Jahnke E., and Emde, F. TABLES OF FUNCTIONS WITH FORMULAE and CURVES. (Funktionentafeln.) Fourth revised edition. Text in German and English. Index. 212 illustrations. xv + 362pp. 5½ x 8½. $133 Paperbound $2.00

Kamke, E. THEORY OF SETS. Translated by F. Bagemihl from the second German edition. Bibliography. Index. viii + 152pp. 5½ x 8. $141 Paperbound $1.35

Kellogg, O. D. FOUNDATIONS OF POTENTIAL THEORY. Index. ix + 384pp. 5½ x 8. $133 Paperbound $2.00

Khintchin, A. I. MATHEMATICAL FOUNDATIONS OF STATISTICAL MATHEMATICS. Translated by G. Gamow. Index. vii + 179pp. 5½ x 8. $146 Clothbound $2.95 $147 Paperbound $1.35

Klein, F. ELEMENTARY MATHEMATICS FROM AN ADVANCED STANDPOINT; Algebra, Arithmetic, Analysis. Translated from the third German edition by E. R. Hedrick and C. A. Noble. Index. 125 illustrations. xiv + 274pp. 5½ x 8. $150 Paperbound $1.75

Klein, F. ELEMENTARY MATHEMATICS FROM AN ADVANCED STANDPOINT; Geometry. Translated from the third German edition by E. R. Hedrick and C. A. Noble. Index. 141 illustrations. ix + 214pp. 5½ x 8. $151 Paperbound $1.75

Klein, F. FAMOUS PROBLEMS OF ELEMENTARY GEOMETRY. Translated by W. W. Beman and D. E. Smith. Second edition, revised and enlarged with notes by R. C. Archibald. xi + 92pp. 5½ x 8. $146 Clothbound $1.50 $129 Paperbound $1.00

Klein, F. LECTURES ON THE ICOSEHEDRON. xvi + 289pp. 5½ x 8. $134 Paperbound $1.85

Knopp, K. INFINITE SEQUENCES AND SERIES. Translated by Frederick Bagemihl. x + 176pp. 5½ x 8. $152 Clothbound $3.30 $153 Paperbound $1.75

Knopp, K. ELEMENTS OF THE THEORY OF FUNCTIONS. Translated by Frederick Bagemihl. Index. 160pp. 5½ x 8. $154 Paperbound $1.35


Knopp, K. THEORY OF FUNCTIONS. Part II: Applications and Continuations of the General Theory. Translated from the fourth German edition by Frederick Bagemihl. Bibliography. Index. 7 illustrations. x + 150pp. 5½ x 8. $157 Paperbound $1.35


Knopp, F. PROBLEM BOOK IN THE THEORY OF FUNCTIONS. Vol. II: Problems in the Advanced Theory of Functions. Translated by Frederick Bagemihl. 144pp. 5½ x 8. $159 Paperbound $1.35

Kober, H. DICTIONARY OF CONFORMAL REPRESENTATIONS. 447 diagrams. xvi + 208pp. 6½ x 9¼. $160 Clothbound $2.95

Kraitchik, M. MATHEMATICAL RECREATIONS. Second revised edition. 181 illustrations. Over 40 tables. 382pp. 5½ x 8. $162 Paperbound $1.75

Langer, S. AN INTRODUCTION TO SYMBOLIC LOGIC. Second revised edition corrected and expanded. 368pp. 5½ x 8. $164 Paperbound $1.75

Levy, H. and Baggott, E. A. NUMERICAL SOLUTIONS OF DIFFERENTIAL EQUATIONS. 18 diagrams. 20 tables. viii + 238pp. 5½ x 8. $168 Paperbound $1.75

Lewin, C. I. and Langford, C. H. SYMBOLIC LOGIC. Index. 8 diagrams. vii + 50pp. 5½ x 8. $170 Clothbound $4.50

Littlewood, J. E. ELEMENTS OF THE THEORY OF REAL FUNCTIONS. Third edition, x + 71pp. 5½ x 8. $171 Clothbound $2.65 $172 Paperbound $1.25

