a first order part: $\dot{l} = -\frac{\partial R}{\partial L} - \frac{\partial}{\partial L}\left(\frac{\mu^2}{2L}\right)$

In the zeroth order:

$$\dot{l} = \frac{\mu^2}{L^3} = n \; ; \quad l = nt + \cdots$$

LECTURE 18 : 4-25-62

For the two planet problem, R is:

$$R = \underset{\substack{\text{in} \\ \text{secular} \\ \text{part}}}{N} = Gm_1 \Big[4C + (e^2+e_1^2) D - 2ee_1 E \cos(\tilde{\omega} - \tilde{\omega}_1)$$

$$- (\gamma^2 + \gamma_1^2) D + 2\gamma\gamma_1 D \cos(\Omega - \Omega_1) \Big]$$

$$\frac{N}{m_1} = \frac{N_1}{m}$$

Take for variables:

$$h = e \sin \tilde{\omega} \quad , \quad k = e \cos \tilde{\omega}$$
$$p = \sin i \sin \Omega \quad , \quad q = \sin i \cos \Omega$$

The Lagrange equations of motion are:

$$\dot{h} = \frac{\cos\phi}{na^2} \frac{\partial R}{\partial k} + \frac{k \tan \frac{1}{2} i}{\gamma na^2 \cos\phi} \left(p \frac{\partial R}{\partial p} + q \frac{\partial R}{\partial q} \right) + A \underset{\substack{\shortparallel \\ 0}}{\frac{\partial R}{\partial \epsilon}}$$

For $\dot{k}$ let $k \rightarrow -h$ above.

$$\gamma = \tan i \quad ; \quad \cos\phi = \sqrt{1-e^2}$$

We will work only to 3rd order, hence we neglect p, q term. Also, let $\cos\phi \rightarrow 1$ otherwise we will have higher order. Then:

$$\dot{h} = \frac{1}{na^2} \frac{\partial N}{\partial k} \quad ; \quad \dot{k} = -\frac{1}{na^2} \frac{\partial N}{\partial h}$$

$$\dot{p} = \frac{1}{na^2} \frac{\partial N}{\partial q} \quad ; \quad \dot{q} = -\frac{1}{na^2} \frac{\partial N}{\partial p}$$

The same equations hold for the other planet. $\dot{a}$ is zero because it does not have any secular terms. The above equation are almost in canonical form. Now:

$$N = Gm_1 D \left[h^2 + k^2 + h_1^2 + k_1^2 - p^2 - q^2 - p_1^2 - q_1^2 + 2pp_1 + 2qq_1 \right]$$
$$- Gm_1 E (hh_1 + kk_1)$$

This gives for the equations of motion:

$$\dot{h} = \alpha k - \beta k_1 \qquad \dot{p} = \alpha(q_1 - q)$$
$$\dot{k} = \beta h_1 - \alpha h \qquad \dot{q} = \alpha(p - p_1)$$
$$\dot{h}_1 = \alpha_1 k_1 - \beta_1 k \qquad \dot{p}_1 = \alpha_1(q - q_1)$$
$$\dot{k}_1 = \beta_1 h - \alpha_1 h_1 \qquad \dot{q}_1 = \alpha_1(p_1 - p)$$

where: $\alpha = \dfrac{2Gm_1D}{na^2}$; $\beta = \dfrac{2Gm_1E}{na^2}$

This set of equations can be solved by letting:

$x = k + ih$; $x_1 = k_1 + ih_1$

This gives:

$$\ddot{x} - i(\alpha - \alpha_1)\dot{x} - (\alpha\alpha_1 - \beta\beta_1)x = 0$$

Take for solution: $x = M e^{i(gt+c)}$

which gives as equation for g which we take to be real:

$$g^2 - (\alpha + \alpha_1)g + (\alpha\alpha_1 - \beta\beta_1) = 0$$

The discriminant is: $(\alpha+\alpha_1)^2 - 4(\alpha\alpha_1 - \beta\beta_1) > 0$
or $(\alpha - \alpha_1)^2 + 4\beta\beta_1 > 0$

because it is now obvious that g is real and also positive as $D > E$.

Hence:

$$x = M_1 e^{i(g_1 t + c_1)} + M_2 e^{i(g_2 t + c_2)}$$

or: $$h = M_1 \sin(g_1 t + c_1) + M_2 \sin(g_2 t + c_2)$$
$$k = M_1 \cos(g_1 t + c_1) + M_2 \cos(g_2 t + c_2)$$

Then:

$$e^2 = h^2 + k^2 = C_1 + C_2 \cos\left[(g_1 - g_2)t + c_1 - c_2\right]$$

$\tan \tilde{\omega} = \dfrac{h}{k}$

Hence we see that e is bounded at least to this order. For Jupiter and Saturn, the period of $g_1 - g_2$ is about 70,000 years.

For the p and q, the secular equation has the structure:

$$g^2 - (\alpha + \alpha_1) g = 0 \quad ; \quad g_1 = 0 \; ; \; g_2 = -(\alpha_1 + \alpha_2)$$

Then: $p = M \sin c_1 + M_2 \sin (g_2 t + c_2)$

$$\gamma^2 = p^2 + q^2 \quad ; \quad \tan \Omega = \frac{p}{q}$$

By showing that e and i are bounded but Ω and $\tilde{\omega}$ may or may not have secular terms, we have indicated that a two planet system is stable at least to 3rd order.

The n-Planet Problem

$$N_i = G \sum_j m_j D_{ij} \left(h_i^2 + k_i^2 - p_i^2 - q_i^2 + h_j^2 + k_j^2 - p_j^2 - q_j^2 + 2 p_i p_j + 2 q_i q_j \right) - 2G \sum_j m_j E_{ij} (h_i h_j + k_i k_j), \; i \neq j$$

The equations of motion are:

$$\dot{h}_i = \frac{1}{n_i a_i^2} \frac{\partial N_i}{\partial k_i} \quad ; \quad \dot{k}_i = -\frac{1}{n_i a_i^2} \frac{\partial N_i}{\partial h_i}$$

Let: $\dfrac{2 G m_j D_{ij}}{n_i a_i^2} = (i, j) \quad ; \quad \dfrac{2 G m_j E_{ij}}{n_i a_i^2} = [i, j]$