Lovitt, W. V. LINEAR INTEGRAL EQUATIONS. Index. 27 diagrams. x + 252pp. 5½ x 8. $175 Clothbound $3.50 $176 Paperbound $1.60

MacRobert, T. M. SPHERICAL HARMONICS; an Elementary Treatise on Harmonic Functions with Applications. Second revised edition. Index. 20 diagrams. vi + 372pp. 5½ x 8½. $179 Clothbound $4.50

Available at your dealer or write Dover Publications, Inc., Department TF1, New York 10, New York. Send for free catalog of all Dover books on mathematics.
ALMOST PERIODIC FUNCTIONS
by A. S. Besicovitch

This important summary by a well-known mathematician covers the theory of almost periodic functions created by Harold Bohr. It examines Bohr's own work, as well as newer, shorter, and more elementary proofs than Bohr's, and also demonstrates extensions of the theory beyond the class of uniformly continuous functions to which Bohr's work was limited. The contributions of Wiener, Weyl, de la Vallée Poussin, Stepanoff, and Bochner are examined, while the author's own work on the piecewise continuous case is also included.

The first portion of this book establishes basic theorems of uniformly a.p. functions, including Bohr's original work, and de la Vallée Poussin's ingenious short proof based on classical theory of purely periodic functions. It considers such matters as summation of Fourier series of u.a.p. functions by partial sums, the Bochner-Fejer summation of u.a.p. functions, particular cases of Fourier series, and u.a.p. functions of two variables.

The second portion of this work covers generalizations and extensions of the original theory, discussing relaxation of continuity restrictions, auxiliary theorems and formulae, the Parseval equation and the Riesz-Fischer theorem, and similar matters. The third chapter discusses analytic a.p. functions, including results in the location of singularities, behavior at infinity, and convergence of series. It opens a way to study a wide class of trigonometric series of the general type and of exponential series (Dirichlet series).

"For those interested in the concrete and calculational aspects of theory," APPLIED MECHANICS REVIEW. "A clear, concise, reasonably self-contained treatment of theory fundamentals," DESIGN NEWS.

Bibliography. xiii + 180pp. 5½ x 8.

S18 Paperbound $1.75
This new Dover edition is a reissue of the first edition republished through special permission of Cambridge University Press. Copyright 1954 by Dover Publications, Inc.

**CONTENTS**

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Preface</td>
<td>vii</td>
</tr>
<tr>
<td>Introduction</td>
<td>ix</td>
</tr>
<tr>
<td><strong>Chapter I. UNIFORMLY ALMOST PERIODIC FUNCTIONS:</strong></td>
<td></td>
</tr>
<tr>
<td>§ 1. Definition and elementary properties</td>
<td>1</td>
</tr>
<tr>
<td>2. Normality of <em>u.a.p.</em> functions</td>
<td>10</td>
</tr>
<tr>
<td>3. Mean values of <em>u.a.p.</em> functions and their Fourier series</td>
<td>12</td>
</tr>
<tr>
<td>4. Fundamental theorem of the theory of <em>u.a.p.</em> functions</td>
<td>21</td>
</tr>
<tr>
<td>5. Polynomial approximation to <em>u.a.p.</em> functions</td>
<td>29</td>
</tr>
<tr>
<td>6. Limit periodic functions</td>
<td>32</td>
</tr>
<tr>
<td>7. Base of <em>u.a.p.</em> functions. Connection of <em>u.a.p.</em> functions with limit periodic functions of several variables</td>
<td>34</td>
</tr>
<tr>
<td>8. Summation of Fourier series of <em>u.a.p.</em> functions by partial sums</td>
<td>38</td>
</tr>
<tr>
<td>9. Bochner-Fejér summation of <em>u.a.p.</em> functions</td>
<td>46</td>
</tr>
<tr>
<td>10. Some particular cases of Fourier series of <em>u.a.p.</em> functions</td>
<td>51</td>
</tr>
<tr>
<td>11. Arithmetical nature of translation numbers</td>
<td>52</td>
</tr>
<tr>
<td><strong>Chapter II. GENERALISATION OF ALMOST PERIODIC FUNCTIONS:</strong></td>
<td></td>
</tr>
<tr>
<td>Introduction</td>
<td>67</td>
</tr>
<tr>
<td>§ 1. Auxiliary theorems and formulae</td>
<td>68</td>
</tr>
<tr>
<td>2. General closures and general almost periodicity</td>
<td>70</td>
</tr>
<tr>
<td>3. <em>S a.p.</em> functions</td>
<td>79</td>
</tr>
<tr>
<td>4. <em>W a.p.</em> functions</td>
<td>82</td>
</tr>
</tbody>
</table>
CONTENTS