Then:

$$\dot{h}_i = \sum_j (i,j) k_i - \sum_j [i,j] k_j$$

$$\dot{k}_i = -\sum_j (i,j) h_i + \sum_j [i,j] h_j$$

Define: $\sum_j (i,j) = C_i$

and: $\sqrt{m_i n_i}\, a_i\, [i,j] = \sqrt{m_j n_j}\, a_j\, B_{ij}$

$$B_{ij} = \frac{\sqrt{m_i n_i}\, a_i\, 2 G m_j E_{ij}}{\sqrt{n_i}\, a_i^2\, \sqrt{m_j n_j}\, a_j}$$

Then: $$B_{ij} = \frac{2 G E_{ij} \sqrt{m_i m_j}}{\sqrt{n_i n_j}\; a_i a_j} = B_{ji}$$

Also define, for canonical purposes:

$$H_i = \sqrt{n_i m_i}\; a_i h_i$$
$$K_i = \sqrt{n_i m_i}\; a_i k_i$$

This gives for the last term of $\dot{h}_i$:

$$-\sum_j \sqrt{n_i m_i}\; a_i \,[i,j]\, k_j = -\sum_j B_{ij} K_j$$

The new equations of motion are:

$$\dot{H}_i - C_i K_i + \sum_j B_{ij} K_j = 0 \qquad \{ H_i$$
$$\dot{K}_i + C_i H_i - \sum_j B_{ij} H_j = 0 \qquad \{ K_i$$

Sum equations and sum on i

This gives:

$$\sum_i (H_i \dot{H}_i + K_i \dot{K}_i) + \underbrace{\sum_{ji} B_{ij} (H_i K_j - K_i H_j)}_{0 \text{ by symmetry of } B_{ij}} = 0$$

Hence:
$$\sum_i (H_i^2 + K_i^2) = \text{constant}$$

or: $$\sum_i n_i m_i a_i^2 e_i^2 = \text{constant}$$

Since all quantities above are positive, this shows that e_i cannot grow above a certain value or that all e's are bounded. Similarly:

$$\sum_i n_i m_i a_i^2 \gamma_i^2 = \text{constant}$$

or that the inclinations are bounded.

To obtain solution, define $U_s = K_s + i H_s$:

Then:

$$\dot{U}_s - i C_s U_s + i \sum_j B_{sj} U_j = 0$$

Try $U_s = M_s e^{i(gt+c)}$ which gives:

$$g M_s - C_s M_s + \sum_j B_{sj} M_j = 0$$

which is a determenential equation for g.
g are the eigenvalues of a real symmetric matrix and are hence all real. The general solution is then:

$$U_s = \sum_j M_{sj} e^{i(g_j t + c_j)}$$

and: $H_s = \sum_j M_{sj} \sin(g_j t + c_j)$, etc.

One of the results of this analysis is that the eccentricity of earth and Venus will at some time be 1.
For the p, q we find a structure like:

$$P_s = M_{s1} \sin C_1 + \sum_{j=2}^{n} M_{sj} \sin(g_j t + c_j)$$

Consider:

$$2N = \sum_i C_i (H_i^2 + K_i^2) - \sum_{ij} B_{ij} (H_i H_j + K_i K_j) = \text{constant}$$

$$\dot{H}_i = \frac{\partial N}{\partial K_i} \quad ; \quad \dot{K}_i = -\frac{\partial N}{\partial H_i}$$

We now make an orthogonal CT given by:

$$H_r = \sum_s A_{rs} L_s \quad ; \quad K_r = \sum_s A_{rs} l_s$$

with $\sum_r A_{rs} A_{rt} = \delta_{st}$

We obtain:

$$2N = \sum_i a_i (L_i^2 + l_i^2) - \sum_{ij} f_{ij} (L_i l_j + L_j l_i)$$

$$f_{ij} = f_{ji}$$

Choose the A_{ij}'s such that the last term vanishes.

Now, $\dot{L}_i = \dfrac{\partial N}{\partial l_i}$; $\dot{l}_i = -\dfrac{\partial N}{\partial L_i}$

and this gives:

$$\dot{L}_i = a_i l_i$$
$$\dot{l}_i = -a_i L_i$$

or: $\ddot{L}_i = -a_i^2 L_i$

whose solutions are:

$$L_i = P_i \sin(a_i t + c_i)$$

Also: $l_i = P_i \cos(a_i t + c_i)$

Hence we get the same solution as before:

$$H_s = \sum_j Q_{sj} \sin(a_j t + c_j)$$

LECTURE 19: 4-30-62

Delauney Treatment of Disturbed Motion:
Take for the disturbing function:

$$R' = \Sigma\left\{-B(LGH) - A(LGH)\cos\theta\right\}$$

$$\theta = i l + j g + k h + q n_1 t + q'$$

where we recall from previous results:

$$L = \sqrt{\mu a} \qquad l = n(t+\beta_1)$$
$$G = \sqrt{\mu a(1-e^2)} \qquad g = \omega$$
$$H = \sqrt{\mu a(1-e^2)}\cos i \qquad h = \Omega$$

$$\dot{L}_i = \frac{\partial R'}{\partial l_i} \; ; \quad \dot{l}_i = -\frac{\partial R'}{\partial L_i} \; ; \quad R' = R + \frac{\mu^2}{2L^2}$$

L, G, H are the Delauney variables or elements.

Now make a CT to a new set of canonical variables, taking into account that e and i are small quantities of the first order:

$$L' = L$$
$$G' = G - L = \sqrt{\mu a}\left(\sqrt{1-e^2} - 1\right) \doteq -\frac{e^2}{2} L$$
$$H' = H - G = \sqrt{\mu a(1-e^2)}\,(\cos i - 1) \doteq \sin^2\frac{i}{2}\, G$$

Hence G' and H' are small quantities of the second order.

The criteria for an extended linear transformation is:

$$Ll + Gg + Hh = L'l' + G'g' + H'h'$$

Now: $L'(l+g+h) = L(l+g+h) = Ll + Lg + Lh + Gg + Hh - Gg - Hh$
$= Ll + Gg + Hh + (L-G)g + (L-H)h$

$$Ll + Gg + Hh = L'(l+g+h) + (G-L)g + (H-L)h$$
$$= L'(l+g+h) + (G-L)(g+h) + (H-G)h$$

Hence: $l' = l+g+h = M+\omega+\Omega = M+\tilde{\omega}$
$g' = g+h = \tilde{\omega}$
$h' = h = \Omega$

The Hamiltonian remains the same as the CT is time independent. We now drop the primes everywhere using the new definition in terms of the old elements.

The Delauney method treats parts of the Hamiltonian successively, taking each previous solution as the basis for the next part, much like multiple perturbation theory in Quantum Mechanics.

In this spirit, treat each periodic term in R' on its own. Consider:

$$R_0 = -B - A\cos\theta$$

$$\theta = i l + j g + k h + q n_1 t + g'$$

Make another extended linear point transformation:

$$l' = i l + j g + k h$$

Hence: $\dot{G}' = \frac{\partial R_0}{\partial g'} = 0$; $\dot{H}' = \frac{\partial R_0}{\partial h'} = 0$

so that we eliminate two variables.

Take $L' = \frac{1}{i}L$: $Ll + Gg + Hh = \frac{1}{i}L(il + jg + kh) + G'g' + H'h'$

$$= Ll + \frac{j}{i}Lg + \frac{k}{i}Lh + G'g' + H'h'$$

$$G'g' + H'h' = (G - \tfrac{j}{i}L)g + (H - \tfrac{k}{i}L)h$$

Then: $L' = \frac{1}{i}L$ $\qquad l' = il + jg + kh$
$G' = G - \frac{j}{i}L$ $\qquad g' = g$
$H' = H - \frac{k}{i}L$ $\qquad h' = h$

Make the substitutions in R_0:

$$R_0 = -B - A\cos\theta = -B(L'G'H') - A(L'G'H')\cos\theta$$

where $G' =$ constant and $H' =$ constant, and $\theta = l' + q n_1 t + g'$.

The HJ equation is then:

$$\frac{\partial S}{\partial t} - B - A \cos\left(\frac{\partial S}{\partial L'} + q n_1 t + q'\right) = 0$$

Eliminate t by the substitution:

$$S = S' - (q n_1 t + q')L'$$

$$\frac{\partial S}{\partial t} = -q n_1 L' + \frac{\partial S'}{\partial t}$$

$$\frac{\partial S}{\partial L'} = \frac{\partial S'}{\partial L'} - (q n_1 t + q')$$

This eliminates explicit dependence on t. We get:

$$\frac{\partial S'}{\partial t} - q n_1 L' - B - A \cos\left(\frac{\partial S'}{\partial L'}\right) = 0$$

Now eliminate the time completely:

$$S' = S_1 + ct$$

$$\frac{\partial S'}{\partial t} = c$$

$$\frac{\partial S'}{\partial L'} = \frac{\partial S_1}{\partial L'}$$

and: $\quad c - q n_1 L' - B - A \cos\left(\frac{\partial S_1}{\partial L'}\right) = 0$

Write: $\quad B_1 = B + q n_1 L'$

$$c - B_1 - A \cos\left(\frac{\partial S_1}{\partial L'}\right) = 0$$

Hence: $\quad S_1 = \int_{x_q}^{L'} \cos^{-1}\left(\frac{c - B_1}{A}\right) dL' + D$

x_q chosen to make lower limit $= 0$

Since a complete solution has the form: $S = S(q_1 q_2 q_3\, \alpha_1 \alpha_2 \alpha_3\, t)$
so we choose for D:

$$D = \alpha_2 G' + \alpha_3 H'$$

The complete generating function is:

$$S = S_1 - (q n_1 t + q')L' + Ct + \alpha_2 G' + \alpha_3 H'$$

The solutions are:

$$\beta_1 = \frac{\partial S}{\partial C} = \frac{\partial S_1}{\partial C} + t$$

$$\beta_2 = \frac{\partial S}{\partial \alpha_2} = G' \;; \qquad \beta_3 = \frac{\partial S}{\partial \alpha_3} = H'$$

$$l' = \frac{\partial S}{\partial L'} = \frac{\partial S_1}{\partial L'} - (q n_1 t + q')$$

$$g' = \frac{\partial S}{\partial G'} = \frac{\partial S_1}{\partial G'} + \alpha_2$$

$$h' = \frac{\partial S}{\partial H'} = \frac{\partial S_1}{\partial H'} + \alpha_3$$

Now: $\theta = l' + q n_1 t + q'$; $C - B_1 = A \cos\theta$

$$\int_x^{L'} \cos^{-1}\left(\frac{C-B_1}{A}\right) dL' = \int_x^{L'} \theta \, dL'$$

We can invert $\beta_1 = \frac{\partial S}{\partial \alpha_1}(q\,\alpha\,t)$ to find $q = q(\alpha\,\beta\,t)$. That is, we can take $S_1 = S_1(L', G', H'; C)$; $t - \beta_1 = -\frac{\partial S_1}{\partial C}$ and find:

$$L' = L'(t - \beta_1, G', H'; C)$$

Consider: $\dot{L}' = \frac{\partial R_0}{\partial l'}$; $\dot{l}' = -\frac{\partial R_0}{\partial L'}$

$R_0 = -B - A\cos\theta$; $\theta = l' + q n_1 t + g'$

Then: $\dot{L}' = A \sin\theta$, $\dot{l}' = \frac{\partial B}{\partial L'} + \frac{\partial A}{\partial L'} \cos\theta$

Now: $\dot{\theta} = \dot{l}' + q n_1$

or: $\dot{\theta} = \frac{\partial B}{\partial L'} + q n_1 + \frac{\partial A}{\partial L'} \cos\theta = \frac{\partial B_1}{\partial L'} + \frac{\partial A}{\partial L'} \cos\theta$

Define, $B' = \frac{\partial B_1}{\partial L'}$: $A' = \frac{\partial A}{\partial L'}$

So the two equations we want to solve are:

$$\dot{L}' = A \sin\theta \quad ; \quad \dot{\theta} = B' + A' \cos\theta$$

B' is the 0^{th} order part of the disturbing function, and A and A' are small quantities ($B = \mu^2/2L^2$). We solve by successive approximations:

$$\dot{L}' = 0 \quad ; \quad \dot{\theta} = B_0' = \theta_0$$
$$L' = L_0$$

so that $\theta = \theta_0(t+c) = \lambda$ is our 1st approximation. This says that the semimajor axis is constant. For the next approximation, take $L = L_0 + L_1$. Then:

$$\dot{L}' = \dot{L}_1' \quad ; \quad \dot{L}_1' = A_0 \sin\lambda \quad ; \quad L_1' = \underbrace{-\frac{A_0}{\theta_0}}_{L_1} \cos\lambda$$

and hence $L' = L_0 + L_1 \cos\lambda$

Treating the θ equation: $\theta = \lambda + \theta_1$

$$\dot{\theta} = \dot{\lambda} + \dot{\theta}_1 = \theta_0 + \dot{\theta}_1$$

Expand B' in a Taylor series:

$$\dot{\theta} = \theta_0 + \dot{\theta}_1 = B_0' + \left(\frac{\partial B'}{\partial L'}\right)_0 L_1' + A_0 \cos\lambda$$

or: $\theta_1 = \theta_1' \sin\lambda$

and: $\theta = \lambda + \theta_1' \sin\lambda$

So in the process of taking many approximations, we obtain:

$$L' = L_0 + \sum_{p=1}^{\infty} L_p \cos p\,\theta_0(t+c)$$

$$\theta = \theta_0(t+c) + \sum_{p=1}^{\infty} \theta_p \sin p\,\theta_0(t+c)$$

We now have effected the inversion to find L'.