§ 1. Some auxiliary theorems in the theory of analytic functions 130

2. Definition of analytic almost periodic functions and their elementary properties 141

3. Dirichlet series 147

4. Behaviour of u.a.p. functions at \( \sigma = \infty \) 158

5. On the behaviour of analytic functions outside the strip of uniform almost periodicity 163

6. On the behaviour of analytic functions on the boundary of the strip of uniform almost periodicity 169

Memoirs referred to in the text 179

PREFACE

The theory of almost periodic functions, created by H. Bohr, has now completed two stages of its development.

Almost periodicity, as a structural property, is a generalisation of pure periodicity and Bohr's original methods for establishing the fundamental results of the theory were always based on reducing the problem to a problem of purely periodic functions. But though the underlying idea of Bohr's method was clear and simple, the actual proofs of the main results were very difficult and complicated. New methods were given by N. Wiener and H. Weyl, by which the results were arrived at in a much shorter way. But these methods have lost the elementary character of Bohr's methods. It was C. de la Vallée Poussin who succeeded in giving a new proof (based partly on H. Weyl's idea), which was very short and at the same time based entirely on classical results in the theory of purely periodic functions.

This represents one stage in the development of the theory of almost periodic functions.

Bohr's theory of a.p. functions was restricted to the class of uniformly continuous functions. Then efforts were directed to generalisations of the theory. Thanks to the work of W. Stepanoff, N. Wiener, H. Weyl, H. Bohr and others, generalisations may be considered to have reached a certain completeness.

This was the second stage in the development of the theory.

These circumstances suggest that the present moment is not unfavourable for writing an account of the theory.

In Chapter I of this account we develop the fundamental part of the theory of a.p. functions of a real variable—the theory of uniformly a.p. functions. In the main problems of this chapter we adopt the methods of H. Bohr, de la Vallée Poussin, Weyl and Bochner.

Chapter II is devoted to a systematic investigation of generalisations of the theory.
PREFACE

In Chapter III we develop the theory of analytic a.p. functions, which in essentials remains unaltered as it was published by H. Bohr.

This account is not encyclopaedic. Our aim is to give the fundamental results of the theory, and we have omitted all discussion of certain special problems. Thus the work of Bohr, Neugebauer, Walter and Bochner on differential equations and difference equations has not been considered in this book, nor has the theory of harmonic a.p. functions developed by J. Favard in his interesting paper. For all these questions the reader is referred to the original papers.

A. S. B.

INTRODUCTION

The theory of almost periodic functions was created and developed in its main features by H. Bohr during the last decade. Like many other important mathematical discoveries it is connected with several branches of the modern theory of functions. On the one hand, almost periodicity as a structural property of functions is a generalisation of pure periodicity; on the other hand, the theory of almost periodic functions opens a way of studying a wide class of trigonometric series of the general type and of exponential series (Dirichlet series), giving in the latter case important contributions to the general problems of the theory of analytic functions.