We can now consider: $\beta_1 = \frac{\partial S_1}{\partial C} + t$

$$S_1 = \int_x^{L'} \theta \, dL'$$

$$\cos\theta = \frac{C - B_1}{A}$$

$$\frac{\partial S_1}{\partial C} = \int_x^{L'} \frac{\partial \theta}{\partial C} dL' + \underbrace{\frac{\partial x}{\partial C}}_{0 \text{ by choice of } x} (\theta)$$

now: $\frac{\partial \theta}{\partial C} = -\frac{1}{A \sin\theta}$

so: $\beta_1 = -\int_x^{L'} \frac{dL'}{A \sin\theta} + t$

But $\dot{L}' = A \sin\theta$

and hence: $\beta_1 = -\int_x^{L'} \frac{dL'}{\dot{L}'} + t = -\int_{t_0}^{t} dt + t$

$= t_0 - t + t$; or $\beta_1 = t_0$

t_0 is the time at which $\theta = 0$ which is $t = -C$
so that $\beta_1 = -C$.

LECTURE 20 : 5-2-62

Summary of Previous Results:

$$L = \sqrt{\mu a} \qquad l = n(t-\tau) + \tilde{\omega}$$
$$G = L\left(\sqrt{1-e^2} - 1\right) \qquad g = \tilde{\omega}$$
$$H = L\sqrt{1-e^2}\,(\cos i - 1) \qquad h = \Omega$$

$$R_0 = -B - A \cos\left(i l + j g + k h + q n_1 t + q'\right)$$

$$L' = \frac{1}{i} L \qquad l' = i l + j g + k h$$
$$G' = G - \frac{j}{i} L \qquad g' = g$$
$$H' = H - \frac{k}{i} L \qquad h' = h$$

$$S = ct + S_1 - (q n_1 t + q')L' + \alpha_2 G' + \alpha_3 H'$$

$$S_1 = \int_x^{L'} \theta \, dL'$$

$$\theta = \cos^{-1}\left(\frac{c - B_1}{A}\right) = l' + q n_1 t + q'$$

$$B_1 = B + q n_1 L' \quad ; \quad \alpha_2 = (g) \quad ; \quad \alpha_3 = (h)$$

Solutions:

$$\beta_1 = t + \frac{\partial S_1}{\partial c}$$
$$\beta_2 = G'$$
$$\beta_3 = H'$$
$$l' = \frac{\partial S_1}{\partial L'} - (q n_1 t + q')$$
$$g' = (g) + \frac{\partial S_1}{\partial G'}$$
$$h' = (h) + \frac{\partial S_1}{\partial H'}$$

$$\dot{L}' = A \sin\theta$$
$$\dot{\theta} = B' + A' \cos\theta$$
$$L' = L_0 + \sum L_p \cos p\,\theta_0(t+c)$$
$$\theta = \theta_0(t+c) + \sum \theta_p \sin p\,\theta_0(t+c)$$

At this point, we have essentially solved for L' and l'. G' and H' are constants, so we also have G and H and L.

Now find g', h':

$$g' = (g) + \frac{\partial S_1}{\partial G'}$$

$$\frac{\partial S_1}{\partial G'} = \int_{*}^{L'} \frac{\partial \theta}{\partial G'} \, dL'$$

$$\cos\theta = \frac{C - B_1}{A}$$

$$-\sin\theta \, \frac{\partial \theta}{\partial G'} = \frac{-A \frac{\partial B_1}{\partial G'} - (C - B_1)\frac{\partial A}{\partial G'}}{A^2}$$

$$A \sin\theta \, \frac{\partial \theta}{\partial G'} = \frac{\partial B_1}{\partial G'} + \frac{C - B_1}{A} \frac{\partial A}{\partial G'}$$

Then: $\quad L'' \dfrac{\partial \theta}{\partial G'} = F(L'; G'; H'; C) = g_0 + \sum D_p \cos p\Theta_0 (t-c)$

Write $\quad \dfrac{dL'}{L''} = dt \quad$ and then:

$$\frac{\partial S_1}{\partial G'} = \int_{-c}^{t} [g_0 + \sum D_p \cos p\Theta_0 (t+c)] \, dt$$

$$= g_0 (t+c) + \sum g_p \sin p\Theta_0 (t+c)$$

Then:

$$g' = (g) + g_0 (t+c) + \sum g_p \sin p\Theta_0 (t+c)$$

$$h' = (h) + h_0 (t+c) + \sum h_p \sin p\Theta_0 (t+c)$$

The next step in the problem is to consider more of the disturbing function. Take:

$C = \alpha_1 \qquad (g) = \alpha_2 \qquad (h) = \alpha_3$

$-c = \beta_1 \qquad G' = \beta_2 \qquad (H') = \beta_3$

$H = H_0 + H_1 \quad ; \quad R = R_0 - (-R_1) \quad ; \quad \dot{\alpha}_n = \dfrac{\partial H_1}{\partial \beta_n} \quad ; \quad \dot{\beta}_n = -\dfrac{\partial H_1}{\partial \alpha_n}$

Hence:

$$\dot{C} = \frac{\partial R_1}{\partial c} \quad ; \quad \dot{G}' = \frac{\partial R_1}{\partial (g)} \quad ; \quad \dot{H}' = \frac{\partial R_1}{\partial (h)}$$

$$\dot{c} = -\frac{\partial R_1}{\partial C} \quad ; \quad (\dot{g}) = -\frac{\partial R_1}{\partial G'} \quad ; \quad (\dot{h}) = -\frac{\partial R_1}{\partial H'}$$

Now R_1 has the form:

$$D(LGH) \cos (l_1 l + j_1 g + h_1 h + q_1 n_1 t + q_1')$$

Upon substitution, D becomes a series:

$$\left[D_0 + \sum D_p \cos p\,\Theta_0 (t+c) \right] \cos \Theta$$

This is the general form of the result.

$$\Theta = l_1 (l) + j_1 (g) + h_1 (h) + (q_1 - i_1 q) n_1 t + q_1' - i q'$$
$$+ (t+c) \left[l_1 l_0 + j_1 g_0 + h_1 h_0 \right] + \sum E_p \sin p\,\Theta_0 (t+c)$$

$$(l) = -\frac{1}{i} \left[j(g) + k(h) \right]$$

$$l_1 = \frac{1}{i} \left[\Theta_0 - j g_0 - k h_0 \right]$$

$$l_p = \frac{1}{i} \left[\Theta_p - j g_p - k h_p \right]$$

$$E_p = i_1 l_p + j_1 g_p + h_1 h_p$$

We will neglect the E_p terms.

Then:

$$\dot{c} = -\frac{\partial R_1}{\partial C} = \frac{\partial D}{\partial C} \cos \Theta + D(t+c) \frac{\partial Q}{\partial C} \sin \Theta$$

where $Q = i_1 l_0 + j_1 g_0 + h_1 h_0 \pm p\,\Theta_0$

with similar results for $(\dot{g}) = -\frac{\partial R_1}{\partial G'}$ and $(\dot{h})$.

now: $t+c = -\frac{\partial S_1}{\partial c}$

$$L' = L'(t+c, G', H'; c)$$

$$S_1 = \int_x^{L'} \theta \, dL'$$

$$S_1 = S_1(L', G', H'; c)$$

$$K(t+c, G', H'; c) = S_1(L', G', H'; c)$$

$$\frac{\partial K}{\partial c} = \frac{\partial S_1}{\partial c} + \frac{\partial S_1}{\partial L'} \frac{\partial L'}{\partial c}$$

$$\frac{\partial K}{\partial c} + t + c = \theta \frac{\partial L'}{\partial c}$$

Evaluate $\theta \frac{\partial L'}{\partial c}$: $\theta = \theta_0(t+c) + \Sigma \theta_p \sin p\, \theta_0(t+c)$
$L' = L_0 + \Sigma L_p \cos p\, \theta_0(t+c)$

$$\frac{\partial L'}{\partial c} = \frac{\partial L_0}{\partial c} + \Sigma \frac{\partial L_p}{\partial c} \cos p\theta_0(t+c) - \Sigma L_p\, p(t+c) \cdot \frac{\partial \theta_0}{\partial c} \sin p\, \theta_0(t+c)$$

$$\theta \frac{\partial L'}{\partial c} = \theta_0(t+c) \frac{\partial L_0}{\partial c} - \frac{1}{2}(t+c) \frac{\partial \theta_0}{\partial c} \Sigma p L_p \theta_p$$

and:

$$\frac{\partial K}{\partial c} + t + c = \theta_0(t+c) \frac{\partial L_0}{\partial c} - \frac{1}{2}(t+c) \frac{\partial \theta_0}{\partial c} \Sigma p L_p \theta_p$$

Convert $\int_x^{L'} \theta \, dL'$ into $\int_{-c}^{t} \theta \frac{dL'}{dt} dt$. Take $\theta_0(t+c) = \lambda$

$$\theta = \lambda + \Sigma \theta_p \sin p\lambda$$
$$L' = L_0 + \Sigma L_p \cos p\lambda$$

$$K = \int_0^{\lambda} \theta \frac{\partial L'}{\partial \lambda} d\lambda$$

$\theta \frac{\partial L'}{\partial \lambda} = -\frac{1}{2} \Sigma p L_p \theta_p$ since $\frac{\partial L'}{\partial \lambda} = -\Sigma p L_p \sin p\lambda$

Finally: $K = -\frac{\theta_0}{2}(t+c)\sum p L_p \theta_p$

Substitution into the equation for $\frac{\partial K}{\partial c}$ gives:

$$1 = \theta_0 \frac{\partial}{\partial C}\underbrace{\left[L_0 + \frac{1}{2}\sum p L_p \theta_p\right]}_{\Lambda}$$

Then:

$$\frac{\partial \Lambda}{\partial C} = \frac{1}{\theta_0} \quad ; \quad \Lambda = \Lambda(C, G'; H')$$

We know examine some of the properties of Λ using:

$$\frac{\partial K}{\partial C} = \frac{\partial S_1}{\partial C} + \theta \frac{\partial L'}{\partial C} \quad \text{and} \quad g' = (g) + \frac{\partial S_1}{\partial G'}$$

since $K(t+c, G', H'; c) = S_1(L', G', H'; c)$

$$\frac{\partial K}{\partial G'} = \frac{\partial S_1}{\partial G'} + \frac{\partial S_1}{\partial L'}\frac{\partial L'}{\partial G'}$$

Then: $g' = (g) + \frac{\partial K}{\partial G'} - \theta \frac{\partial L'}{\partial G'} = (g) + g_0(t+c) + \text{periodic terms}$

$$g_0(t+c) = \frac{\partial K}{\partial G'} - \theta_0 \frac{\partial L'}{\partial G'}$$

or: $\theta_0 \frac{\partial L'}{\partial G'} = -g_0(t+c) + \frac{\partial K}{\partial G'}$

Then: $\frac{\partial \Lambda}{\partial G'} = -\frac{g_0}{\theta_0}$ and similarly: $\frac{\partial \Lambda}{\partial H'} = -\frac{h_0}{\theta_0}$

now: $d\Lambda = \frac{\partial \Lambda}{\partial C}dC + \frac{\partial \Lambda}{\partial G'}dG' + \frac{\partial \Lambda}{\partial H'}dH'$

$$= \frac{1}{\theta_0}dC - \frac{g_0}{\theta_0}dG' - \frac{h_0}{\theta_0}dH'$$

or: $dC = \theta_0\, d\Lambda + g_0\, dG' + h_0\, dH'$

Also: $dC = \frac{\partial C}{\partial \Lambda}d\Lambda + \frac{\partial C}{\partial G'}dG' + \frac{\partial C}{\partial H'}dH'$

Equating coefficients give:

$$\theta_0 = \frac{\partial C}{\partial \Lambda} \; ; \quad g_0 = \frac{\partial C}{\partial G'} \; ; \quad h_0 = \frac{\partial C}{\partial H'}$$

LECTURE 21: 5-7-62

Recapitulation:

$$\dot{C} = \frac{\partial R_1}{\partial c} \qquad \dot{c} = -\frac{\partial R_1}{\partial C}$$

$$\dot{G}' = \frac{\partial R_1}{\partial (g)} \qquad (\dot{g}) = -\frac{\partial R_1}{\partial G'}$$

$$\dot{H}' = \frac{\partial R_1}{\partial (h)} \qquad (\dot{h}) = -\frac{\partial R_1}{\partial H'}$$

$\Lambda = L_0 + \frac{1}{2}\sum_p p L_p \Theta_0$ is to be taken as the new canonic variable, keeping G' and H'.