Almost periodicity is a generalisation of pure periodicity: the general property can be illustrated by means of the particular example

$$f(x) = \sin 2\pi x + \sin 2\pi x \sqrt{2}.$$  

This function is not periodic: there exists no value of $\tau$ which satisfies the equation

$$f(x + \tau) = f(x)$$

for all values of $x$. But we can establish the existence of numbers for which this equation is approximately satisfied with an arbitrary degree of accuracy. For given any $\varepsilon > 0$ as small as we please we can always find an integer $\tau$ such that $\tau \sqrt{2}$ differs from another integer by less than $\varepsilon/2\pi$. It can be proved that there exist infinitely many such numbers $\tau$, and that the difference between two consecutive ones is bounded. For each of these numbers we have

$$f(x + \tau) = \sin 2\pi (x + \tau) + \sin 2\pi (x + \tau) \sqrt{2}$$

$$= \sin 2\pi x + \sin (2\pi x \sqrt{2} + \theta \varepsilon)$$  

$$= f(x) + \theta' \varepsilon.$$  

$$|\theta'| \leq 1$$
Almost periodicity of a function \( f(x) \) in general is defined by this property:

The equation

\[
f(x + \tau) = f(x)
\]

is satisfied with an arbitrary degree of accuracy by infinitely many values of \( \tau \), these values being spread over the whole range from \(-\infty\) to \(+\infty\) in such a way as not to leave empty intervals of arbitrarily great length.

Almost periodicity is a deep structural property of functions which is invariant with respect to the operations of addition (subtraction) and multiplication, and also in many cases with respect to division, differentiation, integration and other limiting processes.

To the structural affinity between almost periodic functions and purely periodic functions may be added an analytical similarity. To any almost periodic function corresponds a "Fourier series" of the type of a general trigonometric series

\[
f(x) \sim \sum_{n=1}^{\infty} A_n e^{i\lambda_n x} \tag{1}
\]

(\( A_n \) being real numbers and \( A_n \) real or complex): it is obtained from the function by the same formal process as in the case of purely periodic functions (namely, by the method of undetermined coefficients and term-by-term integration). The series (1) need not converge to \( f(x) \), but there is a much closer connection between the series and the function than we have yet seen. In the first place Parseval's equation is true, i.e.

\[
M \{ |f(x)|^2 \} = \sum |A_n|^2,
\]

from which follows at once the uniqueness theorem, according to which there exists at most one almost periodic function having a given trigonometric series for its Fourier series. Parseval's equation constitutes the fundamental theorem of the theory of almost periodic functions. Further, the series (1) is "summable to \( f(x) \)," in the sense that there exists a sequence of polynomials

\[
\sum_{n=1}^{\infty} p_n^{(k)} A_n e^{i\lambda_n x} \quad (k = 1, 2, \ldots)
\]

(where \( 0 \leq p_n \leq 1 \), and where for each \( k \) only a finite number of the factors \( p \) differ from zero) which

(a) converge to \( f(x) \) uniformly in \( x \), and

(b) converge formally to the series (1),

by which is meant that for each \( n \)

\[
p_n^{(k)} \rightarrow 1, \quad \text{as} \quad k \rightarrow \infty.
\]

Conversely, any trigonometric polynomial is an almost periodic function, and so is the uniform limit of a sequence of trigonometric polynomials. It is easily proved that the Fourier series of such a limit function is the formal limit of the sequence of trigonometric polynomials. Thus the class of Fourier series of almost periodic functions consists of all trigonometric series of the general type

\[
\sum A_n e^{i\lambda_n x},
\]

to which correspond uniformly convergent sequences of polynomials of the type

\[
\sum_{k=1}^{\infty} p_n^{(k)} A_n e^{i\lambda_n x} \quad (k = 1, 2, \ldots)
\]

formally convergent to the series. Thus the theory of almost periodic functions opened up for study a class of general trigonometric series: the extent of this class will be discussed later on.