$\lambda = \Theta_0(t+c) - q u_1 t - g'$ is the non-periodic part of l'.

$\chi = (g) + g_0(t+c)$ is the non-periodic part of g'.

Similarly: $\eta = (h) + h_0(t+c)$.

Also:

$$\frac{\partial \Lambda}{\partial C} = \frac{1}{\Theta_0}\ ; \quad \frac{\partial \Lambda}{\partial G'} = -\frac{g_0}{\Theta_0}\ ; \quad \frac{\partial \Lambda}{\partial H'} = -\frac{h_0}{\Theta_0}$$

$$\frac{\partial C}{\partial \Lambda} = \Theta_0\ ; \quad \frac{\partial C}{\partial G'} = g_0\ ; \quad \frac{\partial C}{\partial H'} = h_0$$

Our disturbing function is:

$$R_1 = R_1(C, G', H', c, (g), (h)) = R'(\Lambda, G', H', \lambda, \chi, \eta)$$

Check the equations for $\dot{G}'$, $\dot{H}'$:

$$\frac{\partial R_1}{\partial (g)} = \frac{\partial R'}{\partial \chi}\frac{\partial \chi}{\partial (g)} = \frac{\partial R'}{\partial \chi} = \dot{G}'$$

$$\dot{H}' = \frac{\partial R'}{\partial \eta}$$

Now look at $\dot{C}$:

$$\dot{C} = \frac{\partial R_1}{\partial c} = \frac{\partial R'}{\partial \lambda}\frac{\partial \lambda}{\partial c} + \frac{\partial R'}{\partial \chi}\frac{\partial \chi}{\partial c} + \frac{\partial R'}{\partial \eta}\frac{\partial \eta}{\partial c}$$

$$= \frac{\partial R'}{\partial \lambda}\Theta_0 + \frac{\partial R'}{\partial \chi} g_0 + \frac{\partial R'}{\partial \eta} h_0$$

$$= \frac{\partial C}{\partial \Lambda}\dot{\lambda} + \frac{\partial C}{\partial G'}\dot{G}' + \frac{\partial C}{\partial H'}\dot{H}$$

Matching coefficients gives:

$$\dot{\lambda} = \frac{\partial R'}{\partial \lambda} \quad ; \quad \text{because} \quad \dot{c} = \Theta_0 \dot{\lambda} + g_0 \dot{G}' + h_0 \dot{H}'$$

This gives half of the canonic variables.
Now:

$$\dot{x} = (\dot{g}) + \dot{g}_0 (t+c) + g_0 + g_0 \dot{c}$$

$$= -\frac{\partial R_1}{\partial G'} + \dot{g}_0 (t+c) + g_0 - g_0 \frac{\partial R_1}{\partial c}$$

Consider g_0. Originally $g_0 = g_0(C, G', H')$ and on transforming: $g_0 = g_0(\Lambda, G', H')$. Then:

$$\dot{g}_0 = \frac{\partial g_0}{\partial \Lambda}\dot{\Lambda} + \frac{\partial g_0}{\partial G'}\dot{G}' + \frac{\partial g_0}{\partial H'}\dot{H}'$$

Now: $\dfrac{\partial g_0}{\partial \Lambda} = \dfrac{\partial^2 C}{\partial G' \partial \Lambda} = \dfrac{\partial \Theta_0}{\partial G'}$, hence:

$$\dot{g}_0 = \frac{\partial \Theta_0}{\partial G'}\dot{\Lambda} + \frac{\partial g_0}{\partial G'}\dot{G}' + \frac{\partial h_0}{\partial G'}\dot{H}'$$

Also:

$$\dot{g}_0 = \frac{\partial \Theta_0}{\partial G'}\frac{\partial R'}{\partial \lambda} + \frac{\partial g_0}{\partial G'}\frac{\partial R'}{\partial x} + \frac{\partial h_0}{\partial G'}\frac{\partial R'}{\partial \eta}$$

Returning to $R_1 = R'$:

$$\frac{\partial R_1}{\partial G'} + \frac{\partial R_1}{\partial C}\frac{\partial C}{\partial G'} = \frac{\partial R'}{\partial G'} + \frac{\partial R'}{\partial \lambda}\frac{\partial \lambda}{\partial G'} + \frac{\partial R'}{\partial x}\frac{\partial x}{\partial G'} + \frac{\partial R'}{\partial \eta}\frac{\partial \eta}{\partial G'}$$

$$= \frac{\partial R'}{\partial G'} + (t+c)\left[\frac{\partial \Theta_0}{\partial G'}\frac{\partial R'}{\partial \lambda} + \frac{\partial g_0}{\partial G'}\frac{\partial R'}{\partial x} + \frac{\partial h_0}{\partial G'}\frac{\partial R'}{\partial \eta}\right]$$

$$= \frac{\partial R'}{\partial G'} + (t+c)\dot{g}_0$$

Substitute in $\dot{x}$:

$$\dot{x} = -\frac{\partial R'}{\partial G'} + \frac{\partial C}{\partial G'} = -\frac{\partial (R'-c)}{\partial G'}$$

Similarly: $$\dot{\eta} = -\frac{\partial (R'-c)}{\partial H'}$$

Changing to the new disturbing function $R'-C$ does not affect $\dot{\Lambda}$, $\dot{G}'$, $\dot{H}'$ and $C \neq C(\lambda, \chi, \eta)$ is the reason. Now examine $\dot{\lambda}$:

$$\dot{\lambda} = \dot{\Theta}_0 (t+c) + \Theta_0 + \Theta_0 \dot{c} - q n_1$$

$$\dot{\Theta}_0 = \frac{\partial \Theta_0}{\partial \Lambda} \dot{\Lambda} + \frac{\partial \Theta_0}{\partial G'} \dot{G}' + \frac{\partial \Theta_0}{\partial H'} \dot{H}'$$

$$= \frac{\partial \Theta_0}{\partial \Lambda} \dot{\Lambda} + \frac{\partial g_0}{\partial \Lambda} \dot{G}' + \frac{\partial h_0}{\partial \Lambda} \dot{H}'$$

Go back to $R_1 = R'$:

$$\frac{\partial R_1}{\partial C} \frac{\partial C}{\partial \Lambda} = \frac{\partial R'}{\partial \Lambda} + \frac{\partial R'}{\partial \lambda} \frac{\partial \lambda}{\partial \Lambda} + \frac{\partial R'}{\partial \chi} \frac{\partial \chi}{\partial \Lambda} + \frac{\partial R'}{\partial \eta} \frac{\partial \eta}{\partial \Lambda}$$

since: $\Theta_0 = \Theta_0 (C, G', H')$
$\Lambda = \Lambda (G, G', H')$
$\Theta_0 = \Theta_0 (\Lambda, G', H')$, hence

$$\frac{\partial R_1}{\partial C} \frac{\partial C}{\partial \Lambda} = \frac{\partial R'}{\partial \Lambda} + (t+c)\dot{\Theta}_0$$

Substitute in $\dot{\lambda}$:

$$\dot{\lambda} = -\frac{\partial R'}{\partial \Lambda} + \frac{\partial C}{\partial \Lambda} - q n_1 = -\frac{\partial (R' - C + q n_1 \Lambda)}{\partial \Lambda}$$

We can define $R'' = R' - C + q n_1 \Lambda$ without disturbing the previous results. This is equivalent to adding $\frac{\mu^2}{2L^2}$ before to keep the canonical form. This generally happens whenever we try to eliminate periodic terms. The new disturbing function has the structure:

$$R'' = -B - \sum A \cos (i\lambda + j\chi + k\eta + q n_1 t + q')$$

We can repeat the process by taking the secular term plus one periodic term by taking:

$\lambda' = i\lambda + j\chi + k\eta$; $\Lambda' = \frac{1}{i}\Lambda$

Continuing will use up all the periodic terms and give the results in terms of the elements, or the Hamiltonian becomes completely secular, that is, $R = -B(L, G, H)$.

The Hamilton-Jacobi equation is then:

$$-B + \frac{\partial S}{\partial t} = 0$$

$$S = Bt + (l)L + (g)G + (h)H$$

$$l = \frac{\partial S}{\partial L} = \frac{\partial B}{\partial L} t + (l)$$

and the problem is solved.

In practice, instead of continuously transforming the disturbing function, we keep things in terms of the original elements:

$$L' = A \sin\theta \quad ; \quad \dot{\theta}' = A' + B' \cos\theta \quad ; \quad L' = \frac{1}{i} L$$

$$\left.\begin{aligned} L &= \sqrt{\mu a} \\ G &= \sqrt{\mu a}\,(\sqrt{1-e^2} - 1) \\ H &= \sqrt{\mu a(1-e^2)}\,(-\tfrac{1}{2}\sin^2\tfrac{i}{2}) \end{aligned}\right\} \begin{aligned} & L' = L'(e, G') \; ; a = a(e, G') \\ & \text{because } G = L(\sqrt{1-e^2} - 1) \end{aligned}$$

Then: $\dfrac{d(e^2)}{dt} = A(e, G'; H') \sin\theta$

because we can change $A(e, a, \gamma) \rightarrow A(e, G; H')$.
In this way we can obtain:

$$e^2 = e_0^2 + \Sigma \cos p\theta_0(t+c) \quad ; \quad \theta = \theta_0(t+c) + \Sigma \sin p\,\theta_0(t+c)$$

similarly: $a = a_0 + \Sigma PT$; $\gamma = \gamma_0 + \Sigma PT$, since: $\gamma = \gamma(e, G; H')$ so we can obtain all the elements this way.

This concludes Delauney's Lunar Theory.

LECTURE 22 : 5-9-62

Artificial Satellites

Recall the potential of an axially symmetric body:

$$U = \frac{\mu}{r}\left[1 + \sum_i \frac{J_i}{r^i} P_i(\sin\delta)\right]$$

Do two body problem : $U = \frac{\mu}{r}$, rest is R.
Also :

$$\sin\delta = \sin i \; \sin(v+\omega)$$

now:
$$\frac{1}{r^n} = \left(\frac{1 + e\cos v}{p}\right)^n$$

Expanding this along with $P(\sin\delta)$ results in a Fourier series of the form: $\cos(nv + n\omega)$
or:

$$R = \sum A_{ij}\cos(iM + j\omega)$$

$$\frac{1}{2\pi}\int_0^{2\pi} R\,dM = \sum A_{0j}\cos j\omega$$

Consider $\bar{R}$ as the secular part of R and R^* the long-period part. Then the short period part is given by $R - \bar{R} - R^*$.

Here we will only consider the J_2 term. The orders of the terms are:

J_2 : $O(1)$

$J_{3,4}$: $O(2)$

$J_{5,6,7}$: $O(3)$

Consider:
$$R = \frac{GMJ_2}{a^3}\left(\frac{a}{r}\right)^3\left(\frac{1}{3} - \sin^2\delta\right)$$

$$= \frac{GMJ_2}{a^3}\left(\frac{a^3}{r^3}\right)\left[\frac{1}{3} - \frac{1}{2}\sin^2 i + \frac{1}{2}\sin^2 i\cos(2v+2\omega)\right]$$

Consider $\frac{1}{2\pi}\int R\, dM$:

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{a}{r}\right)^3 dM = (1-e)^{-3/2}$$

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{a}{r}\right)^3 \begin{matrix}\sin\\ \cos\end{matrix}(2v)\, dM = 0$$

We see that there are then no longer-period terms left in ω. In fact:

$$\bar{R} = \frac{J_2 G M}{a^3}\left(\frac{1}{3} - \frac{1}{2}\sin^2 i\right)(1-e)^{3/2} \qquad \text{(secular part)}$$

$$R^* = R - \bar{R} \qquad \text{(short period part)}$$

Now use Lagrange planetary equations:

$$\frac{di}{dt} = \frac{\cos i}{na^2\sqrt{1-e^2}\,\sin i}\frac{\partial R}{\partial \omega}$$

Use angular momentum equation to change variables:

$$r^2\dot{v} = na^2\sqrt{1-e^2}\ ; \quad dt = \left(\frac{r}{a}\right)^2 \frac{dv}{n\sqrt{1-e^2}}$$

Then:

$$di_{sp} = \frac{1}{n^2a^2(1-e^2)\tan i}\int \frac{\partial}{\partial\omega}(R-\bar{R})\left(\frac{r}{a}\right)^2 dv$$

Do this process for each of the elements to get the short period parts. For the secular parts, we use the method of Kozai, using M as a variable instead of t.

$$\frac{dM}{dt} = n - \frac{1-e^2}{na^2e}\frac{\partial R}{\partial e} - \frac{2}{na}\frac{\partial R}{\partial a}$$

Then can do this for the rest of the elements.