The first investigations of trigonometric series other than purely periodic ones were carried out by Bohl. He considered the class of functions represented by series of the form

\[
\sum_{n=1}^{\infty} A_{n_1, n_2, \ldots, n_k} e^{i(a_1 + a_2 x + \ldots + a_k x)} \tag{2}
\]

where \( a_1, a_2, \ldots, a_k \) are arbitrary real numbers, and \( A_{n_1, n_2, \ldots, n_k} \) real or complex numbers. The necessary and sufficient conditions that a function is so representable are that it possesses certain quasi-periodic properties which are at first glance very similar to almost periodicity; but Bohl's restriction on the exponents of the trigonometric series places his problem in the class of those whose solution follows in a more or less natural way from existing theories rather than of those giving rise to an entirely new theory.

A quite new way of studying trigonometric series is opened up by Bohr's theory of almost periodic functions. We indicated
above the class of trigonometric series which correspond to almost periodic functions: we have not yet indicated how wide the class is. It is not possible to give any direct test for a series to be the Fourier series of an almost periodic function, nor can the similar problem be solved for the class of purely periodic continuous functions. But when the property of almost periodicity is properly generalised, then the corresponding class of Fourier series acquires a rather definite character of completeness.

The original work of Bohr was confined to the almost periodicity defined above. Thereafter work was done in the way of generalisation of the property by Stepanoff, Wiener, Weyl, Bohr and others. The new types of almost periodic functions were represented by new classes of trigonometric series

\[ \sum A_n e^{i\lambda_n z} \]

As before, to a series of one of the new classes still corresponds a convergent series of polynomials

\[ \sum \beta_n e^{i\lambda_n z} \]

but to each new type of almost periodicity corresponds a different kind of convergence of this sequence—not uniform convergence, although the convergence always has some features of uniformity. In fact there exists a strict reciprocity between the kind of almost periodicity of a function and the kind of convergence of the corresponding sequence of polynomials.

When these generalisations are taken into consideration some answer can be made to the above question of the extent of the class of trigonometric series which are Fourier series of almost periodic functions. This answer is given by the Riesz-Fischer Theorem:

**Any trigonometric series** \( \sum A_n e^{i\lambda_n z} \), **subject to the single condition** that the series \( \sum |A_n|^2 \) is convergent, **is the Fourier series of an almost periodic function.**

This is completely analogous to the Riesz-Fischer Theorem for the case of purely periodic functions. Generalisations of this result similar to those for purely periodic functions are possible.

Thus the Fourier series of all almost periodic functions form as large a subset of the class of all trigonometric series of the general type as the Fourier series of purely periodic functions do of the class of all trigonometric series of the ordinary type.

There is no doubt whatever that a trigonometric series of the general type

\[ \sum A_n e^{i\lambda_n z} \]

(with no restriction on the coefficients), in general does not represent a function (is not "summable") in any natural way, and it may be that almost periodicity is the decisive test for a non-artificial summability.

Almost periodicity is generalised in a natural way to the class of analytic functions in a strip \( a < Rez < b \) by the condition that the approximate equation

\[ f(z + \tau i) = f(z), \quad (\tau \text{ real}) \]

must be satisfied in the whole strip. One of the main features of the theory of analytic almost periodic functions is the existence of the "Dirichlet series" \( \sum A_n e^{\lambda_n z} \), which corresponds to the Fourier series of almost periodic functions of a real variable. The consequence is the same as in the case of almost periodic functions of a real variable. We get a possibility of enlarging the class of exponential series accessible to investigation. While in the case of ordinary Dirichlet series \( \sum A_n e^{\lambda_n z} \) the exponents are subject to the condition of forming a monotone sequence, there is no restriction of this kind on Dirichlet series of almost periodic functions. In fact any set of real numbers may form Dirichlet exponents of an analytic almost periodic function.

The connection between analytic functions and their Dirichlet series is even deeper than between almost periodic functions of a real variable and their Fourier series. The behaviour of almost periodic functions and the character of their singularities at infinity are defined by the nature of their Dirichlet series.

The study of harmonic and doubly periodic functions has also brought interesting and important results.

Applications of almost periodic functions have been made to linear differential and difference equations, and undoubtedly further development of the theory will lead to wider applications.