The results for the secular parts are; using $\mu = a_0^3 n_0^2$, $p = a(1-e^2)$ and:

$$\bar{n} = n_0 \left[1 + \frac{J_2}{p^2} \left(1 - \frac{3}{2}\sin^2 i\right) \sqrt{1-e^2} \right]$$

$$\bar{M} = M_0 + \bar{n} t$$

$$\bar{\Omega} = \Omega_0 - \left[\frac{J_2}{p^2} \bar{n} \cos i \right] t$$

$$\bar{\omega} = \omega_0 + \frac{J_2 \bar{n}}{p^2} \left(2 - \frac{5}{2}\sin^2 i\right) t$$

We see that the node regresses. For a polar orbit, the perturbation vanishes and is greatest for an equatorial orbit.

The above is the strongest corrections to a satellite orbiting about a point mass.

A more elegant way would be to find a potential close to the true potential that would give an exactly solvable problem to which we could add perturbations as desired. We need to search for a potential which leads to a separable Hamilton - Jacobi equation. The condition for separability is due to Stäckel. Choose spherical coordinates. If the potential is of the form:

$$V = f_1(r) + \frac{1}{r^2} f(\phi)$$

then the problem will be separable. The elements obtained will be different than those of the two body problem. Take for V:

$$V = \frac{\mu}{r}(1 + C_1) + \frac{C_2}{r^2} + \frac{C_3 \sin^2 \delta}{r^2}$$

where C_1 is to taken to vary the mass while C_2 is choosen to give the proper precessing ellipse. C_3 determines the secular perturbation for the node.

The solution involves two circular integrals and two elliptic integrals, and the six canonic constants.

Say that the correct equation is given by $H = H_0 - H_1$ where H_0 is V and H is the old Hamiltonian. Then we can find H_1 and proceed by the usual perturbation methods.

Another method is due to Vinti where we take:

$$U = \frac{\mu}{r}\left[1 + \sum_1 (-J_2)^n \left(\frac{a_\oplus}{r}\right)^{2n} P_{2n}(\sin\delta)\right]$$

This implies $J_4 = -J_2^2$, Actually $J_4 \cong -2J_2^2$ for the Earth. If we can solve this $O(1)$ part exactly, we also have $O(2)$ part exactly. The problem has to be solved in oblate spheroidal coordinates. These coordinates are:

$$x = \sqrt{\rho^2 + c^2}\,\sqrt{1-\sigma^2}\,\cos\alpha$$

$$y = \sqrt{\rho^2 + c^2}\,\sqrt{1-\sigma^2}\,\sin\alpha$$

$$z = \rho\sigma$$

The idea for this probably arose from a QM problem.

The Von Zipple method - similar to Delauney method. Suppose we have a Hamiltonian $F(LGHlgh)$. Can we find a transformation:

$$F(LGHlgh) \rightarrow F'(L'G'H')$$

The Delauney method removed periodic terms one at a time by a canonical transformation. This method does this all at once via a generating function. The unprimed and primed quantities differ only by first quantities so that the generating function differs from unity by only $O(1)$ quantities. Hence we choose:

$$S = L'l + G'g + H'h + S_1$$

H is a constant.

Do in two steps: $F(LGlg) \rightarrow F'(L'G'g')$ which removes the long term perturbation and then remove g' or the short period part thus leaving only secular terms. Take:

$$S = L'l + G'g + S_1 + S_2$$

Now: $$L = \frac{\partial S}{\partial l} = L' + \frac{\partial S_1}{\partial l} + \frac{\partial S_2}{\partial l} = L' + \Delta L$$

$$G = \frac{\partial S}{\partial g} = G' + \frac{\partial S_1}{\partial g} + \frac{\partial S_2}{\partial g} = G' + \Delta G$$

$$l' = l + \frac{\partial S_1}{\partial L'} \text{ (do not need } S_2 \text{ term)} = l + \Delta l$$

$$g' = g + \frac{\partial S_1}{\partial G'} = g + \Delta g$$

Then:

$$F(LGlg) = F(L'+\Delta L, G'+\Delta G, l, g)$$

Expand in a Taylor series:

$$F(LGlg) = F(L'G'lg) + F_{L'}\Delta L + F_{G'}\Delta G + \tfrac{1}{2} F_{L'L'}(\Delta L)^2$$

Split F up into 3 parts: $F_0 = \frac{\mu^2}{2L^2}$; $F_1 = R_1$; $F_2 = R_2$

Now: $F_{0L} = -n$; $F_{0LL} = \frac{3n}{L}$

Always working to second order in ΔL:

$$F = F(L'G'lg) + (F_{0L'} + F_{1L'})\Delta L + F_{1G'}\Delta L + \tfrac{1}{2} F_{0LL}(S_{1l})^2$$

$$= F_0 + F_1 + F_2 + F_{0L}(S_{1l} + S_{2l}) + F_{1L} S_{1l}$$

$$+ F_{1G} S_{1g} + \tfrac{1}{2} F_{0LL} S_{1l}^2 \quad ; \quad L = L',\ G = G'$$

$$= F' = F_0' + F_1' + F_2'$$

Now equate orders of magnitude.

Look only at $O(1)$ parts:

$$F_0' = F_0 = \frac{\mu^2}{2L'}$$

$$F_1' = F_1 + F_{0L} S_{1\ell} = F_1 + (-n) S_{1\ell}$$

Now $S_{1\ell}$ contains only short period terms:

$$\overline{F_1'} = \overline{F_1} = \overline{R_1}$$

$$F_1'^{*} = F_1^{*} = R_1^{*} = 0$$

Hence: $F'_{sp} = F' - \overline{F'} - F^{*} = 0$

$$F_1' = \overline{F_1} + F_1^{*} \quad ; \quad \overline{R_1} = R_1 - n S_{1\ell}$$

$$S_1 = \frac{1}{n}\int (R_1 - \overline{R_1})\, d\ell$$

which gives the $O(1)$ part of S. Our Hamiltonian now has the structure $F'(G'g')$ which we want to transform to $F''(G'')$ by the same process as above. When we go thru this transformation, we find:

$$S_1^{*} = -\frac{1}{R_{1G''}}\int (R_2^{*} + \Phi^{*})\, dg$$

where Φ is the second order part of above.

Now:

$$\dot{\overline{g}}'' = -\frac{\partial \overline{R_1}}{\partial G''}$$

$\dot{g}''$ in motion of perigee ω but this vanishes at 63°.4. However, this is a mathematical singularity and does not appear in Vinti's method.

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